Deriving Boltzmann equations from Kadanoff-Baym equations in curved space-time

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In this paper we consider the transition from the Kadanoff-Baym equations to the quantum kinetic equations and then to the Boltzmann equation in curved space-time. As one expects from general considerations, the derived equations appear to be covariant generalizations of the corresponding equations in Minkowski space-time. The formalism will be applied to test approximations commonly made in the computation of the baryon asymmetry in the leptogenesis scenario.

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I. INTRODUCTION

As has been shown by Sakharov [1], the observed baryon asymmetry of the Universe can be generated dynamically, provided that the following three conditions are fulfilled: violation of baryon (or baryon minus lepton) number; violation of C and CP; and deviation from thermal equilibrium.

The third Sakharov condition raises the question of how to describe a quantum system out of thermal equilibrium. The usual choice is the Boltzmann equation [2–5]. However, it is known to have several shortcomings. In particular classical Boltzmann equations neglect off-shell effects, introduce irreversibility, and feature spurious constants of motion. A quantum mechanical generalization of the Boltzmann equation, free of the mentioned problems, has been developed by Kadanoff and Baym [6]. Direct numerical computations demonstrate that already for simple systems far from thermal equilibrium the Kadanoff-Baym and Boltzmann equations do lead to quantitatively, and in some cases even qualitatively, different results [7–12]. Studying processes responsible for the generation of the asymmetry in the framework of the Kadanoff-Baym formalism is therefore of considerable scientific interest.

The application of the Kadanoff-Baym equations to the computation of the lepton and baryon asymmetries in the so-called leptogenesis scenario [13] has been studied at different levels of approximation by several authors [14,15] and lead to qualitatively new and interesting results. However, issues related to the rapid expansion of the Universe, which drives the required deviation from thermal equilibrium, have not been addressed there.

The modification of the Kadanoff-Baym formalism in curved space-time has been considered in [16–19], where it was applied to a model with quartic self-interactions and a $O(N)$ model, though the dynamics of quantum field theoretical models with CP violation remained uninvestigated.

Our goal is to develop a consistent description of leptogenesis in the Kadanoff-Baym and Boltzmann approaches and to test approximations commonly made in the computation of the lepton and baryon asymmetries. In particular, we want to find out how the dense background plasma affects the collision terms of processes contributing to the generation and washout of the asymmetry, check the applicability of the real intermediate state subtraction procedure in the case of resonant leptogenesis [20,21], and investigate the time dependence of the CP-violating parameter in the expanding Universe [15]. Mostly, the phenomenological study of leptogenesis relies on the Boltzmann approach. Testing the underlying approximations and procedures is therefore of substantial practical importance.

Since this is a rather ambitious goal, we first study a simple toy model of leptogenesis containing two real and one complex scalar fields, which mimic the heavy right-handed Majorana neutrinos and leptons respectively [22]. The starting point of our analysis is the generating functional for the (connected) Green's functions. Performing a Legendre transformation we obtain the effective action, which we use to derive the Schwinger-Dyson equations. The latter ones are equivalent to a system of Kadanoff-Baym equations for the spectral function and the statistical propagator. Employing a first-order gradient expansion and a Wigner transformation leads to a system of quantum kinetic equations. Finally, neglecting the Poisson brackets and making use of the quasiparticle approximation, we obtain the Boltzmann equations. Our derivation is manifestly covariant in every step.

The peculiarities of the calculation, related to the presence of a gravitational field, are determined only by transformation properties of the quantum fields—scalar fields in this case. For this reason, in the present paper, we use a model of a single real scalar field with quartic self-interactions, minimally coupled to gravity, to illustrate the main points. That is, we use the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\varphi\partial_{\mu}\varphi - \frac{1}{2}M^2\varphi^2 - \frac{\lambda}{4!}\varphi^4, \quad (1)$$

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which does also have the advantage, that one can compare the derived equations with their Minkowski space-time counterparts [9] and with the results obtained in [16–19]. The formalism presented here will be used to analyze the toy model of leptogenesis [22].

The outline of the paper is as follows. In Sec. II we consider the two point irreducible (2PI) effective action in curved space-time and derive the system of Schwinger-Dyson equations. The explicit form of the 2PI effective action, corresponding to the two-, three-, and four-loop contributions, is given in Sec. III. In Sec. IV we derive the system of the Kadanoff-Baym equations from the Schwinger-Dyson equations. Introducing center and relative coordinates and performing a Taylor expansion we derive the quantum kinetic equations in Sec. V. Some further approximations lead to the system of Boltzmann equations, which are considered in Sec. VI. We summarize our results and draw the conclusions in Sec. VII.

II. SCHWINGER-DYSON EQUATIONS

In the derivation of the Schwinger-Dyson equations we employ results from [18,23,24]. Our starting point is the generating functional for Green's functions with local and bilocal external scalar sources $J(x)$ and $K(x, y)$,

$$\mathcal{Z}[J, K] = \int \mathcal{D}\varphi \exp\left[i\left(S + J\varphi + \frac{1}{2}\varphi K\varphi\right)\right], \tag{2}$$

where the action S is given by the integral of the Lagrange density over space. The Minkowski space-time volume element d^4x is replaced in curved space-time by the invariant volume element $\sqrt{-g}d^4x$, where $\sqrt{-g}$ is the square root of the determinant of the metric:

$$S = \int \sqrt{-g}d^4x\mathcal{L}.$$

In the Friedmann-Robertson-Walker (FRW) universe we have $\sqrt{-g} = a^4(\eta)$, where a is the cosmic scale factor and η denotes conformal time. The invariant volume element enters also in the scalar products of the sources and the scalar field

$$J\varphi \equiv \int \sqrt{-g}d^4xJ(x)\varphi(x), \tag{3a}$$

$$\varphi K\varphi \equiv \iint \sqrt{-g}d^4x\sqrt{-g}d^4y\varphi(x)K(x, y)\varphi(y). \tag{3b}$$

The functional integral measure is modified in curved space-time as well. For scalar densities of zero weight it reads

$$\mathcal{D}\varphi = \prod_x d[(-g)^{1/4}\varphi(x)].$$

The evolution of the quantum system out of thermal equilibrium is performed in the Schwinger-Keldysh formalism [25,26]. In this approach the field and the external sources are defined on the positive and negative branches

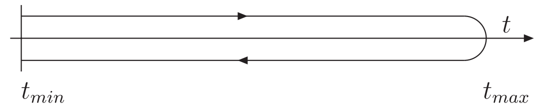

FIG. 1. Closed real-time path $\mathcal{C}$.

of a closed real-time contour, see Fig. 1, the functions[1] on the positive branch being independent[2] of the functions on the negative branch. This applies also to the metric tensor, i.e. $g^+_{\mu\nu} \neq g^-_{\mu\nu}$ in general.

In realistic models of leptogenesis the contribution of the heavy right-handed neutrinos to the energy density of the Universe is less than 5% and can safely be neglected. In other words, leptogenesis takes place in a space-time with a metric, whose time development is (in this approximation) independent of the decays of the right-handed neutrinos and determined by the contributions of the ultrarelativistic standard model species. Correspondingly, in our analysis of the toy model of leptogenesis, we will also neglect the impact of the scalar fields on the expansion of the Universe.[3] This implies, in particular, that the metric tensor on the positive and negative branches is determined only by the external processes, and one can set $g^+_{\mu\nu} = g^-_{\mu\nu} = g_{\mu\nu}$. To shorten the notation we will also suppress the branch indices of the scalar field and the sources.

The existence of the two branches also affects the definition of the delta function: $\delta(x, y)$ is always zero if its arguments lie on different branches [28]. In curved space-time it is further generalized to fulfill the relation

$$\int d^4y\sqrt{-g}f(y)\delta^g(x, y) = f(x), \tag{4}$$

where the integration is performed over the closed contour. The solution to this equation is given by [23]

$$\delta^g(x, y) = (-g_x)^{-(1/4)}\delta(x, y)(-g_y)^{-(1/4)}. \tag{5}$$

The generalized delta function is used to define functional differentiation in curved space-time [29]

$$\frac{\delta\mathcal{F}[\phi]}{\delta\phi(y)} \equiv \lim_{\varepsilon\to 0}\frac{\mathcal{F}[\phi(x) + \varepsilon\delta^g(x, y)] - \mathcal{F}[\phi(x)]}{\varepsilon}. \tag{6}$$

From the definition (6) it follows immediately that

[1]In particular there are two local (J_+ and J_-) and four bilocal (K_{++}, K_{+-}, K_{-+}, and K_{--}) sources. Analogously, the field value on the two branches is denoted by φ_+ and φ_- respectively, whereas the two-point function components are denoted by G_{++}, G_{+-}, G_{-+}, and G_{--} [27].

[2]This is not true for the point $t = t_{max}$.

[3]A theoretical analysis of the backreaction of the fields on the gravitational field has been performed in [16]. An analysis, with very interesting numerical results, of a model with quartic self-interactions in the Friedmann-Robertson-Walker universe has been carried out in [19].

$$\frac{\delta J(x)}{\delta J(y)} = \delta^g(x, y), \qquad \frac{\delta K(x, y)}{\delta K(u, v)} = \delta^g(x, u)\delta^g(y, v). \quad (7)$$

The functional derivatives of the generating functional for connected Green's functions

$$\mathcal{W}[J, K] = -i \ln \mathcal{Z}[J, K] \quad (8)$$

with respect to the external sources read

$$\frac{\partial \mathcal{W}[J, K]}{\partial J(x)} = \Phi(x), \quad (9a)$$

$$\frac{\partial \mathcal{W}[J, K]}{\partial K(x, y)} = \frac{1}{2}[G(y, x) + \Phi(x)\Phi(y)], \quad (9b)$$

where Φ denotes expectation value of the field and G is the propagator. The effective action is the Legendre transform of the generating functional for connected Green's functions,

$$\Gamma[\Phi, G] \equiv \mathcal{W}[J, K] - J\Phi - \tfrac{1}{2}\mathrm{tr}[KG] - \tfrac{1}{2}\Phi K \Phi. \quad (10)$$

Its functional derivatives with respect to the expectation value and the propagator reproduce the external sources:

$$\frac{\delta \Gamma[G, \Phi]}{\delta \Phi(x)} = -J(x) - \int \sqrt{-g} d^4 z K(x, z)\Phi(z), \quad (11a)$$

$$\frac{\delta \Gamma[G, \Phi]}{\delta G(x, y)} = -\frac{1}{2} K(y, x). \quad (11b)$$

Next, we shift the field by its expectation value

$$\varphi \to \varphi + \Phi.$$

The action can then be written as a sum of two terms

$$S[\varphi] \to S_{\mathrm{cl}}[\Phi] + S[\varphi, \Phi]. \quad (12)$$

S_{cl} denotes the classical action, which depends only on Φ, whereas $S[\varphi, \Phi] = S_0[\varphi] + S_{\mathrm{int}}[\varphi, \Phi]$ contains terms quadratic, cubic, and quartic in the shifted field φ. The free field action can be written in the form

$$S_0 = \frac{1}{2} \iint \sqrt{-g_x} d^4 x \sqrt{-g_y} d^4 y \varphi(i \mathcal{G}^{-1})\varphi, \quad (13)$$

where $\mathcal{G}^{-1}$ is the zero-order inverse propagator

$$\mathcal{G}^{-1}(x, y) = i(\Box_x + M^2)\delta^g(x, y), \qquad \Box_x \equiv g_{\mu\nu}\nabla_x^\mu \nabla_x^\nu. \quad (14)$$

Since the integration measure in the path integral is translationally invariant, the effective action can be rewritten in the form

$$\Gamma[\Phi, G] = -i \ln \int \mathcal{D}\varphi \exp\Big[i\Big(S + J\varphi + \frac{1}{2}\varphi K \varphi\Big)\Big] + S_{\mathrm{cl}}[\Phi] - \frac{1}{2}\mathrm{tr}[KG]. \quad (15)$$

Now we tentatively write the effective action in the form

$$\Gamma[\Phi, G] \equiv S_{\mathrm{cl}}[\Phi] + \frac{i}{2} \ln \det[G^{-1}] + \frac{i}{2}\mathrm{tr}[\mathcal{G}^{-1} G] + \Gamma_2[\Phi, G], \quad (16)$$

defining the functional Γ_2. The third term on the right-hand side is defined by

$$\mathrm{tr}[\mathcal{G}^{-1} G] \equiv \iint \sqrt{-g_x} d^4 x \sqrt{-g_y} d^4 y \mathcal{G}^{-1}(x, y) G(y, x),$$

whereas the second term on the right-hand side is defined by the path integral

$$\det\Big[\frac{G^{-1}}{2\pi}\Big] \equiv \int \mathcal{D}\varphi \exp(\varphi G^{-1} \varphi).$$

Using (11) we can find the functional derivatives of Γ. Differentiation of $\mathrm{tr}[\mathcal{G}^{-1} G]$ with respect to G is straightforward and gives

$$\frac{\delta}{\delta G(x, y)} \mathrm{tr}[\mathcal{G}^{-1} G] = \mathcal{G}^{-1}(y, x). \quad (17)$$

To calculate the functional derivative of $\ln \det[G^{-1}]$ we take into account that in curved space-time

$$\int \sqrt{-g} d^4 z G^{-1}(u, z) G(z, v) = \delta^g(u, v). \quad (18)$$

After some algebra and use of (18) we obtain a result analogous to that in Minkowski space-time

$$\frac{\delta}{\delta G(x, y)} \ln \det[G^{-1}] = -G^{-1}(y, x). \quad (19)$$

The functional derivative of (16) with respect to G then reads

$$\frac{\delta \Gamma[G, \Phi]}{\delta G(x, y)} = -\frac{i}{2} G^{-1}(y, x) + \frac{i}{2}\mathcal{G}^{-1}(y, x) + \frac{\delta \Gamma_2[G, \Phi]}{\delta G(x, y)} = -\frac{1}{2} K(y, x). \quad (20)$$

Solving (20) with respect to K and substituting it into (16), we can rewrite the effective action in the form

$$\Gamma_2[G, \Phi] = -i \ln \int \mathcal{D}\varphi \exp\left[i\left(S + J\varphi - \varphi \frac{\delta \Gamma_2}{\delta G}\varphi\right)\right] + \mathrm{tr}\left[\frac{\delta\Gamma_2}{\delta G} G\right] - \frac{i}{2} \ln \det[G^{-1}] + \text{const}, \tag{21}$$

where again $S = S_0 + S_{\text{int}}$, but now with S_0 given by

$$S_0 = \frac{1}{2} \iint \sqrt{-g_x} d^4x \sqrt{-g_y} d^4y \varphi(iG^{-1})\varphi. \tag{22}$$

This implies that $i\Gamma_2$ is the sum of all 2PI vacuum diagrams with vertices as given by $\mathcal{L}_{\text{int}}$ and internal lines representing the complete connected propagators G [30].

Physical situations correspond to vanishing sources. Introducing the self-energy

$$\Pi(x, y) \equiv 2i \frac{\delta\Gamma_2[G, \Phi]}{\delta G(y, x)}, \tag{23}$$

we can then rewrite (20) in the form

$$G^{-1}(x, y) = \mathcal{G}^{-1}(x, y) - \Pi(x, y). \tag{24}$$

Thus the above calculation yields the Schwinger-Dyson (SD) equation. Because the (inverse) propagators and the self-energy are scalars, the derived equation has exactly the same form as in Minkowski space-time.

III. 2PI EFFECTIVE ACTION

The structure of the Schwinger-Dyson equation is determined only by the particle content of the model (here a single real scalar field) and completely independent of the particular form of the interaction Lagrangian. The latter determines the form of the 2PI effective action. The lowest order contribution is due to the two-loop diagram in Fig. 2, which only takes into account local effects and cannot describe thermalization. Thus one usually also considers the three-loop diagram, which describes $2 \leftrightarrow 2$ scattering. In addition we take into account the four-loop contribution. As is demonstrated below, in the Boltzmann approximation it describes the one-loop correction for $2 \leftrightarrow 2$ scattering. The resulting expression for the effective action is similar to that given in [8,9,18,31]:

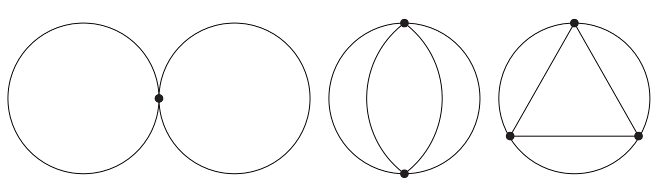

FIG. 2. Two-, three-, and four-loop contributions to the 2PI effective action.

$$\begin{aligned}
i\Gamma_2[G] &= \sum_n i\Gamma_2^{(n)}[G],\\
i\Gamma_2^{(2)}[G] &= -\frac{i\lambda}{8}\int \sqrt{-g_x} d^4x G^2(x, x),\\
i\Gamma_2^{(3)}[G] &= -\frac{\lambda^2}{48}\int \sqrt{-g_x} d^4x \sqrt{-g_y} d^4y G^2(x, y) G^2(y, x),\\
i\Gamma_2^{(4)}[G] &= \frac{i\lambda^3}{48}\int \sqrt{-g_x} d^4x \sqrt{-g_y} d^4y \sqrt{-g_z} d^4z\\
&\quad \times G^2(y, x) G^2(x, z) G^2(z, y).
\end{aligned} \tag{25}$$

Note, however, the presence of the $\sqrt{-g}$ factors which ensure invariance of the effective action under coordinate transformations.

Using the definition of the self-energy (23) and the functional differentiation rule in curved space-time we obtain

$$\begin{aligned}
\Pi(x, y) &= \sum_n \Pi^{(n)}(x, y),\\
\Pi^{(2)}(x, y) &= -i\delta^g(x, y)\frac{\lambda}{2} G(x, x),\\
\Pi^{(3)}(x, y) &= -\frac{\lambda^2}{6} G(y, x) G(x, y) G(x, y),\\
\Pi^{(4)}(x, y) &= \frac{i\lambda^3}{4} G(y, x) \int \sqrt{-g_z} d^4z G^2(x, z) G^2(z, y).
\end{aligned} \tag{26}$$

It is worth mentioning that the appearance of the generalized delta function in the first *local* term is a consequence of the form of the effective action and the functional differentiation rule (6). For each vertex in the loop diagrams there is a corresponding integral in the effective action. Because of the appearance of the generalized δ functions two of the integrals can be carried out trivially after functional differentiation. Further integrals persist in the self-energy. That is, four- and higher-loop contributions to $\Pi(x, y)$ contain integrations over space-time with the corresponding number of $\sqrt{-g}$ factors to ensure the invariance of the self-energy.

IV. KADANOFF-BAYM EQUATIONS

Convolving the Schwinger-Dyson equations (24) with G from the right and using (18) we obtain

$$i[\Box_x + M^2] G(x, y) = \delta^g(x, y) + \int \sqrt{-g} d^4z \Pi(x, z) G(z, y). \tag{27}$$

Next, we define the spectral function

$$G_\rho(x, y) = i\langle[\varphi(x), \varphi(y)]_-\rangle, \tag{28}$$

and the statistical propagator

$$G_F(x, y) = \tfrac{1}{2}\langle[\varphi(x), \varphi(y)]_+\rangle. \tag{29}$$

As is clear from the definitions, the statistical propagator of real scalar field is symmetric whereas the spectral function is antisymmetric with respect to permutation of its arguments. For a real scalar field $G_F(x, y)$ and $G_\rho(x, y)$ are real-valued functions [7]. The full Feynman propagator can be decomposed into a statistical and a spectral part

$$G(x, y) = G_F(x, y) - \frac{i}{2}\mathrm{sign}(x^0 - y^0)G_\rho(x, y). \tag{30}$$

Upon use of the sign- and δ-function differentiation rules, the action of the $\Box_x$ operator on the second term on the right-hand side of (30) gives a product of $g^{00}\delta(x^0, y^0)$ and $\nabla_0^x G_\rho(x, y)$. Using the definition (28) and the canonical commutation relations in curved space-time [32]

$$\lim_{y^0\to x^0}[\varphi(x^0, \vec{x}), \pi(x^0, \vec{y})]_- = i\delta(\vec{x}, \vec{y}), \tag{31}$$

where[4] $\pi = g^{00}\sqrt{-g}\nabla_0\varphi$, we find for the derivative of the spectral function

$$\nabla_0^x G_\rho(x, y) = \frac{\delta(\vec{x}, \vec{y})}{g^{00}\sqrt{-g}}. \tag{32}$$

Multiplication of (32) by $g^{00}\delta(x^0, y^0)$ then gives the generalized delta function $\delta^g(x, y)$, which cancels the delta function on the right-hand side of (27).

The local term of the self-energy (26), proportional to the delta function, can be absorbed in the effective mass

$$M^2(x) \equiv M^2 + \frac{\lambda}{2}G(x, x). \tag{33}$$

The remaining part of the self-energy can also be split into a spectral part $\Pi_\rho(x, y)$ and a statistical part $\Pi_F(x, y)$ in complete analogy to (30).

Integrating along the closed time path in the direction indicated in Fig. 2, and taking into account that any point of the negative branch is considered as a later instant than any point of the positive branch, we finally obtain the system of Kadanoff-Baym equations:

$$[\Box_x + M^2(x)]G_F(x, y) = \int_0^{y^0} \sqrt{-g}d^4z\Pi_F(x, z)G_\rho(z, y) - \int_0^{x^0} \sqrt{-g}d^4z\Pi_\rho(x, z)G_F(z, y), \tag{34a}$$

$$[\Box_x + M^2(x)]G_\rho(x, y) = -\int_{y^0}^{x^0} \sqrt{-g}d^4z\Pi_\rho(x, z)G_\rho(z, y). \tag{34b}$$

Comparing with the Kadanoff-Baym equations presented in [7,9], we conclude that (34) appear to be the covariant generalization of the Kadanoff-Baym equations in Minkowski space-time.

Equations (34) are exact equations for the quantum dynamical evolution of the statistical propagator and spectral function. It is important that, due to the characteristic memory integrals on the right-hand sides, the dynamics of the system depends on the history of its evolution [34].

To complete this section we derive explicit expressions for the spectral and statistical self-energies. Using symmetry (antisymmetry) of the spectral and statistical propagators with respect to permutation of the arguments, we obtain for the three-loop contribution to the self-energy components

$$\Pi_F^{(3)}(x, y) = -\frac{\lambda^2}{6}\left[G_F(x, y)G_F(x, y)G_F(x, y) - \frac{3}{4}G_F(x, y)G_\rho(x, y)G_\rho(x, y)\right], \tag{35a}$$

$$\Pi_\rho^{(3)}(x, y) = -\frac{\lambda^2}{6}\left[3G_F(x, y)G_F(x, y)G_\rho(x, y) - \frac{1}{4}G_\rho(x, y)G_\rho(x, y)G_\rho(x, y)\right]. \tag{35b}$$

Four- and higher-loop contributions to the self-energy components contain integrations over space-time with x^0 and y^0 as the integration limits. Introducing

$$G_{4F}(x, y) = \int_0^{x^0} \sqrt{-g}d^4zG_F(x, z)G_\rho(x, z)\left[G_F^2(z, y) - \frac{1}{4}G_\rho^2(z, y)\right] + \{x \leftrightarrow y\}, \tag{36a}$$

$$G_{4\rho}(x, y) = \int_0^{x^0} \sqrt{-g}d^4zG_F(x, z)G_\rho(x, z)[2G_F(z, y)G_\rho(z, y)] - \{x \leftrightarrow y\}, \tag{36b}$$

we can write the four-loop contribution to the statistical and spectral components of the self-energy as

[4] To simplify the calculation we set $g_{0i} = 0$. The off-diagonal components of the metric tensor can always be set to zero by an appropriate choice of the coordinate system [33]. Examples are the longitudinal and synchronous gauges.

$$\Pi_F^{(4)}(x,y) = \frac{\lambda^3}{2}\left[G_F(x,y)G_{4F}(x,y) - \frac{1}{4}G_\rho(x,y)G_{4\rho}(x,y)\right], \tag{37a}$$

$$\Pi_\rho^{(4)}(x,y) = \frac{\lambda^3}{2}[G_F(x,y)G_{4\rho}(x,y) + G_\rho(x,y)G_{4F}(x,y)]. \tag{37b}$$

Of course, all quantities entering the Kadanoff-Baym equations must be renormalized. A consistent renormalization procedure in Minkowski space-time has been developed in [35–38]. A renormalization procedure at the tadpole order in the Gaussian scheme has been applied to the analysis of the Kadanoff-Baym equations in [19].

V. QUANTUM KINETICS

Introducing the retarded and advanced propagators

$$G_R(x,y) \equiv \theta(x^0 - y^0)G_\rho(x,y), \tag{38a}$$

$$G_A(x,y) \equiv -\theta(y^0 - x^0)G_\rho(x,y), \tag{38b}$$

and the corresponding definitions for the self-energies, one can rewrite the system of Kadanoff-Baym equations in the form

$$[\Box_x + M^2(x)]G_F(x,y) = -\int \sqrt{-g}d^4z\theta(z^0)[\Pi_F(x,z)G_A(z,y) + \Pi_R(x,z)G_F(z,y)], \tag{39a}$$

$$[\Box_x + M^2(x)]G_\rho(x,y) = -\int \sqrt{-g}d^4z\theta(z^0)[\Pi_\rho(x,z)G_A(z,y) + \Pi_R(x,z)G_\rho(z,y)]. \tag{39b}$$

The system (39) should be supplemented by the analogous equations for the retarded (advanced) propagators; they can be derived from (34b) upon use of (32)

$$[\Box_x + M^2(x)]G_{R(A)}(x,y) = \delta^g(x,y) - \int \sqrt{-g}d^4z\Pi_{R(A)}(x,z)G_{R(A)}(z,y). \tag{40}$$

Let us now interchange x and y on both sides of the Kadanoff-Baym equations (39). Using the relation $G_R(x,y) = G_A(y,x)$ and symmetry (antisymmetry) of the statistical (spectral) propagators with respect to interchange of the arguments, we obtain the following expressions for the *differences* of the original and resulting equations:

$$[\Box_x - \Box_y + M^2(x) - M^2(y)]G_F(x,y) = -\int \sqrt{-g}d^4z\theta(z^0)[\Pi_F(x,z)G_A(z,y) - G_R(x,z)\Pi_F(z,y) + \Pi_R(x,z)G_F(z,y) - G_F(x,z)\Pi_A(z,y)], \tag{41a}$$

$$[\Box_x - \Box_y + M^2(x) - M^2(y)]G_\rho(x,y) = -\int \sqrt{-g}d^4z\theta(z^0)[\Pi_\rho(x,z)G_A(z,y) - G_R(x,z)\Pi_\rho(z,y) + \Pi_R(x,z)G_\rho(z,y) - G_\rho(x,z)\Pi_A(z,y)]. \tag{41b}$$

Interchanging x and y on both sides of the equation for the advanced propagator, and adding it to the equation for the retarded propagator, we obtain

$$[\Box_x + \Box_y + M^2(x) + M^2(y)]G_R(x,y) = 2\delta^g(x,y) - \int \sqrt{-g}d^4z[\Pi_R(x,z)G_R(z,y) + G_R(x,z)\Pi_R(z,y)]. \tag{42}$$

Next, we introduce center and relative coordinates. In Minkowski space-time they are given by half of the sum and by the difference of x and y, respectively [9]. In other words the center coordinate lies in the middle of the geodesic connecting x and y, whereas the relative coordinate gives the length of the "curve"[5] connecting the two points.

Consider now curved space-time. Let ς be the affine parameter of the geodesic connecting x and y (see Fig. 3) and $\xi(\varsigma)$ a function mapping ς onto the points of the geodesic, with

$$x^\alpha = \xi(\varsigma'), \qquad y^\alpha = \xi(\varsigma''). \tag{43}$$

The center coordinate lies in the middle of the geodesic, i.e. it corresponds to $\varsigma_X \equiv \frac{1}{2}(\varsigma' + \varsigma'')$. The relative coordinate is given by the sum of the infinitesimal distance

[5]In Minkowski space-time geodesics are straight lines.

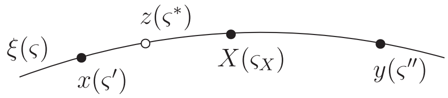

FIG. 3. Arrangement of the points along the geodesic.

vectors $d\xi^\alpha$ along the geodesic, all of which must have been submitted to parallel transfer to ς_X from the integration point on the curve.[6] According to [40] this implies

$$X^\alpha \equiv X^\alpha_{xy} = \xi^\alpha(\varsigma_X), \qquad s^\alpha \equiv s^\alpha_{xy} = (\varsigma' - \varsigma'')u^\alpha(\varsigma_X). \tag{44}$$

All quantities in equations (41) are now recast in terms of X^α and s^α. Up to higher order, proportional to the curvature tensor terms, the Laplace-Beltrami operator is given by [40]

$$\Box_{x,y} \approx \frac{1}{4}D^\alpha D_\alpha + \frac{\partial^2}{\partial s^\alpha \partial s_\alpha} \pm D^\alpha \frac{\partial}{\partial s_\alpha}, \tag{45}$$

where D_α is the covariant derivative

$$D_\alpha \equiv \frac{\partial}{\partial X^\alpha} - \Gamma^\beta_{\alpha\gamma}s^\gamma \frac{\partial}{\partial s^\beta}. \tag{46}$$

Note that in (45) we have neglected the corrections proportional to the Riemann and Ricci tensors. Next, we Taylor expand the effective masses to first order around the center coordinate X

$$M^2 \approx M^2(X) \pm \tfrac{1}{2}s^\alpha D_\alpha M^2(X), \tag{47}$$

where the minus sign corresponds to y whereas the plus sign corresponds to x. The propagators on the left-hand side of (41) can also be reparametrized in terms of the center and relative coordinates: $G_F(x, y) \equiv \tilde{G}_F(X, s)$ and $G_\rho(x, y) \equiv \tilde{G}_\rho(X, s)$.

On the right-hand sides we have convolutions of functions of x and z and functions of z and y. That is, we have to introduce the corresponding center and relative coordinates and perform the integration. Making use of the identity

$$(\varsigma' + \varsigma^*) = (\varsigma' + \varsigma'') + (\varsigma^* - \varsigma'') = 2\varsigma_X + (\varsigma^* - \varsigma'')$$

and Taylor expanding around ς_X, we obtain to first order

$$\Pi_F(x, z) \equiv \tilde{\Pi}_F(X_{xz}, s_{xz}) \approx \tilde{\Pi}_F(X, s_{xz}) + \left(\frac{\partial \tilde{\Pi}_F}{\partial \xi^\alpha}\frac{d\xi^\alpha}{d\varsigma} + \frac{\partial \tilde{\Pi}_F}{\partial u^\alpha}\frac{du^\alpha}{d\varsigma}\right) \times \frac{\varsigma^* - \varsigma''}{2}. \tag{48}$$

Using furthermore the definition of the four-velocity and the geodesic equation

$$\frac{d\xi^\alpha}{d\varsigma} = u^\alpha, \qquad \frac{du^\alpha}{d\varsigma} = -\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma, \tag{49}$$

we can rewrite (48) in the form

$$\Pi_F(x, z) \approx \tilde{\Pi}_F(X, s_{xz}) + \tfrac{1}{2}s^\alpha_{zy}D_\alpha \tilde{\Pi}_F(X, s_{xz}), \tag{50}$$

where $s^\alpha_{zy} \equiv (\varsigma^* - \varsigma'')u^\alpha(\varsigma_X)$. Making use of the identity

$$(\varsigma'' + \varsigma^*) = (\varsigma' + \varsigma'') - (\varsigma' - \varsigma^*) = 2\varsigma_X - (\varsigma' - \varsigma^*),$$

we get a similar expression for the functions of z and y

$$G_A(z, y) \approx \tilde{G}_A(X, s_{zy}) - \tfrac{1}{2}s^\alpha_{xz}D_\alpha \tilde{G}_A(X, s_{zy}). \tag{51}$$

To perform the integration of the product of (50) and (51), we shift the coordinate center to ς_X and replace the integration with respect to z by integration with respect to distance s_{X_z} from X to z along the geodesic. Moreover, we approximate[7] $\sqrt{-g_z}$ by its value at the origin $\sqrt{-g_X}$.

The Kadanoff-Baym equations describe the dynamics of a system in terms of the spectral function and statistical propagator. The latter ones are functions of two coordinates in the four-dimensional space-time. By introducing center and relative coordinates we have traded one set of coordinates for another one. Performing the so-called Wigner transformation, one can also trade one of the arguments defined in the coordinate space for an argument defined in the momentum space. In curved space-time [40]

$$\tilde{G}_F(X, p) = \sqrt{-g_X}\int d^4s\, e^{ips}\tilde{G}_F(X, s), \tag{52a}$$

$$\tilde{G}_F(X, s) = \frac{1}{\sqrt{-g_X}}\int \frac{d^4p}{(2\pi)^4}e^{-ips}\tilde{G}_F(X, p). \tag{52b}$$

Note that in (52) and in the rest of the paper we use *contravariant* components of the space-time coordinates and *covariant* components of the momenta. Let us also note that

$$d\Pi^4_p \equiv \frac{1}{\sqrt{-g_X}}\frac{d^4p}{(2\pi)^4}$$

[6]Calzetta and Hu [17,39] have employed a different method based on the use of Riemann normal coordinates and the momentum representation of the propagators. Their approach has some advantages for the study of the quantum kinetics equations. Here we are mainly interested in the Kadanoff-Baym and Boltzmann equations and consider the derivation of the quantum kinetic equations as an intermediate step connecting both of them. For this reason, we adopt the covariant definitions of the midpoint and distance vectors introduced by Winter [40], which allow us to keep the analysis manifestly covariant in every step.

[7]The next-to-leading term of the Taylor expansion is proportional to the convolution of the Christoffel symbol [33], $\sqrt{-g_z} \approx \sqrt{-g_X}(1 + \Gamma^\nu_{\alpha\nu}s^\alpha)$. This correction can in principle be taken into account and would induce additional terms proportional to $i\partial/\partial p^\alpha$ on the right-hand side of the quantum kinetic equation. Since such terms are neglected in the Boltzmann approximation, the collision terms do not receive any corrections.

is the invariant volume element in momentum space. The definition of the Wigner transform of $\tilde{G}_\rho(X, s)$ differs from (52a) by a factor of $-i$ so that $\tilde{G}_\rho(X, p)$ is again real valued.

As follows from (52b), differentiation with respect to s^α is replaced after the Wigner transformation by p^α

$$\frac{\partial}{\partial s^\alpha} \to -ip^\alpha. \tag{53}$$

Upon integration by parts we also see that s^α is then replaced by differentiation with respect to p^α:

$$s^\alpha \to -i\frac{\partial}{\partial p^\alpha}. \tag{54}$$

Consequently, the Wigner-transformed covariant derivative reads

$$D_\alpha \to \mathcal{D}_\alpha = \frac{\partial}{\partial X^\alpha} + \Gamma^\beta_{\alpha\gamma} p_\beta \frac{\partial}{\partial p_\gamma}. \tag{55}$$

Correlations between earlier and later times are exponentially suppressed, which leads to a gradual loss of the dependence on the initial conditions [9,34]. Exploiting this fact, one can drop the θ function from the integrals in the difference equations (41). Furthermore we let the relative-time coordinate s^0 range from $-\infty$ to ∞ in order to perform the Wigner transformation, see [34,41] for a detailed discussion of these approximations. Then using (54) and (55) we obtain for the Wigner transform of the first term on the right-hand side of (41a):

$$\int \sqrt{-g_z} d^4 z \Pi_F(x, z) G_A(z, y) \to \tilde{\Pi}_F(X, p)\tilde{G}_A(X, p) + \frac{i}{2}\{\tilde{\Pi}_F(X, p), \tilde{G}_A(X, p)\}_{PB}, \tag{56}$$

where the Poisson brackets are defined by

$$\{\tilde{A}(X, p), \tilde{B}(X, p)\}_{PB} \equiv \frac{\partial}{\partial p_\alpha}\tilde{A}(X, p)\mathcal{D}_\alpha \tilde{B}(X, p) - \mathcal{D}_\alpha \tilde{A}(X, p)\frac{\partial}{\partial p_\alpha}\tilde{B}(X, p). \tag{57}$$

Comparing (57) to its Minkowski-space counterpart we see that the derivatives with respect to X are replaced by the covariant derivatives, just as one would expect.

Wigner transforming the rest of the terms, we obtain a rather lengthy expression which can be substantially simplified with the help of the relations between $\tilde{G}_R(X, p)$, $\tilde{G}_A(X, p)$, and $\tilde{G}_\rho(X, p)$. Recalling the Fourier transform of the θ function

$$\int ds^0 \exp(iws^0)\theta(\pm s^0) = \lim_{\epsilon\to 0}\frac{\pm i}{\omega \pm i\epsilon},$$

we find that

$$\tilde{G}_R(X, p) = -\int \frac{d\omega}{2\pi}\frac{\tilde{G}_\rho(X, \vec{p}, \omega)}{p_0 - \omega + i\epsilon}, \tag{58a}$$

$$\tilde{G}_A(X, p) = -\int \frac{d\omega}{2\pi}\frac{\tilde{G}_\rho(X, \vec{p}, \omega)}{p_0 - \omega - i\epsilon}. \tag{58b}$$

From comparison of (58a) and (58b) it follows that

$$\tilde{G}_A(X, p) = \tilde{G}^*_R(X, p). \tag{59}$$

Recalling furthermore that the δ function can be approximated by

$$\delta(\omega) = \lim_{\epsilon\to 0}\frac{\epsilon}{\pi(\omega^2 + \epsilon^2)},$$

we also find that

$$\tilde{G}_R(X, p) - \tilde{G}_A(X, p) = i\tilde{G}_\rho(X, p). \tag{60}$$

Analogous relations also hold for the retarded and advanced components of the self-energy.

As can be inferred from (45) and (47), the Wigner transform of the left-hand side of (41) reads[8]

$$\Box_x - \Box_y + M^2(x) - M^2(y) \to -i\Big(2p^\alpha \mathcal{D}_\alpha + D_\alpha M^2 \frac{\partial}{\partial p_\alpha}\Big). \tag{61}$$

Introducing the quantity

$$\tilde{\Omega}(X, p) \equiv p^\mu p_\mu - M^2(X) - \Re[\tilde{\Pi}_R(X, p)] \tag{62}$$

and collecting the terms on the right-hand side of the difference equation, one can write the kinetic equation for the Wigner transform of the statistical propagator in the compact form:

$$\{\tilde{\Omega}(X, p), \tilde{G}_F(X, p)\}_{PB} = \tilde{G}_F(X, p)\tilde{\Pi}_\rho(X, p) - \tilde{\Pi}_F(X, p)\tilde{G}_\rho(X, p) + \{\tilde{\Pi}_F(X, p), \Re[\tilde{G}_R(X, p)]\}_{PB}. \tag{63}$$

The same procedure leads also to a kinetic equation for the Wigner transform of the spectral function

$$\{\tilde{\Omega}(X, p), \tilde{G}_\rho(X, p)\}_{PB} = \{\tilde{\Pi}_\rho(X, p), \Re[\tilde{G}_R(X, p)]\}_{PB}. \tag{64}$$

As has been mentioned in the previous section, the exact quantum dynamical evolution of the system depends on its whole evolution history. Mathematically, this manifests itself in the memory integrals on the right-hand sides of

[8]Additional contributions arising from the decomposition of the Laplace-Beltrami operator are proportional to Riemann and Ricci tensors and to the curvature [see Eq. (4.40) in [40]] and may be relevant in strong gravitational fields. Since all these terms contain at least one $i\partial/\partial p_\alpha$ derivative, they do not contribute in the Boltzmann approximation.

(34). In fact, performing the linear order Taylor expansion around X, we take into account only a very short part of the history of the evolution. Since the expansion coefficients are defined at X, after the integration we obtain equations which are *local* in time.

Next we consider the Wigner transform of (42). On the left-hand side we have $\Box_x + \Box_y = 2\partial_{s^\alpha}\partial_{s_\alpha}$, to first order in the covariant derivative, whereas $M^2(x) + M^2(y) \approx M^2(X)$. On the right-hand side the Poisson brackets cancel out and only the product of $\tilde{\Pi}_R(X,p)$ and $\tilde{G}_R(X,p)$ remains. Finally, the Wigner transform of the generalized δ function is just unity. Therefore, we get an *algebraic* equation for the Wigner transform of the retarded propagator

$$[p^\mu p_\mu - M^2(X) - \tilde{\Pi}_R(X,p)]\tilde{G}_R(X,p) = -1. \tag{65}$$

Taking the real part of its solution, we obtain for the real part of the retarded propagator:

$$\Re[\tilde{G}_R(X,p)] = \frac{-\tilde{\Omega}(X,p)}{\tilde{\Omega}^2(X,p) + \frac{1}{4}\tilde{\Pi}_\rho^2(X,p)}. \tag{66}$$

As follows from (59) and (60), the Wigner transform of the spectral function is twice the imaginary part of the retarded propagator,

$$\tilde{G}_\rho(X,p) = \frac{-\tilde{\Pi}_\rho(X,p)}{\tilde{\Omega}^2(X,p) + \frac{1}{4}\tilde{\Pi}_\rho^2(X,p)}. \tag{67}$$

To complete this section, we have to express the Wigner transforms of the spectral and statistical self-energies in terms of the Wigner transforms of the spectral function and statistical propagator. Using the definitions of the Wigner transformation and its inverse we find for the Wigner transform of a product of functions of the same arguments:

$$\begin{aligned} f_1(x,y)\ldots f_n(x,y) \to \widetilde{f_1\ldots f_n}(X,p) &\equiv \int d\Pi^4_{p_1}\ldots d\Pi^4_{p_n}(2\pi)^4\sqrt{-g_X}\delta^4 \\ &\quad\times(-p + p_1 + \ldots p_n) \\ &\quad\times \tilde{f}(X,p_1)\ldots\tilde{f}(X,p_n). \end{aligned} \tag{68}$$

Note that $\delta_g(q) \equiv \sqrt{-g_X}\delta(q)$ represents the momentum-space generalization of the δ function, invariant under coordinate transformations (this can be checked with help of the scaling property of the δ function). Keeping in mind that the definition of $\tilde{G}_\rho(X,p)$ contains an additional factor of $-i$ we can then write the Wigner transforms of (35) in the form

$$\tilde{\Pi}_F^{(3)}(X,p) = -\frac{\lambda^2}{6}\left[\tilde{G}_F^3(X,p) + \frac{3}{4}\widetilde{G_F G_\rho^2}(X,p)\right], \tag{69a}$$

$$\tilde{\Pi}_\rho^{(3)}(X,p) = -\frac{\lambda^2}{6}\left[3\widetilde{G_F^2 G_\rho}(X,p) + \frac{1}{4}\tilde{G}_\rho^3(X,p)\right]. \tag{69b}$$

The expression for the Wigner transform of the three-loop retarded self-energy can be obtained from (69) by replacing one of the $\tilde{G}_\rho$ by $\tilde{G}_R$. The Wigner transforms of the four-loop contributions (35) can be written in a similar way

$$\tilde{\Pi}_F^{(4)}(X,p) = \frac{\lambda^3}{2}\left[\widetilde{G_F G_{4F}}(X,p) + \frac{1}{4}\widetilde{G_\rho G_{4\rho}}(X,p)\right], \tag{70a}$$

$$\tilde{\Pi}_\rho^{(4)}(X,p) = \frac{\lambda^3}{2}\left[\widetilde{G_F G_{4\rho}}(X,p) + \widetilde{G_\rho G_{4F}}(X,p)\right]. \tag{70b}$$

Note, however, that $\tilde{G}_{4F}$ and $\tilde{G}_{4\rho}$ are Wigner transforms of *convolutions* of four two-point functions,

$$\begin{aligned} G_{4F}(x,y) = \int\sqrt{-g}d^4z G_F(x,z)G_R(x,z) \\ \times\left[G_F^2(z,y) - \frac{1}{4}G_\rho^2(z,y)\right] + \{x\leftrightarrow y\}, \end{aligned} \tag{71a}$$

$$\begin{aligned} G_{4\rho}(x,y) = \int\sqrt{-g}d^4z G_F(x,z)G_R(x,z)[2G_F(z,y)G_\rho(z,y)] \\ - \{x\leftrightarrow y\}, \end{aligned} \tag{71b}$$

where we have used the definitions of the retarded and advanced propagators and dropped again the $\theta(z^0)$ factor. Proceeding as in Eq. (56) and making use of the relations (59) and (60), we obtain for the Wigner transforms of G_{4F} and $G_{4\rho}$

$$\begin{aligned} \tilde{G}_{4F}(X,p) &= 2[\tilde{G}_F^2(X,p) + \tfrac{1}{4}\tilde{G}_\rho^2(X,p)]\widetilde{G_F\Re[G_R]}(X,p) \\ &\quad + \tfrac{1}{2}\{\tilde{G}_F^2(X,p) + \tfrac{1}{4}\tilde{G}_\rho^2(X,p), \widetilde{G_F G_\rho}(X,p)\}_{P.B.}, \end{aligned} \tag{72a}$$

$$\tilde{G}_{4\rho}(X,p) = 4\widetilde{G_F G_\rho}(X,p)\widetilde{G_F\Re[G_R]}(X,p). \tag{72b}$$

Finally, the expression for the Wigner transform of the four-loop retarded self-energy can be obtained from (70b) by replacing $\tilde{G}_\rho$ with $\tilde{G}_R$ and $\tilde{G}_{4\rho}$ with $\tilde{G}_{4R}$. The latter one is related to $\tilde{G}_{4\rho}$ by Eq. (58a).

VI. BOLTZMANN KINETICS

In order to derive the Boltzmann equation from the quantum kinetic equations, one has to discard the Poisson brackets on the right-hand sides of (63) and (64) and set the effective mass M to constant. Effectively, we take into account only the first term of the Taylor expansion around X and *completely* neglect the previous evolution of the system, i.e. the memory effects. Physically this corresponds to the *Stosszahlansatz* of Boltzmann.

In this approximation, Eq. (64) for the spectral function simplifies to

$$p^\alpha\mathcal{D}_\alpha\tilde{G}_\rho(X,p) = 0 \tag{73}$$

and admits a quasiparticle solution (on-shell form with zero width, see [6] for a detailed discussion of this *ansatz* in the nonrelativistic case):

$$\tilde{G}_\rho(X,p) = 2\pi \text{sign}(p_0)\delta(g_{\mu\nu}p^\mu p^\nu - M^2). \tag{74}$$

Note that Eqs. (73) and (74) state that the effective mass M of the field quanta does not change as they move along the geodesic, just like it is the case for particles. The quasiparticle approximation for the statistical propagator, which is usually referred to as the Kadanoff-Baym ansatz, reads [6,31]

$$\tilde{G}_F(X,p) = [n(X,p) + \tfrac{1}{2}]\tilde{G}_\rho(X,p). \tag{75}$$

Symmetry (antisymmetry) of the statistical (spectral) propagator with respect to permutation of its arguments and the definition of the Wigner transformation imply that

$$\tilde{G}_F(X,p) = \tilde{G}_F(X,-p), \qquad \tilde{G}_\rho(X,p) = -\tilde{G}_\rho(X,-p). \tag{76}$$

Therefore, for a single real scalar field, we have

$$n(X,-p) = -[n(X,p)+1]. \tag{77}$$

It is also convenient to introduce

$$\tilde{G}_{\gtrless}(X,p) \equiv \tilde{G}_F(X,p) \pm \tfrac{1}{2}\tilde{G}_\rho(X,p) \tag{78}$$

and their self-energy analogs $\Pi_{\gtrless}$. From Eqs. (63) and (73) it then follows that

$$[p^\alpha \mathcal{D}_\alpha n(X,p)]\tilde{G}_\rho(X,p) = \tfrac{1}{2}[\tilde{\Pi}_>(X,p)\tilde{G}_<(X,p) - \tilde{G}_>(X,p)\tilde{\Pi}_<(X,p)]. \tag{79}$$

Explicit expressions for $\tilde{\Pi}_{\gtrless}(X,p)$ can be obtained after some algebra from Eqs. (69) and (70). It is however more convenient to first derive $\Pi_{\gtrless}(x,y)$ and then perform the Wigner transformation. Using the decomposition

$$G(x,y) = \theta(x^0-y^0)G_>(x,y) + \theta(y^0-x^0)G_<(x,y), \tag{80}$$

we obtain for the three-loop contribution

$$\Pi^{(3)}_{\gtrless}(x,y) = -\frac{\lambda^2}{6}G_{\gtrless}(x,y)G_{\gtrless}(x,y)G_{\gtrless}(x,y). \tag{81}$$

Its Wigner transform reads

$$\tilde{\Pi}^{(3)}_{\gtrless}(X,p) = -\frac{\lambda^2}{6}\int d\Pi^4_k d\Pi^4_q d\Pi^4_t (2\pi)^4 \times \delta_g(-p-t+k+q)\tilde{G}_{\lessgtr}(X,t) \times \tilde{G}_{\gtrless}(X,k)\tilde{G}_{\gtrless}(X,q), \tag{82}$$

where we have used the relation $\tilde{G}_{\gtrless}(X,t) = \tilde{G}_{\lessgtr}(X,-t)$, which follows from Eqs. (76) and (78). It describes $2 \leftrightarrow 2$ scattering and corresponds to the tree-level Feynman diagram in Fig. 4.

Expression for the four-loop contribution contains integration over the contour

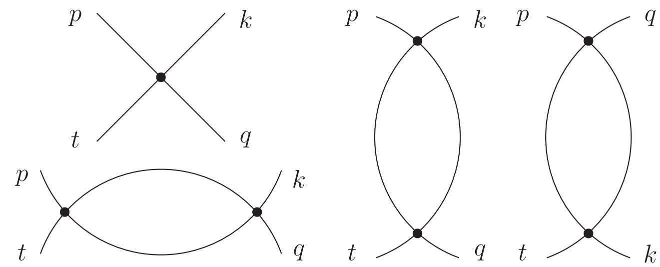

FIG. 4. Feynman diagrams of $2 \leftrightarrow 2$ scattering at tree and one-loop levels.

$$\tilde{\Pi}^{(4)}_{\gtrless}(x,y) = \frac{\lambda^3}{2}G_{\gtrless}(x,y)\int \sqrt{-g_z}d^4z\,\theta(z^0) \times [G_F(x,z)G_R(x,z)G^2_{\gtrless}(z,y) + G^2_{\gtrless}(x,z)G_A(z,y)G_F(z,y)]. \tag{83}$$

After some algebra we obtain for the Wigner transform of (83) in the Boltzmann approximation (that is, with the Poisson brackets neglected)

$$\tilde{\Pi}^{(4)}_{\gtrless}(X,p) = \frac{\lambda^3}{2}\int d\Pi^4_k d\Pi^4_q d\Pi^4_t (2\pi)^4 \delta_g(-p-t+k+q) \times \tilde{G}_{\lessgtr}(X,t)\tilde{G}_{\gtrless}(X,k)\tilde{G}_{\gtrless}(X,q)L(X,k+q), \tag{84}$$

where

$$L(X,p) \equiv \int d\Pi^4_k d\Pi^4_q (2\pi)^4 \delta_g(-p+k+q)\tilde{G}_F(X,k) \times [\tilde{G}_R(X,q) + \tilde{G}_A(X,q)]. \tag{85}$$

From (84) it follows that $L(p)$ is the same for the forward and inverse processes. As is demonstrated in Appendix A it corresponds to the integrals of the one-loop Feynman diagrams in Fig. 4. Let us note at this point that the contribution of a particular term of the 2PI effective action to the Boltzmann equation can be deduced by removing one of the vertices in the 2PI diagrams. Removing one of the vertices in the three-loop contribution, we obtain the diagram of $2 \leftrightarrow 2$ scattering at tree level, whereas removing one of the vertices in the four-loop contribution we obtain the one-loop correction for this process.

The quasiparticle approximation (74) for $\tilde{G}_F(X,k)$ in (85) forces one of the intermediate states in the loop to be on the mass shell. Combining relation (58a) with the quasiparticle ansatz, we obtain the familiar expression for the retarded propagator

$$\tilde{G}_R(X,p) = \frac{1}{p^2 - M^2 + 2i\varepsilon p_0}. \tag{86}$$

The expression for the advanced propagator differs from (86) by complex conjugation. The sum of $\tilde{G}_R$ and $\tilde{G}_A$, which describes the second intermediate state in the loop, vanishes on the mass shell. That is, the *real intermediate state* contributions ($2 \to 2$ scattering into two *on-shell* states followed by another $2 \to 2$ scattering) are *automatically* subtracted from the four-loop self-energies. Performing the integration and taking into account that one of the intermediate states is on-shell, we obtain the following expression for the loop integral:

$$L(X, p) = \lim_{\epsilon \to 0} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{2n(X, \mathbf{k}) + 1}{2E_k} \bigg[\frac{p^2 - 2pk}{(p^2 - 2pk)^2 + \epsilon^2} + \frac{p^2 + 2pk}{(p^2 + 2pk)^2 + \epsilon^2} \bigg], \tag{87}$$

where $k = (E_k, \mathbf{k})$ is the on-shell four-momentum expressed in terms of the "physical" components. In (87) the background plasma "affects" only one of the internal lines; the other one is off-shell and we cannot associate the particle number density with it.

Next, we integrate the left- and right-hand side of (79) over p_0 and choose the positive energy solution of (74) on the left-hand side. On the right-hand side both, the positive and the negative energy, solutions contribute. For positive p_0 momentum-energy conservation allows the following three combinations:

$$\begin{array}{llll} (a) & k_0 > 0, & q_0 > 0, & t_0 > 0, \\ (b) & k_0 > 0, & q_0 < 0, & t_0 < 0, \\ (c) & k_0 < 0, & q_0 > 0, & t_0 < 0. \end{array}$$

As far as the three-loop self-energy (82) is concerned, each combination leads to the same result, i.e. an overall factor of 3 appears. For the four-loop self-energy the arising terms are not equal due to the presence of the loop integral L in (84). After some algebra, use of (77) and redefinition of the momenta we finally arrive at the Boltzmann equation for the distribution function,

$$p^\alpha \mathcal{D}_\alpha n(X, \mathbf{p}) = -\frac{\pi}{16} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{d\mathbf{q}}{(2\pi)^3} \frac{d\mathbf{t}}{E_k E_q E_t} \delta(E_p + E_t - E_q - E_k) \delta(\mathbf{p} + \mathbf{t} - \mathbf{q} - \mathbf{k}) \Lambda^2(X, \mathbf{k}, \mathbf{q}, \mathbf{t}) \{ n(X, \mathbf{p}) n(X, \mathbf{t}) \times [n(X, \mathbf{k}) + 1][n(X, \mathbf{q}) + 1] - [n(X, \mathbf{p}) + 1][n(X, \mathbf{t}) + 1] n(X, \mathbf{k}) n(X, \mathbf{q}) \}, \tag{88}$$

where, for instance, $E_p \equiv p_0/\sqrt{g_{00}}$ denotes the "physical" energy, and $\mathbf{p}$ the "physical" momentum.

The effective coupling in the external gravitational background field at nonzero particle number density at one-loop level reads[9]

$$\Lambda^2(X, \mathbf{k}, \mathbf{q}, \mathbf{t}) \equiv \lambda^2 (1 - \lambda[L(X, k + q) + L(X, k - t) + L(X, q - t)]). \tag{89}$$

Note that the only remnant of the curved structure of space-time is the covariant derivative on the left-hand side. All the $\sqrt{-g_X}$ factors have disappeared due to the introduction of the "physical" momenta and energies. Let us also mention that in the FRW universe the left-hand side of (88) takes the form

$$p^\alpha \mathcal{D}_\alpha n = \frac{E}{a} \bigg(\frac{\partial}{\partial \eta} - \frac{\mathbf{p}^2}{E} \mathcal{H} \frac{\partial}{\partial E} \bigg) n, \qquad \mathcal{H} \equiv \frac{a'}{a}. \tag{90}$$

As for the right-hand side, it is remarkable that if only pointlike interactions (i.e. only the three-loop contribution to the 2PI effective action in the considered case) are taken into account, Eq. (88) coincides with the classical Boltzmann equation with the collision term calculated in *vacuum*. The inclusion of four- (and higher-loop) corrections to the effective potential induces further terms in the Boltzmann equation. These terms correspond to the remnant space-time integrals in the self-energy and involve additional momentum integrals over the distribution functions.

[9]Note, however, that a self-consistently dressed description of the vertex in $\lambda\varphi^4$ theory requires use of the 4PI effective action.

VII. SUMMARY AND CONCLUSIONS

In this paper we have considered the dynamics of an out-of-equilibrium quantum system in a background gravitational field. As one would expect, the resulting equations turned out to be covariant generalizations of their Minkowski-space counterparts.

As starting point we have used the generating functional for the (connected) Green's functions with invariantly defined space-time and field integration measures. Performing the Legendre transform we have defined the effective action. The latter one has been used to obtain the Schwinger-Dyson equations. Since the propagator and the self-energy are scalars, the SD equations has exactly the same form as in the Minkowski space-time.

From the SD equations we have derived the system of Kadanoff-Baym equations for the statistical propagator and the spectral function. As one would expect, it differs from its Minkowski space-time equivalent by the invariantly defined space-time integrals and the d'Alembert operator replaced by the Laplace-Beltrami operator.

Using the covariant definition of the center and relative coordinates X and s, Taylor expanding the propagators and self-energies around X and performing the Wigner transformation we obtained the quantum kinetic equations. In curved space-time the derivative with respect to the space-time coordinates is replaced by the covariant derivative.

Introducing the quasiparticle approximation and making use of the Kadanoff-Baym ansatz, we finally arrived at the Boltzmann equation. Remarkably, the only remnant of the curved structure of the space-time is the covariant derivative on its left-hand side. Furthermore, if only the tree-level processes are taken into account, then the resulting equation coincides with the Boltzmann equation with the collision term calculated in *vacuum*. Our result for the Boltzmann equation is analogous to the ones describing the evolution of the right-handed neutrino distribution functions in leptogenesis. Consequently, we have shown that, as long as the decays and inverse decays are considered at tree level, the usual vacuum approximation for the collision term is justified. Processes described by loop diagrams, which induce corrections to the self-coupling, involve additional momentum integrals over the distribution functions, so that the resulting collision terms no longer coincide with those calculated in vacuum. Interestingly, in used formalism loop corrections (i.e. processes with *off-shell* states) can be taken into account even if the quasiparticle ansatz is applied. It is also important that contributions of the *real intermediate states* to the loop diagrams are automatically subtracted.

Since the peculiarities of the calculation, related to the presence of a background gravitational field, are determined only by transformation properties of the fields—scalar fields in the present case—the developed formalism can be applied to arbitrary systems of scalar fields without any modifications. In [22] we study further implications of the formalism for leptogenesis and address the remaining two approximations in the framework of a toy model, which, by satisfying the Sakharov conditions for $U(1)$ charges assigned to the scalar fields and interpreted as lepton number, qualitatively reproduces the features of popular leptogenesis models.

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APPENDIX A: 2 ↔ 2 SCATTERING

The tree-level amplitude of 2 ↔ 2 scattering (see Fig. 4) in Minkowski space-time is given by

$$M_{fi}^{\text{tree}} = -i\lambda. \tag{A1}$$

There are also three one-loop diagrams which contribute to the scattering amplitude; their contribution reads

$$M_{fi}^{\text{loop}} = \frac{-\lambda^2}{2(2\pi)^4} \int \frac{d^4\xi d^4\eta \delta(-\sigma + \xi + \eta)}{[\xi^2 - M^2 + i\epsilon][\eta^2 - M^2 + i\epsilon]}, \tag{A2}$$

where σ is equal to $k + q$, to $k - t$, or $q - t$ (see Fig. 4). Because of the presence of the δ function one of the integrations (for instance, over η) can be performed trivially. Calculating residues of the integrand we can perform the integration over $d\xi_0$. The result of the integration reads

$$M_{fi}^{\text{loop}} = \frac{i\lambda^2}{2(2\pi)^3} \int \frac{d\xi^3}{2E_\xi}\left[\frac{1}{\xi^2 + 2\xi\sigma} + \frac{1}{\xi^2 - 2\xi\sigma}\right]. \tag{A3}$$

The quantity which enters the right-hand side of the Boltzmann equation is the amplitude modulo squared. To leading order in small λ it is given by

$$|M_{fi}|^2 = \lambda^2(1 - \lambda[L^{\text{vac}}(k+q) + L^{\text{vac}}(k-t) + L^{\text{vac}}(q-t)]), \tag{A4}$$

where $L^{\text{vac}}(\sigma)$ coincides with (87) if $n(X, \mathbf{k})$ and ϵ are set to zero. The former condition arises from the fact that in this Appendix we calculate the scattering amplitudes in *vacuum*, whereas the latter one is related to the fact that we have not subtracted the contributions of *real intermediate states* to the one-loop amplitude. Comparing (A4) with (89) we conclude that $L(X, p)$ indeed describes the integrals of the one-loop diagrams. Note also that, as can be inferred from comparison of (A3) and (87), one can easily generalize results of the calculation in vacuum to the case of nonzero particle densities.

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