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Remarks on Fermi liquid from holography

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We investigate the signatures of Fermi liquid formation in the $\mathcal{N} = 4$ super Yang-Mills theory coupled to fundamental hypermultiplet at nonvanishing chemical potential for the global $U(1)$ vector symmetry. At strong 't Hooft coupling the system can be analyzed in terms of the D7-brane dynamics in the $AdS_5 \times S^5$ background. The phases with vanishing and finite charge density are separated at zero temperature by a quantum phase transition. In the case of vanishing hypermultiplet mass, Karch, Son, and Starinets discovered a gapless excitation whose speed equals the speed of sound. We find that this zero sound mode persists to all values of the hypermultiplet mass, and its speed vanishes at the point of phase transition. The value of critical exponent and the ratio of the velocities of zero and first sounds are consistent with the predictions of Landau Fermi liquid theory at strong coupling.

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I. INTRODUCTION AND SUMMARY

Gauge/string duality is a spin-off of string theory which allows access to the dynamics of strongly coupled field theories by relating them to string theories in certain backgrounds. In principle, this relation provides a unique way of understanding phenomena at strong coupling which cannot be analyzed by other means. One of the fundamentally interesting questions (with many practical applications) concerns the behavior of the Fermi systems at strong coupling. Here we attempt to analyze the possibility of the formation of Fermi liquid in a very specific system—N_f $\mathcal{N} = 2$ fundamental hypermultiplets coupled to $SU(N_c)$ $\mathcal{N} = 4$ super Yang-Mills theory. As usual, gauge/string duality methods are applicable in the regime of $N \to \infty$, N_f finite, and large four-dimensional 't Hooft coupling λ. Recent work on this system in the strong coupling regime includes [1–13].

The hypermultiplet field content includes two Weyl fermions and two complex scalars. The global symmetry of the theory is $U(N_f)$, but we will be mostly concerned with its $U(1)_B$ subgroup[1] and our results are not going to depend on the value of N_f. We will be interested in the situation where nonvanishing chemical potential for $U(1)_B$ is turned on. In the case of free fermions, such a setup would result in the formation of a degenerate gas of fermions, whose Fermi energy is equal to the value of the chemical potential μ. As the interaction is turned on, possible scenarios include formation of the Fermi liquid, superfluid, superconductor, or, possibly, some other exotic phase.

Fermi liquid is a state whose properties can be described in terms of the dynamics of quasiparticles of Fermi statistics taking place in the narrow region around Fermi surface. The momentum and energy of a quasiparticle in the degenerate Fermi liquid at $T = 0$ are bounded from above by the Fermi momentum k_F and Fermi energy. Nonrelativistic Fermi liquids are characterized by a number of properties. Among them are heat capacity proportional to T and the existence of zero sound. In [14] the D3-D7 system with vanishing hypermultiplet mass m has been investigated. It has been noted that the heat capacity of the system is proportional to T^6 (unlike the normal Fermi liquid). However, on the basis of the existence of a gapless excitation (arising as a massless pole in the density-density correlator) [14] argued that a novel kind of quantum liquid might be formed. This massless excitation is associated with zero sound in [14], and we discuss some issues related to this identification in this paper.

We generalize the work of [14] to the finite value of m and find that the gapless excitation persists and the dispersion relation has an interesting dependence on m. In the relativistic regime, $\mu \gg m$, we reproduce the results of [14] where the speed of zero sound u_0 was found to be equal to the speed of regular sound. On the other hand, in the nonrelativistic regime, $\mu - m \ll m$, we find that u_0 is proportional to $(\mu - m)^{1/2}$, and differs from the value of the first sound, u_1 by an overall multiplicative constant. Such a behavior is consistent with the prediction of Landau Fermi liquid theory. In particular, the value of the critical exponent is precisely the expected $1/2$, while the ratio of u_0 and u_1 is expected to go to a constant in the limit of strong coupling.

The rest of the paper is organized as follows. In the next section we give a brief review of the relevant parts of Landau's theory of Fermi liquids. In Sec. III we review the thermodynamics of the D3-D7 system. At large value of the 't Hooft coupling the dynamics of fundamental matter are captured by the Dirac-Born-Infeld (DBI) action for D7-branes propagating in the $AdS_5 \times S^5$ background [15]. At low temperatures, increasing the value of chemical potential results in a phase transition between the phases

[1] The subscript here stands for "baryon symmetry."

with vanishing and nonvanishing charge density. In the brane setup this corresponds to the brane embedding intersecting the horizon becoming thermodynamically preferred. In the limit of small temperature, the thermodynamics can be studied analytically [16].

In Sec. IV we derive equations of motion for the fluctuating fields on the brane. We largely follow [15,17], with the new ingredient being the finite value of the hypermultiplet mass m. We then proceed to analyze the equations in the regime of small frequency and momentum, find the massless excitation, and compute its velocity as a function of the hypermultiplet mass m. We discuss our results in Sec. V. Some technical results appear in the appendixes.

II. LANDAU FERMI LIQUID THEORY

In this section we review the basics of Landau Fermi liquid theory and quote the results for the velocities of first and zero sound. We also show that the ratio of these velocities goes to a constant for generic strong interactions between the quasiparticles. Nonrelativistic Fermi liquid at zero temperature involves the dynamics of quasiparticles of Fermi statistics whose dispersion relation is not necessarily equal to that of free fermions. The momentum distribution is described by a step function $\theta(|\mathbf{k}| - k_F)$, where k_F is the value of the Fermi momentum, which is related to the particle density via

$$n \equiv \frac{N}{V} = \frac{k_F^3}{3\pi^3\hbar^3}. \tag{2.1}$$

The quasiparticle description is, in fact, only valid in the small vicinity of the Fermi surface. The change in the total energy of the system due to the change in the distribution function $\delta n(\mathbf{k})$ is

$$\delta E = \int \epsilon(k)\delta n(\mathbf{k})d\tau + \int f(\mathbf{k}, \mathbf{k}')\delta n(\mathbf{k})\delta n(\mathbf{k}')d\tau d\tau', \tag{2.2}$$

where $d\tau = Vd^3k/(2\pi\hbar)^3$ is the element of the phase space and $\epsilon(k)$ is the energy of a quasiparticle with momentum $\mathbf{k}$ (whose absolute value we denote by k). In Eq. (2.2) the second term describes interaction of the quasiparticles which takes place only in the narrow region near the Fermi surface. It is natural then to assume that in $f(\mathbf{k}, \mathbf{k}')$ the quasiparticle momenta lie on the Fermi surface. The function $f(\mathbf{k}, \mathbf{k}')$ therefore only depends on the relative angle ϑ.

It is convenient to introduce the Fermi velocity and the effective quasiparticle mass by

$$v_F = \frac{\partial \epsilon(k)}{\partial k}\bigg|_{k=k_F}, \qquad m^* = \frac{k_F}{v_F} \tag{2.3}$$

and write

$$f(\mathbf{k}, \mathbf{k}') = \frac{k_F m^*}{\pi^2\hbar^3}F(\vartheta), \tag{2.4}$$

where $F(\vartheta)$ is a dimensionless function which can be expanded in terms of Legendre polynomials,

$$F(\vartheta) = \sum_l (2l+1)F_l P_l(\cos\vartheta). \tag{2.5}$$

In the right-hand side of (2.4) the prefactor is equal to the density of states on the Fermi surface. According to [18], the quasiparticle mass is related to the bare mass via

$$\frac{m^*}{m} = 1 + \frac{F_1}{3}. \tag{2.6}$$

An interesting property of the Fermi liquid at zero temperature is its compressibility,

$$u^2 = \frac{\partial P}{\partial \rho} = \frac{N}{m}\frac{\partial \mu}{\partial N}, \tag{2.7}$$

where P is the pressure, ρ is the density, and N is the number of particles. The value of u also defines the speed of (normal) sound in the Fermi liquid. According to [18], it is given by

$$u^2 = \frac{k_F^2}{3mm^*}(1+F_0) = \frac{v_F^2}{3}(1+F_0)\left(1+\frac{F_1}{3}\right), \tag{2.8}$$

where in the second equality we used (2.3) and (2.6). Another interesting property of the Fermi liquid is the existence of a massless excitation in the limit of zero temperature. This excitation, called zero sound, corresponds to the deformation of the shape of the Fermi surface. The speed of zero sound u_0 is bounded from below by, and is generally proportional to, the Fermi velocity v_F. To compute the ratio $s \equiv u_0/v_F$ one is instructed to solve the following integral equation:

$$(s - \cos\vartheta)\nu(\vartheta, \varphi) = \cos\vartheta \int F(\vartheta')\nu(\vartheta', \varphi')\frac{d\Omega'}{2\pi}. \tag{2.9}$$

In Eq. (2.9) the integral is over the solid angle, and function $\nu(\vartheta, \varphi)$ parametrizes the displacement of the spherical Fermi surface.

In particular, when $F(\vartheta)$ contains only the zeroth F_0 and first F_1 harmonic [19], one has

$$\frac{s}{2}\log\left(\frac{s+1}{s-1}\right) - 1 = \frac{\frac{1}{3}F_1 + 1}{F_0 + \frac{1}{3}F_0F_1 + F_1 s^2}, \tag{2.10}$$

where s is the ratio between the speed of zero sound and the Fermi velocity.[2] In the limit of noninteracting quasiparticles, the speed of zero sound is equal to the Fermi velocity. In other words, $s = 1$ and $u_0^2/u^2 = 3$. [This result is independent of the particular form of $F(\vartheta)$.] Another

[2]Here zero sound velocity refers to the velocity of the mode for which $\nu(\vartheta, \varphi)$ is isotropic in the plane perpendicular to its momentum.

interesting case is the limit of strongly interacting quasi-particles with F_0 and F_1 considerably larger than unity. From (2.10) we infer $s^2 \simeq \frac{F_0F_1}{9}$ which combined with (2.8) leads to

$$\frac{u_0^2}{u^2} \simeq 1. \tag{2.11}$$

For generic and large F_l one can argue that the ratio between the speed of zero and first sound goes to a constant in the limit of strong coupling. This is because for a generic function $F(\vartheta) \sim F$ the right-hand side of (2.9) scales like F and therefore $s \sim F$. But according to (2.8) $u \sim F$ as well, which justifies our assertion.

This argument may fail for some nongeneric values of F_l. In particular, when only F_0 is nonvanishing and large, Eq. (2.10) implies $s \approx \sqrt{F_0/3}$. But in this case (2.8) again leads to $u_0/u \approx 1$.

III. D3-D7 SYSTEM

In this section we review the D3-D7 system at zero temperature with nonvanishing chemical potential. Consider N_c D3-branes extended along $x^0, \ldots, x^3$ directions and N_f D7-branes extended along $x^0, \ldots, x^7$ directions. Separating these branes in the x^8-x^9 plane gives a mass to the hypermultiplet whose degrees of freedom are massless excitations of the fundamental string stretched between the D3- and D7-branes. Taking the near horizon limit, $N_c \to \infty$, $g_s \to 0$ with λ fixed but large and $N_f \ll N_c$, we obtain the $AdS_5 \times S^5$ geometry with N_c units of Ramond-Ramond five-form flux and D7-branes propagating in this background. Studying the holographic dual of the D3-D7 system with nonvanishing chemical potential involves analyzing the DBI action for the D7-branes embedded in $AdS_5 \times S^5$ with the electric field flux on their world volume turned on. The value of the chemical potential is equal to the asymptotic value of the gauge field on the brane.

In the following we review the results of [16], where analytic results for the zero temperature limit of the D3-D7 system were first obtained.[3] The background metric can be written as

$$ds^2 = \left(\frac{\rho}{L}\right)^2(-dt^2 + dx^i dx_i) + \left(\frac{\rho}{L}\right)^{-2}(dr^2 + r^2 d\Omega_3^2 + dR^2 + R^2 d\phi^2), \tag{3.1}$$

where t is time, $i = 1, 2, 3$ denote the spatial directions along the D3-brane, and $d\Omega_3^2$ is the metric of a unit three-sphere within S^5. In Eq. (3.1) ρ is the radial coordinate transverse to the D3-branes which is further expressed as

$$\rho^2 = r^2 + R^2. \tag{3.2}$$

In this parametrization R and ϕ are the polar coordinates in the x^8-x^9 plane, and U(1) symmetry allows one to set $\phi =$ const. Hence, the brane embedding is specified by a single function $R(r)$. The induced metric is then

$$ds_{D7}^2 = \left(\frac{\rho}{L}\right)^2(-dt^2 + dx^i dx_i) + \left(\frac{\rho}{L}\right)^{-2}[(1 + R'(r)^2)dr^2 + r^2 d\Omega_3^2], \tag{3.3}$$

where $R'(r) = \frac{\partial R}{\partial r}$. The DBI action for this configuration is now relatively simple. Warp factors drop out and we have

$$S_{D7} = -\mathcal{N} \int dr r^3 \sqrt{1 + R'(r)^2 - A_0'(r)^2}, \tag{3.4}$$

where $\mathcal{N} = \frac{1}{(2\pi)^4}\frac{1}{2\lambda} N_c N_f$. Note that we divided by the volume of $R^{1,3}$; therefore (3.4) is actually an action density. Moreover, we fixed the gauge by choosing $A_r = 0$ and rescaled A_0 according to

$$A_0 \to 2\pi A_0. \tag{3.5}$$

Here and in the rest of the paper we set the string length to one. Given that (3.4) contains only first derivatives of the fields $R(r)$ and $A_0(r)$ there are two conserved charges:

$$-r^3 \frac{R'}{\sqrt{1 + R'^2 - (A_0')^2}} = -c, \qquad r^3 \frac{A_0'}{\sqrt{1 + R'^2 - (A_0')^2}} = d. \tag{3.6}$$

One can reexpress $R'(r)$, $A_0'(r)$ in terms of c and d:

$$R' = \frac{c}{\sqrt{r^6 + d^2 - c^2}}, \qquad A_0' = \frac{d}{\sqrt{r^6 + d^2 - c^2}}. \tag{3.7}$$

The bare quark mass m and chemical potential μ are related to the asymptotic values of $R(r \to \infty)$ and $A_0(r \to \infty)$ as

$$R(r \to \infty) = \tilde{m} = 2\pi m, \qquad A_0(r \to \infty) = \tilde{\mu} = 2\pi\mu. \tag{3.8}$$

Possible phases of this system are classified by the values of integration constants c and d. For $c = d = 0$, both $R(r)$ and $A_0(r)$ are constant. In this phase the condensate and the charge density vanish, and the values of R and A_0 correspond to the quark mass and the chemical potential, respectively. This is the zero temperature limit of the "Minkowski" embedding. Solutions of (3.7) for all other values of c and d satisfying $d^2 - c^2 > 0$ are the zero temperature limits of the finite temperature black hole

[3]Note that our notations are slightly different from those of [16].

embeddings. They are characterized by the boundary conditions

$$R(r=0)=0, \qquad A_0(r=0)=0 \tag{3.9}$$

and nonvanishing values for both the condensate and the charge density. Equation (3.7) in this case yields

$$\begin{aligned} R(r) &= \frac{1}{6}c(d^2-c^2)^{-1/3}\mathcal{B}\Big[\frac{r^6}{r^6+d^2-c^2};\frac{1}{6},\frac{1}{3}\Big],\\ A_0(r) &= \frac{1}{6}d(d^2-c^2)^{-1/3}\mathcal{B}\Big[\frac{r^6}{r^6+d^2-c^2};\frac{1}{6},\frac{1}{3}\Big],\end{aligned} \tag{3.10}$$

where $\mathcal{B}[\frac{r^6}{r^6+d^2-c^2};\frac{1}{6},\frac{1}{3}]$ denotes the incomplete beta function, while the values of c and d are related to the physical variables $m=\frac{\tilde{m}}{2\pi}$ and $\mu=\frac{\tilde{\mu}}{2\pi}$ through

$$c=\gamma\tilde{m}(\tilde{\mu}^2-\tilde{m}^2), \qquad d=\gamma\tilde{\mu}(\tilde{\mu}^2-\tilde{m}^2), \qquad \gamma=(\tfrac{1}{6}\mathcal{B}[\tfrac{1}{6},\tfrac{1}{3}])^{-3}. \tag{3.11}$$

Equation (3.11) can be inverted to yield

$$\begin{aligned} \tilde{m} &= c\gamma^{-1/3}(d^2-c^2)^{-1/3},\\ \tilde{\mu} &= d\gamma^{-1/3}(d^2-c^2)^{-1/3},\\ k_F^0 &\equiv \tilde{\mu}^2-\tilde{m}^2=\gamma^{-2/3}(d^2-c^2)^{1/3}.\end{aligned} \tag{3.12}$$

Combined with $d^2-c^2>0$, (3.12) shows that black hole embeddings are only realized when the chemical potential is greater than the bare quark mass, $\mu>m$.

The phase structure of the D3-D7 system at $T=0$ has been analyzed in [16] with the following results. When $\mu<m$ there is only one configuration available, the Minkowski embedding. In this phase, the system is characterized by zero condensate and charge density. For $\mu>m$ the configuration with brane falling to $R=r=0$, the black hole embedding, is thermodynamically preferred. The two phases are separated by a quantum phase transition at $\mu=m$. The black hole embeddings are distinguished by the nonvanishing values of both the condensate and the charge density.

$$\langle\mathcal{O}_m\rangle=2\pi c\mathcal{N}, \qquad \langle J_0\rangle=2\pi d\mathcal{N}, \tag{3.13}$$

where c and d are related to m and μ via (3.11). Note that the expectation values in Eq. (3.13), and, in particular, the charge density, vanish at the point of phase transition $\mu=m$. In the rest of the paper we will be concerned with the black hole embeddings.

IV. ZERO SOUND

We will be interested in finding the massless excitation in the D3-D7 system at strong coupling. This requires analyzing linearized equations of motion which follow from the DBI action for the D7-brane propagating in $AdS_5\times S^5$. In this section we only quote a few key results. For technical details the reader is encouraged to consult Appendix A.

The DBI action can be written as[4]

$$S_{D7}\sim\int d\Omega_3\int d^4x\int dr\sqrt{-\det\mathcal{G}}, \tag{4.1}$$

where $\mathcal{G}=G+\mathcal{F}$ is the sum of the induced metric and gauge field strength. Here we will only be interested in perturbations which are independent of the coordinates on S^3. We therefore expand the D7-brane embedding coordinate $R=R_0+\xi$, the gauge field $\mathcal{A}_0=A_0^{(0)}+A_0$, $\mathcal{A}_i=A_i$ and the gauge field strength $\mathcal{F}_{0r}=F_{0r}^{(0)}+F_{0r}$, $\mathcal{F}_{ij}=F_{ij}$, where $i,j=1,2,3$ and the superscript (0) denotes the background values of the corresponding fields (whose profile follows from the results of the previous section.) In addition, we set $\mathcal{A}_\theta=0$, $\forall\ \theta\in S^3$ and choose the $\mathcal{A}_r=0$ gauge. To expand the DBI action to quadratic order in fluctuations, one can make use of the formula

$$\sqrt{\det\mathcal{G}}=\sqrt{\det\mathcal{G}_0}(1+\tfrac{1}{8}(\mathrm{tr}\mathcal{G}_0^{-1}\delta\mathcal{G})^2 - \tfrac{1}{4}\mathrm{tr}(\mathcal{G}_0^{-1}\delta\mathcal{G}\mathcal{G}_0^{-1}\delta\mathcal{G})), \tag{4.2}$$

where we write $\mathcal{G}=\mathcal{G}_0+\delta\mathcal{G}$ and $\delta\mathcal{G}$ stands for the fluctuations. In (4.2) there is of course a linear term, but it vanishes by equations of motion. It is convenient to keep the fluctuations of the brane embedding ξ (but not derivatives of ξ) as a part of $\mathcal{G}_0$. Both the leading term and a term linear in fluctuations do not contain ξ, which means it does not appear at quadratic order in fluctuations.

The action for fluctuations is given by [see also Eq. (A1)]

$$\begin{aligned} S_{D7,fl} = -\frac{1}{2}\mathcal{N}\int d^4x\int dr g(r)\Big[&\sum_i F_{ir}^2 - f_1(r)F_{0r}^2 \\ &- L^4 f_2(r)\sum_i F_{0i}^2 + L^4 f_3(r)\sum_{i<j}F_{ij}^2 + f_5(r)(\partial_r\xi)^2 \\ &- \frac{L^4}{\rho_0^4}(\partial_0\xi)^2 + L^4 f_4(r)\sum_i(\partial_i\xi)^2 \\ &- 2L^4 f_6\sum_i F_{0i}(\partial_i\xi) - 2f_7(r)F_{0r}(\partial_r\xi)\Big],\end{aligned} \tag{4.3}$$

where the functions f_i, $i=1,\ldots,7$ can be found in (A2) and (A3). It is convenient to use the momentum space representation,

$$A_M(x^\mu,r)=\int\frac{d^4k}{(2\pi)^4}e^{ik_\mu x^\mu}\tilde{A}_M(k^\mu,r) \tag{4.4}$$

and

[4]The full action contains a Chern-Simons term. It can be shown that it does not contribute to quadratic order in the fluctuations for the field configurations considered below.

$$\xi(x^\mu, r) = \int \frac{d^4k}{(2\pi)^4} e^{ik_\mu x^\mu} \tilde{\xi}(k^\mu, r). \tag{4.5}$$

We can also choose the direction of wave propagation by setting $k_\mu = (-\omega, 0, 0, q)$. It is important to work with the gauge invariant combination,

$$E = \tilde{q}\tilde{A}_0 + \tilde{\omega}\tilde{A}_\parallel. \tag{4.6}$$

We can use Gauss's law (equation of motion for A_r),

$$\tilde{q}\tilde{A}'_\parallel + \tilde{\omega} f_1 \tilde{A}'_0 - \tilde{\omega} f_7 \tilde{\xi}' = 0, \tag{4.7}$$

to express A'_0 (or $A'_\parallel$) in terms of E' and $\tilde{\xi}'$. Here prime denotes differentiation with respect to r. In these equations and in the following we define $\tilde{q} = qL^2 = q\sqrt{2\lambda}$ and $\tilde{\omega} = \omega L^2 = \omega\sqrt{2\lambda}$. Equations of motion are in general quite complicated. In particular, the equations for the longitudinal components of the gauge field are coupled to the equations for the embedding fluctuation ξ. It turns out, however, that they can be solved analytically in the two overlapping regimes, much like in [14].

As explained in Appendix A, in the vicinity of the horizon equations for E and $\tilde{\xi}$ decouple:

$$\ddot{E} + \frac{2}{z}\dot{E} + \Omega^2 E = 0, \tag{4.8}$$

$$\ddot{\tilde{\xi}} + \frac{2}{z}\dot{\tilde{\xi}} + \Omega^2 \tilde{\xi} = 0, \tag{4.9}$$

where $\Omega^2 \equiv \tilde{\omega}^2 \frac{d^2 - c^2}{d^2}$ and the dot denotes differentiation with respect to $z = 1/r$. The solutions of (4.8) and (4.9) are

$$E(z) = A\frac{e^{+i\Omega z}}{z}, \qquad \tilde{\xi}(z) = B\frac{e^{+i\Omega z}}{z}. \tag{4.10}$$

The positive exponent is singled out by incoming boundary conditions at the horizon [14,20–23]. In the limit of small $\tilde{\omega}$, $\Omega z \ll 1$ (4.10) becomes

$$E(z) = i\Omega A + \frac{A}{z}, \qquad \tilde{\xi} = i\Omega B + \frac{B}{z}. \tag{4.11}$$

On the other hand, for sufficiently small $\tilde{\omega}$ and $\tilde{q}$ equations of motion simplify again:

$$\dot{Z} + \left(\frac{2}{z} + \frac{1}{hf_5}\frac{\tilde{q}^2 \dot{f}_1}{\tilde{q}^2 - \tilde{\omega}^2 f_1}\right)Z - \left(\frac{\dot{f}_7}{hf_5}\frac{\tilde{q}^2 - \tilde{\omega}^2}{\tilde{q}^2 - \tilde{\omega}^2 f_1}\right)Y = 0,$$
$$\dot{Y} + \left(\frac{2}{z} + \frac{\dot{f}_5}{hf_5}\frac{\tilde{q}^2 - \tilde{\omega}^2}{\tilde{q}^2 - \tilde{\omega}^2 f_1}\right)Y + \left(\frac{\dot{f}_7}{hf_5}\frac{\tilde{q}^2}{\tilde{q}^2 - \tilde{\omega}^2 f_1}\right)Z = 0. \tag{4.12}$$

where $Z = g(z)\dot{E}(z)$, $Y = \tilde{q}g(z)\dot{\tilde{\xi}}(z)$, and $h(z) = (1 + (d^2 - c^2)z^6)/(1 - c^2 z^6)$. The solution of (4.12) is given by

$$\dot{E} = -\frac{z}{(1 + (d^2 - c^2)z^6)^{3/2}}[C_1[(1 - c^2 z^6)\tilde{q}^2 - (1 + (d^2 - c^2)z^6)\tilde{\omega}^2] + C_2 cdz^6 \tilde{q}^2],$$
$$\dot{\tilde{\xi}} = \frac{qz}{(1 + (d^2 - c^2)z^6)^{3/2}}[C_1 cdz^6 - C_2(1 + d^2 z^6)], \tag{4.13}$$

where C_1 and C_2 are arbitrary integration constants. Equation (4.13) can also be obtained directly from Eq. (A6) by neglecting all nonderivative terms, performing trivial integration, and using Gauss's law to express $\tilde{A}'_0$ in terms of E' and $\tilde{\xi}'$. Equation (4.13) can in turn be integrated to yield

$$E = C_0 + \int_0^z \frac{-xdx}{(1 + (d^2 - c^2)x^6)^{3/2}}[C_1[(1 - c^2 x^6)\tilde{q}^2 - (1 + (d^2 - c^2)x^6)\tilde{\omega}^2] + C_2 cdx^6 \tilde{q}^2],$$
$$\tilde{\xi} = \tilde{C}_0 + \int_0^z \frac{qxdx}{(1 + (d^2 - c^2)x^6)^{3/2}} \times [C_1 cdx^6 - C_2(1 + d^2 x^6)]. \tag{4.14}$$

The integrals can be expressed in terms of hypergeometric functions. We do not need this representation, since only small and large z asymptotics are important for our purposes. In particular, near the boundary

$$E = C_0 + \mathcal{O}(z^2), \qquad \xi = \tilde{C}_0 + \mathcal{O}(z^2). \tag{4.15}$$

This implies that the spectrum of quasinormal modes can be obtained by requiring $C_0 = \tilde{C}_0 = 0$.

In the region $z \gg 1$ the leading asymptotic behavior of (4.14) is a constant whose value we can infer by performing integration from $z = 0$ to $z = \infty$. The subleading $1/z$ term can be extracted from the expression for the derivatives (4.13). The results are

$$E = C_0 + b_1 C_1 + b_2 C_2 + \frac{a_1 C_1}{z} + \frac{a_2 C_2}{z}, \tag{4.16}$$

where

$$a_1 = -\frac{c^2\tilde{q}^2 + (d^2 - c^2)\tilde{\omega}^2}{(d^2 - c^2)^{3/2}}, \qquad a_2 = \frac{cd\tilde{q}^2}{(d^2 - c^2)^{3/2}} \tag{4.17}$$

and

$$b_1 = \frac{\Gamma(\frac{7}{6})\Gamma(\frac{4}{3})}{(d^2 - c^2)^{4/3}\sqrt{\pi}}[(3c^2 - d^2)\tilde{q}^2 + 3(d^2 - c^2)\tilde{\omega}^2],$$
$$b_2 = -2cd\tilde{q}^2 \frac{\Gamma(\frac{7}{6})\Gamma(\frac{4}{3})}{(d^2 - c^2)^{4/3}\sqrt{\pi}}. \tag{4.18}$$

Likewise,

$$\tilde{\xi} = \tilde{C}_0 + \tilde{b}_1 C_1 + \tilde{b}_2 C_2 + \frac{\tilde{a}_1 C_1}{z} + \frac{\tilde{a}_2 C_2}{z}, \tag{4.19}$$

where

$$\tilde{a}_1 = -\frac{cd}{(d^2 - c^2)^{3/2}}, \qquad \tilde{a}_2 = \frac{d^2}{(d^2 - c^2)^{3/2}} \tag{4.20}$$

and

$$\tilde{b}_1 = \frac{2cd\Gamma(\frac{7}{6})\Gamma(\frac{4}{3})}{(d^2 - c^2)^{4/3}\sqrt{\pi}}, \qquad \tilde{b}_2 = -\frac{\Gamma(\frac{7}{6})\Gamma(\frac{4}{3})}{(d^2 - c^2)^{4/3}\sqrt{\pi}}(3d^2 - c^2). \tag{4.21}$$

We will focus on the quasinormal mode with the linear dispersion relation in the regime of small $\tilde{\omega}$ and $\tilde{q}$. In this case we can match the near horizon solutions (4.11) to (4.16) and (4.19). Requiring $C_0 = \tilde{C}_0 = 0$ and neglecting terms of higher order in $\tilde{\omega}$ and $\tilde{q}$, we obtain

$$b_1C_1 + b_2C_2 = 0, \qquad \tilde{b}_1\tilde{C}_1 + \tilde{b}_2\tilde{C}_2 = 0. \tag{4.22}$$

These equations can be simultaneously solved with non-vanishing C_1 and C_2 whenever $b_1\tilde{b}_2 - b_2\tilde{b}_1 = 0$. Using (4.18) and (4.21) we arrive at

$$\tilde{\omega}^2 = u_0^2\tilde{q}^2, \qquad u_0^2 = \frac{d^2 - c^2}{3d^2 - c^2}. \tag{4.23}$$

Using (3.11) we can write the expression for the speed of zero sound as

$$u_0^2 = \frac{\mu^2 - m^2}{3\mu^2 - m^2}. \tag{4.24}$$

This is the main result of this paper. In the relativistic limit $\mu \gg m$ we recover the result of [14], $u_0^2 = 1/3$. In the vicinity of the phase transition, where $\delta\mu = \mu - m \ll m$, the speed of zero sound vanishes as $u_0^2 \sim \delta\mu/m$. As discussed below, precisely such a behavior is expected from the Landau's theory of Fermi liquids at strong coupling.

V. DISCUSSION

Before discussing the significance of (4.24) let us make a comment regarding the identification of the gapless mode with zero sound. At first sight, our calculations imply that the dispersion relation is modified by a term of the type $-iq^2$. In particular, in the simpler case of vanishing mass considered in [14], the dispersion relation looks like

$$\tilde{\omega} = \pm\frac{\tilde{q}}{3} - i\frac{\tilde{q}^2}{d^{1/3}}. \tag{5.1}$$

Note that in [14] the $-iq^2$ behavior of the imaginary part of the pole was important for identification of the massless mode with zero sound. In the derivation of dispersion relation it has been assumed that $\tilde{\omega}z \ll 1$ and $d^{1/3}z \gg 1$ regions overlap. More precisely, the solutions (4.8), (4.9), and (4.13) are valid up to terms $\mathcal{O}(d^{-2}z^{-6})$ and $\mathcal{O}(w^2z^2)$, respectively. This seems to imply that (5.1) is only corrected at higher order in q.

A number of quantities in the Fermi liquid setup can be computed if the particle density is known as a function of $\delta\mu = \mu - m$. In principle, we can compute the charge density using

$$\rho = \mathcal{N}2\pi d, \qquad \mathcal{N} = \frac{N_f N_c}{(2\pi)^4 2\lambda}. \tag{5.2}$$

It would be equal to the quark density n, if the quarks were the only degrees of freedom. This would allow one to determine k_F as a function of μ. However, the situation is more complicated, as both the fermions and the scalars in the fundamental hypermultiplet are charged under $U(1)_B$. At vanishing coupling the system is unstable due to the condensation of bosons. However, the Lagrangian contains a term which is quartic in scalar fields. Hence, at large λ the condensation of bosons can be stabilized, and indeed the solution for the brane embedding does not exhibit instabilities. At large coupling one can use the holographic dictionary to compute the expectation value of the operator which contains the scalar bilinear. And, as noted in [16], the dependence of this condensate on $\delta\mu$ is consistent with the quartic potential for the scalars and a quadratic term of the form $(m^2 - \mu^2)qq^*$. In summary, using (5.2) to determine k_F would yield $k_F \sim (d/\lambda)^{1/3} \sim (m^2\delta\mu/\lambda)^{1/3}$, but this result is unreliable due to the contribution of the boson condensate to the charge density. However, the expression for ρ can still serve as an upper bound on n.

We can deduce the critical behavior of various quantities using the results reviewed in Sec. II. Writing (2.7) in the form

$$u^2 = \frac{n}{m}\frac{\partial\delta\mu}{\partial n} \tag{5.3}$$

and noting that n vanishes near the critical point faster than $\delta\mu$, we infer $u^2 \sim \delta\mu$ and, consecutively, $v_F \sim \delta\mu^{1/2}$. This implies a dispersion relation of the form $\epsilon(k) = k^2/2m^* + \mathcal{O}(k^3)$. Hence, in the vicinity of the critical point the density of quarks behaves like $n \sim k_F^3 \sim \delta\mu^{3/2}$. This, in turn, leads to

$$u^2 \approx \frac{2\delta\mu}{3m}. \tag{5.4}$$

As reviewed in Sec. II, the speed of zero sound is proportional to Fermi velocity as well, and must therefore vanish at the critical point as $u_0 \sim \delta\mu^{1/2}$. This is precisely what Eq. (4.24) implies in the limit $\delta\mu \ll m$:

$$u_0^2 \approx \frac{\delta\mu}{m}. \tag{5.5}$$

Hence, we observe that our string theoretic computation reproduces (in a rather nontrivial fashion) the critical exponent predicted by the phenomenological theory.

What about the coefficient in (5.5)? As reviewed in Sec. II, in the limit of noninteracting fermions $u_0^{(0)} = v_F$ or, in other words, $(u_0^{(0)})^2 = 2\delta\mu/m$. This implies $u_0^{(0)}/u^{(0)} = \sqrt{3}$. However, this ratio is expected to be modified when the coupling is strong. Indeed, our result $u_0/u = \sqrt{3/2}$ differs from the value at weak coupling by a factor of $\sqrt{2}$. This is not surprising since, as discussed in Sec. II, the ratio of the velocities of first and zero sound goes to a constant which is $\mathcal{O}(1)$ for generic strong interaction between quasiparticles.

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APPENDIX A: FLUCTUATIONS

The action is

$$\begin{aligned} S_{\mathrm{D7},fl} = -\frac{1}{2}\mathcal{N}\int d^4x \int dr g(r)\Big[\sum_i F_{ir}^2 - f_1(r)F_{0r}^2 \\ - L^4 f_2(r)\sum_i F_{0i}^2 + L^4 f_3(r)\sum_{i<j} F_{ij}^2 + f_5(r)(\partial_r\xi)^2 \\ - \frac{L^4}{\rho_0^4}(\partial_0\xi)^2 + L^4 f_4(r)\sum_i(\partial_i\xi)^2 \\ - 2L^4 f_6(r)\sum_i F_{0i}(\partial_i\xi) - 2f_7(r)F_{0r}(\partial_r\xi)\Big], \end{aligned} \tag{A1}$$

where the functions f_i with $i = 1, \cdots 7$ are defined in terms of the background fields as follows:

$$\begin{aligned} g(r) &= \frac{r^3}{\sqrt{1 + R'^2 - F_{0r}^{(0)2}}}, \qquad f_1(r) = \frac{1 + R'^2}{1 + R'^2 - F_{0r}^{(0)2}}, \\ f_2(r) &= \frac{1 + R'^2}{\rho_0^4}, \qquad f_3(r) = \frac{1 + R'^2 - F_{0r}^{(0)2}}{\rho_0^4}, \\ f_4(r) &= \frac{1 - F_{0r}^{(0)2}}{\rho_0^4}, \qquad f_5(r) = \frac{1 - F_{0r}^{(0)2}}{1 + R'^2 - F_{0r}^{(0)2}}, \\ f_6(r) &= -\frac{R_0' F_{0r}^{(0)}}{\rho_0^4}, \qquad f_7(r) = -\frac{R_0' F_{0r}^{(0)}}{1 + R'^2 - F_{0r}^{(0)2}}. \end{aligned} \tag{A2}$$

Using (3.7) we can rewrite these functions as

$$\begin{aligned} g(r) &= \sqrt{r^6 + d^2 - c^2}, \qquad f_1(r) = 1 + \frac{d^2}{r^6}, \\ f_2(r) &= \frac{r^6 + d^2}{r^6 + d^2 - c^2}\frac{1}{\rho_0^4}, \qquad f_3(r) = \frac{r^6}{r^6 + d^2 - c^2}\frac{1}{\rho_0^4}, \\ f_4(r) &= \frac{r^6 - c^2}{r^6 + d^2 - c^2}\frac{1}{\rho_0^4}, \qquad f_5(r) = 1 - \frac{c^2}{r^6}, \\ f_6(r) &= \frac{cd}{r^6 + d^2 - c^2}\frac{1}{\rho_0^4}, \qquad f_7(r) = \frac{cd}{r^6}, \end{aligned} \tag{A3}$$

with $\rho_0(r)$ given by (3.2). It is useful to note that not all of these functions are independent from one another. For instance

$$f_2 = f_3 f_1, \qquad f_6 = f_3 f_7, \qquad f_4 = f_3 f_5. \tag{A4}$$

These identities will be particularly useful in expressing the equations of motion in a compact manner. The field equations are more conveniently expressed in the momentum space representation

$$A_M(x^\mu, r) = \int \frac{d^4k}{(2\pi)^4} e^{ik_\mu x^\mu}\tilde{A}_M(k^\mu, r) \tag{A5}$$

and

$$\xi(x^\mu, r) = \int \frac{d^4k}{(2\pi)^4} e^{ik_\mu x^\mu}\tilde{\xi}(k^\mu, r). \tag{A6}$$

Further choosing $k_\mu = (-\omega, 0, 0, q)$ we find

$$\begin{aligned} \partial_r[g f_1(\partial_r\tilde{A}_0) - g f_7(\partial_r\tilde{\xi})] - \tilde{\omega}\,\tilde{q}\, g f_2\tilde{A}_\| - \tilde{q}^2 g f_2\tilde{A}_0 + \tilde{q}^2 g f_6\tilde{\xi} &= 0, \\ \partial_r[g(\partial_r\tilde{A}_\|)] + \tilde{\omega}\,\tilde{q}\, g f_2\tilde{A}_0 + \tilde{\omega}^2 g f_2\tilde{A}_\| - \tilde{\omega}\,\tilde{q}\, g f_6\tilde{\xi} &= 0, \\ \partial_r[g f_5(\partial_r\tilde{\xi}) + g f_7(\partial_r\tilde{A}_0)] + \tilde{\omega}^2\frac{g}{\rho_0^4}\tilde{\xi} - \tilde{q}^2 g f_4\tilde{\xi} - \tilde{q}\,\tilde{\omega}\, g f_6\tilde{A}_\| - \tilde{q}^2 g f_6\tilde{A}_0 &= 0, \\ \partial_r[g(\partial_r\tilde{A}_\perp)] + g[\tilde{\omega}^2 f_2 - \tilde{q}^2 f_3]\tilde{A}_\perp &= 0. \end{aligned} \tag{A7}$$

Here we have absorbed the 't Hooft coupling constant λ into the variables $\tilde{\omega}$, $\tilde{q}$ defined as $\tilde{q} = qL^2 = q\sqrt{2\lambda}$ and $\tilde{\omega} = \omega L^2 = \omega\sqrt{2\lambda}$. To fix the residual gauge invariance we impose Gauss law

$$\tilde{q}\tilde{A}'_{\parallel} + \tilde{\omega} f_1 \tilde{A}'_0 - \tilde{\omega} f_7 \tilde{\xi}' = 0. \tag{A8}$$

Choosing the gauge invariant combination

$$E = \tilde{q}\tilde{A}_0 + \tilde{\omega}\tilde{A}_{\parallel} \tag{A9}$$

and using (A8) to express the first derivative of $\tilde{A}_0$ in terms of E' and $\tilde{\xi}'$,

$$\tilde{A}'_0 = \frac{\tilde{q}}{\tilde{q}^2 - \tilde{\omega}^2 f_1}\left(E' - \frac{\tilde{\omega}^2}{\tilde{q}} f_7 \tilde{\xi}'\right) \tag{A10}$$

leads to

$$\begin{aligned} &E'' + \left(\frac{g'}{g} + \frac{f_1'}{f_1} + \frac{\tilde{\omega}^2 f_1'}{\tilde{q}^2 - \tilde{\omega}^2 f_1}\right)E' + f_3(\tilde{\omega}^2 f_1 - \tilde{q}^2)E - \frac{f_7}{f_1}\left[X'' + \left(\frac{g'}{g} + \frac{f_7'}{f_7} + \frac{\tilde{\omega}^2 f_1'}{\tilde{q}^2 - \tilde{\omega}^2 f_1}\right)X' + f_3(\tilde{\omega}^2 f_1 - \tilde{q}^2)X\right] = 0, \\ &\frac{f_7}{f_5}\frac{\tilde{q}^2}{\tilde{q}^2 - h\tilde{\omega}^2}\left[E'' + \left(\frac{g'}{g} + \frac{f_7'}{f_7} + \frac{\tilde{\omega}^2 f_1'}{\tilde{q}^2 - \tilde{\omega}^2 f_1}\right)E' + f_3(\tilde{\omega}^2 f_1 - \tilde{q}^2)\right]E + X'' \\ &\qquad + \left(\frac{g'}{g} + \frac{f_5'}{f_5} + \frac{\tilde{\omega}^2 f_1'}{\tilde{q}^2 - \tilde{\omega}^2 f_1} - \frac{\tilde{\omega}^2 h'}{\tilde{q}^2 - \tilde{\omega}^2 h}\right)X' + f_3(\tilde{\omega}^2 f_1 - \tilde{q}^2)X = 0, \end{aligned} \tag{A11}$$

where

$$h \equiv \frac{1}{f_4 \rho_0^4} = \frac{f_7^2 + f_1 f_5}{f_5} = \frac{r^6 + (d^2 - c^2)}{r^6 - c^2} \tag{A12}$$

and we defined $X = q\tilde{\xi}$. Performing a change of variable from r to $z = \frac{1}{r}$ (A11) is recast into

$$\begin{aligned} &\ddot{E} + \left(\frac{2}{z} + \frac{\dot{g}}{g} + \frac{\dot{f}_1}{f_1} + \frac{\tilde{\omega}^2 \dot{f}_1}{\tilde{q}^2 - \tilde{\omega}^2 f_1}\right)\dot{E} + \frac{f_3}{z^4}(\tilde{\omega}^2 f_1 - \tilde{q}^2)E - \frac{f_7}{f_1}\left[\ddot{X} + \left(\frac{2}{z} + \frac{\dot{g}}{g} + \frac{\dot{f}_7}{f_7} + \frac{\tilde{\omega}^2 \dot{f}_1}{\tilde{q}^2 - \tilde{\omega}^2 f_1}\right)\dot{X} + \frac{f_3}{z^4}(\tilde{\omega}^2 f_1 - \tilde{q}^2)X\right] = 0, \\ &\frac{f_7}{f_5}\frac{\tilde{q}^2}{\tilde{q}^2 - h\tilde{\omega}^2}\left[\ddot{E} + \left(\frac{2}{z} + \frac{\dot{g}}{g} + \frac{\dot{f}_7}{f_7} + \frac{\tilde{\omega}^2 \dot{f}_1}{\tilde{q}^2 - \tilde{\omega}^2 f_1}\right)\dot{E} + \frac{f_3}{z^4}(\tilde{\omega}^2 f_1 - \tilde{q}^2)E\right] + \ddot{X} \\ &\qquad + \left(\frac{2}{z} + \frac{\dot{g}}{g} + \frac{\dot{f}_5}{f_5} + \frac{\tilde{\omega}^2 \dot{f}_1}{\tilde{q}^2 - \tilde{\omega}^2 f_1} - \frac{\tilde{\omega}^2 \dot{h}}{\tilde{q}^2 - \tilde{\omega}^2 h}\right)\dot{X} + \frac{f_3}{z^4}(\tilde{\omega}^2 f_1 - \tilde{q}^2)X = 0, \end{aligned} \tag{A13}$$

where dots indicated differentiation with respect to the variable z. Observe that (A13) reduces to two independent equations for E and X near the boundary since the prefactors

$$\frac{f_7}{f_1} = \frac{cdz^6}{1 + d^2 z^6}, \qquad \frac{f_7}{f_5}\frac{\tilde{q}^2}{\tilde{q}^2 - h\tilde{\omega}^2} \simeq \frac{cdz^6}{1 - c^2 z^6}\frac{\tilde{q}^2}{\tilde{q}^2 - \tilde{\omega}^2} \tag{A14}$$

vanish when $z \to 0$. We are then left with

$$\begin{aligned} &\ddot{E}(z) - \frac{1}{z}\dot{E}(z) + (\tilde{\omega}^2 - \tilde{q}^2)E(z) = 0, \\ &\ddot{X}(z) - \frac{1}{z}\dot{X}(z) + (\tilde{\omega}^2 - \tilde{q}^2)X(z) = 0. \end{aligned} \tag{A15}$$

The general solution of this equation can be written in terms of Bessel functions of the first kind as

$$\begin{aligned} E(z) &= \mathcal{A}_1 Y_1[z\sqrt{\tilde{\omega}^2 - \tilde{q}^2}] + \mathcal{B}_1 J_1[z\sqrt{\tilde{\omega}^2 - \tilde{q}^2}], \\ X(z) &= \mathcal{A}_2 Y_1[z\sqrt{\tilde{\omega}^2 - \tilde{q}^2}] + \mathcal{B}_2 J_1[z\sqrt{\tilde{\omega}^2 - \tilde{q}^2}]. \end{aligned} \tag{A16}$$

Expanding around $z = 0$ yields

$$\begin{aligned} E(z) &\simeq \mathcal{A}_1\left[1 + \frac{1}{4}(\tilde{\omega}^2 - \tilde{q}^2) \times \left(1 - 2\tilde{\gamma} - \ln\left[\frac{\tilde{\omega}^2 - \tilde{q}^2}{4}z^2\right]\right)z^2\right] + \mathcal{B}_1 z^2, \\ X(z) &\simeq \mathcal{A}_2\left[1 + \frac{1}{4}(\tilde{\omega}^2 - \tilde{q}^2) \times \left(1 - 2\tilde{\gamma} - \ln\left[\frac{\tilde{\omega}^2 - \tilde{q}^2}{4}z^2\right]\right)z^2\right] + \mathcal{B}_2 z^2. \end{aligned} \tag{A17}$$

This identifies the constants $\mathcal{A}_i$ and $\mathcal{B}_i$ for $i = 1, 2$ with the coefficients of the non-normalizable and the normalizable mode, respectively.

In the vicinity of the horizon on the other hand notice that

$$\frac{\dot g}{g} \simeq -\frac{3}{z}\frac{1}{(d^2-c^2)z^6}, \qquad \frac{\dot f_1}{f_1} \simeq \frac{6}{z},$$
$$\frac{\tilde\omega^2 \dot f_1}{\tilde q^2 - \tilde\omega^2 f_1} \simeq -\frac{6}{z}, \qquad \frac{\dot f_5}{f_5} \simeq \frac{6}{z}, \qquad \frac{\dot f_7}{f_7} = \frac{6}{z}, \tag{A18}$$
$$\frac{\tilde\omega^2 \dot h}{\tilde q^2 - \tilde\omega^2 h} \simeq \frac{6}{z}\frac{d^2}{c^4 z^6}\frac{\tilde\omega^2}{\tilde q^2 + \frac{d^2-c^2}{c^2}\tilde\omega^2},$$

whereas

$$\frac{f_3}{z^4}(\tilde\omega^2 f_1 - \tilde q^2) \simeq \tilde\omega^2 \frac{d^2-c^2}{d^2} \tag{A19}$$

since $R_0(z)$ behaves for large z as $\frac{c}{\sqrt{d^2-c^2}}\frac{1}{z}$. It follows that Eq. (A13) reduces to

$$\ddot E + \frac{2}{z}\dot E + \Omega^2 E - \frac{c}{d}\Big[\ddot X + \frac{2}{z}\dot X + \Omega^2 X\Big] = 0,$$
$$\ddot X + \frac{2}{z}\dot X + \Omega^2 X + \frac{d}{c}\frac{\tilde q^2}{\tilde q^2 + \frac{d^2-c^2}{c^2}\tilde\omega^2}\Big[\ddot E + \frac{2}{z}\dot E + \Omega^2 E\Big] = 0, \tag{A20}$$

where we defined $\Omega = \tilde\omega\sqrt{\frac{d^2-c^2}{d^2}}$. Multiplying either of the equations with appropriate factors and adding them up yields the following system of decoupled equations near the "horizon":

$$\ddot E + \frac{2}{z}\dot E + \Omega^2 E = 0, \qquad \ddot X + \frac{2}{z}\dot X + \Omega^2 X = 0, \tag{A21}$$

with solutions

$$E(z) = A\frac{e^{i\Omega z}}{z}, \qquad X(z) = B\frac{e^{i\Omega z}}{z}. \tag{A22}$$

Note that the incoming wave boundary condition singled out the positive exponent in (A22).

Equation (A13) can be simplified even further.

$$\ddot E + \Big(\frac{2}{z} + \frac{\dot g}{g} + \frac{1}{hf_5}\frac{\tilde q^2 \dot f_1}{\tilde q^2 - \tilde\omega^2 f_1}\Big)\dot E + \frac{f_3}{z^4}(\tilde\omega^2 f_1 - \tilde q^2)E - \Big(\frac{\dot f_7}{hf_5}\frac{\tilde q^2 - \tilde\omega^2}{\tilde q^2 - \tilde\omega^2 f_1}\Big)\dot X = 0,$$
$$\ddot X + \Big(\frac{2}{z} + \frac{\dot g}{g} + \frac{\dot f_5}{hf_5}\frac{\tilde q^2 - \tilde\omega^2}{\tilde q^2 - \tilde\omega^2 f_1}\Big)\dot X + \frac{f_3}{z^4}(\tilde\omega^2 f_1 - \tilde q^2)X + \Big(\frac{\dot f_7}{hf_5}\frac{\tilde q^2}{\tilde q^2 - \tilde\omega^2 f_1}\Big)\dot E = 0. \tag{A23}$$

To arrive at (A23) multiply either of the equations with the relevant factor and add it to the other one. Note that the equations for E and X are now coupled only through first derivative terms of the fields. This form is particularly useful when considering the region $\Omega z \ll 1$. In this case, terms without derivatives of the fields E and X can be neglected to yield

$$\dot Z + \Big(\frac{2}{z} + \frac{1}{hf_5}\frac{\tilde q^2 \dot f_1}{\tilde q^2 - \tilde\omega^2 f_1}\Big)Z - \Big(\frac{\dot f_7}{hf_5}\frac{\tilde q^2 - \tilde\omega^2}{\tilde q^2 - \tilde\omega^2 f_1}\Big)Y = 0, \qquad \dot Y + \Big(\frac{2}{z} + \frac{\dot f_5}{hf_5}\frac{\tilde q^2 - \tilde\omega^2}{\tilde q^2 - \tilde\omega^2 f_1}\Big)Y + \Big(\frac{\dot f_7}{hf_5}\frac{\tilde q^2}{\tilde q^2 - \tilde\omega^2 f_1}\Big)Z = 0, \tag{A24}$$

where $Z = g(z)\dot E(z)$ and $Y = g(z)\dot X(z)$. Solutions for Z and Y are given by

$$Z = \frac{d^2 \hat C_1[\tilde q^2 - \tilde\omega^2(1 + (d^2-c^2)z^6)] + \tilde q^2 \hat C_2[-\tilde q^2(1 - c^2 z^6) + \tilde\omega^2(1 + (d^2-c^2)z^6)]}{cd\tilde q^2 z^2[1 + (d^2-c^2)z^6]},$$
$$Y = \frac{1}{z^2}\frac{\hat C_1}{1 + (d^2-c^2)z^6} + z^4\frac{\hat C_2}{1 + (d^2-c^2)z^6}, \tag{A25}$$

where the integration constants $\hat C_1$ and $\hat C_2$ are related to the ones appearing in the main text through

$$\hat C_1 = -C_2 \tilde q^2, \qquad \hat C_2 = -d(dC_2 - cC_1)\tilde q^2. \tag{A26}$$

Equation (A25) can be easily integrated to yield an expression for E and X in terms of hypergeometric functions.

APPENDIX B: THE MATCHING TECHNIQUE IN DETAIL

Here we investigate in detail the regions of applicability of the matching technique used in Sec. V. Let us focus on understanding the precise conditions under which Eq. (A23) reduces to Eq. (A21) or (A24).

Observe that in the vicinity of the horizon, when z is large enough so that $z \gg (d^2-c^2)^{-1/6} > d^{-1/3}$, the following simplifications occur:

$$\frac{\dot g}{g} \simeq -\frac{3}{z}\frac{1}{(d^2-c^2)z^6},$$
$$\frac{1}{hf_5}\frac{\tilde q^2 \dot f_1}{\tilde q^2-\tilde\omega^2 f_1} \simeq \frac{6}{z}\frac{d^2}{d^2-c^2}\frac{1}{1-\frac{\tilde\omega^2}{\tilde q^2}d^2z^6},$$
$$\frac{\dot f_5}{hf_5}\frac{\tilde q^2-\tilde\omega^2}{\tilde q^2-\tilde\omega^2 f_1} \simeq -\frac{6}{z}\frac{c^2}{d^2-c^2}\frac{1-\frac{\tilde\omega^2}{\tilde q^2}}{1-\frac{\tilde\omega^2}{\tilde q^2}d^2z^6},$$
$$\frac{\dot f_7}{hf_5}\frac{\tilde q^2-\tilde\omega^2}{\tilde q^2-\tilde\omega^2 f_1} \simeq \frac{6}{z}\frac{cd}{d^2-c^2}\frac{1-\frac{\tilde\omega^2}{\tilde q^2}}{1-\frac{\tilde\omega^2}{\tilde q^2}d^2z^6},$$
$$\frac{\dot f_7}{hf_5}\frac{\tilde q^2}{\tilde q^2-\tilde\omega^2 f_1} \simeq \frac{6}{z}\frac{cd}{d^2-c^2}\frac{1}{1-\frac{\tilde\omega^2}{\tilde q^2}d^2z^6}. \tag{B1}$$

Moreover, given that $R(z)$ for large z behaves like $R(z) \simeq \frac{c}{\sqrt{d^2-c^2}}\frac{1}{z}$,

$$\frac{f_3}{z^4}(\tilde\omega^2 f_1-\tilde q^2) \simeq \frac{\tilde q^2(d^2-c^2)}{d^2}\frac{1}{d^2z^6}\left(\frac{\tilde\omega^2}{\tilde q^2}d^2z^6-1\right), \tag{B2}$$

and if we additionally assume that $z \gg d^{-1/3}(\frac{\tilde q}{\tilde\omega})^{1/3}$, Eq. (A23) reduces to (A21).

$$\ddot E+\frac{2}{z}\dot E+\Omega^2 E=0, \qquad \ddot X+\frac{2}{z}\dot X+\Omega^2 X=0, \tag{B3}$$

with Ω defined to be $\Omega^2 \equiv \tilde\omega^2\frac{d^2-c^2}{d^2}$. The solution of (B3) is given by

$$E(z)\sim\frac{e^{i\Omega z}}{z}, \qquad X(z)\sim\frac{e^{i\Omega z}}{z}. \tag{B4}$$

When furthermore $z \ll \frac{1}{\Omega}$, the solution becomes

$$E(z)=Ai\Omega+\frac{A}{z}, \qquad E(z)=Bi\Omega+\frac{B}{z}, \tag{B5}$$

with A, B overall multiplicative constants. In summary, (B5) consistently describes the behavior of the solution of Eq. (A23) as long as z lies within the annulus $\mathrm{Max}[(d^2-c^2)^{-1/6}, d^{-1/3}(\frac{\tilde q}{\tilde\omega})^{1/3}] \ll z \ll \frac{1}{\Omega}$. This in turn implies that we are effectively exploring the region of parameter space in which

$$\mathrm{Max}\left[(d^2-c^2)^{-1/6}, d^{-1/3}\left(\frac{\tilde q}{\tilde\omega}\right)^{1/3}\right] \ll \frac{1}{\Omega}. \tag{B6}$$

Now let us investigate the behavior of (A23) in the regime $\Omega z \ll 1$ first. Note that terms without derivatives always appear with the prefactor $\frac{f_3}{z^4}(\tilde\omega^2 f_1-\tilde q^2)$ as compared to the two derivative terms. Moreover, this prefactor is given by the difference of two monotonic functions. Its absolute value is therefore bounded above by the absolute value of their sum. The latter is a monotonic function which tends to a constant for both large and small z. In particular,

$$h(z) \equiv \left|\frac{f_3}{z^4}(\tilde\omega^2 f_1+\tilde q^2)\right| = \begin{cases}\Omega^2 & z \gg \mathrm{Max}[(d^2-c^2)^{-1/6}, (\frac{\tilde q}{\tilde\omega})^{1/3}d^{-1/3}]\\ |\tilde\omega^2+\tilde q^2| & z \ll \mathrm{Min}[\frac{1}{\tilde m}, (d^2-c^2)^{-1/6}, (\frac{\tilde q}{\tilde\omega})^{1/3}d^{-1/3}].\end{cases} \tag{B7}$$

It follows that terms without derivatives can be neglected as long as they are small compared to the two derivative terms. This means that for $z \ll \mathrm{Min}[\frac{1}{\Omega}, \frac{1}{\tilde\omega}|1+\frac{\tilde q^2}{\tilde\omega^2}|^{-1/2}]$ Eq. (A23) consistently reduces to

$$\dot Z+\left(\frac{2}{z}+\frac{1}{hf_5}\frac{\tilde q^2\dot f_1}{\tilde q^2-\tilde\omega^2 f_1}\right)Z-\left(\frac{\dot f_7}{hf_5}\frac{\tilde q^2-\tilde\omega^2}{\tilde q^2-\tilde\omega^2 f_1}\right)Y=0,$$
$$\dot Y+\left(\frac{2}{z}+\frac{\dot f_5}{hf_5}\frac{\tilde q^2-\tilde\omega^2}{\tilde q^2-\tilde\omega^2 f_1}\right)Y+\left(\frac{\dot f_7}{hf_5}\frac{\tilde q^2}{\tilde q^2-\tilde\omega^2 f_1}\right)Z=0, \tag{B8}$$

with $Z=g(z)\dot E(z)$ and $Y=g(z)\dot X(z)$.

As previously explained, we would like (A24) and (B3) to be simultaneously valid. This implies that we can investigate (A13) in the following region of parameter space

$$\mathrm{Max}\left[(d^2-c^2)^{-1/6}, d^{-1/3}\left(\frac{\tilde q}{\tilde\omega}\right)^{1/3}\right] \ll \mathrm{Min}\left[\frac{1}{\Omega}, \frac{1}{\tilde\omega}\left|1+\frac{\tilde q^2}{\tilde\omega^2}\right|^{-1/2}\right]. \tag{B9}$$

It is manifest from (B9) that the analysis of Sec. V covers the region of small $\tilde\omega$, $\tilde q$.

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