PHYSICAL REVIEW D **78,** 085022 (2008)

Subcritical string and large N QCD

Charles B. Thorn*

Institute for Fundamental Theory, Department of Physics, University of Florida, Gainesville Florida 32611, USA

(Received 5 August 2008; published 24 October 2008)

We pursue the possibility of using subcritical string theory in 4 spacetime dimensions to establish a string dual for large N QCD. In particular we study the even G-parity sector of the 4 dimensional Neveu-Schwarz dual resonance model as the natural candidate for this string theory. Our point of view is that the open string dynamics given by this model will *determine* the appropriate subcritical closed string theory, a tree level background of which should describe the sum of planar multiloop open string diagrams. We examine the one-loop open string diagram, which contains information about the closed string spectrum at weak coupling. Higher loop open string diagrams will be needed to determine closed string interactions. We also analyze the field theory limit of the one-loop open string diagram and recover the correct running coupling behavior of the limiting gauge theory.

DOI: 10.1103/PhysRevD.78.085022 PACS numbers: 11.15.Bt, 11.15.Pg, 11.25.Pm, 11.25.Tq

I. INTRODUCTION

The underlying logic for field/string duality does not strictly involve supersymmetry, although that symmetry plays a very important practical role in the tractability of Maldacena's original $\mathcal{N} = 4$ Yang-Mills/IIB String on $AdS_5 \times S^5$ equivalence [1]. This logic involves three basic facts about string theory:

(1) The low energy limit ($\alpha' \to 0$) of open string dual resonance models (DRM) is generically the tree approximation of some flat space matrix quantum field theory (QFT) [with $SU(N)$ Chan-Paton factors, this QFT is, more specifically, a non-Abelian gauge theory with gauge group $SU(N)$].
(2) The sum of planar open string multiloop diagrams has the low energy limit of the sum of planar diagrams in the QFT, which gives its large N limit [2].
(3) A planar open string loop can be interpreted as a tree emission of a closed string which is absorbed into the vacuum.

Notice that, from the closed string point of view, the sum of planar diagrams is just a tree level shift of the vacuum. If we tried to describe the low energy closed string dynamics by an effective quantum field theory, this vacuum shift could be accomplished by solving classical field equations. In the case of $\mathcal{N} = 4$ such an effective field theory description is valid in the large 't Hooft coupling limit, which is the regime that has been most systematically studied over the last decade.

However, it is not meaningful to make a strong coupling approximation in QCD, because it is asymptotically free. Thus any attempt to apply an effective field theory analysis to a string dual of QCD should be taken with a grain of salt. It *might* reflect some qualitative feature of QCD, but it could just as probably be completely misleading. Thus we expect that even after finding the dual string theory for QCD, we will have to deal with the vacuum shift representing the sum of planar diagrams as a true string theory, not an effective field theory.

There has been a huge effort to adapt the AdS/CFT paradigm to construct a string dual to QCD. The mainstream approach to this problem has been to introduce schemes that break the symmetries of the $\mathcal{N} = 4$ theory down to those of QCD. This approach was first proposed by Witten [3], who found a way to break the supersymmetries by replacing the AdS space on the string side with an Einstein manifold that was a black hole embedded in AdS.

Here we follow another path, which is to base the dual QCD string construction on the original Neveu-Schwarz (NS) dual resonance model in four spacetime dimensions [4], with all odd G-parity states projected out. For brevity we shall call this model NS+ in this article.[1] It describes an open string theory whose low energy limit has long been known to be precisely Yang-Mills theory in four spacetime dimensions [8], essentially because the lowest state of the open NS+ string is a massless gauge particle. There is no open string tachyon in the even G-parity sector. The application of this model to the construction of a string formulation of QCD was first explicitly suggested by Polyakov [9]. Note that $N \to \infty$ suppresses the coupling of fields in the fundamental representation of $SU(N)$ so that infinite N QCD is the same as infinite N pure Yang-Mills theory. In the following when we refer to QCD, we mean this infinite

*thorn@phys.ufl.edu

[1] The internal consistency of this NS+ model at the level of open strings has been appreciated at least since May 1971: just after Halpern and I discovered a 5 dimensional modification of the NS model with no tachyons [5], Mandelstam pointed out to us that this NS+ model is a much simpler (indeed the simplest) tachyon free dual resonance model [6]. Later I tried to stimulate interest in this model for its own sake at the Santa Fe meeting [7].

N QCD, which involves only the purely gluonic sector of QCD.

The $D = 10$ version of this model has come to be known as the type-0 string model because it has no supersymmetry [7,10,11]. Since it is formulated in the critical dimension its consistent coupling to closed strings is known. With the introduction of D3-branes, one can engineer its low energy limit to be Yang-Mills coupled to 6 adjoint scalar fields [12].

For our purposes, though, we take $D = 4 < 10$, so we work directly with the subcritical string, rather than try to embed QCD in a 10 dimensional critical string theory. Subcritical string theory is not well understood. It is believed that its consistent realization will involve a new scalar (Liouville) world sheet field which can be designed to cancel the conformal anomaly [13–15]. However, we still lack a completely satisfactory formulation of such theories. Recall that the unresolved issues are associated with our imperfect understanding of the closed string sector in subcritical theories. On the other hand the subcritical open string dual models are not only self-consistent and well understood, but they are also known to *imply* the existence and dynamics of closed strings via unitarity. We therefore adopt the working hypothesis that the appropriate closed string theory we seek can eventually be extracted from the open string multiloop diagrams [16,17].

Incidentally, although this is not usually done, we could put the $\mathcal{N} = 4/\mathrm{AdS}_5 \times \mathrm{S}^5$ correspondence in this same setting. We would first "lift" the $\mathcal{N} = 4$ theory to its simplest open string parent, in this case the Neveu-Schwarz-Ramond open superstring [4,18–20], vibrating in 10 dimensions but with ends fixed to a stack of D3-branes. Then we would "discover" the closed strings and their dynamics in the nonplanar diagrams, and finally we would interpret the sum of multiloop planar open string diagrams as a closed string background sourced by the D3-branes. Of course, one would still need to further recognize that the strong 't Hooft coupling limit coupled with $\alpha' \to 0$ would validate an effective field theory determination of this background to be $\mathrm{AdS}_5 \times \mathrm{S}^5$.

This article initiates a program to find the subcritical closed string theory that consistently couples to the four dimensional even G-parity NS open string, by analyzing the open string multiloop diagrams. We take the first step in this direction by reinterpreting the 1 loop diagrams in terms of closed strings. As observed in [16] the so-called "unitarity violating" pomeron cut that arises in these diagrams can be interpreted as a continuous mass spectrum for the closed strings. Alternatively one can associate this continuous mass spectrum with a holographic fifth dimension, suggesting that the closed string theory we seek is best formulated in at least five spacetime dimensions. The interactions between closed strings will only be revealed in diagrams with two or more loops.

The rest of the paper is organized as follows. In Sec. II we briefly review the construction of NS tree amplitudes for the scattering of any number of gluons. Section III is devoted to a study of the 1 loop gluon amplitudes in general. We begin by quoting the general formula for the M gluon 1 loop amplitude in the NS+ model, with enough derivation details to clearly establish the notation and meaning of the formula. We also include in this section a brief description of a very useful regularization of these formally divergent expressions due to Goddard, Neveu, and Scherk. Finally, we discuss the closed string interpretation including an explanation of the proper way to understand the "Pomeron cut." In Sec. IV we analyze the field theory limit of the one-loop diagram in enough detail to extract the renormalization group one-loop beta function coefficient. Though the coefficient for the NS+ model is the same as the one obtained earlier for the bosonic string by Metsaev and Tseytlin, the details of the calculation are sufficiently different to merit a complete treatment. We close the paper with Sec. V which contains further discussion of our results.

II. BRIEF REVIEW OF NS GLUON TREE AMPLITUDES

For our purposes in this article, we shall only need the old operator formalism of the dual resonance models. The Neveu-Schwarz model [4,21] makes use of integer moded bosonic oscillators $a_n^\mu = (a_{-n}^\mu)^\dagger$, with $a_0^\mu = \sqrt{2\alpha'}p^\mu$, and half integer moded fermionic oscillators $b_r^\mu = (b_{-r}^\mu)^\dagger$

$$[a_n^\mu, a_m^\nu] = \eta^{\mu\nu} n \delta_{n,-m}, \qquad \{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r,-s}. \quad (1)$$

Here and in the following, r, s will always be understood to be half odd integers and m, n to be integers. The string mass spectrum is given in terms of the Virasoro generator

$$L_0 = \sum_{n=1}^\infty a_{-n}\cdot a_n + \sum_{r=1/2}^\infty r b_{-r}\cdot b_r + \alpha' p^2 \equiv R + \alpha' p^2. \quad (2)$$

The physical string eigenstates satisfy

$$(L_0 - 1/2)|\mathrm{Phys}\rangle = L_n|\mathrm{Phys}\rangle = G_r|\mathrm{Phys}\rangle = 0, \qquad n, r > 0, \quad (3)$$

in the picture 2 Fock space [21]. We do not need the explicit forms for L_n, G_r in this paper. G-parity in picture 2 is just $G = -(-)^{2R}$. Thus the even G-parity states have the spectrum $\alpha' m_e^2 = -\alpha' p^2 = 0, 1, \ldots,$ and the odd G-parity states the spectrum $\alpha' m_o^2 = -\alpha' p^2 = -1/2, 1/2, 3/2, \ldots$. The lowest mass even G-parity state is $\epsilon \cdot b_{-1/2}|0, k\rangle$ with $k^2 = 0$ and $k \cdot \epsilon = 0$. This massless gauge particle state will be called the gluon in this article.

Vertex operators are constructed from the following world sheet fields, defined on the upper half complex plane $z = x + iy$, $y > 0$:

$$\mathcal{P}(z) = \sum_n a_n z^{-n}, \qquad H(z) = \sum_r b_r z^{-r},$$
$$V_0(k, z) = z^{\alpha' p^2} :e^{ik\cdot(q+i\sqrt{2\alpha'}\sum_{n\neq 0} a_n z^{-n}/n)} : z^{-\alpha' p^2}. \tag{4}$$

The gluon vertex operator is

$$V_\epsilon =: [\epsilon \cdot \mathcal{P}(1) + \sqrt{2\alpha'} k \cdot H(1) \epsilon \cdot H(1)] V_0(k, 1): \tag{5}$$

and the M gluon tree amplitude is then (in picture 2)

$$T_M = \langle 0, -k_1 | \epsilon_1 \cdot b_{1/2} V_{\epsilon_2} \frac{1}{L_0 - 1/2} V_{\epsilon_3} \cdots \frac{1}{L_0 - 1/2} \times V_{\epsilon_{M-1}} \epsilon_M \cdot b_{-1/2} | 0, k_M \rangle. \tag{6}$$

Note that because $k^2 = k \cdot \epsilon = 0$, the normal ordering of V_ϵ in the definition is not really necessary. Also notice that because the vertex operator commutes with G-parity, the poles in T_M only reveal even G-parity states: the odd G-parity states automatically decouple in these trees.

III. ONE-LOOP MULTIGLUON AMPLITUDES IN THE NS+ MODEL

The one-loop amplitudes for the Neveu-Schwarz model were first constructed by Goddard and Waltz [22], who evaluated the planar and nonplanar 1 loop diagrams for any number of odd G-parity tachyons, with vertex operator $ik \cdot HV_0(k, 1)$. The calculation is easily adapted to the gluon case by (1) using the gluon vertex operator and (2) projecting the trace onto even G-parity states by inserting the projector $P = (1 + G)/2$. We do this for the planar case in some detail.

$$\mathcal{M}_{\rm P} = \int_0^1 du_1 \cdots du_M \int \frac{d^D p}{(2\pi)^D} {\rm Tr} V_1 u_1^{R+\alpha' p_0^2 - 3/2} \cdots V_M u_M^{R+\alpha' p_0^2 - 3/2} \frac{1-(-)^{2R}}{2} \tag{7}$$

$$= \int dw \int \frac{d^D p}{(2\pi)^D} w^{\alpha' p^2 - 3/2} \prod_{i=2}^M \frac{dy_i}{y_i} \prod_i y_i^{2\alpha' p \cdot k_i} \prod_{i<j} y_j^{-2\alpha' k_i \cdot k_j} {\rm Tr} V_1(y_1) \cdots V_M(y_M) w^R \frac{1-(-)^{2R}}{2}. \tag{8}$$

Recall that here all external particles are massless. The even G-parity projection is easily handled by doing the calculation without the projection and then subtracting from it the expression obtained by reversing the signs of all the w^r with r half integral, and dividing the difference by 2. In the following we complete the calculation without the projector.

The integral over p is easily performed:

$$\int \frac{d^D p}{(2\pi)^D} \exp\Big\{\alpha' p^2 \ln w + 2\alpha' p \cdot \sum_i k_i \ln y_i\Big\} = \left(\frac{-1}{4\pi\alpha' \ln w}\right)^{D/2} \exp\Big\{-\alpha' \frac{(\sum_i k_i \ln y_i)^2}{\ln w}\Big\}. \tag{9}$$

We also need

$$\int \frac{d^D p}{(2\pi)^D} \exp\{\alpha' p^2 \ln w\} p^{\mu_1} \cdots p^{\mu_k} \equiv \left(\frac{-1}{4\pi\alpha' \ln w}\right)^{D/2} \langle p^{\mu_1} \cdots p^{\mu_k} \rangle, \tag{10}$$

where $\langle p^{\mu_1} \cdots p^{\mu_k} \rangle$ can be evaluated with a Wick expansion with contractions

$$\langle p^\mu p^\nu \rangle = \frac{-\eta^{\mu\nu}}{2\alpha' \ln w}. \tag{11}$$

Because $k_i^2 = 0$, we have

$$\Big(\sum_i k_i \ln y_i\Big)^2 = \frac{1}{2} \sum_{i\neq j} k_i \cdot k_j \Big(-\ln^2 \frac{y_i}{y_j} + \ln^2 y_i + \ln^2 y_j\Big) = -\sum_{i<j} k_i \cdot k_j \ln^2 \frac{y_i}{y_j}, \tag{12}$$

so

$$\mathcal{M}_{\rm P} = \int \frac{dw}{w} \prod_{i=2}^M \frac{dy_i}{y_i} w^{-1/2} \left(\frac{-1}{4\pi\alpha' \ln w}\right)^{D/2} \times \exp\Big\{\alpha' \sum_{i<j} k_i \cdot k_j \frac{\ln^2 y_i/y_j}{\ln w}\Big\} \prod_{i<j} y_j^{-2\alpha' k_i \cdot k_j} \times \langle {\rm Tr} V_1(y_1) \cdots V_M(y_M) w^R \rangle. \tag{13}$$

The variables y_i are given by

$$y_1 = 1, \qquad y_i = u_1 u_2 \cdots u_{i-1}, \qquad w = u_1 u_2 \cdots u_M \tag{14}$$

$$0 < w < y_M < y_{M-1} < \cdots < y_2 < y_1 = 1 \tag{15}$$

$$du_1 \cdots du_M = \frac{dy_2}{y_2} \cdots \frac{dy_M}{y_M} dw. \tag{16}$$

The gluon vertex operator is $V = e^{ik\cdot x}(\epsilon \cdot \mathcal{P} + \sqrt{2\alpha'} k \cdot H \epsilon \cdot H) \equiv e^{ik\cdot x} \hat{\mathcal{P}}$. Then

$$\langle \mathrm{Tr} V_1(y_1)\cdots V_M(y_M) w^R\rangle = \langle \hat{\mathcal{P}}(y_1)\cdots\hat{\mathcal{P}}(y_M)\rangle \frac{\prod_r(1+w^r)^D}{\prod_n(1-w^n)^D}\prod_{i<j}\Big[\Big(1-\frac{y_j}{y_i}\Big)\prod_n\frac{(1-w^n\frac{y_i}{y_j})(1-w^n\frac{y_j}{y_i})}{(1-w^n)^2}\Big]^{2\alpha' k_i\cdot k_j}$$
$$= \langle \hat{\mathcal{P}}(y_1)\cdots\hat{\mathcal{P}}(y_M)\rangle \frac{\prod_r(1+w^r)^D}{\prod_n(1-w^n)^D}\prod_{i<j} y_j^{2\alpha' k_i\cdot k_j}\prod_{i<j}\Big[2i\frac{\theta_1(\frac{1}{2i}\ln\frac{y_i}{y_j},\sqrt{w})}{\theta_1'(0,\sqrt{w})}\Big]^{2\alpha' k_i\cdot k_j}. \quad (17)$$

Here the $\langle\cdots\rangle$ is a correlator of a finite number of $\mathcal{P}$ and H world sheet fields determined by its Wick expansion with the following contraction rules

$$\langle\mathcal{P}(y_l)\rangle = \sqrt{2\alpha'}\sum_i k_i\Big[-\frac{\ln(y_i/y_l)}{\ln w}+\frac{1}{2}\frac{y_i+y_l}{y_l-y_i}+\sum_{n=1}^\infty\Big(\frac{y_i w^n}{y_l-y_i w^n}-\frac{y_l w^n}{y_i-y_l w^n}\Big)\Big]$$
$$\langle\mathcal{P}^\mu(y_i)\mathcal{P}^\nu(y_l)\rangle = \langle\mathcal{P}^\mu(y_i)\rangle\langle\mathcal{P}^\nu(y_l)\rangle + \eta^{\mu\nu}\Big[-\frac{1}{\ln w}+\frac{y_i y_l}{(y_i-y_l)^2}+\sum_{n=1}^\infty\Big(\frac{y_i y_l w^n}{(y_l-y_i w^n)^2}+\frac{y_i y_l w^n}{(y_i-y_l w^n)^2}\Big)\Big]$$
$$\langle H^\mu(y_i)H^\nu(y_j)\rangle^+ = \eta^{\mu\nu}\sum_r\frac{(y_j/y_i)^r+(wy_i/y_j)^r}{1+w^r} \quad (18)$$
$$\langle H^\mu(y_i)H^\nu(y_j)\rangle^- = \eta^{\mu\nu}\sum_r\frac{(y_j/y_i)^r-(wy_i/y_j)^r}{1-w^r}.$$

The $\pm$ superscript on the H contractions distinguishes the two types of traces over the b_r oscillators: for $+$ odd and even G-parity states contribute with the same sign, whereas for $-$ they contribute with opposite signs. In picture 2, the difference of the two traces projects out the odd G-parity states.

The Jacobi function θ_1 has the expansions

$$\theta_1(z,q) = -i\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}e^{(2n+1)iz}(-)^n \quad (19)$$

$$= 2q^{1/4}\sin z\prod_{n=1}^\infty(1-q^{2n})\prod_{n=1}^\infty(1-q^{2n}e^{2iz})(1-q^{2n}e^{-2iz}) \quad (20)$$

$$\frac{\theta_1(z,q)}{\theta_1'(0,q)} = \sin z\prod_{n=1}^\infty\frac{(1-q^{2n}e^{2iz})(1-q^{2n}e^{-2iz})}{(1-q^{2n})^2}. \quad (21)$$

Putting $q=e^{i\pi\tau}$, $\theta_1(z|\tau)\equiv\theta_1(z,q)$, the imaginary transform reads

$$\theta_1(z|\tau) = i(-i\tau)^{-1/2}e^{z^2/\pi i\tau}\theta_1\Big(\frac{z}{\tau}\Big|-\frac{1}{\tau}\Big). \quad (22)$$

We apply this formula with $w=e^{2i\pi\tau}$

$$2i\exp\Big(\frac{1}{2\ln w}\ln^2\frac{y_i}{y_j}\Big)\frac{\theta_1(\frac{1}{2i}\ln\frac{y_i}{y_j}|\tau)}{\theta_1'(0|\tau)} = \frac{\ln w}{\pi}\frac{\theta_1(\frac{\pi}{\ln w}\ln\frac{y_i}{y_j}|-\frac{1}{\tau})}{\theta_1'(0|-1/\tau)} \quad (23)$$

$$\frac{\theta_1(\frac{\pi}{\ln w}\ln\frac{y_i}{y_j}|-\frac{1}{\tau})}{\theta_1'(0|-1/\tau)} = \sin\frac{\theta_{ij}}{2}\times\prod_{n=1}^\infty\frac{(1-q^{2n}e^{i\theta_{ij}})(1-q^{2n}e^{-i\theta_{ij}})}{(1-q^{2n})^2}, \quad (24)$$

where $\theta_i\equiv 2\pi\ln y_i/\ln w$, $\theta_{ij}=\theta_i-\theta_j$, and $q=e^{-\pi i/\tau}$. Then $dy_i/y_i=\frac{\ln w}{2\pi}d\theta_i$ and $dw/w=-\ln^2 w\,dq/2\pi^2 q$. Thus

$$\frac{dw}{w}\frac{dy_2}{y_2}\cdots\frac{dy_M}{y_M} = \frac{-\ln w}{\pi}\Big[-\frac{\ln w}{2\pi}\Big]^M\frac{dq}{q}d\theta_2\cdots d\theta_M. \quad (25)$$

Because all external legs are massless, we have $2\sum_{i<j}k_i\cdot k_j=(\sum_i k_i)^2=0$ by momentum conservation. This means that constant factors raised to this power can be dropped: the factor $\ln w/\pi$ in the above formula can therefore be dropped when it is inserted into the amplitude integrand.

$$\langle\mathcal{P}(y_l)\rangle = \frac{2\pi}{-\ln w}\sqrt{2\alpha'}\sum_i k_i\Big[\frac{1}{2}\cot\frac{\theta_{il}}{2}+\sum_{n=1}^\infty\frac{2q^{2n}\sin\theta_{il}}{1-2q^{2n}\cos\theta_{il}+q^{4n}}\Big]$$
$$= \frac{2\pi}{-\ln w}\sqrt{2\alpha'}\sum_i k_i\Big[\frac{1}{2}\cot\frac{\theta_{il}}{2}+\sum_{n=1}^\infty\frac{2q^{2n}}{1-q^{2n}}\sin n\theta_{il}\Big]$$
$$= \frac{2\pi}{-\ln w}\sqrt{2\alpha'}\sum_i k_i\Big[\sum_{n=1}^\infty\frac{1+q^{2n}}{1-q^{2n}}\sin n\theta_{il}\Big] \quad (26)$$

$$\langle \mathcal{P}(y_i)\mathcal{P}(y_l)\rangle - \langle \mathcal{P}(y_i)\rangle\langle \mathcal{P}(y_l)\rangle = \frac{4\pi^2}{\ln^2 w}\left[\frac{1}{4}\csc^2\frac{\theta_{il}}{2} + \sum_{n=1}^{\infty}\frac{2q^{2n}(2q^{2n}-[1+q^{4n}]\cos\theta_{il})}{(1-2q^{2n}\cos\theta_{il}+q^{4n})^2}\right]$$
$$= \frac{4\pi^2}{\ln^2 w}\left[\frac{1}{4}\csc^2\frac{\theta_{il}}{2} - \sum_{n=1}^{\infty} n\frac{2q^{2n}}{1-q^{2n}}\cos n\theta_{il}\right] = \frac{4\pi^2}{\ln^2 w}\left[-\sum_{n=1}^{\infty} n\frac{1+q^{2n}}{1-q^{2n}}\cos n\theta_{il}\right] \quad (27)$$

$$\langle H(y_i)H(y_j)\rangle^+ = -\frac{2\pi}{\ln w}\left[\frac{1}{2\sin(\theta_{ji}/2)} + 2\sin\frac{\theta_{ji}}{2}\sum_{n=1}^{\infty}(-)^n\frac{q^n(1+q^{2n})}{1-2q^{2n}\cos\theta_{ji}+q^{4n}}\right]$$
$$= -\frac{2\pi}{\ln w}\left[\frac{1}{2\sin(\theta_{ji}/2)} - 2\sum_r\frac{q^{2r}\sin r\theta_{ji}}{1+q^{2r}}\right] = \frac{2\pi i}{\ln w}\sum_r\frac{e^{ir\theta_{ji}}+q^{2r}e^{-ir\theta_{ji}}}{1+q^{2r}} = -\frac{2\pi}{\ln w}\sum_r\frac{1-q^{2r}}{1+q^{2r}}\sin r\theta_{ji} \quad (28)$$

$$\langle H(y_i)H(y_j)\rangle^- = -\frac{2\pi}{\ln w}\left[\frac{\cos(\theta_{ji}/2)}{2\sin(\theta_{ji}/2)} - 2\sum_n\frac{q^{2n}\sin n\theta_{ji}}{1+q^{2n}}\right]. \quad (29)$$

In these expressions we have suppressed the spacetime indices carried by the operators on the left as well as the $\eta^{\mu\nu}$ factors on the right. Note that the first forms of each contraction show a singularity at $\theta = 0$, whereas this singular behavior is hidden in the second forms. Since these singularities correspond to poles in the invariants of the process, it is tempting to associate them with one particle reducible diagrams and drop their contributions when extracting the 1PIR contributions. This procedure actually seems to work in the case of the bosonic string. However, for the Neveu-Schwarz model we are considering here, some of these apparently "reducible" contributions must be included in the "1PIR" answer. This is because in constructing the one-loop diagrams in the picture 2 formalism, one has implicitly carried out some integrations by parts, and rearranged what one calls reducible and irreducible.

Notice that after the Jacobi transform the correlators all acquire factors of $-2\pi/\ln w$ in such a way that each contribution to $\langle\cdots\rangle$ acquires the same factor $(-2\pi/\ln w)^M$, where M is the number of external legs in the loop diagram. These factors compensate factors from the Jacobian of the change of integration variables. Thus

$$\langle\cdots\rangle_{y,w}\frac{dw}{w}\frac{dy_2}{y_2}\cdots\frac{dy_M}{y_M} = \frac{-\ln w}{\pi}\langle\cdots\rangle_{\theta,q}\frac{dq}{q}d\theta_2\cdots d\theta_M, \quad (30)$$

where $\langle\cdots\rangle_{\theta,q}$ is computed without the $-2\pi/\ln w$ factors.

The various partition functions have the following transformation properties:

$$w^{1/24}\prod_n(1-w^n) = \left(-\frac{\ln w}{2\pi}\right)^{-1/2} q^{1/12}\prod_n(1-q^{2n}) \quad (31)$$

$$w^{-1/48}\prod_r(1+w^r) = q^{-1/24}\prod_r(1+q^{2r}) \quad (32)$$

$$w^{1/24}\prod_n(1+w^n) = \frac{1}{\sqrt{2}}q^{-1/24}\prod_r(1-q^{2r}) \quad (33)$$

$$w^{-1/48}\prod_r(1-w^r) = \sqrt{2}q^{1/12}\prod_n(1+q^{2n}), \quad (34)$$

where $n = 1, 2, \cdots$, $r = 1/2, 3/2, \cdots$. The partition function factor in the loop integrand is

$$\frac{\prod_r(1+w^r)^{D-2}}{\prod_n(1-w^n)^{D-2}} = w^{(D-2)/16}q^{-(D-2)/8}\left(-\frac{\ln w}{2\pi}\right)^{(D-2)/2} \times \frac{\prod_r(1+q^{2r})^{D-2}}{\prod_n(1-q^{2n})^{D-2}} \quad (35)$$

in the critical dimension (here $D = 10$) after removal of spurious states. In projecting out the odd G-parity states we also need the partition function with $w^r \to -w^r$:

$$\frac{\prod_r(1-w^r)^{D-2}}{\prod_n(1-w^n)^{D-2}} = w^{(D-2)/16}2^{(D-2)/2}\left(-\frac{\ln w}{2\pi}\right)^{(D-2)/2} \times \frac{\prod_n(1+q^{2n})^{D-2}}{\prod_n(1-q^{2n})^{D-2}}, \quad (36)$$

also in the critical dimension (here $D = 10$).

For $D < 10$ the physical state conditions eliminate fewer states than in the critical dimension, though all the physical states still have positive norm [23]. In this case the methodology for removing spurious states from loops is that of Brower and Thorn [24], adapted to the Neveu-Schwarz case in [22]. In the subcritical case the null spurious states are all of the form $G_{-1/2}|\mathrm{Phys}, L_0 = 0\rangle$. Consequently, as in [24] the partition function power is reduced from D to $D - 1$ and, because of this restricted form of the null states, there is a further factor of $1 - w^{1/2} = (1 - w)/(1 + w^{1/2})$. Roughly speaking, we may say that only one component of a_n^μ and one component of $b_{n-1/2}^\mu$ are removed when $n > 1$, but two components of both a_1^μ and $b_{1/2}^\mu$ are removed:

$$(1 - w^{1/2})\frac{\prod_r(1+w^r)^{D-1}}{\prod_n(1-w^n)^{D-1}} = (1 - w^{1/2})w^{(D-1)/16}q^{-(D-1)/8}\left(-\frac{\ln w}{2\pi}\right)^{(D-1)/2}\frac{\prod_r(1+q^{2r})^{D-1}}{\prod_n(1-q^{2n})^{D-1}}, \tag{37}$$

and for $w^r \to -w^r$

$$(1 + w^{1/2})\frac{\prod_r(1-w^r)^{D-1}}{\prod_n(1-w^n)^{D-1}} = (1 + w^{1/2})w^{(D-1)/16}2^{(D-1)/2}\left(-\frac{\ln w}{2\pi}\right)^{(D-1)/2}\frac{\prod_n(1+q^{2n})^{D-1}}{\prod_n(1-q^{2n})^{D-1}}. \tag{38}$$

After the change of integration variables to q, θ, the left over factors of w and $\ln w$ from Eqs. (13), (37), and (38) are as follows for $D < 10$:

$$w^{-1/2}(1 \mp w^{1/2})w^{(D-1)/16} = (1 \mp w^{1/2})w^{(D-9)/16} \tag{39}$$

$$\left(\frac{-1}{4\pi\alpha' \ln w}\right)^{D/2}\left(-\frac{\ln w}{2\pi}\right)^{(D-1)/2}\frac{-\ln w}{\pi} = 2\left(\frac{1}{8\pi^2\alpha'}\right)^{D/2}\left(-\frac{\ln w}{2\pi}\right)^{1/2} = 2\left(\frac{1}{8\pi^2\alpha'}\right)^{D/2}\left(-\frac{\pi}{\ln q}\right)^{1/2}. \tag{40}$$

In contrast for the critical dimension all the w dependence of these factors cancels:

$$w^{-1/2}w^{(D-2)/16} = w^{(D-10)/16} \to 1 \tag{41}$$

$$\left(\frac{-1}{4\pi\alpha' \ln w}\right)^{D/2}\left(-\frac{\ln w}{2\pi}\right)^{(D-2)/2}\frac{-\ln w}{\pi} = 2\left(\frac{1}{8\pi^2\alpha'}\right)^{D/2} \to 2\left(\frac{1}{8\pi^2\alpha'}\right)^{5}. \tag{42}$$

Incidentally, for the subcritical bosonic string ($D < 26$) these extra factors are

$$w^{-1}(1-w)w^{(D-1)/24} = (1-w)w^{(D-25)/24} \tag{43}$$

$$\left(\frac{-1}{4\pi\alpha' \ln w}\right)^{D/2}\left(-\frac{\ln w}{2\pi}\right)^{(D-1)/2}\frac{-\ln w}{\pi} = 2\left(\frac{1}{8\pi^2\alpha'}\right)^{D/2}\left(-\frac{\ln w}{2\pi}\right)^{1/2} = 2\left(\frac{1}{8\pi^2\alpha'}\right)^{D/2}\left(-\frac{\pi}{\ln q}\right)^{1/2}. \tag{44}$$

In particular, the factors of $\ln w$ work out in exactly the same way. Of course, for $D = 26$ the extra factors cancel but now $5 \to 13$.

Our expressions for the one-loop amplitude are formal since the integrals diverge in various regions. To give them meaning, a regularization must be found, and one should then be able to show that divergences can be absorbed in renormalization of parameters. Neveu and Scherk [25], following an earlier suggestion of Goddard [26], showed that the divergence for $q \to 1$ can be regulated by temporarily suspending energy momentum conservation by an amount p: $\sum_i k_i + p = 0$. This works because essentially one is injecting momentum p into the boundary of the planar loop with no particles attached: it can be interpreted as the momentum of a closed string spurion. In the following we shall refer to this procedure as the GNS regularization. It has a very interesting feature that is illustrated by a simple example in field theory in the appendix. With $p \neq 0$ the two legs of an off shell propagator with a self energy insertion would have poles in different variables, say p_1^2 and $p_2^2 = (p + p_1)^2$. When the mass shift is zero, as is the case with a gauge particle, they coalesce to a single pole as $p \to 0$, say $(Z-1)/p_1^2$. But then when $p \neq 0$, the residues of the poles in p_1^2 and p_2^2 are each $(Z-1)/2$. If the self energy insertion is on an external leg of an S-matrix element, one of these legs say p_1 is amputated and put on shell. If $p \neq 0$ this produces a wave function renormalization factor $(Z-1)/2$, *not* the $(Z-1)$ that would arise if $p = 0$ from the start. This factor of $1/2$ is precisely what is needed to end up with a properly normalized scattering amplitude. Thus the GNS regulation is particularly apt for string theory amplitudes which are of necessity always on shell. Using it, one-loop on-shell diagram calculations will automatically be correctly normalized, without the customary $\sqrt{Z}$ adjustments that are required in usual Feynman diagram evaluations!

In summary, we quote the one-loop planar M gluon NS+ amplitude for $D < 10$:

$$\mathcal{M}_{\rm P} = \tfrac{1}{2}(\mathcal{M}_{\rm P}^+ - \mathcal{M}_{\rm P}^-) \tag{45}$$

where, in cylinder variables, $\ln q = 2\pi^2/\ln w$,

$$\mathcal{M}_{\rm P}^+ = \left(\frac{1}{8\pi^2\alpha'}\right)^{D/2}\int\prod_{k=2}^{M}d\theta_k\int_0^1\frac{dq}{q}\sqrt{\frac{-\pi}{\ln q}}q^{-(D-1)/8}(w^{(D-9)/16} - w^{(D-1)/16})\frac{\prod_r(1+q^{2r})^{D-1}}{\prod_n(1-q^{2n})^{D-1}} \times\prod_{l<m}[\psi(\theta_m-\theta_l,q)]^{2\alpha' k_l\cdot k_m}\langle\hat{\mathcal{P}}_1\hat{\mathcal{P}}_2\cdots\hat{\mathcal{P}}_M\rangle^+ \tag{46}$$

$$\mathcal{M}_{\mathrm{P}}^- = \left(\frac{1}{8\pi^2\alpha'}\right)^{D/2}\int\prod_{k=2}^{M} d\theta_k \int_0^1 \frac{dq}{q}\sqrt{\frac{-\pi}{\ln q}}\,2^{(D-1)/2}(w^{(D-9)/16}+w^{(D-1)/16})\frac{\prod_n(1+q^{2n})^{D-1}}{\prod_n(1-q^{2n})^{D-1}}$$
$$\times\prod_{l<m}[\psi(\theta_m-\theta_l,q)]^{2\alpha' k_l\cdot k_m}\langle\hat{\mathcal{P}}_1\hat{\mathcal{P}}_2\cdots\hat{\mathcal{P}}_M\rangle^- \tag{47}$$

$$\psi(\theta,q) = \sin\frac{\theta}{2}\prod_n\frac{(1-q^{2n}e^{i\theta})(1-q^{2n}e^{-i\theta})}{(1-q^{2n})^2}, \qquad \hat{\mathcal{P}} = \epsilon\cdot\mathcal{P} + \sqrt{2\alpha'}k\cdot H\epsilon\cdot H, \tag{48}$$

where the average $\langle\cdots\rangle$ is evaluated with contractions:

$$\langle\mathcal{P}_l\rangle = \sqrt{2\alpha'}\sum_i k_i\left[\frac{1}{2}\cot\frac{\theta_{il}}{2} + \sum_{n=1}^{\infty}\frac{2q^{2n}}{1-q^{2n}}\sin n\theta_{il}\right] \tag{49}$$

$$\langle\mathcal{P}_i\mathcal{P}_l\rangle - \langle\mathcal{P}_i\rangle\langle\mathcal{P}_l\rangle = \frac{1}{4}\csc^2\frac{\theta_{il}}{2} - \sum_{n=1}^{\infty} n\frac{2q^{2n}}{1-q^{2n}}\cos n\theta_{il} \tag{50}$$

$$\langle H_iH_j\rangle^+ = \frac{1}{2\sin(\theta_{ji}/2)} - 2\sum_r\frac{q^{2r}\sin r\theta_{ji}}{1+q^{2r}} \tag{51}$$

$$\langle H_iH_j\rangle^- = \frac{\cos(\theta_{ji}/2)}{2\sin(\theta_{ji}/2)} - 2\sum_n\frac{q^{2n}\sin n\theta_{ji}}{1+q^{2n}}, \tag{52}$$

and we have again suppressed spacetime indices. Finally, the range of integration is

$$0 = \theta_1 < \theta_2 < \cdots < \theta_N < 2\pi. \tag{53}$$

In these formulas r ranges over positive half odd integers, n over positive integers, and $l, m \in [1, \cdots, M]$.

It is useful to visualize the planar loop diagram we have just quoted as in Fig. 1. It shows that the divergence encountered as $q \to 0$ can be interpreted as a closed string emission into the vacuum. It also shows graphically the physical appropriateness of the GNS regularization scheme! To discover the closed string spectrum one can examine the 1 loop nonplanar diagram shown in Fig. 2 This diagram allows the closed string to propagate with nonzero momentum K. The big qualitative difference with the planar 1-loop amplitude [22] is that K^2 now enters the exponent of q [see (46)]:

$$q^{-(D-1)/8} \to q^{-(D-1)/8+\alpha' K^2/2} \tag{54}$$

so that $\mathcal{M}_{\mathrm{NP}}^+$ has a closed string cut starting at $\alpha' K^2 = (D-1)/4$. Interestingly, the closed string cut in $\mathcal{M}_{\mathrm{NP}}^-$ [see (47)] starts instead at $K^2 = 0$.

As shown in [16] we can interpret the "unitarity violating" closed string cut in nonplanar diagrams as simply reflecting a continuous mass spectrum. To see this let us rewrite the new factors in the $D < 10$ nonplanar integrand not present for critical dimension $D = 10$:

$$\sqrt{\frac{-\pi}{\ln q}}(w^{(D-9)/16} \mp w^{(D-1)/16}) = \int\frac{d\mu}{2}q^{\mu^2/4}\left(\cosh\mu\sqrt{\frac{9-D}{16}} \mp \cos\mu\sqrt{\frac{D-1}{16}}\right) = \int d\mu\, q^{\mu^2/4}\left\{\begin{matrix}\sinh\frac{\mu\gamma_+}{2}\ \sinh\frac{\mu\gamma_-}{2}\\ \cosh\frac{\mu\gamma_+}{2}\ \cosh\frac{\mu\gamma_-}{2}\end{matrix}\right. \tag{55}$$

$$\gamma_\pm = \sqrt{\frac{9-D}{16}} \pm i\sqrt{\frac{D-1}{16}}. \tag{56}$$

Thus we can think of the integral over μ as an integral over a "momentum" in a $(D+1)$th dimension. Then the sinh (cosh) factors can be interpreted as momentum space wave functions. Each is a linear combination of two eigenstates of the "position" operator $q \equiv i\partial/\partial\mu$ with eigenvalues $q = \pm i\gamma_+/2$ at one end of the cylinder and $q = \pm i\gamma_-/2$ at the other end. Let us represent $(D+1)$th dimension by a world sheet scalar field ϕ, whose zero mode is q. Then we see that the one-loop diagram is a sum of terms on which Dirichlet conditions on ϕ are imposed: Open strings end

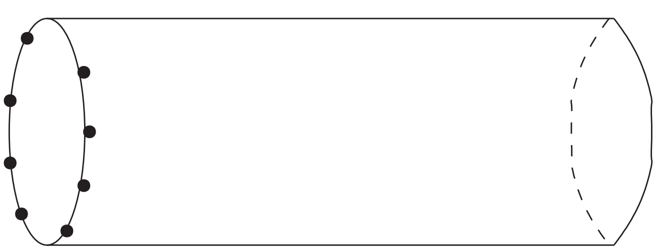

FIG. 1. World sheet of the planar loop represented as a cylinder. The length of the cylinder is proportional to $-\ln q$.

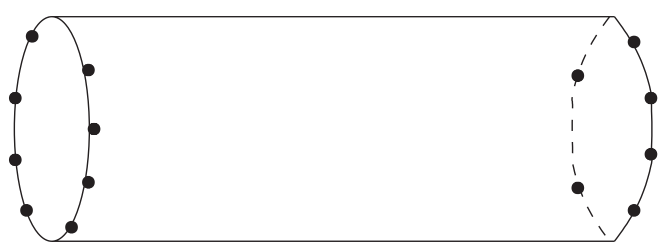

FIG. 2. World sheet for a nonplanar open string loop diagram.

on "Dp-branes" in $D+1$ dimensional closed string theory, with $p = D-1$. We should therefore think of the closed strings as propagating in the $D+1$ dimensional bulk, and we have a holographic interpretation. Then there is a tachyon pole at $\alpha'(K^2+\mu^2)/2 = (D-1)/8$ in $\mathcal{M}^+_{\rm NP}$, but no massless graviton poles. However, there are massless RR closed string poles in $\mathcal{M}^-_{\rm NP}$. Specializing to $D=4$, so that the bulk is 5 dimensional, the RR tensor structures correspond to scalar, vector, and antisymmetric tensor fields $(S, A_\mu, A_{\mu\nu})$. We should expect that the planar diagram sum should resolve the IR issues connected to the tachyon and the RR massless states in an interesting way.

IV. THE FIELD THEORY LIMIT: ASYMPTOTIC FREEDOM

For the bosonic string the field theory limit of the uv divergence structure of the one-loop diagrams has been carefully analyzed by Metsaev and Tseytlin [27], and we follow their logic closely. We shall specialize to the planar case, not only for simplicity, but also because our main interest is the relationship to large N QCD, which only includes planar graphs. It is enough to examine the 2 and 3 gluon scattering amplitudes to extract the one-loop renormalization group coefficient.

A. The two gluon function

The two point function controls the perturbative mass shifts, so that the two gluon function should vanish on mass shell, because gauge particles must remain massless in perturbation theory. Let us examine the θ integration at fixed q. First for the bosonic string, we consider the coefficient of $\epsilon_1\cdot\epsilon_2$:

$$\mathcal{M}_2^{\rm Bose} = \int_0^{2\pi} d\theta\left(\sin\frac{\theta}{2}\prod_{n=1}^\infty\frac{(1-q^{2n}e^{i\theta})(1-q^{2n}e^{-i\theta})}{(1-q^{2n})^2}\right)^{2\alpha' k_1\cdot k_2}\times\left[\frac{1}{4}\csc^2\frac{\theta}{2} - \sum_{n=1}^\infty n\frac{2q^{2n}}{1-q^{2n}}\cos n\theta\right]. \tag{57}$$

With no regularization, $k_2 = -k_1$, $k_1^2 = 0$, $k_1\cdot\epsilon_1 = k_1\cdot\epsilon_2 = 0$, $k_1\cdot k_2 = -k_1^2 = 0$, this expression reduces to

$$\int_0^{2\pi} d\theta \frac{1}{4}\csc^2\frac{\theta}{2},$$

which is decidedly not zero. However, with the Goddard-Neveu-Scherk (GNS) regularization, $k_2 = -k_1 - p$, so $2k_1\cdot k_2 = (k_1+k_2)^2 = p^2$, and we have instead [25]

$$\frac{1}{4}\int_0^{2\pi} d\theta\left(\sin\frac{\theta}{2}\right)^{\alpha' p^2 - 2} = \frac{1}{2}\frac{\Gamma(1/2)\Gamma(-1/2+\alpha' p^2/2)}{\Gamma(\alpha' p^2/2)} \sim -\frac{\pi\alpha' p^2}{2} \to 0 \tag{58}$$

as $p\to 0$. Thus in the GNS regularization the gluon mass shift is zero as it should be. Anticipating integrals done in the next section we quote the $p\to 0$ behavior of

$$\mathcal{M}_2^{\rm Bose} \sim \pi\alpha' p^2\left[-\frac{1}{2} + 4\sum_{n=1}^\infty\frac{q^{2n}}{(1-q^{2n})^2}\right]. \tag{59}$$

We see from this calculation that the original divergence at $\theta = 0, 2\pi$ was just due to the integral representation of a pole at $\alpha' p^2 = 1$. Since there is no pole at $p^2 = 0$ the analytic continuation to $p^2 = 0$ should be finite. The fact that it is actually 0 is very welcome here, and is very much due to the stringy pole structure of the gamma functions.

The 2 gluon function in the NS+ model is a similar story. Since its vanishing follows from the integration over θ the fact that the q dependent factors are different plays no role. One gets a $\mathcal{PP}$ correlator whose integral over θ gives 0 just as in the bosonic string. The new feature is the correlator,

$$2\alpha'\langle k_1\cdot H\epsilon_1\cdot H k_2\cdot H\epsilon_2\cdot H\rangle^\pm \equiv 2\alpha'(k_2\cdot\epsilon_1 k_1\cdot\epsilon_2 - k_1\cdot k_2\epsilon_1\cdot\epsilon_2)C^\pm = 2\alpha'\left(p\cdot\epsilon_1 p\cdot\epsilon_2 - \frac{p^2}{2}\epsilon_1\cdot\epsilon_2\right)C^\pm$$
$$C^+ = \left[\frac{1}{2\sin(\theta/2)} - 2\sum_r\frac{q^{2r}\sin r\theta}{1+q^{2r}}\right]^2 = \left[\frac{1}{4\sin^2(\theta/2)} - \frac{2}{\sin(\theta/2)}\sum_r\frac{q^{2r}\sin r\theta}{1+q^{2r}} + 4\sum_{r,s}\frac{q^{2(r+s)}\sin r\theta\sin s\theta}{(1+q^{2r})(1+q^{2s})}\right] \tag{60}$$

$$C^- = \left[\frac{\cos(\theta/2)}{2\sin(\theta/2)} - 2\sum_n\frac{q^{2n}\sin n\theta}{1+q^{2n}}\right]^2 = \left[\frac{1}{4\sin^2(\theta/2)} - \frac{1}{4} - \frac{2\cos(\theta/2)}{\sin(\theta/2)}\sum_n\frac{q^{2r}\sin n\theta}{1+q^{2n}} + 4\sum_{m,n}\frac{q^{2(m+n)}\sin m\theta\sin m\theta}{(1+q^{2m})(1+q^{2n})}\right]. \tag{61}$$

This expression nominally vanishes as p^2 for $p\to 0$. When it is inserted in the integrand of the two point function, the integral over θ of the first term in square brackets vanishes just as in $\langle\mathcal{PP}\rangle$, and the integral of the remaining terms gives a finite contribution. Thus the $O(p^2)$ estimate for the integrand applies also for the integral over θ. Thus the new contribution in the NS+ case vanishes as $O(p^2)$ as $p\to 0$. Since the coefficient of $\epsilon_1\cdot\epsilon_2$ already has an explicit p^2, the $p\to 0$ behavior is obtained by setting all $k_i\cdot k_j$ in the exponents to zero and using the integrals,

$$\int_0^{2\pi} d\theta \frac{\sin r\theta}{\sin\theta/2} = 2\pi, \qquad \int_0^{2\pi} d\theta \sin r\theta \sin s\theta = \pi\delta_{rs}, \qquad \int_0^{2\pi} d\theta \cot(\theta/2)\sin n\theta = 2\pi, \tag{62}$$

to obtain for the new contribution to the coefficient of $\epsilon_1 \cdot \epsilon_2$

$$-\pi\alpha' p^2\left[-\sum_r \frac{4q^{2r}}{1+q^{2r}} + \sum_r \frac{4q^{4r}}{(1+q^{2r})^2}\right] = -\pi\alpha' p^2\left[-4\sum_r \frac{q^{2r}}{(1+q^{2r})^2}\right]$$
$$-\pi\alpha' p^2\left[-\frac{1}{2} - \sum_n \frac{4q^{2n}}{1+q^{2n}} + \sum_n \frac{4q^{4n}}{(1+q^{2n})^2}\right] = -\pi\alpha' p^2\left[-\frac{1}{2} - 4\sum_n \frac{q^{2n}}{(1+q^{2n})^2}\right]. \tag{63}$$

Combining with the bose contribution gives for the Neveu-Schwarz 2 gluon function

$$\mathcal{M}_2^{\mathrm{NS},+} \sim \pi\alpha' p^2\left[-\frac{1}{2} + 4\sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2} + 4\sum_r \frac{q^{2r}}{(1+q^{2r})^2}\right], \qquad \mathcal{M}_2^{\mathrm{NS},-} \sim \pi\alpha' p^2\left[4\sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2} + 4\sum_n \frac{q^{2n}}{(1+q^{2n})^2}\right]. \tag{64}$$

B. Three gluon function

We focus here on the polarization structure $\epsilon_1 \cdot \epsilon_2\sqrt{2\alpha'}k_1 \cdot \epsilon_3$, which is one cyclic ordering of the polarization structure of the 3 gluon vertex in Yang-Mills theory. For the bosonic string 1 loop 3 gluon function the coefficient of this structure is

$$\begin{aligned}\mathcal{M}_3 = \int[dq]\int_0^{2\pi} d\theta_3 \int_0^{\theta_3} d\theta_2\left[\frac{1}{4}\csc^2\frac{\theta_2}{2} - \sum_{n=1}^{\infty} n\frac{2q^{2n}}{1-q^{2n}}\cos n\theta_2\right] \\ \times\left[\frac{\sin(\theta_2/2)}{2\sin(\theta_3/2)\sin(\theta_{32}/2)} + \sum_{n=1}^{\infty}\frac{2q^{2n}}{1-q^{2n}}(\sin n\theta_{32} - \sin n\theta_3)\right]\left[\sin\frac{\theta_2}{2}\prod_{n=1}^{\infty}\frac{(1-q^{2n}e^{i\theta_2})(1-q^{2n}e^{-i\theta_2})}{(1-q^{2n})^2}\right]^{2\alpha' k_1\cdot k_2} \\ \times\left[\sin\frac{\theta_3}{2}\prod_{n=1}^{\infty}\frac{(1-q^{2n}e^{i\theta_3})(1-q^{2n}e^{-i\theta_3})}{(1-q^{2n})^2}\right]^{2\alpha' k_1\cdot k_3}\left[\sin\frac{\theta_{32}}{2}\prod_{n=1}^{\infty}\frac{(1-q^{2n}e^{i\theta_{32}})(1-q^{2n}e^{-i\theta_{32}})}{(1-q^{2n})^2}\right]^{2\alpha' k_2\cdot k_3},\end{aligned} \tag{65}$$

where we include all momentum independent factors in $[dq]$. Metsaev and Tseytlin [27] extract the uv divergences in the field theory limit by first managing the θ integrals. They identify the 1PIR contribution by setting the exponents to zero and replacing the singular terms in the remaining factors by their formal expansions

$$\frac{1}{4}\csc^2\frac{\theta}{2} \to -\sum_{n=1} n\cos n\theta, \qquad \frac{1}{2}\cot\frac{\theta}{2} \to \sum_{n=1}^{\infty}\sin n\theta. \tag{66}$$

Then the θ integrals are elementary with the result

$$2\pi\sum_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1-q^{2n}}\right)^2 \to 2\pi\left[-\frac{1}{2} + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\right], \tag{67}$$

where the formal sum $\sum_n 1$ has been interpreted as $\zeta(0) = -1/2$.

To complete the calculation we need to extract the reducible contributions, which we do by employing the GNS regularization of the string loop integral. So we introduce a spurion momentum p and take $p + k_1 + k_2 + k_3 = 0$. We send $p \to 0$ at the end of the calculation. With $p \neq 0$ the on-shell condition on the k's allows $k_i \cdot k_j \neq 0$. Let us first extract the pole at $k_1 \cdot k_2 = 0$, which comes from the region $\theta_2 \approx 0$. Doing the integral over the region $0 < \theta_2 < \epsilon$ leads, for the θ integration, to

$$\begin{aligned}\frac{1}{2\alpha' k_1\cdot k_2}\int_0^{2\pi} d\theta_3\left[\frac{1}{4}\csc^2\frac{\theta_3}{2} - \sum_{n=1}^{\infty} n\frac{2q^{2n}}{1-q^{2n}}\cos n\theta_3\right] \\ \times\left[\sin\frac{\theta_3}{2}\prod_{n=1}^{\infty}\frac{(1-q^{2n}e^{i\theta_3})(1-q^{2n}e^{-i\theta_3})}{(1-q^{2n})^2}\right]^{2\alpha'(k_1+k_2)\cdot k_3}\end{aligned}$$

Defining

$$\begin{aligned}f(z) = \int_0^{2\pi} d\theta\left[\frac{1}{4}\csc^2\frac{\theta}{2} - \sum_{n=1}^{\infty} n\frac{2q^{2n}}{1-q^{2n}}\cos n\theta\right] \\ \times\left[\sin\frac{\theta}{2}\prod_{n=1}^{\infty}\frac{(1-q^{2n}e^{i\theta})(1-q^{2n}e^{-i\theta})}{(1-q^{2n})^2}\right]^z,\end{aligned} \tag{68}$$

we are interested in its small z behavior. We can expand the infinite product factors

$$\left[\prod_{n=1}^{\infty}\frac{(1-q^{2n}e^{i\theta})(1-q^{2n}e^{-i\theta})}{(1-q^{2n})^2}\right]^z = 1 + z\sum_{n=1}^{\infty}\ln\frac{(1-q^{2n}e^{i\theta})(1-q^{2n}e^{-i\theta})}{(1-q^{2n})^2} + O(z^2) = 1 + z\sum_{m=1}^{\infty}\frac{1}{m}\frac{2q^{2m}}{1-q^{2m}}(1-\cos m\theta) + O(z^2). \quad (69)$$

Working first with the contributions to the 1 term, we find

$$\int_0^{2\pi} d\theta \frac{1}{4}\csc^2\frac{\theta}{2}\left[\sin\frac{\theta}{2}\right]^z = \frac{1}{2}\frac{\Gamma((z-1)/2)\Gamma(1/2)}{\Gamma(z/2)} = -\frac{\pi}{2}z + O(z^2) \quad (70)$$

$$\int_0^{2\pi} d\theta \cos n\theta\left[\sin\frac{\theta}{2}\right]^z = -\frac{z}{2n}\int_0^{2\pi} d\theta \frac{\sin n\theta}{\sin(\theta/2)} \times \cos\frac{\theta}{2}\left[\sin\frac{\theta}{2}\right]^z \sim -\frac{\pi}{n}z. \quad (71)$$

Thus the 1-term contribution is

$$1-\text{term} \sim \pi z\left[-\frac{1}{2} + \sum_{n=1}^{\infty}\frac{2q^{2n}}{1-q^{2n}}\right] + O(z^2). \quad (72)$$

To find the remaining terms we use

$$\int_0^{2\pi} d\theta \frac{1}{4}(1-\cos m\theta)\csc^2\frac{\theta}{2} = \pi m \quad (73)$$

$$\int_0^{2\pi} d\theta (1-\cos m\theta)\cos n\theta = -\pi\delta_{mn}, \quad (74)$$

to get

$$\text{Remaining terms} = \pi z\left[\sum_{n=1}^{\infty}\frac{2q^{2n}}{1-q^{2n}} + \sum_{n=1}^{\infty}\frac{4q^{4n}}{(1-q^{2n})^2}\right] + O(z^2), \quad (75)$$

so, all together,

$$f(z) = \pi z\left[-\frac{1}{2} + \sum_{n=1}^{\infty}\frac{4q^{2n}}{1-q^{2n}} + \sum_{n=1}^{\infty}\frac{4q^{4n}}{(1-q^{2n})^2}\right] + O(z^2) = \pi z\left[-\frac{1}{2} + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\right] + O(z^2). \quad (76)$$

Now for this contribution $z = 2\alpha' k_3\cdot(k_1+k_2) = -2\alpha' k_3\cdot(p+k_3) = -2\alpha' p\cdot k_3$, whereas $2k_1\cdot k_2 = (k_1+k_2)^2 = (p+k_3)^2 = p^2 + 2p\cdot k_3$. Thus the reducible contribution with pole in $k_1\cdot k_2$ is

$$-\pi\frac{p\cdot k_3}{p\cdot k_3 + p^2/2}\left[-\frac{1}{2} + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\right] \to -\pi\left[-\frac{1}{2} + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\right]. \quad (77)$$

Notice that this is just $-1/2$ times the 1PIR contribution found in [27], as expected for usual schemes for wave function renormalization factors. There are two other reducible contributions to the three gluon amplitude associated with poles in $k_1\cdot k_3$ ($\theta_3-\theta_2\approx 0$) and $k_2\cdot k_3$ ($\theta_3\approx 2\pi$). But inspection of the integrand of the 1 loop three gluon amplitude in these regions shows that these contributions will be identical to the first. Thus the net renormalization for the three gluon scattering amplitude will be $(1-3/2) = -1/2$ times the 1PIR result ($(D-26)/24$) found in [27]:

$$\left(1-\frac{3}{2}\right)\frac{D-26}{24} = -\frac{D-26}{48}. \quad (78)$$

Notice that for $D=4$ this goes to $11/24$: the factor of 11 is just the well-known 11 that occurs in the one-loop Yang-Mills running coupling.

We should obtain this same result in the NS+ model, but the details of the calculation are different in a very interesting way. The measure factors are different of course, but in the field theory limit $w\sim 0$ the difference is that the factor $(1-w)^{-D+2}/w \sim (D-2) + 1/w$ in the bosonic string measure is replaced by $(1+\sqrt{w})^{D-2}/\sqrt{w} \sim (D-2) + 1/\sqrt{w}$ in the Neveu-Schwarz measure. In addition, the NS loop integrand involves a more complicated correlator

$$\langle(\epsilon_1\cdot\mathcal{P} + \sqrt{2\alpha'}k_1\cdot H\epsilon_1\cdot H)(\epsilon_2\cdot\mathcal{P} + \sqrt{2\alpha'}k_2\cdot H\epsilon_2\cdot H)(\epsilon_3\cdot\mathcal{P} + \sqrt{2\alpha'}k_3\cdot H\epsilon_3\cdot H)\rangle = \epsilon_1\cdot\epsilon_2[\langle\mathcal{P}_1\mathcal{P}_2\rangle\epsilon_3\cdot\langle\mathcal{P}_3\rangle - 2\alpha' k_1\cdot k_2\langle H_1H_2\rangle^2\epsilon_3\cdot\langle\mathcal{P}_3\rangle - (2\alpha')^{3/2}k_2\cdot k_3 k_1\cdot\epsilon_3\langle H_1H_2\rangle\langle H_1H_3\rangle\langle H_2H_3\rangle + (2\alpha')^{3/2}k_1\cdot k_3 k_2\cdot\epsilon_3\langle H_1H_2\rangle\langle H_1H_3\rangle\langle H_2H_3\rangle] + \cdots \quad (79)$$

where $\cdots$ represents all the other polarization structures. The first term in square brackets is identical to the correlator encountered in the bosonic string. The remaining terms, because of the explicit factors of $k_i\cdot k_j$ are nominally a factor of p smaller than this first term. However, these factors can be canceled by poles arising from the θ integrals in the respective regions $\theta_2\approx 0$, $\theta_3\approx 0$, or $\theta_3\approx\theta_2$. Thus these contributions look like 1 particle reducible contributions. We examine these contributions, after making some simplifications valid as $p\to 0$.

The poles under consideration are at most $O(p^{-1})$, so we can neglect terms in the square bracket of $O(p^2)$ or smaller. So we can replace $k_2 \cdot \epsilon_3 = -(k_1 + p) \cdot \epsilon_3 \to -k_1 \cdot \epsilon_3$. Furthermore we can write $k_1 \cdot k_2 = (k_1 + k_2)^2/2 = (p + k_3)^2/2 = p^2/2 + k_3 \cdot p \to k_3 \cdot p$. Similarly $k_1 \cdot k_3 \to k_2 \cdot p$ and $k_2 \cdot k_3 \to k_1 \cdot p$. Finally, we can replace

$$\epsilon_3 \cdot \langle \mathcal{P}_3 \rangle \to \sqrt{2\alpha'} k_1 \cdot \epsilon_3 \left[\frac{1}{2} \cot\frac{\theta_3 - \theta_2}{2} - \frac{1}{2} \cot\frac{\theta_3}{2} + \sum_{n=1}^{\infty} \frac{2q^{2n}(\sin n(\theta_3 - \theta_2) - \sin n\theta_3)}{1 - q^{2n}} \right]. \tag{80}$$

With these simplifications we see that the H terms in the square bracket combine into a common factor $2\alpha' k_3 \cdot p \epsilon_1 \cdot \epsilon_2 \sqrt{2\alpha'} k_1 \cdot \epsilon_3$ times

$$- \left[\frac{1}{2} \cot\frac{\theta_3 - \theta_2}{2} - \frac{1}{2} \cot\frac{\theta_3}{2} + \sum_{n=1}^{\infty} \frac{2q^{2n}(\sin n(\theta_3 - \theta_2) - \sin n\theta_3)}{1 - q^{2n}} \right] \left[\frac{1}{2} \csc\frac{\theta_2}{2} - 2\sum_r \frac{q^{2r} \sin r\theta_2}{1 + q^{2r}} \right]^2 + \left[\frac{1}{2} \csc\frac{\theta_2}{2} - 2\sum_r \frac{q^{2r} \sin r\theta_2}{1 + q^{2r}} \right] \left[\frac{1}{2} \csc\frac{\theta_3}{2} - 2\sum_r \frac{q^{2r} \sin r\theta_3}{1 + q^{2r}} \right] \left[\frac{1}{2} \csc\frac{\theta_3 - \theta_2}{2} - 2\sum_r \frac{q^{2r} \sin r(\theta_3 - \theta_2)}{1 + q^{2r}} \right]. \tag{81}$$

By inspection we see that this combination of terms is not singular as either as $\theta_3 \to 0$ or as $\theta_3 \to \theta_2$, so these regions of integration will not produce poles. Moreover the $\theta_2 \to 0$ behavior of the first line is identical to the corresponding limit for the bosonic string, producing a pole whose residue is $O(p)$ and so will not compensate the explicit $p \cdot k_3$ factor. So the only contribution that will survive the $p \to 0$ limit is the $\theta_2 \approx 0$ region of the θ_2 integration of the second line:

$$2\alpha' k_3 \cdot p \int_0^{\epsilon} d\theta_2 \theta_2^{2\alpha' k_3 \cdot p - 1} \left[\frac{1}{2} \csc\frac{\theta_3}{2} - 2\sum_r \frac{q^{2r} \sin r\theta_3}{1 + q^{2r}} \right]^2 \sim \left[\frac{1}{2} \csc\frac{\theta_3}{2} - 2\sum_r \frac{q^{2r} \sin r\theta_3}{1 + q^{2r}} \right]^2. \tag{82}$$

Inserting this last result into the loop integrand, we encounter the same integral as the two gluon amplitude already evaluated, the result being

$$-4\pi \sum_r \frac{q^{2r}}{(1 + q^{2r})^2}, \quad \text{for } +, \tag{83}$$

which is the result for $+$ correlators of H fields. Retracing the derivation for $-$ correlators leads to the result

$$-\frac{\pi}{2} - 4\pi \sum_n \frac{q^{2n}}{(1 + q^{2n})^2}, \quad \text{for } -. \tag{84}$$

To summarize, we have identified three contributions to the q integrand of the 1 loop 3 gluon scattering amplitude in the NS model. The $\langle \mathcal{P}^3 \rangle$ correlator produces a 1PIR contribution

$$I_{\mathcal{P}}^{\rm 1PIR} = 2\pi \left[-\frac{1}{2} + 4\sum_n \frac{q^{2n}}{(1 - q^{2n})^2} \right], \tag{85}$$

and a reducible contribution which is $-3/2$ times the 1PIR piece piece:

$$I_{\mathcal{P}}^{\rm 1PR} = -3\pi \left[-\frac{1}{2} + 4\sum_n \frac{q^{2n}}{(1 - q^{2n})^2} \right]. \tag{86}$$

Finally, there are the contributions involving H correlators which are also reducible

$$I_H^{1PR+} = -4\pi \sum_r \frac{q^{2r}}{(1 + q^{2r})^2}, \qquad I_H^{1PR-} = -\frac{\pi}{2} - 4\pi \sum_n \frac{q^{2n}}{(1 + q^{2n})^2}. \tag{87}$$

Combining all the contributions together gives the simple result

$$I^+ = I_{\mathcal{P}}^{\rm 1PIR} + I_{\mathcal{P}}^{\rm 1PR} + I_H^{1PR+} = -\pi \left[-\frac{1}{2} + 4\sum_n \frac{q^{2n}}{(1 - q^{2n})^2} + 4\sum_r \frac{q^{2r}}{(1 + q^{2r})^2} \right]$$
$$I^- = I_{\mathcal{P}}^{\rm 1PIR} + I_{\mathcal{P}}^{\rm 1PR} + I_H^{1PR-} = -\pi \left[4\sum_n \frac{q^{2n}}{(1 - q^{2n})^2} + 4\sum_n \frac{q^{2n}}{(1 + q^{2n})^2} \right]. \tag{88}$$

As discussed in [27] the field theory limit is controlled by $w \sim 0$ and there it is shown that

$$4\sum_n \frac{q^{2n}}{(1 - q^{2n})^2} = -2q \frac{d}{dq} \sum_n \ln(1 - q^{2n}) = \frac{1}{6} + \frac{\ln w}{2\pi^2} + \frac{\ln^2 w}{24\pi^2} + \frac{\ln^2 w}{\pi^2} \sum_n \ln(1 - w^n) \sim \frac{1}{6} + \frac{\ln w}{2\pi^2} + \frac{\ln^2 w}{24\pi^2} + O(w). \tag{89}$$

In a similar manner it is easily seen that

$$4\sum_r \frac{q^{2r}}{(1+q^{2r})^2} = -2q\frac{d}{dq}\left(\sum_n \ln\frac{1+q^{2n}}{1-q^{2n}} + \sum_r \ln\frac{1-q^{2r}}{1+q^{2r}}\right)$$
$$= -\frac{\ln w}{2\pi^2} + \frac{\ln^2 w}{\pi^2} w \frac{d}{dw} \times \left(\sum_n \ln\frac{1+w^n}{1-w^n} + \sum_r \ln\frac{1-w^r}{1+w^r}\right)$$
$$\sim -\frac{\ln w}{2\pi^2} - w^{1/2}\frac{\ln^2 w}{\pi^2} + O(w) \tag{90}$$
$$4\sum_n \frac{q^{2n}}{(1+q^{2n})^2} = -2q\frac{d}{dq}\left(\sum_r \ln(1+q^{2r})(1-q^{2r}) - \sum_n \ln(1-q^{2n})(1+q^{2n})\right)$$
$$= -\frac{1}{3} - \frac{\ln w}{2\pi^2} + \frac{\ln^2 w}{\pi^2} w \frac{d}{dw} \times \left(\sum_n \ln\frac{1+w^n}{1-w^n} + \sum_r \ln\frac{1+w^r}{1-w^r}\right)$$
$$\sim -\frac{1}{3} - \frac{\ln w}{2\pi^2} + w^{1/2}\frac{\ln^2 w}{\pi^2} + O(w).$$

When considering the field theory limit in the NS+ model, the details are different from the bosonic string. Recall that in the $w \to 0$ limit the factor $D - 2 + 1/w$ in the bosonic measure changes to $D - 2 + 1/\sqrt{w}$ in the + amplitude and to $-D + 2 + 1/\sqrt{w}$ in the − amplitude of the NS+ model. The $1/w$ in the bosonic string case compensates $O(w)$ contributions in the limit of $I_{\mathcal{P}}$. However, in the NS cases we only have a $1/\sqrt{w}$ and these $O(w)$ contributions will vanish. Thus instead of $(D - 26)/24$, the $I_{\mathcal{P}}$ contribute only $(D - 2)/24$ to the irreducible part and $-3(D - 2)/48 = -(D - 2)/16$ to the reducible part. Finally, the $I_H^{\pm}$ contributes 1/2 to the reducible part. Recall that for the NS+ amplitude we need the combination

$$\frac{1}{2}\left(D - 2 + \frac{1}{\sqrt{w}}\right)I^+ - \frac{1}{2}\left(-D + 2 + \frac{1}{\sqrt{w}}\right)I^-, \tag{91}$$

so the "tachyon" singularity $1/\sqrt{w}$ cancels and the $O(w^0)$ terms from the + and − contributions add. Thus the total reducible part is $-(D - 10)/16$. Of course the total contribution is

$$\frac{D-2}{24} - \frac{D-10}{16} = -\frac{D-26}{48}. \tag{92}$$

just as for the bosonic string model. This had to be the case because both the bosonic string and the NS+ string go to the same gauge theory as $\alpha' \to 0$. It is mildly amusing that the reducible contribution to charge renormalization vanishes in the critical dimension for the NS model ($D = 10$). To the extent that we can associate the reducible contribution to wave function renormalization, this would mean that there is none in the critical dimension. However, the fact remains that there is no physically meaningful distinction between reducible and irreducible contributions to on-shell scattering amplitudes. Physics sees only the complete package.

V. DISCUSSION AND CONCLUSION

This article is only the beginning of a substantial program. We have studied the one-loop NS+ diagram in enough detail to confirm that it shows the correct renormalization group properties in the field theory limit as well as the mass spectrum of the closed string that couples to it. Along the way we have appreciated the great utility of the GNS regularization of string loop diagrams.

We take a few lines here to describe how the properties of the closed string revealed so far can be consistently described by a new (Liouville) world sheet field ϕ. First recall the modification of the Virasoro generators discovered by David Fairlie and me independently in 1971 (see [28]). Here we include the easy extension to the NS super Virasoro generators:

$$L_n = i\alpha n a_n^5 + \hat{L}_n, \qquad G_r = 2i\alpha r b_r^5 + \hat{G}_r, \qquad L_0 = \frac{\alpha^2}{2} + \hat{L}_0, \tag{93}$$

where a_n^5, b_r^5 are the bose and fermi oscillators associated with a "fifth" [really $(D + 1)$th] dimension. The hatted generators are the usual flat space generators in $D + 1$ dimensions. These modified operators satisfy the super Virasoro algebra with $c = D + 1 + 8\alpha^2$. Of course, the algebra is doubled to describe closed strings. Vanishing of the conformal anomaly requires $c = 10$ which then determines $\alpha^2 = (9 - D)/8$. Applying the on-shell condition $L_0 = 1/2$ then determines the "D dimensional" mass as

$$\frac{\alpha' M_D^2}{4} = \frac{\alpha^2}{2} - \frac{1}{2} + \frac{\alpha' p_5^2}{4} + R = -\frac{D-1}{16} + \frac{\alpha' p_5^2}{4} + R. \tag{94}$$

This shows a continuous mass spectrum starting at $M^2 = -(D - 1)/4$ just as revealed in the one-loop calculation studied here. We also see that the holographic 5 dimensional mass spectrum is discrete. It is tempting to identify ϕ with the free field incarnation of the Liouville field obtained via the Bäcklund transformation. One further piece of information from the one-loop analysis in favor of this interpretation is the fact that the eigenfunctions $\sinh\gamma\mu$, $\cosh\gamma\mu$ are eigenstates of the zero mode parity operation $\mu \to -\mu$. This restriction was essential to the success of the quantum Bäcklund transformation constructed in [29]. However, it would be a bit hyperbolic to claim that these coincidences establish the validity of the Liouville interpretation.

In our study of the field theory limit of the one-loop amplitude, we found the Goddard-Neveu-Scherk regularization indispensable, since it respects the proper normalization of scattering amplitudes in on-shell perturbation theory. At a more fundamental level as applied to the sum of planar diagrams, it simply reflects the validity of interpreting that sum as tree emission of closed strings into the vacuum. We think this is an interesting and valuable insight that string theory brings to quantum field theory.

There is clearly much work that remains to be done. We have just scratched the surface in determining the subcritical closed string dynamics implied by the even G-parity 4D Neveu-Schwarz model. Multiloop diagrams have yet to be determined, let alone analyzed for their closed string content. This is a major challenge for the immediate future. Even at the one-loop level there is more to be understood. We have only analyzed the field theory limit of two and three gluon amplitudes, which serve to determine a single renormalization group coefficient. It would be instructive to extend the analysis, at the very least, to four gluon amplitudes.

We already know from the one-loop analysis that the closed string spectrum includes tachyonic and massless states, which signal a breakdown of the perturbative vacuum. Could the resolution of this instability explain confinement in large N QCD? We must await the determination of the closed string effective field theory to address this question.

ACKNOWLEDGMENTS

I should like to thank Oren Bergman, André Neveu, and Arkady Tseytlin for very helpful discussions. I also thank the [Department of Energy's] Institute for Nuclear Theory at the University of Washington, where the research described in Sec. IV was initiated, for its hospitality and the Department of Energy for partial support. This research was also supported in part by the Department of Energy under Grant No. DE-FG02-97ER-41029.

APPENDIX: QFT MODEL OF GNS REGULARIZATION

Introduce a neutral scalar field ϕ with interaction term

$$-\tfrac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}\lambda\phi. \tag{A1}$$

Then the vertex Feynman rules are

$$\begin{aligned}
&-ig[\eta_{\mu_1\mu_2}(p_1-p_2)^{\mu_3}+\eta_{\mu_3\mu_1}(p_3-p_1)^{\mu_2}+\eta_{\mu_2\mu_3}(p_2-p_3)^{\mu_1}], \qquad \text{3G}\\
&i\lambda[\eta_{\mu_1\mu_2}p_1\cdot p_2-p_{1\mu_2}p_{2\mu_1}], \qquad \text{2G}-\phi\\
&-i\lambda g[\eta_{\mu_1\mu_2}(p_1-p_2)^{\mu_3}+\eta_{\mu_3\mu_1}(p_3-p_1)^{\mu_2}+\eta_{\mu_2\mu_3}(p_2-p_3)^{\mu_1}], \qquad \text{3G}-\phi.
\end{aligned} \tag{A2}$$

A ϕ insertion models a loop insertion in a string tree diagram. So the regularized 1 loop 2 gluon function is given by the $p \to 0$ limit of the 1 ϕ two gluon vertex ($p_1 + p_2 + p = 0$):

$$i\lambda(p_1\cdot p_2\epsilon_1\epsilon_2 - p_1\cdot\epsilon_2 p_2\cdot\epsilon_1)\frac{(-i)^2}{p_1^2p_2^2}. \tag{A3}$$

We can take $p \to 0$ by first setting p^+, $\mathbf{p}$ to 0, and at the same time take the light-cone gauge $\epsilon_i^+ = 0$, so that $p^2 = 0$ and $p\cdot\epsilon_i = 0$. Then we have

$$\frac{i\lambda}{2}\frac{p_1^2+p_2^2}{p_1^2p_2^2}\epsilon_1\epsilon_2 \to \frac{i\lambda}{p_1^2}\epsilon_1\epsilon_2, \quad \text{for } p_2\to p_1. \tag{A4}$$

This shows that the gluon wave function renormalization is $Z = 1 - \lambda$. Notice that the finite momentum p has separated the poles on the two legs of the two point function, in such a way that if one of them is put on shell first, as would be the case for an on-shell external leg the wave function renormalization correction is reduced by a factor of $1/2$. This is in fact precisely what is required by the proper application of the reduction formalism: a factor of $\sqrt{Z}$ should be associated with each external leg!

[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[2] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[3] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).
[4] A. Neveu and J. H. Schwarz, Nucl. Phys. **B31**, 86 (1971).
[5] M. B. Halpern and C. B. Thorn, Phys. Lett. **35B**, 441 (1971).
[6] S. Mandelstam (private communication).
[7] C. B. Thorn, Santa Fe Institute workshop, November 8-10, 1985 (unpublished).
[8] A. Neveu and J. Scherk, Nucl. Phys. **B36**, 155 (1972).
[9] A. M. Polyakov, Int. J. Mod. Phys. A **14**, 645 (1999).
[10] N. Seiberg and E. Witten, Nucl. Phys. **B276**, 272 (1986).
[11] L. J. Dixon and J. A. Harvey, Nucl. Phys. **B274**, 93 (1986).
[12] I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. **B546**, 155 (1999).
[13] A. M. Polyakov, Phys. Lett. **103B**, 207 (1981); **103B**, 211 (1981).

[14] T. L. Curtright and C. B. Thorn, Phys. Rev. Lett. **48**, 1309 (1982); **48**, 1768(E) (1982).

[15] J. L. Gervais and A. Neveu, Nucl. Phys. **B209**, 125 (1982).

[16] C. B. Thorn, Phys. Lett. B **242**, 364 (1990).

[17] C. R. Preitschopf and C. B. Thorn, Nucl. Phys. **B349**, 132 (1991).

[18] P. Ramond, Phys. Rev. D **3**, 2415 (1971).

[19] C. B. Thorn, Phys. Rev. D **4**, 1112 (1971); A. Neveu and J. H. Schwarz, Phys. Rev. D **4**, 1109 (1971).

[20] F. Gliozzi, J. Scherk, and D. I. Olive, Phys. Lett. **65B**, 282 (1976); Nucl. Phys. **B122**, 253 (1977).

[21] A. Neveu, J. H. Schwarz, and C. B. Thorn, Phys. Lett. **35B**, 529 (1971).

[22] P. Goddard and R. E. Waltz, Nucl. Phys. **B34**, 99 (1971).

[23] P. Goddard and C. B. Thorn, Phys. Lett. **40B**, 235 (1972).

[24] R. C. Brower and C. B. Thorn, Nucl. Phys. **B31**, 163 (1971).

[25] A. Neveu and J. Scherk, Nucl. Phys. **B36**, 317 (1972).

[26] P. Goddard, Nuovo Cimento Soc. Ital. Fis. A **4**, 349 (1971).

[27] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. **B298**, 109 (1988).

[28] A. Chodos and C. B. Thorn, Nucl. Phys. **B72**, 509 (1974).

[29] E. Braaten, T. Curtright, and C. B. Thorn, Phys. Lett. **118B**, 115 (1982); Ann. Phys. (N.Y.) **147**, 365 (1983).