

Low-lying quasinormal modes of topological AdS black holes and hydrodynamics

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We analytically calculate the low-lying gravitational quasinormal modes of a topological AdS black hole of arbitrary dimension in the high temperature limit. We show that they are in agreement with corresponding results from the hydrodynamics of the gauge theory plasma on the boundary, as required by the AdS/CFT (anti-de Sitter/conformal field theories) correspondence. For some of these modes, we obtain a lifetime which is comparable to or longer than the longest lifetime of perturbations of spherical black holes. These modes might play a significant role in the late-time behavior of the gauge theory plasma.

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I. INTRODUCTION

The anti-de Sitter/conformal field theories (AdS/CFT) correspondence [1,2] has provided a path for understanding a gauge theory in terms of a dual gravitational description in one higher dimension. The gauge field in four dimensions is an $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory, which is not QCD as there is no confinement among other aspects. However, once the limit of confinement is breached, $\mathcal{N} = 4$ SYM results seem to be relevant to QCD. Moreover, the plasma seen at the Relativistic Heavy Ion Collider (RHIC) is thought to be strongly interacting [3] and progress with QCD is difficult to achieve, but AdS/CFT does not rely on the same techniques and is therefore not hampered by the strength of the coupling. This has cultivated interest in the string theory–RHIC scenarios and progress is rapidly developing [4–7].

Via the AdS/CFT correspondence, information for the plasma in the strong coupling regime is gained by studying the dual theory of a supergravity solution in AdS space. From the AdS metric one may calculate the stress-energy tensor of the CFT using various techniques [8]. The simplest AdS solution to study is the Schwarzschild metric, which has a dual static CFT on the boundary. This may be extended by looking at small deformations of the Schwarzschild metric, i.e., quasinormal modes which dictate the late-time behavior of the black hole [9]. Calculating these modes has been held in high importance and thereby studied in vast detail (see [10] and references therein). According to the AdS/CFT correspondence, the lowest frequency modes govern the hydrodynamic behavior of the conformal field theory on the boundary [11]. However, these modes are difficult to find and may be missed by some quasinormal mode techniques [12].

In [9,13] the lowest-lying gravitational quasinormal modes for an AdS Schwarzschild solution were numeri-

cally calculated in four and five dimensions and were shown to be in agreement with hydrodynamic perturbations of the gauge theory plasma on the AdS boundary. For AdS₅ this was understood as a finite “conformal soliton flow” after the spherical AdS₅ boundary one obtains in global coordinates was conformally mapped to the physically relevant flat Minkowski spacetime. The perturbations also allowed for calculations of the elliptic flow of the plasma and its thermalization time—two of the observables at RHIC. While there is still work to be done, the calculations compared well with what has been found experimentally.

An alternative to a spherical AdS black hole would be to choose one with a hyperbolic horizon [14–18]. They are usually referred to as topological AdS black holes because they possess topologically nontrivial horizons. Our aim is to elucidate their effect on the gauge theory plasma on the AdS boundary. By studying gravitational perturbations, we shall show that they possess quasinormal modes whose lifetime is comparable to or longer than their counterparts in the case of horizons with positive curvature (spherical black holes). These results are in agreement with those obtained by studying the hydrodynamics of the gauge theory plasma on the boundary. Therefore, topological AdS black holes might have a significant effect on the behavior of the quark-gluon plasma in heavy ion collisions at RHIC and the LHC via the AdS/CFT correspondence.

In Sec. II we discuss the scalar, vector, and tensor gravitational perturbations of a topological AdS black hole in d dimensions. We calculate analytically the lowest-lying quasinormal modes using the procedure of Ref. [12]. In Sec. III we study the hydrodynamics of a gauge theory plasma on a hyperbolic space in $d - 1$ dimensions extending the results of Ref. [13]. We show that the frequencies obtained from hydrodynamics are in agreement with their counterparts obtained from black hole perturbations in Sec. II. We summarize our conclusions in Sec. IV.

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II. TOPOLOGICAL ADS BLACK HOLES

The Einstein equations for vacuum anti-de Sitter space allows for three separate maximally symmetric solutions parameterized with a single parameter K taking the values $0, \pm 1$. For $K = 0$ we have a flat horizon whereas for $K = +1$ the horizon is a compact sphere. The case $K = -1$ yields a horizon which is a hyperbolic space and has been much less studied. Nevertheless, in the context of the AdS/CFT correspondence all solutions to the Einstein equations should be taken into account. Here we concentrate on the case of black holes with a hyperbolic horizon ($K = -1$) aiming at elucidating their effect on the gauge theory plasma on the AdS boundary.

The metric of an AdS black hole with $K = -1$ in d spacetime dimensions takes the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_{d-2}^2, \quad (1)$$

$$f(r) = r^2 - 1 - \frac{2\mu}{r^{d-3}},$$

where we have chosen units in which the AdS radius is $R = 1$. The horizon radius is found from

$$2\mu = r_H^{d-1} \left(1 - \frac{1}{r_H^2} \right). \quad (2)$$

The Hawking temperature is

$$T_H = \frac{(d-1)r_H^2 - (d-3)}{4\pi r_H}. \quad (3)$$

The area of the horizon is rendered finite by introducing identifications in the hyperbolic space which make the horizon topologically nontrivial. Thus $\Sigma_{d-2} = \mathbb{H}^{d-2}/\Gamma$ where Γ is a discrete group of isometries of the hyperbolic space \mathbb{H}^{d-2} . Various choices of Γ were eagerly studied in the late nineties in preparation for the Wilkinson Microwave Anisotropy Probe as it was thought to possibly describe the type of universe we live in [19]. For example, in $d = 4$ the boundary may be compactified with periodic boundaries around an octagon specified by Γ ; in higher dimensions the fundamental domain becomes a generalization of the octagon.

The mass and entropy of the hole are given, respectively, by [17]

$$M = (d-2)(r_H^2 - 1) \frac{r_H^{d-3}}{16\pi G} V_{d-2}, \quad S = \frac{r_H^{d-2}}{4G} V_{d-2}, \quad (4)$$

where V_{d-2} is the volume of the hyperbolic space Σ_{d-2} .

Through the AdS/CFT correspondence, the black hole is mapped onto a gauge theory fluid on the boundary of AdS which in this case is the open Einstein static universe, $\Sigma_{d-2} \times \mathbb{R}$. On the other hand, the quark-gluon plasma created in heavy ion collisions lives in Minkowski space ($\mathbb{R}^{d-2,1}$ for $d = 5$). To understand experimental results, we need to choose a different foliation near the boundary of AdS consisting of hypersurfaces which become asymptotically flat. This is achieved through an appropriate coordinate transformation in the bulk which amounts to a conformal transformation between part of the open Einstein static universe, $\Sigma_{d-2} \times \mathbb{R}$, and the Minkowski space, $\mathbb{R}^{d-2,1}$. This transforms the static plasma on $\Sigma_{d-2} \times \mathbb{R}$ to a flow of finite extend (soliton) on $\mathbb{R}^{d-2,1}$. In the case of a spherical horizon, this flow was dubbed ‘‘conformal soliton flow’’ [9]. Understanding the behavior of plasma in our case would entail numerical techniques due to the complexity of $\Sigma_{d-2} \times \mathbb{R}$. We shall leave such a detailed calculation for future work.

Here we concentrate on perturbations of the conformal soliton whose characteristic frequencies and lifetimes are determined by the quasinormal modes of the black hole (real and imaginary parts, respectively).

For the study of perturbations, we need to understand the behavior of harmonic functions on Σ_{d-2} . In general, they obey

$$(\nabla^2 + k^2)\mathbb{T} = 0. \quad (5)$$

Without identifications (i.e., in \mathbb{H}^{d-2}), the spectrum is continuous. We obtain [20]

$$k^2 = \xi^2 + \left(\frac{d-3}{2} \right)^2 + \delta, \quad (6)$$

where ξ is arbitrary and $\delta = 0, 1, 2$ for scalar, vector, and tensor perturbations, respectively. When a compactification scheme is chosen, the spectrum becomes discrete. Depending on the choice of Γ , the discretized eigenvalues ξ may be made as small as desired, i.e., zero is an accumulation point of the spectrum of ξ [19]. As $\xi \rightarrow 0$, the complexity of the set of isometries Γ increases and the volume V_{d-2} of the hyperbolic space Σ_{d-2} diverges (hence also the mass and entropy of the hole). This ought to be studied numerically for a detailed comparison with experimental data in heavy ion collisions at the RHIC through a generalization of the approach of [9].

Having understood the harmonics on Σ_{d-2} , we may write the wave equation for gravitational perturbations in the general Schrödinger-like form [21]

$$-\frac{d^2\phi}{dr_*^2} + V[r(r_*)]\phi = \omega^2\phi \quad (7)$$

in terms of the tortoise coordinate r_* defined by

$$\frac{dr_*}{dr} = \frac{1}{f(r)} \quad (8)$$

where $f(r)$ is defined in (1). The potential takes different forms for different types of perturbation. We shall study each case separately.

A. Vector perturbations

The vector potential is given by

$$V_V = \frac{f(r)}{r^2} \left(k_V^2 - 1 + \frac{(d-2)(d-4)}{4} (r^2 - 1) - \frac{3(d-2)^2 \mu}{r^{d-3}} \right), \quad (9)$$

where k_V^2 is an eigenvalue of a vector harmonic [Eq. (6) with $\delta = 1$].

It is convenient to introduce the variable

$$u = \left(\frac{r_H}{r} \right)^{d-3}. \quad (10)$$

The wave Eq. (7) takes the form

$$-(d-3)^2 u^{(d-4)/(d-3)} \hat{f}(u) \partial_u (u^{(d-4)/(d-3)} \hat{f}(u) \partial_u \phi) + \hat{V}_V(u) \phi = \hat{\omega}^2 \phi, \quad (11)$$

where

$$\begin{aligned} \mathcal{A} &= -(d-3)^2 u^{(2d-8)/(d-3)} (1 - u^{(d-1)/(d-3)}) + \frac{2(d-3)^2}{r_H^2} u^2 (1-u), \\ \mathcal{B} &= -(d-3) [d-4 - (2d-5)u^{(d-1)/(d-3)}] u^{(d-5)/(d-3)} - 2(d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{(2d-8)/(d-3)} (1 - u^{(d-1)/(d-3)})}{1-u} \\ &\quad + \frac{d-3}{r_H^2} u \left[(d-3)(2-3u) - (d-1) \frac{1-u}{1 - u^{(d-1)/(d-3)}} u^{(d-1)/(d-3)} \right], \\ \mathcal{C} &= \hat{k}_V^2 + \frac{(d-2)[d-4 - 3(d-2)u^{(d-1)/(d-3)}]}{4} u^{-2/(d-3)} - (d-3) \frac{i\hat{\omega}}{d-1} \frac{[d-4 - (2d-5)u^{(d-1)/(d-3)}] u^{(d-5)/(d-3)}}{1-u} \\ &\quad - (d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{(2d-8)/(d-3)} (1 - u^{(d-1)/(d-3)})}{(1-u)^2} - \frac{d-2}{2r_H^2} \left[d-4 - (2d-5)u - (d-1) \right. \\ &\quad \left. \times \frac{1-u}{1 - u^{(d-1)/(d-3)}} u^{(d-1)/(d-3)} \right]. \end{aligned} \quad (15)$$

Expanding the wave function,

$$F = F_0 + F_1 + \dots \quad (16)$$

we may solve the wave equation (14) perturbatively.

The zeroth order wave equation

$$\mathcal{H}_0 F_0 = 0 \quad (17)$$

$$\begin{aligned} \hat{V}_V(u) &= \hat{f}(u) \left[\hat{k}_V^2 + \frac{(d-2)(d-4)}{4} u^{2/(3-d)} - \frac{3(d-2)^2}{4} u \right. \\ &\quad \left. - \frac{1}{r_H^2} \left(1 + \frac{(d-2)(d-4)}{4} - \frac{3(d-2)^2}{4} u \right) \right] \\ \hat{f}(u) &= \frac{f(r)}{r^2} = 1 - u^{2/(d-3)} \left(u + \frac{1-u}{r_H^2} \right), \end{aligned}$$

$$\hat{\omega}^2 = \frac{\omega^2}{r_H^2}, \quad \hat{k}_V^2 = \frac{k_V^2}{r_H^2}. \quad (12)$$

With $\hat{\omega}$ and \hat{k}_V fixed, to leading order in $1/r_H$ this is the same equation as the case of a flat horizon studied in [11] and also coincides with the leading order equation in the case of the spherical horizon studied in [12]. The curvature of the horizon only comes into play at $\mathcal{O}(1/r_H^2)$, as expected, since the horizon becomes flat in the limit $r_H \rightarrow \infty$. Following the perturbative analysis performed in [12], we shall solve the wave equation in the $r_H \rightarrow \infty$ limit and add the $\mathcal{O}(1/r_H^2)$ contributions as perturbative corrections [treating $\hat{\omega}, \hat{k}_V \sim \mathcal{O}(1/r_H^2)$].

Factoring out the behavior of ϕ as it approaches the horizon ($u = 1$),

$$\phi(u) = (1-u)^{-i(\hat{\omega}/(d-1))} F(u) \quad (13)$$

so that the wave equation in the large r_H limit [including $\mathcal{O}(1/r_H^2)$ contributions] becomes

$$\mathcal{H}F \equiv \mathcal{A}F'' + \mathcal{B}F' + \mathcal{C}F = 0, \quad (14)$$

where

is obtained in the limit $\hat{\omega}, \hat{k}_V, 1/r_H^2 \rightarrow 0$. Explicitly,

$$\mathcal{H}_0 F_0 = \mathcal{A}_0 F_0'' + \mathcal{B}_0 F_0' + \mathcal{C}_0 F_0, \quad (18)$$

where

$$\begin{aligned}\mathcal{A}_0 &= -(d-3)^2 u^{(2d-8)/(d-3)} (1 - u^{(d-1)/(d-3)}), \\ \mathcal{B}_0 &= -(d-3)[d-4 - (2d-5)u^{(d-1)/(d-3)}] u^{(d-5)/(d-3)}, \\ \mathcal{C}_0 &= \frac{(d-2)[d-4 - 3(d-2)u^{(d-1)/(d-3)}]}{4} u^{-2/(d-3)}.\end{aligned}\quad (19)$$

The zeroth order wave equation (17) has the two exact solutions

$$\begin{aligned}F_0 &= u^{(d-2)/(2(d-3))}, \\ \check{F}_0 &= u^{-(d-4)/(2(d-3))} {}_2F_1\left(1, -\frac{d-3}{d-1}, \frac{2}{d-1}; u^{(d-1)/(d-3)}\right).\end{aligned}\quad (20)$$

The former is well behaved at both the horizon ($u \rightarrow 1$) and the boundary ($u \rightarrow 0$) but the latter diverges at both ends; therefore, it is unacceptable.

The constraint for $\hat{\omega}$ comes from the first order equation which accounts for the $\mathcal{O}(1/r_H^2)$ terms in (14)

$$\mathcal{H}_0 F_1 + \mathcal{H}_1 F_0 = 0 \quad (21)$$

solved by

$$F_1 = -F_0 \int \frac{\check{F}_0 \mathcal{H}_1 F_0}{\mathcal{A}_0 \mathcal{W}_0} + \check{F}_0 \int \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}_0 \mathcal{W}_0}, \quad (22)$$

where \mathcal{W}_0 is the zeroth order Wronskian

$$\mathcal{W}_0 = \frac{1}{u^{(d-4)/(d-3)} (1 - u^{(d-1)/(d-3)})}. \quad (23)$$

The second term in the expression for F_1 is ill-behaved at both the boundary and the horizon. If we choose one of the limits of integration at the boundary ($u = 0$), then the second term becomes regular there. However, at the horizon it diverges due to the behavior of \check{F}_0 . This is avoided if the coefficient of \check{F}_0 vanishes as $u \rightarrow 1$. This requirement yields the constraint

$$\int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}_0 \mathcal{W}_0} = 0, \quad (24)$$

which is a linear equation in $\hat{\omega}$ whose solution is

$$\hat{\omega} = -i \frac{\hat{k}_V^2 + \frac{d-3}{r_H^2}}{d-1}. \quad (25)$$

This is the frequency of the lowest-lying vector quasinor-

mal mode. It can be written as

$$\omega = -i \frac{\xi^2 + (\frac{d-1}{2})^2}{(d-1)r_H}. \quad (26)$$

This mode is inversely proportional to the radius of the horizon and will dictate the hydrodynamics of the dual gauge theory. We obtain an upper bound for the lifetime of this mode which may be written in terms of the temperature $T_H \approx \frac{d-1}{4\pi} r_H$ [Eq. (3) in the large r_H limit and in units in which the AdS radius is $R = 1$],

$$\tau = \frac{1}{|\omega|} < \frac{16\pi}{(d-1)^2} T_H. \quad (27)$$

In the physically interesting case of $d = 5$, this reads $\tau < \pi T_H$. To compare this with the case of a spherical horizon, note that the frequency is given by [12]

$$\omega^{S^{d-2}} = -i \frac{(l+d-2)(l-1)}{(d-1)r_H}, \quad (28)$$

which yields a maximum lifetime

$$\tau_{\max}^{S^{d-2}} = \frac{4\pi}{d} T_H \quad (29)$$

and in the case $d = 5$, we obtain an upper bound of $\frac{4\pi}{5} T_H$ which is lower than the upper bound in the hyperbolic case (πT_H).

To assess the relevance of this result to heavy ion collisions, one ought to relate the lifetime (27) to the thermalization time of the plasma which lives on flat Minkowski space $\mathbb{R}^{d-2,1}$. For spherical horizons, the latter is roughly half the former [9]. In our case, the relationship will be determined once the conformal map from part of the open Einstein static universe $\Sigma_{d-2} \times \mathbb{R}$ onto $\mathbb{R}^{d-2,1}$ is found. As discussed earlier, this will require the use of numerical techniques and is deferred to future work. Nevertheless, the large value of the lifetime (27) indicates that these modes may play a role in determining the behavior of the quark-gluon plasma.

B. Scalar perturbations

We now turn our attention to scalar perturbations for which the master equation can be cast into the same form as (11) but with a new potential,

$$\begin{aligned}\hat{V}_S(u) &= \frac{u^{-2/(d-3)} - u - \frac{1}{r_H^2}(1-u)}{4(\hat{m} + u)^2} \left\{ (-6+d)(-4+d)\hat{m}^2 - 6(-4+d)(-2+d)\hat{m}u + (-2+d)du^2 - 3(-6+d) \right. \\ &\quad \times (-2+d)\hat{m}^2 u^{(d-1)/(d-3)} + 2(18+d(-11+2d))\hat{m}u^{(2(d-2))/(d-3)} + (-2+d)^2 u^{(3d-7)/(d-3)} + 2(-2+d) \\ &\quad \times (-1+d)\hat{m}^3 u^{2/(-3+d)} - \frac{u^{2/(d-3)}}{r_H^2} [(-2+d)\hat{m}^2(d+2(-1+d)\hat{m}) - 3(-2+d)\hat{m}(-8-6\hat{m}+d(2+\hat{m}))u \\ &\quad \left. + (24+36\hat{m}+d(-10+d-22\hat{m}+4d\hat{m}))u^2 + (-2+d)^2 u^3 \right\},\end{aligned}\quad (30)$$

where

$$\hat{m} = 2 \frac{k_S^2 + d - 2}{(d-1)(d-2)(r_H^2 - 1)} \quad (31)$$

and k_S^2 is an eigenvalue of a scalar harmonic [Eq. (6) with $\delta = 0$].

A new singularity at $u = -\hat{m}$ arises in the scalar potential. It is best to factor out the behavior at this point in addition to the behavior at the horizon and boundary. We see again that the effect of the curvature enters at $\mathcal{O}(1/r_H^2)$ and the wave equation matches the spherical case [12] at leading order first in $1/r_H$.

Defining

$$\phi(u) = (1-u)^{-i(\hat{\omega}/(d-1))} \frac{u^{(d-4)/(2(d-3))}}{\hat{m} + u} F(u) \quad (32)$$

as in the vector case we obtain a wave equation for F which may be solved perturbatively. In the vector case, we had $\hat{\omega}$, $\hat{k}_V^2 \sim \mathcal{O}(1/r_H^2)$, so keeping terms to $\mathcal{O}(1/r_H^2)$ we could drop terms which were quadratic in $\hat{\omega}$. In the scalar case, the frequency has a real part which is related to the speed of sound in the gauge theory fluid [11]. In a conformal fluid, the speed of sound is $\frac{1}{\sqrt{d-2}}$. Therefore, in the limit $r_H \rightarrow \infty$ we expect $\omega \sim \mathcal{O}(1)$, consequently terms which are quadratic in $\hat{\omega} = \frac{\omega}{r_H}$ must be kept and will contribute at first order in $1/r_H^2$.

The zeroth order wave equation ought to coincide with the case of a spherical horizon, because the curvature plays no role at leading order. Following [12], we choose

$$\mathcal{H}_0 F_0 = \mathcal{A}_0 F_0'' + \mathcal{B}_0 F_0' + \mathcal{C}_0 F_0 = 0, \quad (33)$$

where

$$\begin{aligned} \mathcal{A}_0 &= -(d-3)^2 u^{(2d-8)/(d-3)} (1-u)^{(d-1)/(d-3)}, \\ \mathcal{B}_0 &= -(d-3) u^{(2d-8)/(d-3)} (1-u)^{(d-1)/(d-3)} \\ &\quad \times \left[\frac{d-4}{u} - \frac{2(d-3)}{\hat{m} + u} \right] - (d-3) \\ &\quad \times [d-4 - (2d-5)u^{(d-1)/(d-3)}] u^{(d-5)/(d-3)}, \\ \mathcal{C}_0 &= 0, \end{aligned} \quad (34)$$

This zeroth order wave equation has two linearly independent solutions,

$$F_0 = 1, \quad (35)$$

which is well behaved at all points and a singular one which can be written in terms of the Wronskian,

$$\check{F}_0 = \int \mathcal{W}_0, \quad \mathcal{W}_0 = \frac{(\hat{m} + u)^2}{u^{(2d-8)/(d-3)} (1-u)^{(d-1)/(d-3)}}. \quad (36)$$

Care must be exercised in the case $d = 4$ where \check{F}_0 does not lead to a singularity at the boundary; however, the

boundary conditions ought to be altered to Robin boundary conditions [12,13].

Proceeding as with vector perturbations, a constraint similar to (24) is found by including terms up to $\mathcal{O}(1/r_H^2)$ which also account for the contributions of $\hat{m} \sim \mathcal{O}(1/r_H^2)$ and $\hat{\omega} \sim \mathcal{O}(1/r_H)$. After some tedious algebra, we arrive at a quadratic equation for $\hat{\omega}$,

$$\begin{aligned} \frac{d-1}{2} \frac{1 + (d-2)\hat{m}}{(1+\hat{m})^2} - \frac{1}{r_H^2} \left(\frac{1}{\hat{m}} + \mathcal{O}(1) \right) \\ - i\hat{\omega} \frac{d-3}{(1+\hat{m})^2} - \hat{\omega}^2 \left(\frac{1}{\hat{m}} + \mathcal{O}(1) \right) = 0. \end{aligned} \quad (37)$$

The two solutions for small \hat{m} are

$$\hat{\omega} = \pm \sqrt{\frac{d-1}{2} \hat{m} - \frac{1}{r_H^2}} - i \frac{d-3}{2} \hat{m}, \quad (38)$$

which may also be written as

$$\omega_0 = \pm \frac{k_S}{\sqrt{d-2}} - i \frac{d-3}{(d-1)(d-2)r_H} [k_S^2 + d - 2]. \quad (39)$$

The real part gives the correct speed of sound ($\frac{1}{\sqrt{d-2}}$) whereas the imaginary part yields the lifetime

$$\tau = \frac{1}{|\Im \omega|} = \frac{4\pi(d-2)}{(d-3)(\xi^2 + (\frac{d-1}{2})^2)} T_H. \quad (40)$$

This is bounded by

$$\tau < \frac{16\pi(d-2)}{(d-3)(d-1)^2} T_H \quad (41)$$

to be compared with the maximum lifetime of a scalar mode in the spherical horizon case [12]

$$\tau_{\max}^{S^{d-2}} = \frac{4(d-2)\pi}{(d-3)d} T_H. \quad (42)$$

In the physically interesting case $d = 5$, the bound for a hyperbolic horizon is $\frac{3\pi}{2} T_H$ which is higher than the maximum lifetime for a spherical horizon, $\frac{6\pi}{5} T_H$, as well as the upper bounds of vector modes.

These modes may be important in understanding the quark-gluon plasma, but again, as with vector modes, their significance will be determined after one computes their relationship to the thermalization time of the quark-gluon plasma. These modes correspond to perturbations of a static fluid on the open Einstein universe $\Sigma_{d-2} \times \mathbb{R}$ and need to be transformed to the Minkowski space ($\mathbb{R}^{d-2,1}$,

where $d = 5$ for comparison with experimental data) in which the quark-gluon plasma lives.

C. Tensor perturbations

The remaining quasinormal modes come from tensor perturbations. Following [21], the wave equation may be cast in the same form as (11) with the potential in the large r_H limit,

$$\hat{V}_T(u) = \frac{d-2}{4} \left(du^{-2/(d-3)} - (d-2)u^{(2(d-2))/(d-3)} - 2u \right) + \hat{k}_T^2 (1 - u^{(d-1)/(d-3)}), \quad (43)$$

where $\hat{k}_T = k_T/r_H$ and k_T is the tensor harmonic eigenvalue given by Eq. (6) with $\delta = 2$.

The zeroth order wave equation can be solved as in the spherical case [12] to find the two independent solutions

$$\begin{aligned} \phi_0 &= u^{-(d-2)/(2(d-3))}, \\ \hat{\phi}_0 &= u^{-(d-2)/(2(d-3))} \ln(1 - u^{(d-1)/(d-3)}). \end{aligned} \quad (44)$$

Both can be seen to diverge at the boundary ($u \rightarrow 0$) and the horizon ($u \rightarrow 1$). Therefore, there are no low frequency tensor modes. The lowest modes are expected to have frequencies $\omega \sim \mathcal{O}(r_H)$ and cannot be found using the same perturbative technique as with vector and scalar modes. We are not interested in finding the tensor modes in this case because they do not contribute to the hydrodynamic behavior of the gauge theory plasma.

III. HYDRODYNAMICS

In the previous section we calculated the lowest-lying quasinormal modes whose imaginary part was inversely proportional to the radius of the horizon (and therefore their lifetime was proportional to the Hawking temperature of the black hole). Based on the analysis in [10], the overtones do not exhibit this behavior; their frequencies are all proportional to the radius of the horizon for large black holes. This leads to the interpretation of the lowest-lying modes corresponding to the hydrodynamics on the dual gauge theory plasma [11], and the subsequent overtones to its microscopic behavior. In this section, we study the hydrodynamics in the linearized regime of a $d-1$ dimensional fluid with dissipative effects taken into account. The fluid lives on the boundary with topology $\mathbb{R} \times \Sigma_{d-2}$ where $\Sigma_{d-2} = \mathbb{H}^{d-2}/\Gamma$, i.e., the quotient of the hyperbolic space \mathbb{H}^{d-2} with the discrete group of isometries Γ . We thus extend earlier results for a spherical boundary [13].

Using μ, ν running over the boundary with metric

$$ds_{\text{boundary}}^2 = -dt^2 + d\Sigma_{d-2}^2 \quad (45)$$

and i, j over only the hyperbolic space Σ_{d-2} , the hydrodynamic equations for the conformal fluid follow in a standard manner,

$$\begin{aligned} T^{\mu\nu} &= (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu} \\ &\quad - \eta \left(\Delta^{\mu\lambda} \nabla_\lambda u^\nu + \Delta^{\nu\lambda} \nabla_\lambda u^\mu - \frac{2}{d-2} \Delta^{\mu\nu} \nabla_\lambda u^\lambda \right) \\ &\quad - \zeta \Delta^{\mu\nu} \nabla_\lambda u^\lambda, \\ \nabla_\mu T^{\mu\nu} &= 0, \quad T^\mu_\mu = 0, \end{aligned} \quad (46)$$

where $\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ and ϵ, p, η , and ζ represent the energy density, pressure, shear viscosity, and bulk viscosity, respectively, of the conformal field theory. Two constraints on the parameters immediately follow,

$$\epsilon = (d-2)p, \quad \zeta = 0, \quad (47)$$

u^μ is the velocity field of the conformal fluid. The reference frame is chosen so that $u^\mu u_\mu = -1$. In the rest frame of the fluid, $u^\mu = (1, 0, 0, 0)$. Perturbations introduce small disturbances,

$$u^\mu = (1, u^i), \quad (48)$$

where u^i is small and also allow for small corrections to the pressure so that

$$p = p_0 + \delta p. \quad (49)$$

Applying (46), we obtain the set of hydrodynamic equations

$$\begin{aligned} 0 &= \nabla_\mu T^{\mu t} = (d-2)\partial_t \delta p + (d-1)p_0 \nabla_i u^i, \\ 0 &= \nabla_\mu T^{\mu i} \\ &= (d-1)p_0 \partial_t u^i + \partial^i \delta p \\ &\quad - \eta \left[\nabla^j \nabla_j u^i - (d-3)u^i + \frac{d-4}{d-2} \partial^i (\nabla_j u^j) \right], \end{aligned} \quad (50)$$

where we used $R_{ij} = -(d-3)g_{ij}$.

Looking first at vector perturbations of the fluid, the appropriate ansatz is [13]

$$\delta p = 0, \quad u^i = \mathcal{A}_V e^{-i\Omega t} \mathbb{V}^i, \quad (51)$$

where \mathbb{V}^i is a vector harmonic.

The first hydrodynamic equation is trivially satisfied and the second becomes

$$-i\Omega(d-1)p_0 + \eta[k_V^2 + d-3] = 0. \quad (52)$$

This can be solved for the frequency Ω characterizing the deviation from a perfect fluid. The solution may be written in terms of the parameters of the dual black hole. Using Eqs. (4) and (47), we obtain

$$\frac{\eta}{p_0} = \frac{4\pi\eta}{s} \frac{r_H}{r_H^2 - 1}, \quad (53)$$

where s is the entropy density. With $\frac{\eta}{s} = \frac{1}{4\pi}$ [11] and for large r_H we arrive at the expression for the frequency of vector perturbations

$$\Omega = -i \frac{k_V^2 + d-3}{(d-1)r_H}. \quad (54)$$

This is in agreement with the frequency of vector modes of the black hole, Eq. (26), on account of the definition (6).

Turning now to scalar hydrodynamic perturbations, we should allow for deviations in pressure as well as the velocity field. The appropriate ansatz is [13]

$$u^i = \mathcal{A}_S e^{-i\Omega t} \partial^i \mathbb{S}, \quad \delta p = \mathcal{B}_S e^{-i\Omega t} \mathbb{S}, \quad (55)$$

where \mathbb{S} is a scalar harmonic. The hydrodynamic equations become

$$(d-2)i\Omega \mathcal{B}_S + (d-1)p_0 k_S^2 \mathcal{A}_S = 0, \\ \mathcal{B}_S + \mathcal{A}_S \left[-i\Omega(d-1)p_0 + 2(d-3)\eta + 2\eta k_S^2 \frac{d-3}{d-2} \right] = 0. \quad (56)$$

This is a linear system of homogeneous equations. To be compatible, their determinant must vanish,

$$\det \begin{pmatrix} (d-2)i\Omega & (d-1)p_0 k_S^2 \\ 1 & -i\Omega(d-1)p_0 + 2(d-3)\eta + 2\eta k_S^2 \frac{d-3}{d-2} \end{pmatrix} = 0, \quad (57)$$

which imposes a constraint on the frequency Ω . Working along the same lines as for the vector perturbation, we arrive at the expression for Ω ,

$$\Omega = \pm \frac{k_S}{\sqrt{d-2}} - i \frac{d-3}{(d-1)(d-2)r_H} [k_S^2 + d-2], \quad (58)$$

which is in exact agreement with the quasinormal frequency of scalar gravitational perturbations (39).

Finally, an ansatz cannot be built to describe tensor perturbations with the associated harmonics because of the tracelessness and zero divergence of tensor spherical harmonics. This is consistent with the negative conclusion reached in Sec. II on tensor modes of gravitational perturbations of the black hole.

IV. CONCLUSION

We analytically calculated the low-lying quasinormal modes of topological AdS black holes in arbitrary dimension in the high temperature limit. These are black holes with hyperbolic horizons of nontrivial topology. We considered all three different types of perturbations (scalar, vector, and tensor) and solved the wave equation [21] in each case by applying the method of Ref. [12]. We obtained quasinormal frequencies which were in agreement with the frequencies obtained by considering perturbations of the gauge theory fluid on the boundary, thus extending results obtained in the case of black holes with spherical horizons [13].

In the physically interesting case of five dimensions, we showed that the lifetimes of some of these modes exceed the longest lifetime of the modes of a black hole with spherical horizon [9,12]. Therefore, they play an important role in the late-time behavior of the gauge theory fluid and may contribute to the properties of the quark-gluon plasma produced in heavy ion collisions. Further work is required for a detailed comparison with experimental data which will determine how a topological AdS black hole scenario may be applicable to the RHIC and the LHC. Following the analysis of [9], one needs to map the hyperbolic boundary of the topological black holes onto flat Minkowski space via a conformal map and study the resulting flow of the gauge theory fluid. Unlike in the case of a spherical boundary, this procedure cannot be carried out analytically for topological AdS black holes owing to the complexity of the (topologically nontrivial) boundary [19]. Instead, one needs to resort to numerical techniques. Work in this direction is in progress.

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