New relations for gauge-theory amplitudes

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We present an identity satisfied by the kinematic factors of diagrams describing the tree amplitudes of massless gauge theories. This identity is a kinematic analog of the Jacobi identity for color factors. Using this we find new relations between color-ordered partial amplitudes. We discuss applications to multiloop calculations via the unitarity method. In particular, we illustrate the relations between different contributions to a two-loop four-point QCD amplitude. We also use this identity to reorganize gravity tree amplitudes diagram by diagram, offering new insight into the structure of the Kawai-Lewellen-Tye (KLT) relations between gauge and gravity tree amplitudes. This insight leads to similar but novel relations. We expect this to be helpful in higher-loop studies of the ultraviolet properties of gravity theories.

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I. INTRODUCTION

Gauge and gravity scattering amplitudes have a far simpler and richer structure than apparent from Feynman rules or from their respective Lagrangians. Striking tree-level examples include the Parke-Taylor maximally helicity violating (MHV) amplitudes in QCD [1], the delta-function support of amplitudes on polynomial curves in twistor space [2,3], and the Kawai-Lewellen-Tye (KLT) relations [4–6] between gravity and gauge-theory tree amplitudes. Besides their intrinsic theoretical value, these structures have led to a number of useful computational advances, as described in various reviews [7–10].

In particular, tree-level color-ordered partial amplitudes satisfy simplifying relations dictated by the color algebra. Adjoint representation amplitudes must vanish whenever an external gluon is replaced by a color neutral photon, giving a "photon decoupling identity" [7] (also referred to as the subcyclic property). More generally, the Kleiss-Kuijf relations [11,12] reduce the number of independent n-point tree partial amplitudes from $(n-1)!$ to $(n-2)!$ partial amplitudes.

In this paper we propose a kinematic identity that further constrains the n-point color-ordered partial amplitudes at tree level. This is based on the observation that gauge-theory amplitudes can be rearranged into a form where the kinematic factors of diagrams describing the amplitudes satisfy an identity analogous to the Jacobi identity obeyed by the color factors associated with the same diagrams. At four points this kinematic identity has been used previously to explain certain zeroes in cross sections [13]. By solving the generated set of equations at higher points, we obtain new nontrivial relations amongst color-ordered tree amplitudes, reducing the number of independent partial amplitudes to $(n-3)!$.

The existence of such an identity is unobvious from the Feynman diagrams contributing to higher-point tree amplitudes. Indeed, color-ordered Feynman diagrams, in a generic choice of gauge, will *not* satisfy this new identity in isolation beyond four points. Rather the kinematic factors satisfying the identity only appear after rearranging terms between contributing Feynman diagrams into convenient representations of the amplitudes. In this paper we do not present a complete proof that this rearrangement is always possible. However, because of the many explicit tree-level checks that we have performed, and because of its close connection to the color Jacobi identity, we expect this new n-point identity to hold for tree-level color-ordered Yang-Mills amplitudes. In this paper we will focus on gluonic amplitudes.

Although the amplitude relations derived from the new kinematic identity may be helpful in tree and one-loop calculations, powerful computational methods are already available, or are under development [7,8,10,14]. In the past decade important progress has also been made in computing higher-loop scattering amplitudes both for phenomenological and theoretical purposes. For example, on the phenomenological side, even fully differential cross sections for processes as complicated as $e^+e^- \to 3$ jets at next-to-next-to-leading order are now computable [15]. Much of this progress relies on various improvements in loop integration techniques [16]. On the theoretical side, multiloop calculations of scattering amplitudes have become important as a means of studying fundamental issues in gauge and gravity theories [17–20]. Various useful relations aiding computations in higher-loop maximally supersymmetric theories have been discussed in Refs. [18,21–25]. To go beyond this, a greatly improved understanding of the structure of multiloop scattering amplitudes will likely be important.

The unitarity method [26] gives us a means of transferring properties of amplitudes from tree level to loop level. Since this approach constructs loop diagrams out of tree-level amplitudes, we can apply the relations following from the kinematic identity to help simplify multiloop calculations. Specifically, we will show that it induces nontrivial relations between planar and nonplanar loop-level contributions. Its application to the construction of the four-loop four-point amplitude of $\mathcal{N} = 4$ super-Yang-Mills theory in terms of loop integrals—including nonplanar contributions—will be given elsewhere [27]. The planar contributions at four and five loops have already been given in Refs. [18,24].

Besides applications to gauge theories, the identity also suggests a natural reorganization of gravity tree amplitudes so that the numerator of each kinematic pole in the amplitude is given by a product of two gauge-theory kinematic numerators. As we will show, this is closely connected to the KLT relations between color-ordered gauge-theory and gravity amplitudes. The new representations for gravity tree amplitudes can be used in loop calculations via generalized unitarity [19,20,22,28]. There may also be a connection to other recently uncovered relations between maximally helicity violating gravity and gauge-theory amplitudes [6], but this requires further study.

This paper is organized as follows. In Sec. II we review some pertinent properties needed later in the paper, as well as establish notation. In Sec. III, we derive the identity at four points, guiding our higher-point construction. Then in Sec. IV we motivate the higher-point generalization of the identity, discussing the five-point case in some detail. In Sec. V we work out multiloop QCD examples of its application. In Sec. VI we present implications for gravity amplitudes, showing how the identity clarifies the KLT relations and can be used to derive new representations for gravity tree amplitudes in terms of gauge-theory ones.

II. REVIEW

In this section we set up the terminology, notation, and review a number of pertinent results directly relevant for our subsequent discussion.

A. Gauge-theory color structure

At tree level, with particles all in the adjoint representation of $SU(N_c)$, the full tree amplitude can be decomposed as

$$\mathcal{A}_n^{\text{tree}}(1, 2, 3, \ldots, n) = g^{n-2} \sum_{\mathcal{P}(2,3,\ldots,n)} \text{Tr}[T^{a_1}T^{a_2}T^{a_3}\cdots T^{a_n}]A_n^{\text{tree}}(1, 2, 3, \ldots, n), \tag{2.1}$$

where A_n^{tree} is a tree-level color-ordered n-leg partial amplitude. The T^a's are color-group generators, encoding the color of each external leg $1, 2, 3 \ldots n$. The sum is over all noncyclic permutations of legs, which is equivalent to all permutations keeping leg 1 fixed. Helicities and polarizations are suppressed. At higher loops, detailed color decompositions of gauge-theory amplitudes have not been given though general properties are clear. For example, at L loops the corresponding decomposition involves up to $L + 1$ color traces per term. Discussions of such color decompositions at tree level and at one loop may be found in Refs. [7,29].

Other color decompositions involve using the f^{abc} group structure constants. At tree level and at one loop a decomposition of this type was given in Refs. [12,30]. For tree level, this decomposition is similar to the one in Eq. (2.1) using, instead, color matrices in the adjoint representation (the f^{abc}'s), and summing over fewer partial amplitudes.

Color-ordered tree-level amplitudes satisfy a set of well-known relations. The simplest of these are the cyclic and reflection properties,

$$\begin{aligned} A_n^{\text{tree}}(1, 2, \ldots, n) &= A_n^{\text{tree}}(2, \ldots, n, 1), \\ A_n^{\text{tree}}(1, 2, \ldots, n) &= (-1)^n A_n^{\text{tree}}(n, \ldots, 2, 1). \end{aligned} \tag{2.2}$$

Next there is the "photon"-decoupling identity (or subcyclic property) [7,11],

$$\sum_{\sigma\in\text{cyclic}} A_n^{\text{tree}}(1, \sigma(2, 3, \ldots, n)) = 0, \tag{2.3}$$

where the sum runs over all cyclic permutations of legs $2, 3, 4, \ldots n$. This identity follows by replacing $T^{a_1} \to 1$ in the full amplitude (2.1) corresponding to replacing leg 1 with a photon. The amplitude must then vanish since photons cannot couple directly to adjoint representation particles.

Other important relations are the Kleiss-Kuijf relations [11]:

$$A_n^{\text{tree}}(1, \{\alpha\}, n, \{\beta\}) = (-1)^{n_\beta} \sum_{\{\sigma\}_i\in \text{OP}(\{\alpha\},\{\beta^T\})} A_n^{\text{tree}}(1, \{\sigma\}_i, n), \tag{2.4}$$

where the sum is over the "ordered permutations" $\text{OP}(\{\alpha\}, \{\beta^T\})$, that is, all permutations of $\{\alpha\} \cup \{\beta^T\}$ that maintains the order of the individual elements belonging to each set within the joint set. The notation $\{\beta^T\}$ represents the set $\{\beta\}$ with the ordering reversed, and n_β is the number of β elements. These relations were conjectured in Ref. [11] and proven in Ref. [12].

Consider, as an example, a five-point tree amplitude. For $A_5^{\text{tree}}(1, \{2, 3\}, 5, \{4\})$ we have

$$\begin{aligned} A_5^{\text{tree}}(1, 2, 3, 5, 4) = &-A_5^{\text{tree}}(1, 2, 3, 4, 5) - A_5^{\text{tree}}(1, 2, 4, 3, 5) \\ &- A_5^{\text{tree}}(1, 4, 2, 3, 5). \end{aligned} \tag{2.5}$$

The other five-point relations are given by permuting legs 2, 3, 4 and using the cyclic and reflection properties. This means that the six amplitudes $A_5^{\text{tree}}(1, \mathcal{P}\{2, 3, 4\}, 5)$—where $\mathcal{P}\{2, 3, 4\}$ signifies all permutations over legs $\{2, 3, 4\}$—form a basis in which the remaining five-point partial amplitudes can be expressed. More generally, for multiplicity n, the Kleiss-Kuijf relations can be used to rewrite any color-ordered partial amplitude in terms of only $(n - 2)!$ basis partial amplitudes, where two legs are held fixed in the ordering.

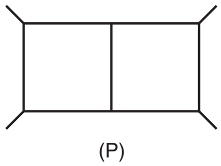

FIG. 1. The parent diagrams for the two-loop four-point identical-helicity amplitudes of QCD and $\mathcal{N}=4$ super-Yang-Mills theory. All other diagrams appearing in the amplitude are obtained by collapsing propagators. These parent diagrams also determine the color factors appearing in Eq. (2.6), by dressing them with $\tilde{f}^{abc}$s in a clockwise ordering.

For the two-loop four-point identical-helicity pure gluon and two-loop MHV amplitudes in maximally supersymmetric Yang-Mills theory, a color decomposition in terms of f^{abc} has been given in terms of "parent" diagrams establishing both the color and kinematic structure [22,31],

$$\mathcal{A}_4^{\text{2-loop}}(1,2,3,4) = g^6[C^{\rm P}_{1234}A^{\rm P}_{1234} + C^{\rm P}_{3421}A^{\rm P}_{3421} + C^{\rm NP}_{12;34}A^{\rm NP}_{12;34} + C^{\rm NP}_{34;21}A^{\rm NP}_{34;21} + \mathcal{C}(234)], \tag{2.6}$$

where "$+\mathcal{C}(234)$" signifies that one should add the two cyclic permutations of 2, 3, 4. The $A^{\rm P}$ and $A^{\rm NP}$ are primitive amplitudes stripped of color. The values of the color coefficients $C^{\rm P}$ and $C^{\rm NP}$ may be read off directly from the parent diagrams in Fig. 1. For example, $C^{\rm P}_{1234}$ is the color factor obtained from diagram (a) by dressing each vertex with an $\tilde{f}^{abc}$, where

$$\tilde{f}^{abc} \equiv i\sqrt{2}f^{abc} = \text{Tr}([T^a, T^b]T^c), \tag{2.7}$$

and dressing each internal line with a δ^{ab}. In Sec. V, we will make use of this decomposition in a two-loop example. As discussed in Ref. [12], general representations in terms of parent diagrams can be constructed by making repeated use of the Jacobi identity, but this has not been studied in full generality.

A key property of the $\tilde{f}^{abc}$'s are that they satisfy the Jacobi identity illustrated in Fig. 2,

$$c_u \equiv \tilde{f}^{a_4a_2b}\tilde{f}^{ba_3a_1}, \qquad c_s \equiv \tilde{f}^{a_1a_2b}\tilde{f}^{ba_3a_4}, \qquad c_t \equiv \tilde{f}^{a_2a_3b}\tilde{f}^{ba_4a_1}, \qquad c_u = c_s - c_t. \tag{2.8}$$

The main result of this paper is that the kinematic factors corresponding to n-point tree diagrams can be rearranged to satisfy an analogous identity giving nontrivial constraints on the form of tree amplitudes. This then has useful consequences at loop level and for corresponding gravity amplitudes.

B. Higher-loop integral representation and the unitarity method

Any multiloop amplitude can be expanded in a set of loop integrals with rational coefficients,

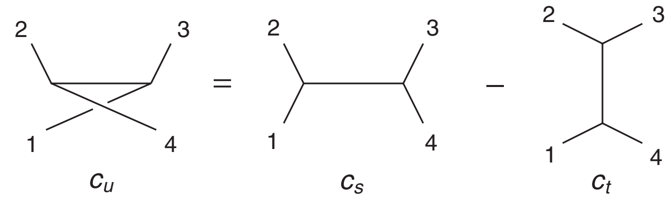

FIG. 2. The Jacobi identity relating the color factors of the u, s, t channel "color diagrams." The color factors are given by dressing each vertex with an $\tilde{f}^{abc}$ following a clockwise ordering.

$$A_n^{(L)} = \sum_i a_i I_n^{(L),i}, \tag{2.9}$$

where the a's are the coefficients, i runs over a list of integrals $I_n^{(L),i}$, and $A_n^{(L)}$ is a generic n-point L-loop amplitude, not necessarily color decomposed. Here, we neither demand that the integrals form a linear independent basis under integral reductions, nor that integrals vanishing under integration be removed from the set.

In this paper we consider representations of n-point L-loop integrals that can be written in a D-dimensional Feynman-like manner, schematically,

$$I_n^{(L),i} = \int\left(\prod_{m=1}^{L}\frac{d^Dl_m}{(2\pi)^D}\right)\frac{N_i}{\prod_j p_j^2}, \tag{2.10}$$

where the propagators, $1/p_j^2$, specify the structure of the graph. The numerator factors N_i in general depend on the loop momenta and also on external kinematics and polarizations. The number of powers of loop momenta that can appear in the numerator factors depends on the theory under consideration. The l_m are the L independent variables of loop momenta, usually picked from the set of propagator momenta p_j.

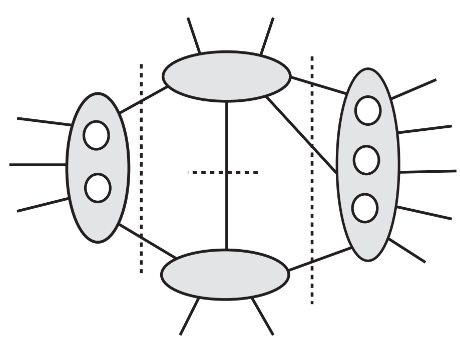

FIG. 3. The unitarity method constructs multiloop amplitudes from lower-loop amplitudes. The blobs represent amplitudes, the white holes loops, and the dotted lines indicate cuts which replace propagators with on-shell delta functions. Generalized cuts which decompose loop amplitude solely in terms of tree amplitudes are particularly useful in carrying out multiloop calculations.

To carry over the newly uncovered tree-level relations to loop level we use the unitarity method [21,22,26,28,29,32] to construct complete loop-level amplitudes, at the level of the integrands, prior to carrying out any loop integration. In this method higher-loop-level integrands are constructed by taking the product of lower-loop or tree amplitudes and imposing on-shell conditions on intermediate legs. As illustrated in Fig. 3, the unitarity cuts are given by a product of lower-loop amplitudes,

$$\sum_{\text{states}} A_{(1)} A_{(2)} A_{(3)} \cdots A_{(m)}, \tag{2.11}$$

where the sum runs over all particle types and physical states that can propagate on the internal cut lines. A complete integrand is then found by systematically constructing an ansatz that has the correct cuts in all channels. Overviews of the unitarity method may be found in Refs. [29]. A systematic description of the merging procedure for constructing multiloop amplitudes from the cuts may be found in Ref. [32]. Examples of explicit higher-loop calculations using the unitarity method may be found in Refs. [21,22].

For the purposes of this paper, the generalized cuts give us a means of applying the new tree-level relations and identities directly at higher-loop level, since it allows us to construct amplitudes entirely from tree-level amplitudes.

A particularly useful set of cuts are the maximal cuts [24,25,33] where the maximum number of propagators are put on shell.[1] In this case, the cut is given by a sum over products between three-point tree amplitudes, isolating a particular parent diagram (modulo any contact terms). We build complete amplitudes by systematically releasing on-shell conditions to identify all relevant contact terms. This procedure has been described in some detail in Ref. [24]. Such maximal cuts may be exploited in D-dimension, as we do in Sec. V.

C. Gravity amplitudes

At tree level, gravity amplitudes satisfy a remarkable relation with gauge-theory amplitudes, first uncovered in string theory by Kawai, Lewellen, and Tye [4–6]. These relations also hold in field theory, as the low-energy limit of string theory. In this limit, the KLT relations for four-, five-, and six-point amplitudes are

$$M_4^{\text{tree}}(1,2,3,4) = -is_{12} A_4^{\text{tree}}(1,2,3,4)\tilde{A}_4^{\text{tree}}(1,2,4,3), \tag{2.12}$$

$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34} A_5^{\text{tree}}(1,2,3,4,5)\tilde{A}_5^{\text{tree}}(2,1,4,3,5) + is_{13}s_{24} A_5^{\text{tree}}(1,3,2,4,5)\tilde{A}_5^{\text{tree}}(3,1,4,2,5), \tag{2.13}$$

$$M_6^{\text{tree}}(1,2,3,4,5,6) = -is_{12}s_{45} A_6^{\text{tree}}(1,2,3,4,5,6)[s_{35}\tilde{A}_6^{\text{tree}}(2,1,5,3,4,6) + (s_{34}+s_{35})\tilde{A}_6^{\text{tree}}(2,1,5,4,3,6)] + \mathcal{P}(2,3,4). \tag{2.14}$$

Here the M_n's are amplitudes in a gravity theory stripped of couplings, the A_n's and $\tilde{A}_n$'s are the color-ordered amplitudes in two, possibly different, gauge theories (the gravity states are direct products of gauge-theory states for each external leg) [7,34], $s_{ij} \equiv s_{i,j} = (k_i + k_j)^2$ with k_i being the outgoing momentum of leg i, and $\mathcal{P}(2,3,4)$ signifies a sum over all permutations of the labels 2, 3, and 4. An n-point generalization of the KLT relations is [35],

$$M_n^{\text{tree}}(1,2,\ldots,n) = i(-1)^{n+1}\Big[A_n^{\text{tree}}(1,2,\ldots,n)\sum_{\text{perms}} f(i_1,\ldots,i_j)\bar{f}(l_1,\ldots,l_{j'})\tilde{A}_n^{\text{tree}}(i_1,\ldots,i_j,1,n-1,l_1,\ldots,l_{j'},n)\Big] + \mathcal{P}(2,\ldots,n-2), \tag{2.15}$$

where the sum is over all permutations $\{i_1,\ldots,i_j\} \in \mathcal{P}\{2,\ldots,\lfloor n/2\rfloor\}$ and $\{l_1,\ldots,l_{j'}\} \in \mathcal{P}\{\lfloor n/2\rfloor+1,\ldots,n-2\}$ with $j=\lfloor n/2\rfloor-1$ and $j'=\lfloor n/2\rfloor-2$, which gives a total of $(\lfloor n/2\rfloor-1)!(\lfloor n/2\rfloor-2)!$ terms inside the square brackets. The notation "$+\mathcal{P}(2,\ldots,n-2)$" signifies a sum over the preceding expression for all permutations of legs $2,\ldots,n-2$. The functions f and $\bar{f}$ are given by

$$f(i_1,\ldots,i_j) = s_{1,i_j}\prod_{m=1}^{j-1}\Big(s_{1,i_m} + \sum_{k=m+1}^{j} g(i_m,i_k)\Big),$$
$$\bar{f}(l_1,\ldots,l_{j'}) = s_{l_1,n-1}\prod_{m=2}^{j'}\Big(s_{l_m,n-1} + \sum_{k=1}^{m-1} g(l_k,l_m)\Big), \tag{2.16}$$

and the function g is

$$g(i,j) = \begin{Bmatrix} s_{i,j} & \text{if } i>j \\ 0 & \text{else} \end{Bmatrix}. \tag{2.17}$$

[1]We use the terminology of Ref. [24], not Refs. [25] where maximal cuts include additional hidden singularities as well.

The full gravity amplitudes are obtained by multiplying with gravity coupling constants,

$$\mathcal{M}_n^{\rm tree} = \left(\frac{\kappa}{2}\right)^{n-2} M_n^{\rm tree}. \tag{2.18}$$

III. AN IDENTITY AT FOUR POINTS

In this section we discuss an identity that kinematic numerator factors of color-ordered four-point gauge-theory amplitudes satisfy. Specifically, we show that four-point gauge-theory amplitudes can be decomposed in terms of numerators n_s, n_t, n_u of kinematic poles of the Mandelstam variables, s, t, and u. As we shall see these numerator factors satisfy an identity analogous to the Jacobi identity (2.8) for adjoint representation color factors. For four-point tree amplitudes this identity may appear to be a curiosity, but as we will see below, the consequences at higher points and loops will be rather nontrivial. Interestingly, the identity had been noted almost three decades ago at the four-point level [13].

To derive the identity we will utilize general properties of adjoint representation gluonic amplitudes. As explained in Sec. II A, color-ordered tree-level amplitudes satisfy a photon-decoupling identity [7]. At four points we have

$$A_4^{\rm tree}(1,2,3,4) + A_4^{\rm tree}(1,3,4,2) + A_4^{\rm tree}(1,4,2,3) = 0. \tag{3.1}$$

To exploit this equation, we note that tree amplitudes in general are rational functions of polarization vectors, spinors, momenta, and Mandelstam invariants, $s = (k_1 + k_2)^2$, $t = (k_1 + k_4)^2$, and $u = (k_1 + k_3)^2$, where the k_i are outgoing massless momenta, corresponding to each external leg i. Because the decoupling identity (3.1) does not rely on the specific polarizations or space-time dimension, the cancellation is entirely due to the amplitude's dependence on the Mandelstam variables. In particular, it cannot rely on four-dimensional spinor identities. We recognize that the only nontrivial way this can happen is if the sum in Eq. (3.1) is equivalent to the vanishing of $s + t + u$ as follows:

$$\begin{aligned} &A_4^{\rm tree}(1,2,3,4) + A_4^{\rm tree}(1,3,4,2) + A_4^{\rm tree}(1,4,2,3) \\ &\quad = (s+t+u)\chi = 0, \end{aligned} \tag{3.2}$$

where χ is a shared factor that depends on the polarizations and momenta. From Eq. (3.2), the partial amplitudes should be proportional to each other. Furthermore, since $A_4^{\rm tree}(1,2,3,4)$ treats any factors of s and t the same, its contribution to Eq. (3.2) should be proportional to $u = -(s+t)$. We therefore make the following identification:

$$A_4^{\rm tree}(1,2,3,4) = u\chi. \tag{3.3}$$

Similar considerations give

$$A_4^{\rm tree}(1,3,4,2) = t\chi, \qquad A_4^{\rm tree}(1,4,2,3) = s\chi, \tag{3.4}$$

consistent with Eq. (3.2). After eliminating χ we obtain the following relations between four-point amplitudes,

$$\begin{aligned} tA_4^{\rm tree}(1,2,3,4) &= uA_4^{\rm tree}(1,3,4,2), \\ sA_4^{\rm tree}(1,2,3,4) &= uA_4^{\rm tree}(1,4,2,3), \\ tA_4^{\rm tree}(1,4,2,3) &= sA_4^{\rm tree}(1,3,4,2). \end{aligned} \tag{3.5}$$

We note that the well-known four-point tree-level helicity amplitudes in $D = 4$ explicitly satisfy this. For example, pure gluon amplitudes in the two color orders,

$$\begin{aligned} A_4^{\rm tree}(1^-,2^-,3^+,4^+) &= i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \\ &= -i\frac{\langle 12\rangle^2[34]^2}{st}, \\ A_4^{\rm tree}(1^-,4^+,2^-,3^+) &= i\frac{\langle 12\rangle^4}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle} \\ &= -i\frac{\langle 12\rangle^2[34]^2}{tu}, \end{aligned} \tag{3.6}$$

satisfy the relations (3.5). The $\langle ab\rangle$ and $[ab]$ are spinor inner products of Weyl spinors, using notation of Refs. [7].

To obtain the kinematic analog of the Jacobi identity we exploit the fact that the color-ordered tree amplitudes can be expanded in a convenient representation in terms of the poles that appear,

$$\begin{aligned} A_4^{\rm tree}(1,2,3,4) &\equiv \frac{n_s}{s} + \frac{n_t}{t}, \\ A_4^{\rm tree}(1,3,4,2) &\equiv -\frac{n_u}{u} - \frac{n_s}{s}, \\ A_4^{\rm tree}(1,4,2,3) &\equiv -\frac{n_t}{t} + \frac{n_u}{u}. \end{aligned} \tag{3.7}$$

Practically, this can be done in terms of Feynman diagrams (by absorbing the quartic contact terms into the cubic diagrams). One may also think of the numerators as unknown until solved for, and thus Eq. (3.7) defines the n_i's. The sign flipping is due to the antisymmetry of color-ordered Feynman rules. (The overall signs of the numerators depend on our choice of conventions. Of course, trivial redefinitions of the type $n_i \to -n_i$, can be used to modify the signs used to define the n_i, but this cannot change the actual value of the diagram or residues that the numerators correspond to.)

Comparing Eq. (3.5) to Eq. (3.7) gives us the desired kinematic numerator identity,

$$n_u = n_s - n_t. \tag{3.8}$$

With the chosen sign conventions in Eq. (3.7), this is exactly of the same form as the Jacobi identity for the color factors given in Eq. (2.8).

Using the definition of the fully dressed amplitude Eq. (2.1) and the partial amplitude form Eq. (3.7), after converting to the $\tilde{f}^{abc}$'s, we obtain a color-dressed representation,

$$\mathcal{A}_4^{\rm tree} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right), \qquad (3.9)$$

where the color factors are given in (2.8). With our sign conventions the signs in this color-dressed diagram representation are all positive. Note that the form of Eq. (3.9) is very similar to the usual expansion in terms of Feynman diagrams, except that we decomposed the four-point contact terms according to their color factors. In absorbing the contact terms into Eq. (3.9) we must ensure that no cross terms, such as $n_s c_t/s$, appear.

We may take the kinematic numerators to be local in the external polarizations and momenta. A natural question is whether these objects are unique gauge-invariant quantities. One check would be to consider the limit $s \to 0$ of the $A_4^{\rm tree}(1, 2, 3, 4)$ amplitude, where the four-point amplitude factorizes into two three-point amplitudes, each of which are manifestly gauge invariant. From this we would conclude that n_s is gauge invariant in the $s \to 0$ factorization limit. (To make n_s nonvanishing in the limit we may use complex momenta [2,36].) However, we should think of n_s/s as essentially an s-channel Feynman diagram, and of course individual Feynman diagrams are not gauge invariant, so we would properly conclude that n_s is gauge dependent for nonzero values of s and, more generally, dependent on the choice of field variables. Under the assumption that n_s is local, this freedom corresponds to all possible terms that can be added to n_s which cancel the $1/s$ pole. We will call this a "gauge freedom," because the gauge transformations are a familiar concept corresponding to a freedom of moving terms between diagrams.[2] We may parametrize this freedom as

$$n_s' = n_s + \alpha(k_i, \varepsilon_i)s, \qquad (3.10)$$

where $\alpha(k_i, \varepsilon_i)$ is local. This corresponds to a contact term ambiguity, and it does not change the residue of the s-pole. However, to keep the value of the tree amplitudes in Eq. (3.7) unchanged we must simultaneously change n_t and n_u,

$$n_t' = n_t - \alpha(k_i, \varepsilon_i)t, \qquad n_u' = n_u - \alpha(k_i, \varepsilon_i)u. \qquad (3.11)$$

Notice that this is exactly what is needed to make the sum of the shifted numerators vanish,

$$-n_s' + n_t' + n_u' = -n_s + n_t + n_u - \alpha(k_i, \varepsilon_i)(s + t + u) = 0. \qquad (3.12)$$

[2]We use the words "gauge freedom" loosely here; the freedom does not necessarily mean that a gauge transformation exists that causes a particular rearrangement of terms.

Therefore the transformation in Eqs. (3.10) and (3.11) has the effect of moving contact terms between the s, t, and u channel diagrams without altering the numerator identity (3.8). Although the numerators depend on the gauge choices, the identity (3.8) remains true for all gauges. As we shall see in Sec. IV, this property is special to four points. At higher points, only specific choices of numerators will satisfy the analogous identities and the form obtained from generic gauge Feynman diagrams will not.

In general we can choose the numerators to be local. It should be noted, however, that even if we allow $\alpha(k_i, \varepsilon_i)$ to be nonlocal, the kinematic identity equation (3.8) remains true. This follows from the observation that we did not use any locality constraints on the n_i's or $\alpha(k_i, \varepsilon_i)$ in arriving at either Eq. (3.8) or Eq. (3.12). So one can choose the n_i to be nonlocal without affecting the value of the amplitudes. We can even set any one of the n_i to zero. For example, choosing $\alpha(k_i, \varepsilon_i) = n_u/u$ we get $n_u' = 0$. This then puts a u pole into the numerators n_s' and n_t', making them nonlocal. In fact this choice takes us back to the relations found in the beginning of this section in Eq. (3.5). In the next section, we will obtain nontrivial relations between higher-point tree-level partial amplitudes, by choosing nonlocal numerators.

IV. HIGHER-POINT GENERALIZATION

In this section we generalize the relations found at four points to higher points. To do this we will promote Eq. (3.8) to be the master identity of this paper. We will use the close analogy between this kinematic numerator identity and the color-group Jacobi identity to apply it to higher-point tree-level amplitudes. We will find that this is indeed possible given that certain extra conditions are satisfied. Specifically, given three dependent color factors c_α, c_β, c_γ associated with tree-level color diagrams, we propose that color-ordered scattering amplitudes can always be decomposed into kinematic diagrams with numerator factors n_α, n_β, n_γ that obey the analogous numerator identity,

$$c_\alpha - c_\beta + c_\gamma = 0, \quad \Rightarrow \quad n_\alpha - n_\beta + n_\gamma = 0. \qquad (4.1)$$

A. The five-point kinematic identities

First consider the five-point case. We may again represent the amplitudes in terms of diagrams with purely cubic interactions. These diagrams specify the poles, examples of which are given in Fig. 4. As in the four-point case we absorb any contact terms into the numerator factors of these diagrams. In total there are 15 independent diagrams with two kinematic poles. In order for the four-point numerator identity to generalize we require that the numerators of the 15 independent diagrams can be arranged so that they satisfy exactly the same identities as the $\tilde{f}^{abc}$ composed color factors.

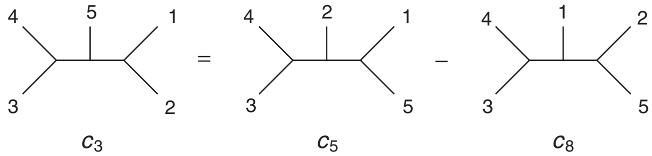

FIG. 4. The Jacobi identity at five points. These diagrams can be interpreted as relations for color factors, where each color factor is obtained by dressing the diagrams with $\tilde{f}^{abc}$ at each vertex in a clockwise ordering. Alternatively it can be interpreted as relations between the kinematic numerator factors of corresponding diagrams, where the diagrams are nontrivially rearranged compared to Feynman diagrams.

For example, consider the diagrams in Fig. 4. Interpreting these diagrams as color diagrams we immediately see that as a consequence of the Jacobi identity they satisfy the color-factor identity,

$$c_3 = c_5 - c_8, \tag{4.2}$$

where

$$\begin{aligned} c_3 &\equiv \tilde{f}^{a_3a_4b}\tilde{f}^{ba_5c}\tilde{f}^{ca_1a_2}, \\ c_5 &\equiv \tilde{f}^{a_3a_4b}\tilde{f}^{ba_2c}\tilde{f}^{ca_1a_5}, \\ c_8 &\equiv \tilde{f}^{a_3a_4b}\tilde{f}^{ba_1c}\tilde{f}^{ca_2a_5}. \end{aligned} \tag{4.3}$$

If the kinematic numerators are to satisfy a corresponding identity we must ensure that contact term contributions are associated with the proper numerator. This is not automatic, as we can easily check. Even color-ordered Feynman diagrams at five points, in isolation, do not generally satisfy the identity. Rather the numerator factors correspond to rearrangements of the Feynman diagrams.

In general, a color-ordered five-point tree amplitude consist of five diagrams,

$$A_5^{\rm tree}(1,2,3,4,5) = \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}, \tag{4.4}$$

where we use the notation $s_{ij} = (k_i + k_j)^2$, and the numerators are simply labeled n_i for $i = 1, 2, \ldots, 15$. Again we take the external momenta, k_i, to be massless and outgoing. The color-ordered five-point amplitudes are symmetric under cyclic permutations and antisymmetric under reflections. This means that there are at most 12 color orders that are not trivially related. We have not yet made use of the Kleiss-Kuijf relations [11,12] given in Eq. (2.5). Doing so reduces the number of independent amplitudes down to only six,

$$\begin{aligned} A_5^{\rm tree}(1,2,3,4,5) &\equiv \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}, \\ A_5^{\rm tree}(1,4,3,2,5) &\equiv \frac{n_6}{s_{14}s_{25}} + \frac{n_5}{s_{43}s_{51}} + \frac{n_7}{s_{32}s_{14}} + \frac{n_8}{s_{25}s_{43}} + \frac{n_2}{s_{51}s_{32}}, \\ A_5^{\rm tree}(1,3,4,2,5) &\equiv \frac{n_9}{s_{13}s_{25}} - \frac{n_5}{s_{34}s_{51}} + \frac{n_{10}}{s_{42}s_{13}} - \frac{n_8}{s_{25}s_{34}} + \frac{n_{11}}{s_{51}s_{42}}, \\ A_5^{\rm tree}(1,2,4,3,5) &\equiv \frac{n_{12}}{s_{12}s_{35}} + \frac{n_{11}}{s_{24}s_{51}} - \frac{n_3}{s_{43}s_{12}} + \frac{n_{13}}{s_{35}s_{24}} - \frac{n_5}{s_{51}s_{43}}, \\ A_5^{\rm tree}(1,4,2,3,5) &\equiv \frac{n_{14}}{s_{14}s_{35}} - \frac{n_{11}}{s_{42}s_{51}} - \frac{n_7}{s_{23}s_{14}} - \frac{n_{13}}{s_{35}s_{42}} - \frac{n_2}{s_{51}s_{23}}, \\ A_5^{\rm tree}(1,3,2,4,5) &\equiv \frac{n_{15}}{s_{13}s_{45}} - \frac{n_2}{s_{32}s_{51}} - \frac{n_{10}}{s_{24}s_{13}} - \frac{n_4}{s_{45}s_{32}} - \frac{n_{11}}{s_{51}s_{24}}, \end{aligned} \tag{4.5}$$

where the 15 numerator factors are distinguished by the propagator structure that accompany them. The relative signs are again due to the antisymmetry of the color-ordered Feynman vertices, or alternatively the antisymmetry of the color factors.

It is important to note that the diagram expansion on the right-hand side of Eq. (4.5) will satisfy the Kleiss-Kuijf relations. In fact, the Kleiss-Kuijf relations can be understood from this perspective: Any color-ordered amplitudes that have an expansion in terms of diagrams with only totally antisymmetric cubic vertices will automatically satisfy the Kleiss-Kuijf relations.

The full color-dressed amplitudes can also be expressed in terms of the kinematic numerators n_i and color factors via

$$\begin{aligned} \mathcal{A}_5^{\rm tree} = g^3\bigg(&\frac{n_1c_1}{s_{12}s_{45}} + \frac{n_2c_2}{s_{23}s_{51}} + \frac{n_3c_3}{s_{34}s_{12}} + \frac{n_4c_4}{s_{45}s_{23}} + \frac{n_5c_5}{s_{51}s_{34}} \\ &+ \frac{n_6c_6}{s_{14}s_{25}} + \frac{n_7c_7}{s_{32}s_{14}} + \frac{n_8c_8}{s_{25}s_{43}} + \frac{n_9c_9}{s_{13}s_{25}} + \frac{n_{10}c_{10}}{s_{42}s_{13}} \\ &+ \frac{n_{11}c_{11}}{s_{51}s_{42}} + \frac{n_{12}c_{12}}{s_{12}s_{35}} + \frac{n_{13}c_{13}}{s_{35}s_{24}} + \frac{n_{14}c_{14}}{s_{14}s_{35}} + \frac{n_{15}c_{15}}{s_{13}s_{45}}\bigg), \end{aligned} \tag{4.6}$$

where the color factors are explicitly given by

$$
\begin{aligned}
&c_1 \equiv \tilde f^{a_1a_2b}\tilde f^{ba_3c}\tilde f^{ca_4a_5}, && c_2 \equiv \tilde f^{a_2a_3b}\tilde f^{ba_4c}\tilde f^{ca_5a_1},\\
&c_3 \equiv \tilde f^{a_3a_4b}\tilde f^{ba_5c}\tilde f^{ca_1a_2}, && c_4 \equiv \tilde f^{a_4a_5b}\tilde f^{ba_1c}\tilde f^{ca_2a_3},\\
&c_5 \equiv \tilde f^{a_5a_1b}\tilde f^{ba_2c}\tilde f^{ca_3a_4}, && c_6 \equiv \tilde f^{a_1a_4b}\tilde f^{ba_3c}\tilde f^{ca_2a_5},\\
&c_7 \equiv \tilde f^{a_3a_2b}\tilde f^{ba_5c}\tilde f^{ca_1a_4}, && c_8 \equiv \tilde f^{a_2a_5b}\tilde f^{ba_1c}\tilde f^{ca_4a_3},\\
&c_9 \equiv \tilde f^{a_1a_3b}\tilde f^{ba_4c}\tilde f^{ca_2a_5}, && c_{10} \equiv \tilde f^{a_4a_2b}\tilde f^{ba_5c}\tilde f^{ca_1a_3},\\
&c_{11} \equiv \tilde f^{a_5a_1b}\tilde f^{ba_3c}\tilde f^{ca_4a_2}, && c_{12} \equiv \tilde f^{a_1a_2b}\tilde f^{ba_4c}\tilde f^{ca_3a_5},\\
&c_{13} \equiv \tilde f^{a_3a_5b}\tilde f^{ba_1c}\tilde f^{ca_2a_4}, && c_{14} \equiv \tilde f^{a_1a_4b}\tilde f^{ba_2c}\tilde f^{ca_3a_5},\\
&c_{15} \equiv \tilde f^{a_1a_3b}\tilde f^{ba_2c}\tilde f^{ca_4a_5}.
\end{aligned} \tag{4.7}
$$

The number of independent color factors always coincides with the number of partial amplitudes that are independent under the Kleiss-Kuijf relations. This can, for example, be seen in the color decomposition of Ref. [12]. There are six independent five-point partial amplitudes, corresponding to the six independent color factors c_i. The other color factors are related to these six by the Jacobi identity. At five points, this means that there can be no more than six independent n_i's, if the corresponding numerator identity is to generalize. If these coincide—as they do—we might expect that we should pick exactly one basis numerator per amplitude, for example, the numerator of the first term of every amplitude in Eq. (4.5). However, this choice is likely not optimal for our purposes. Naively, such a diagram basis would correspond to six choices of gauge-dependent quantities which may not necessarily be mutually compatible. If we instead pick n_1, n_2, n_3, n_4 and then define n_5 and n_6, using

$$
n_5 \equiv s_{51}s_{34}\bigg(A_5^{\rm tree}(1,2,3,4,5) - \frac{n_1}{s_{12}s_{45}} - \frac{n_2}{s_{23}s_{51}} - \frac{n_3}{s_{34}s_{12}} - \frac{n_4}{s_{45}s_{23}}\bigg), \tag{4.8}
$$

$$
n_6 \equiv s_{14}s_{25}\bigg(A_5^{\rm tree}(1,4,3,2,5) - \frac{n_5}{s_{43}s_{51}} - \frac{n_7}{s_{32}s_{14}} - \frac{n_8}{s_{25}s_{43}} - \frac{n_2}{s_{51}s_{32}}\bigg), \tag{4.9}
$$

then fewer gauge-dependent choices are made, and the consistency of amplitudes $A_5^{\rm tree}(1,2,3,4,5)$ and $A_5^{\rm tree}(1,4,3,2,5)$ is automatically guaranteed. (Here we have defined n_6 in terms of n_7 and n_8 but this will not cause any problems, as they turn out to depend only on n_1, n_2, n_3, n_4.)

We require that the remaining numerators, $n_7, n_8, \ldots, n_{15}$, satisfy the same Jacobi identity equations as the corresponding color factors associated with each diagram. For example, we require,

$$
c_3 - c_5 + c_8 = 0, \quad \Rightarrow \quad n_3 - n_5 + n_8 = 0, \tag{4.10}
$$

where the color factors and numerators correspond to Eq. (4.6), and the explicit values of the color factors are given in Eq. (4.7). Working our way through all possible Jacobi identities at five points we find ten numerator identities,

$$
\begin{aligned}
&n_3 - n_5 + n_8 = 0, && n_3 - n_1 + n_{12} = 0,\\
&n_4 - n_1 + n_{15} = 0, && n_4 - n_2 + n_7 = 0,\\
&n_5 - n_2 + n_{11} = 0, && n_7 - n_6 + n_{14} = 0,\\
&n_8 - n_6 + n_9 = 0, && n_{10} - n_9 + n_{15} = 0,\\
&n_{10} - n_{11} + n_{13} = 0, && (n_{13} - n_{12} + n_{14} = 0),
\end{aligned} \tag{4.11}
$$

where the last equation is redundant. Solving for the nine numerators, $n_7, n_8, \ldots, n_{15}$, then gives

$$
\begin{aligned}
&n_7 = n_2 - n_4, \qquad n_8 = -n_3 + n_5,\\
&n_9 = n_3 - n_5 + n_6, \qquad n_{10} = -n_1 + n_3 + n_4 - n_5 + n_6,\\
&n_{11} = n_2 - n_5, \qquad n_{12} = n_1 - n_3,\\
&n_{13} = n_1 + n_2 - n_3 - n_4 - n_6, \qquad n_{14} = -n_2 + n_4 + n_6,\\
&n_{15} = n_1 - n_4.
\end{aligned} \tag{4.12}
$$

We may now replace n_7 and n_8 in our definition of n_6,

$$
\begin{aligned}
n_6 = {}& A_5^{\rm tree}(1,4,3,2,5)s_{14}s_{25}\\
&- A_5^{\rm tree}(1,2,3,4,5)(s_{15}+s_{25})s_{14} + n_1\frac{s_{14}(s_{15}+s_{25})}{s_{12}s_{45}}\\
&+ n_2\frac{s_{23}+s_{35}}{s_{23}} + n_3\frac{s_{14}}{s_{12}} + n_4\frac{s_{14}s_{15}+s_{14}s_{25}+s_{25}s_{45}}{s_{23}s_{45}}.
\end{aligned} \tag{4.13}
$$

Note that both n_5 and n_6 appear to be nonlocal quantities in general, but by adjusting n_1, n_2, n_3, n_4 in Eq. (4.13) we can cancel poles in $A_5^{\rm tree}(1,2,3,4,5)$ and $A_5^{\rm tree}(1,4,3,2,5)$, making all numerators local.

Any one of the five-point color-ordered tree amplitudes $A_5^{\rm tree}$ can now be found by writing down the diagram expansion and plugging in the solutions of the corresponding numerator factors from the numerator identities (4.8), (4.12), and (4.13). The amplitudes will be functions of six parameters: the four gauge-dependent numerators n_1, n_2, n_3, n_4 and the two gauge-independent amplitudes $A_5^{\rm tree}(1,2,3,4,5)$, $A_5^{\rm tree}(1,4,3,2,5)$. However, since n_1, n_2, n_3, n_4 are gauge dependent this construction may seem problematic. Indeed we must check that the construction is consistent with the known properties of amplitudes.

We verify consistency by considering all factorization channels of the five-point tree amplitudes. In these limits each five-point amplitude factorizes into the product of a three-point and a four-point amplitude. In the factorization channels the five-point numerator equations (4.11) reduce to the four-point numerator identity (3.8). Thus, any potential violation of the general numerator identity must come from contact terms that vanish in all of these limits. Similarly, as the five-point numerator equations are gauge invariant in these factorization channels, the relevant terms that potentially break the gauge invariance of the five-point identity must be exactly these contact terms. Therefore, we need to investigate how the five-point numerators behave under the freedom,

$$n_1' = n_1 + \alpha_1 s_{12}s_{45}, \qquad n_2' = n_2 + \alpha_2 s_{23}s_{51},$$
$$n_3' = n_3 + \alpha_3 s_{34}s_{12}, \qquad n_4' = n_4 + \alpha_4 s_{45}s_{23}. \tag{4.14}$$

Let us, for the moment, treat the α's as local objects, but otherwise arbitrary functions of the kinematics and polarizations. By construction $A_5^{\text{tree}}(1, 2, 3, 4, 5)$ and $A_5^{\text{tree}}(1, 4, 3, 2, 5)$ are invariant under this freedom, since n_5 and n_6 are constrained so as to correctly reproduce these amplitudes. From the definition of n_5 and n_6 we obtain the transformations

$$n_5' = n_5 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)s_{51}s_{34},$$
$$n_6' = n_6 + \alpha_3 s_{12}s_{14} + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)s_{14}s_{15} + (\alpha_1 + \alpha_3 + \alpha_4)s_{14}s_{25} - \alpha_2 s_{15}s_{25} + \alpha_4 s_{25}s_{45}. \tag{4.15}$$

Remarkably, we find the four remaining amplitudes in Eq. (4.5) are also invariant under these transformations. Plugging in the shifts of numerators $n_1, n_2, \ldots, n_6$ into the remaining n_i using Eq. (4.12) gives a set of nontrivial cancellations,

$$\Delta A_5^{\text{tree}}(1, 3, 4, 2, 5) = \frac{\Delta n_9}{s_{13}s_{25}} - \frac{\Delta n_5}{s_{34}s_{51}} + \frac{\Delta n_{10}}{s_{42}s_{13}} - \frac{\Delta n_8}{s_{25}s_{34}} + \frac{\Delta n_{11}}{s_{51}s_{42}} = 0,$$
$$\Delta A_5^{\text{tree}}(1, 2, 4, 3, 5) = \frac{\Delta n_{12}}{s_{12}s_{35}} + \frac{\Delta n_{11}}{s_{24}s_{51}} - \frac{\Delta n_3}{s_{43}s_{12}} + \frac{\Delta n_{13}}{s_{35}s_{24}} - \frac{\Delta n_5}{s_{51}s_{43}} = 0,$$
$$\Delta A_5^{\text{tree}}(1, 4, 2, 3, 5) = \frac{\Delta n_{14}}{s_{14}s_{35}} - \frac{\Delta n_{11}}{s_{42}s_{51}} - \frac{\Delta n_7}{s_{23}s_{14}} - \frac{\Delta n_{13}}{s_{35}s_{42}} - \frac{\Delta n_2}{s_{51}s_{23}} = 0,$$
$$\Delta A_5^{\text{tree}}(1, 3, 2, 4, 5) = \frac{\Delta n_{15}}{s_{13}s_{45}} - \frac{\Delta n_2}{s_{32}s_{51}} - \frac{\Delta n_{10}}{s_{24}s_{13}} - \frac{\Delta n_4}{s_{45}s_{32}} - \frac{\Delta n_{11}}{s_{51}s_{24}} = 0, \tag{4.16}$$

where $\Delta n_i = n_i' - n_i$.

As for the four-point case, there is no need to restrict the α parameters to be local. In fact we can pick the α's so that $n_1' = n_2' = n_3' = n_4' = 0$, which implies that the explicit dependence on these parameters in the amplitudes cancel out. Hence, our construction is completely gauge invariant since the amplitudes in Eq. (4.5) depend only on the two gauge-invariant basis amplitudes $A_5^{\text{tree}}(1, 2, 3, 4, 5)$ and $A_5^{\text{tree}}(1, 4, 3, 2, 5)$.

Feeding the numerator solutions in Eqs. (4.8), (4.12), and (4.13) into Eq. (4.5), we find remarkably simple relations,

$$A_5^{\text{tree}}(1, 3, 4, 2, 5) = \frac{-s_{12}s_{45}A_5^{\text{tree}}(1, 2, 3, 4, 5) + s_{14}(s_{24} + s_{25})A_5^{\text{tree}}(1, 4, 3, 2, 5)}{s_{13}s_{24}},$$
$$A_5^{\text{tree}}(1, 2, 4, 3, 5) = \frac{-s_{14}s_{25}A_5^{\text{tree}}(1, 4, 3, 2, 5) + s_{45}(s_{12} + s_{24})A_5^{\text{tree}}(1, 2, 3, 4, 5)}{s_{24}s_{35}},$$
$$A_5^{\text{tree}}(1, 4, 2, 3, 5) = \frac{-s_{12}s_{45}A_5^{\text{tree}}(1, 2, 3, 4, 5) + s_{25}(s_{14} + s_{24})A_5^{\text{tree}}(1, 4, 3, 2, 5)}{s_{35}s_{24}},$$
$$A_5^{\text{tree}}(1, 3, 2, 4, 5) = \frac{-s_{14}s_{25}A_5^{\text{tree}}(1, 4, 3, 2, 5) + s_{12}(s_{24} + s_{45})A_5^{\text{tree}}(1, 2, 3, 4, 5)}{s_{13}s_{24}}, \tag{4.17}$$

independent of n_1, n_2, n_3, n_4. Thus, we find novel nontrivial relations between color-ordered gauge-theory tree amplitudes. Note that these relations should hold for *any* helicity configuration and they should be valid in D dimensions. We explicitly verified these for D-dimensional five-gluon amplitudes. However, we would also like to have a general argument as to why these hold.

This is found by looking at the factorization limits more carefully. The five-point numerator equations reduce to the correct gauge-invariant four-point identity in all factorization limits. Furthermore we know that at four points we can make the numerators local (for example, by using individual color-ordered Feynman diagrams). Similarly, we can make the five-point numerators local, by construction. Specifically, from Eq. (4.8), we can make n_5 local by taking n_1, n_2, n_3, n_4 to be the coefficients of the corresponding poles in the amplitude. When these poles are subtracted, the only remaining poles can be $1/s_{51}s_{34}$, so that n_5 is local. It is a bit trickier to show that n_6 is local. The n_5 and n_2 terms are automatically the correct terms for subtracting poles in $A_5(1, 4, 3, 2, 5)$. However, n_7 and n_8 cannot be adjusted since we demand that these satisfy

Eq. (4.12). Do they have the correct values to subtract the poles? Indeed, they do. This is because in any factorization limit the identities in Eq. (4.12) do hold, since we already demonstrated that they hold at four points. Thus n_6 is also local. The only missing pieces of the amplitudes that cannot be seen in factorization limits are the five-point contact terms. These potentially missing pieces are, of course, local. However from dimensional analysis, a five-point gauge-theory amplitude cannot contain a five-point contact term. Such a term would correspond to a forbidden five-point contact term in the Lagrangian. Thus we can always find a local solution to numerator factors satisfying the identity. The resulting amplitudes have the correct factorizations limits, strongly suggesting the consistency of our construction.

We expect this argument to generalize also to higher-point tree amplitudes, allowing us to do similar rearrangements of the terms in the amplitudes such that the respective kinematic numerator identities are satisfied, with local or nonlocal numerators. What remains to be determined is how much freedom we have in this rearrangement, that is, how many of the numerator factors can be left undetermined and thus gauge dependent, and how many numerators are constrained by extra constraints of the type Eq. (4.9). In the next section we will identify the all-n pattern.

B. Implications for n points

As with the lower-point cases, we expect the numerator identity (3.8) to lead to new constraints between amplitudes for any number of external legs. Under the Kleiss-Kuijf relations we know that there are at most $(n-2)!$ independent color-ordered amplitudes. Here we argue that the kinematic numerator identity imposes additional constraints so that the number of independent color-ordered amplitudes is $(n-3)!$.

Following the four- and five-point discussion we may expand each of these color-ordered amplitudes in terms of diagrams with only cubic and totally antisymmetric vertices, that is, numerators and propagators,

$$A_n^{\text{tree}}(1, 2, 3, \ldots, n) = \sum_j \frac{n_j}{(\prod_m p_m^2)_j}, \tag{4.18}$$

where the sum runs over all distinct ordered diagrams. The number of color-ordered diagrams with a fixed ordering of external legs is given in the first row in Table I. For this we count diagrams with only cubic vertices, and we absorb four-point contact terms into numerator factors that cancel propagators. The total number of distinct diagrams at n points contributing to the full color-dressed amplitudes is $(2n-5)!!$ as listed in the second row of the table.

Applying the same reasoning as to lower points we have explored higher-point properties of the kinematic identity. This leads to the following conjecture for the n-point structure:

(1) The kinematic numerators of gauge-theory tree-level diagrams can always be rearranged to satisfy the numerator identity (3.8) equations, which is in one-to-one correspondence to the Jacobi identity equations satisfied by the color factors of the same diagrams. The numerators can be either local or nonlocal.

(2) One must simultaneously rearrange the diagrams of at least $(n-3)!$ partial amplitudes to ensure gauge invariance of the full amplitude. The remaining partial amplitudes will automatically satisfy the numerator equations, since they are built from the same diagrams.

In Table I, we have collected various numerical counts helpful for understanding the effect of the numerator identity at higher points. The description of the count in each row is given in the caption. For all numbers given in the table we have explicitly constructed the count. The n-point count in the last row remains a conjecture, beyond eight points.

TABLE I. Counts of various diagrams, equations, and amplitudes as a function of the number of external points. We count only diagrams with three vertices. The first row gives the number of diagrams that appear in color-ordered partial amplitudes (where the external legs are cyclically ordered). The second row gives the number of such diagrams that appear in a full color-dressed amplitude. The third row gives the number of numerator (or equivalently Jacobi identity) equations. The fourth row gives the number of such independent equations. The fifth row gives the number of linearly independent (or basis) numerators which are not constrained by these equations. The number of independent amplitudes under the Kleiss-Kuijf relations is given in the sixth row. The number of independent basis partial amplitudes under the new kinematic identity, in terms of which all others can be expressed, is given in the last row. The last row is a conjecture beyond eight points.

External legs	3	4	5	6	7	8	n
ordered diagrams	1	2	5	14	42	132	$\frac{2^{n-2}(2n-5)!!}{(n-1)!}$
diagrams	1	3	15	105	945	10 395	$(2n-5)!!$
numerator equations	0	1	10	105	1260	17 325	$\frac{(n-3)(2n-5)!!}{3}$
indep. numerator eqs.	0	1	9	81	825	9675	$(2n-5)!! - (n-2)!$
basis numerators	1	2	6	24	120	720	$(n-2)!$
Kleiss-Kuijf amplitudes	1	2	6	24	120	720	$(n-2)!$
basis amplitudes	1	1	2	6	24	120	$(n-3)!$

The above conjecture does not address how this arrangement of diagrams is best achieved. The approach we take here is to simply treat the numerators as being unknown variables satisfying an equation system that describes the conjecture,

$$\{n_\alpha = n_\beta - n_\gamma\}, \tag{4.19}$$

$$A_n^{\rm tree}(\mathcal{P}_i\{1,2,3,\ldots,n\}) = \bigg[\sum_j \frac{n_j}{(\prod_m p_m^2)_j}\bigg]_i, \tag{4.20}$$

where Eq. (4.19) represents all possible numerator identities that can be written down at the n-point level corresponding to the color-factor Jacobi identities, and Eq. (4.20) represents the statement that the diagrams dressed with the unknown numerators must sum up to the known partial amplitudes for at least $(n-3)!$ different permutations of the external legs labeled by $i = 1,\ldots,(n-3)!$.

The solution to these equations will give kinematic numerators that are functions of a set of $(n-3)!$ basis amplitudes. The basis amplitudes must be chosen to be independent under the Kleiss-Kuijf relations, since these relations are manifest in our diagrammatic representation, but otherwise the basis is arbitrary. As noted in Table I, there are a total $(2n-5)!!$ diagrams and associated numerators. By counting the equations we see that the solution of Eqs. (4.19) and (4.20) will not fix all these numerators, but it will leave $(n-2)! - (n-3)!$ unspecified. However, our conjecture implies that any amplitude built out of the solution to Eqs. (4.19) and (4.20) will be independent of these free numerators, and will only depend on the gauge-invariant basis amplitudes. In particular, we can set the unspecified numerators to zero without altering any partial amplitudes. We have explicitly checked this through eight points.

Specifically, we solve Eqs. (4.19) and (4.20) using basis amplitudes where legs 1, 2, 3 are fixed $A_n^{\rm tree}(1,2,3,\mathcal{P}\{4,\ldots,n\})$. By feeding the solved numerators into those amplitudes not in the basis, we obtain new tree-level relations. As part of the conjecture we expect that these relations hold for the partial amplitudes with any external polarizations in D dimensions.

Using the fact that the Kleiss-Kuijf relations allow us to always put legs 1 and 2 next to each other, we need only give the formula for the case where leg 3 is separated from leg 2 by a set of legs $\{\alpha\}$ and similarly separated from leg 1 by a set $\{\beta\}$. We order the leg labels in $\{\alpha\}$ and $\{\beta\}$ as

$$\begin{aligned}\{\alpha\} &\equiv \{4,5,\ldots,m-1,m\},\\ \{\beta\} &\equiv \{m+1,m+2,\ldots,n-1,n\},\end{aligned} \tag{4.21}$$

which can always be undone in the final expressions by a permutation of legs $4,\ldots,n$. By extrapolating from the structure of the solutions evaluated up through eight external particles, we obtain an all-n form

$$\begin{aligned}A_n^{\rm tree}(1,2,\{\alpha\},3,\{\beta\}) = &\sum_{\{\sigma\}_j\in {\rm POP}(\{\alpha\},\{\beta\})} A_n^{\rm tree}(1,2,3,\{\sigma\}_j)\\ &\times \prod_{k=4}^{m}\frac{\mathcal{F}(3,\{\sigma\}_j,1|k)}{s_{2,4,\ldots,k}},\end{aligned} \tag{4.22}$$

where the sum runs over "partially ordered permutations" (POP) of the merged $\{\alpha\}$ and $\{\beta\}$ sets. This corresponds to all permutations of $\{\alpha\}\cup\{\beta\}$ that maintains the order of the $\{\beta\}$ elements. Either set may be taken as empty, but if $\{\alpha\}$ is empty the equation becomes trivial. The function $\mathcal{F}$ associated with leg k is given by

$$\mathcal{F}(3,\sigma_1,\sigma_2,\ldots,\sigma_{n-3},1|k) \equiv \mathcal{F}(\{\rho\}|k) = \begin{Bmatrix} \sum_{l=t_k}^{n-1}\mathcal{G}(k,\rho_l) & \text{if } t_{k-1}<t_k \\ -\sum_{l=1}^{t_k}\mathcal{G}(k,\rho_l) & \text{if } t_{k-1}>t_k\end{Bmatrix} + \begin{Bmatrix} s_{2,4,\ldots,k} & \text{if } t_{k-1}<t_k<t_{k+1} \\ -s_{2,4,\ldots,k} & \text{if } t_{k-1}>t_k>t_{k+1} \\ 0 & \text{else}\end{Bmatrix}, \tag{4.23}$$

and where t_k is the position of leg k in the set $\{\rho\}$, except for t_3 and t_{m+1} which are always defined to be[3]

$$t_3 \equiv t_5, \qquad t_{m+1} \equiv 0. \tag{4.24}$$

(Note that for $m=4$ this implies $t_3 = t_{m+1} = 0$.) The function $\mathcal{G}$ is given by

$$\mathcal{G}(i,j) = \begin{Bmatrix} s_{i,j} & \text{if } i<j \quad \text{or} \quad j=1,3 \\ 0 & \text{else}\end{Bmatrix}. \tag{4.25}$$

[3]An alternative choice is $t_3 \equiv \infty$, $t_{m+1}\equiv 0$ which is equivalent to Eq. (4.24) by momentum conservation.

Finally, the kinematic invariants are

$$s_{i,j} = (k_i+k_j)^2, \qquad s_{2,4,\ldots,i} = (k_2+k_4+\ldots+k_i)^2, \tag{4.26}$$

where the momenta are massless and outgoing. We have explicitly confirmed in $D=4$ that all MHV amplitudes through 12 points satisfy Eq. (4.22). We have also checked that gluon amplitudes for all helicity configurations through eight points satisfy this.

The four-, five-, and six-point relations generated by this formula are

$$A_4^{\rm tree}(1,2,\{4\},3) = \frac{A_4^{\rm tree}(1,2,3,4)s_{14}}{s_{24}},$$
$$A_5^{\rm tree}(1,2,\{4\},3,\{5\}) = \frac{A_5^{\rm tree}(1,2,3,4,5)(s_{14}+s_{45}) + A_5^{\rm tree}(1,2,3,5,4)s_{14}}{s_{24}},$$
$$A_5^{\rm tree}(1,2,\{4,5\},3) = \frac{-A_5^{\rm tree}(1,2,3,4,5)s_{34}s_{15} - A_5^{\rm tree}(1,2,3,5,4)s_{14}(s_{245}+s_{35})}{s_{24}s_{245}},$$
$$A_6^{\rm tree}(1,2,\{4\},3,\{5,6\}) = \frac{A_6^{\rm tree}(1,2,3,4,5,6)(s_{14}+s_{46}+s_{45})}{s_{24}} + \frac{A_6^{\rm tree}(1,2,3,5,4,6)(s_{14}+s_{46})}{s_{24}} + \frac{A_6^{\rm tree}(1,2,3,5,6,4)s_{14}}{s_{24}},$$
$$\begin{aligned} A_6^{\rm tree}(1,2,\{4,5\},3,\{6\}) = &-\frac{A_6^{\rm tree}(1,2,3,4,5,6)s_{34}(s_{15}+s_{56})}{s_{24}s_{245}} - \frac{A_6^{\rm tree}(1,2,3,4,6,5)s_{34}s_{15}}{s_{24}s_{245}} \\ &- \frac{A_6^{\rm tree}(1,2,3,6,4,5)(s_{34}+s_{46})s_{15}}{s_{24}s_{245}} - \frac{A_6^{\rm tree}(1,2,3,5,4,6)(s_{14}+s_{46})(s_{245}+s_{35})}{s_{24}s_{245}} \\ &- \frac{A_6^{\rm tree}(1,2,3,5,6,4)s_{14}(s_{245}+s_{35})}{s_{24}s_{245}} - \frac{A_6^{\rm tree}(1,2,3,6,5,4)s_{14}(s_{245}+s_{35}+s_{56})}{s_{24}s_{245}}, \end{aligned}$$

$$\begin{aligned} A_6^{\rm tree}(1,2,\{4,5,6\},3) = &-\frac{A_6^{\rm tree}(1,2,3,4,5,6)s_{34}(s_{245}+s_{56}+s_{15})s_{16}}{s_{24}s_{245}s_{2456}} + \frac{A_6^{\rm tree}(1,2,3,4,6,5)s_{34}s_{15}(s_{2456}+s_{36})}{s_{24}s_{245}s_{2456}} \\ &+ \frac{A_6^{\rm tree}(1,2,3,6,4,5)(s_{34}+s_{46})s_{15}(s_{2456}+s_{36})}{s_{24}s_{245}s_{2456}} - \frac{A_6^{\rm tree}(1,2,3,5,4,6)(s_{14}+s_{46})s_{35}s_{16}}{s_{24}s_{245}s_{2456}} \\ &- \frac{A_6^{\rm tree}(1,2,3,5,6,4)s_{14}s_{35}s_{16}}{s_{24}s_{245}s_{2456}} + \frac{A_6^{\rm tree}(1,2,3,6,5,4)s_{14}(s_{245}+s_{35}+s_{56})(s_{2456}+s_{36})}{s_{24}s_{245}s_{2456}}. \end{aligned} \tag{4.27}$$

We introduce the brackets "{" and "}" to emphasize the connection to Eq. (4.22)—they carry no other significance.

One amusing point is that this solution allows us to express any partial amplitude with three negative helicities as a linear combination of the "split helicity" cases where the three negative helicity legs are nearest neighbors in the color ordering [37].

As for four- and five-point amplitudes, we expect the higher-point numerator identities to lead to interesting consequences at loop level, via the unitarity method. In the next section we will address these consequences, albeit only for the simplest case of the four-point identity.

V. HIGHER-LOOP APPLICATIONS

The four-point identity (3.8) appears to be rather innocuous compared to the higher-point relations, but as we now discuss even this case has interesting consequences at higher loops.

A. Near-maximal cuts

Consider a near-maximal cut where we leave only one four-point tree blob uncut, but where all other propagators are cut, as illustrated in Fig. 5(a). As in Sec. III, we are interested in relating different color orderings, hence we label the cut similarly,

$$C_4(1,2,3,4) \sim A_4^{\rm tree}(1,2,3,4). \tag{5.1}$$

To be specific we can work with the cut in Fig. 5(a), but the structure of the cut outside of the four-point blob is unimportant, since it will play no role in the analysis. The blob appearing in the cut, up to the polarizations, spins, and particle types being summed over, is equivalent to a color-ordered four-point tree amplitude. Therefore, if the tree amplitudes satisfy the relations given in the previous section, the cuts must as well,

$$\begin{aligned} C_4(1,2,3,4) + C_4(1,3,4,2) + C_4(1,4,2,3) &= 0, \\ tC_4(1,2,3,4) &= uC_4(1,3,4,2), \\ sC_4(1,2,3,4) &= uC_4(1,4,2,3), \\ tC_4(1,4,2,3) &= sC_4(1,3,4,2), \end{aligned} \tag{5.2}$$

where the Mandelstam variables are now understood to be for the cut internal loop momenta, $s = (l_1 + l_2)^2$, $t = (l_2 + l_3)^2$, and $u = (l_1 + l_3)^2$, not the external momenta.

We can use the factorization properties of the four-point tree amplitudes appearing in the blob to express the cut as

$$\begin{aligned} C_4(1,2,3,4) &= \frac{n_s}{s} + \frac{n_t}{t}, \\ C_4(1,3,4,2) &= -\frac{n_u}{u} - \frac{n_s}{s}, \\ C_4(1,4,2,3) &= \frac{n_u}{u} - \frac{n_t}{t}, \end{aligned} \tag{5.3}$$

where the numerators n_s, n_t are the s-channel and t-channel numerators, respectively. Following the same steps as for the tree-level discussion in Sec. III we arrive

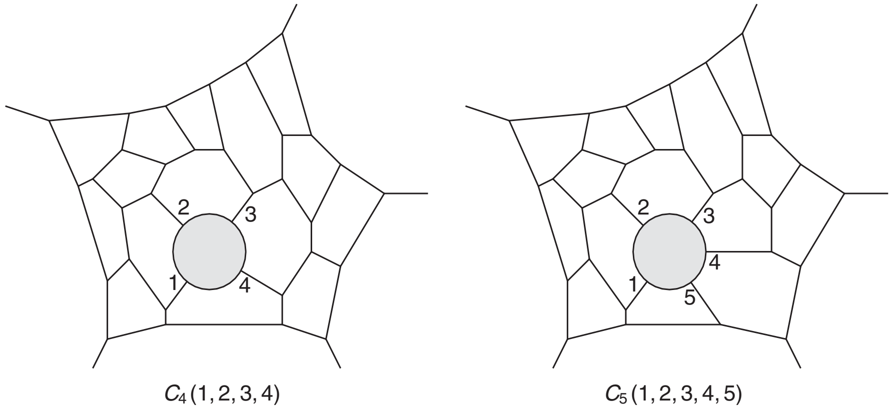

FIG. 5. Near-maximal cuts with specified color order given by the uncut blob. All visible lines are cut, thus on shell. The blobs are tree amplitudes with implied sums over all helicity and particle types entering and leaving the blobs.

at the same kinematic identity for the numerators,

$$n_u = n_s - n_t. \tag{5.4}$$

As for the tree-level case, the numerator factors in the cuts also have a freedom similar to Eqs. (3.10) and (3.11).

We can now apply the four-point identity (5.4) to any higher-loop amplitude. In Fig. 6 we give various examples of such applications. The idea is that if one computes the numerator contributions of the diagrams on the right-hand side, all numerator terms are determined on the left-hand side, up to the cut conditions. The diagrams in the figure specify the propagator structure of contributions under study. We expect the relations in Fig. 6 to hold for any gauge theory. In each of these diagrams the associated color factors are just those obtained by dressing each diagram by $\tilde{f}^{abc}$'s at each vertex.

We note that the construction presented here based on near-maximal cuts with a four-point blob, as shown in

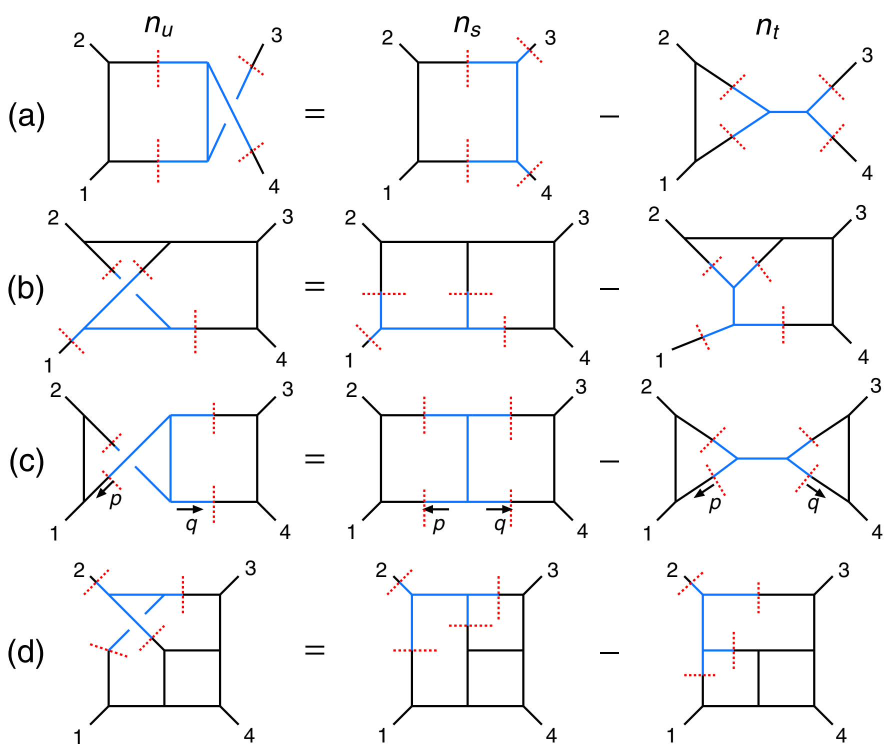

FIG. 6 (color online). Applications of the four-point numerator identity to loop amplitudes. The equalities should be interpreted either as equalities for the color factors obtained by dressing the diagrams with $\tilde{f}^{abc}$s at each vertex, or as equalities between numerators of cut diagrams and hold for any gauge theory. All lines are cut except for the propagator inside the (blue) four-point diagram indicated by the dotted lines. Our construction here is similar to Fig. 5, where we consider near-maximal cuts.

Fig. 5(a), generalizes to higher-point blobs. For example, we can consider instead a near-maximal cut with a five-point blob, as illustrated in Fig. 5(b). This cut obeys the same identities and relations as the five-point amplitude $A_5^{\rm tree}(1, 2, 3, 4, 5)$ in Sec. IVA. We can continue to relax the cut conditions obtaining further generalizations based on higher-point blobs. With further relaxation of cut conditions we can also have generalized cuts with several independent tree blobs. Each n-point blob will now have its own n-point identity and relations. In general, any generalized cut will contain nontrivial relations between the constituent diagrams analogous to the tree-level relations presented in previous sections.

As simple checks of the relations, we used the one- [38], two-[21,22], and three-loop [20,21] expressions for the four-point amplitudes of $\mathcal{N} = 4$ super-Yang-Mills theory to confirm all examples in Fig. 6. To see how this works in practice with more physical theories, here we work out a two-loop example in QCD.

B. Two-loop QCD examples

As a nontrivial example of the four-point numerator identity in QCD we consider the two-loop four-gluon all-plus helicity amplitude of pure Yang-Mills theory. This amplitude has been worked out in Ref. [31] in terms of the color decomposition given in Eq. (2.6). The result is that the planar primitive amplitude is

$$A^{\rm P}_{1234} = i\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\bigg\{s_{12}I_4^{\rm P}[(D_s-2)(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2 + \lambda_q^2\lambda_{p+q}^2) + 16((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2)](s_{12},s_{23}) + 4(D_s-2)I_4^{\rm bow\ tie}[(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)](s_{12}) + \frac{(D_s-2)^2}{s_{12}} \times I_4^{\rm bow\ tie}[\lambda_p^2\lambda_q^2((p+q)^2+s_{12})](s_{12},s_{23})\bigg\}. \quad (5.5)$$

Similarly, the nonplanar primitive amplitude is

$$A^{\rm NP}_{12;34} = i\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}s_{12}I_4^{\rm NP}[(D_s-2)(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2 + \lambda_q^2\lambda_{p+q}^2) + 16((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2)](s_{12},s_{23}). \quad (5.6)$$

The vectors $\vec{\lambda}_p$, $\vec{\lambda}_q$ represent the (-2ϵ)-dimensional components of the loop momenta p and q and $(D_s - 2)$ counts the number of gluon states circulating in the loop—in four dimensions $D_s = 4$. The planar double-box integral, whose corresponding diagram is depicted in Fig. 7(a), is defined as

$$I_4^{\rm P}[N(\lambda_i,p,q,k_i)](s_{12},s_{23}) \equiv \int\frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{N(\lambda_i,p,q,k_i)}{p^2q^2(p+q)^2(p-k_1)^2(p-k_1-k_2)^2(q-k_4)^2(q-k_3-k_4)^2}, \quad (5.7)$$

where $N(\lambda_i, p, q, k_i)$ represents the numerator factor, which is a polynomial in the momenta. Similarly, the nonplanar double-box integral, depicted in Fig. 7(b), is given by

$$I_4^{\rm NP}[N(\lambda_i,p,q,k_i)](s_{12},s_{23}) \equiv \int\frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{N(\lambda_i,p,q,k_i)}{p^2q^2(p+q)^2(p-k_1)^2(q-k_2)^2(p+q+k_3)^2(p+q+k_3+k_4)^2}. \quad (5.8)$$

Note that $A^{\rm NP}_{12;34}$ is symmetric under $k_1 \leftrightarrow k_2$, and under $k_3 \leftrightarrow k_4$. The "bow-tie" integral $I_4^{\rm bow\ tie}$ shown in Fig. 7(c) is defined by

$$I_4^{\rm bow\ tie}[N(\lambda_i,p,q,k_i)](s_{12}) \equiv \int\frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{N(\lambda_i,p,q,k_i)}{p^2q^2(p-k_1)^2(p-k_1-k_2)^2(q-k_4)^2(q-k_3-k_4)^2}. \quad (5.9)$$

Note that in the color decomposition (2.6), the color factors associated with the bow-tie integral are those of the parent double-box integral.

The amplitude has a simple feature which is rather surprising from the Feynman diagram point of view. Namely, the planar and nonplanar numerators have a very similar structure. We may use the cut-loop numerator identity displayed in Fig. 6(b) to explain this feature. To compare the numerators we need to relabel the cut momenta in the nonplanar double-box integral so that it matches the cut momenta of the planar double-box integral. This relabeling amounts to swapping $\lambda_q \leftrightarrow \lambda_{p+q}$ in the nonplanar integrals. Since the nonplanar numerator is invariant under this swap, we may directly read off the numerator from the nonplanar contribution (5.6). Up to an overall prefactor the numerator is

$$n_u = (D_s-2)s_{12}(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2) + 16s_{12}((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2). \quad (5.10)$$

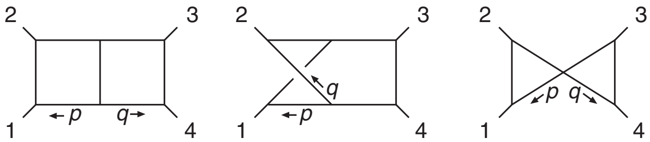

FIG. 7. The diagrams corresponding to integrals contributing to the identical-helicity two-loop four-point QCD amplitude. The numerator factors are given in the text.

Similarly from the planar integral we may read off the numerator factor for the planar integral. This then gives us the numerator

$$n_s = (D_s - 2)s_{12}(\lambda_p^2\lambda_q^2 + \lambda_p^2\lambda_{p+q}^2 + \lambda_q^2\lambda_{p+q}^2) + 16s_{12}((\lambda_p \cdot \lambda_q)^2 - \lambda_p^2\lambda_q^2). \tag{5.11}$$

The bow-tie integrals do not contribute to the indicated cuts in Fig. 6(b). Since there is no contribution with propagators corresponding to the second diagram on the right-hand side of Fig. 6(c) the corresponding numerator vanishes,

$$n_t = 0. \tag{5.12}$$

Thus we see that the numerator identity (5.4) corresponding to Fig. 6(b) is indeed satisfied. It also explains the previously mysterious identical structures of the nonplanar and planar double-box numerators.

We may also use the numerator identity, as applied in Fig. 6(c) to constrain the bow-tie contributions in $A_{1234}^{\rm P}$. First consider the terms proportional to $(D_s - 2)$. Since there are no $1/s_{12}$ contributions we may take

$$n_t = 0, \tag{5.13}$$

where n_t corresponds to the last term in Fig. 6(c) (where s, t, u refer to the figure and not the external kinematics). Reading off n_s from A_{1234} we have

$$n_s = s_{12}(D_s - 2)(\lambda_p^2\lambda_q^2 + \lambda_p^2\lambda_{p+q}^2 + \lambda_q^2\lambda_{p+q}^2) + 16s_{12}((\lambda_p \cdot \lambda_q)^2 - \lambda_p^2\lambda_q^2) + 4(D_s - 2)(\lambda_p^2 + \lambda_q^2)(\lambda_p \cdot \lambda_q)(p+q)^2, \tag{5.14}$$

where the last term comes from the $D_s - 2$ bow-tie contribution. Similarly from $A_{2134}^{\rm P} = A_{3421}^{\rm P}$ we can read off

$$n_u = s_{12}(D_s - 2)(\lambda_p^2\lambda_q^2 + \lambda_p^2\lambda_{p-q}^2 + \lambda_q^2\lambda_{p-q}^2) + 16s_{12}((\lambda_p \cdot \lambda_q)^2 - \lambda_p^2\lambda_q^2) - 4(D_s - 2)(\lambda_p^2 + \lambda_q^2)(\lambda_p \cdot \lambda_q)(p - q + k_3 + k_4)^2, \tag{5.15}$$

where we adjusted the momentum labels to ensure that the cut momenta are the same as in A_{1234}. We have

$$n_s - n_u = 4(D_s - 2)(\lambda_p^2 + \lambda_q^2)(\lambda_p \cdot \lambda_q) \times (s_{12} + (p+q)^2 + (p - q + k_3 + k_4)^2) = 0, \tag{5.16}$$

where we made use of the cut conditions. Thus, from here and from Eq. (5.13), we find that the numerator identity (5.4) is satisfied.

We have also confirmed that the $(D_s - 2)^2$ terms of the bow-tie contributions satisfy the numerator identity in Fig. 6(c). This is somewhat trickier than for the other terms, because the color factors of the terms with a $1/s_{12}$ pole need to be rearranged so they correspond to the last diagram of Fig. 6(c), not the planar double-box diagrams, before the identity is apparent.

The systematics of how we use the numerator identity (3.8) in conjunction with the method of maximal cuts [24] will be discussed elsewhere [27]. For the special case of $\mathcal{N} = 4$ super-Yang-Mills theory, various additional relations between different contributions, including some between different loop orders, may be found in Refs. [21,22,24,25].

VI. IMPLICATIONS FOR GRAVITY AMPLITUDES

The KLT relations [4] tell us that gravity tree amplitudes can be expressed directly in terms of gauge-theory tree amplitudes. These relations were originally derived in string theory and hold in field theory, since the low-energy limit of string theory is field theory. However, from a purely field-theoretic viewpoint, starting from the Einstein-Hilbert and Yang-Mills Lagrangians, these relations have remained obscure [39]. Some new relations between gravity and gauge-theory MHV amplitudes were recently presented in Ref. [6], adding to the mystery.

In this section we use the identity (3.8) to clarify the relationship between gravity and gauge theories, arguing that the KLT relations are equivalent to a diagram-by-diagram numerator "squaring" relation with gauge theory.

Consider first the four-point color-dressed amplitude in Eq. (3.9). As already noted in Ref. [22], the four-point gravity tree amplitudes can be expressed directly in terms of diagrams whose numerators are squares of the numerators of the corresponding gauge theory. In particular, for pure gravity we have

$$-iM_4^{\rm tree}(1,2,3,4) = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}, \tag{6.1}$$

where the numerators n_i are just the kinematic numerators of the gauge theory. More generally, for other particle contents, the two gauge-theory amplitudes corresponding to each factor of n_i can be different. Distinguishing the two gauge-theory amplitudes with a tilde, we have

$$-iM_4^{\rm tree}(1,2,3,4) = \frac{n_s\tilde{n}_s}{s} + \frac{n_t\tilde{n}_t}{t} + \frac{n_u\tilde{n}_u}{u}. \tag{6.2}$$

It is straightforward to verify that this expression reproduces the correct amplitude using the four-point KLT form in Eq. (2.12) together with the definition of the numerators in Eq. (3.7). It is essential here to use the fact that the gauge-theory numerators satisfy the identities

$$n_u = n_s - n_t, \qquad \tilde{n}_u = \tilde{n}_s - \tilde{n}_t. \tag{6.3}$$

Can we generalize this behavior to higher points? Indeed it is straightforward to check that the five-point KLT relation (2.13) is equivalent to a sum over all 15 diagrams defined in Eq. (4.5), but with a product of two gauge-theory numerators,

$$\begin{aligned} -iM_5^{\text{tree}}(1,2,3,4,5) = {}& \frac{n_1\tilde{n}_1}{s_{12}s_{45}} + \frac{n_2\tilde{n}_2}{s_{23}s_{51}} + \frac{n_3\tilde{n}_3}{s_{34}s_{12}} + \frac{n_4\tilde{n}_4}{s_{45}s_{23}} \\ &+ \frac{n_5\tilde{n}_5}{s_{51}s_{34}} + \frac{n_6\tilde{n}_6}{s_{14}s_{25}} + \frac{n_7\tilde{n}_7}{s_{32}s_{14}} + \frac{n_8\tilde{n}_8}{s_{25}s_{43}} \\ &+ \frac{n_9\tilde{n}_9}{s_{13}s_{25}} + \frac{n_{10}\tilde{n}_{10}}{s_{42}s_{13}} + \frac{n_{11}\tilde{n}_{11}}{s_{51}s_{42}} \\ &+ \frac{n_{12}\tilde{n}_{12}}{s_{12}s_{35}} + \frac{n_{13}\tilde{n}_{13}}{s_{35}s_{24}} + \frac{n_{14}\tilde{n}_{14}}{s_{14}s_{35}} \\ &+ \frac{n_{15}\tilde{n}_{15}}{s_{13}s_{45}}. \end{aligned} \tag{6.4}$$

Again, for this to hold, it is important that the n_i's satisfy the numerator identities in Eq. (4.11), and that the $\tilde{n}_i$'s satisfy corresponding ones.

Using Eq. (6.4) we can obtain new relations between gravity and gauge-theory amplitudes simply by altering the basis amplitudes when solving for the kinematic numerators n_i, $\tilde{n}_i$. For example, if we use $A_5^{\text{tree}}(1,2,3,4,5)$ and $A_5^{\text{tree}}(1,3,2,4,5)$ for the basis amplitudes for the left (tildeless) gauge theory and $\tilde{A}_5^{\text{tree}}(2,1,4,3,5)$ and $\tilde{A}_5^{\text{tree}}(3,1,4,2,5)$ for the basis amplitudes for the right (tilde) gauge theory, we immediately obtain the KLT relation (2.13) from Eq. (6.4). On the other hand if we change the basis amplitudes of both the left and right gauge theories to the same orderings, (1, 2, 3, 4, 5) and (1, 4, 3, 2, 5), we find an alternative left-right symmetric representation of the five-point gravity amplitudes,

$$\begin{aligned} -iM_5^{\text{tree}}(1,2,3,4,5) = {}& \frac{s_{12}s_{45}(s_{12}s_{14}s_{23} + s_{34}(s_{12}+s_{13})(s_{23}+s_{25}))}{s_{13}s_{24}s_{35}} A_5^{\text{tree}}(1,2,3,4,5)\tilde{A}_5^{\text{tree}}(1,2,3,4,5) \\ &- \frac{s_{12}s_{14}s_{25}(s_{13}+s_{35})s_{45}}{s_{13}s_{24}s_{35}}(A_5^{\text{tree}}(1,2,3,4,5)\tilde{A}_5^{\text{tree}}(1,4,3,2,5) + A_5^{\text{tree}}(1,4,3,2,5)\tilde{A}_5^{\text{tree}}(1,2,3,4,5)) \\ &+ \frac{s_{14}s_{25}(s_{12}s_{14}s_{34} + s_{23}(s_{13}+s_{14})(s_{34}+s_{45}))}{s_{13}s_{24}s_{35}} A_5^{\text{tree}}(1,4,3,2,5)\tilde{A}_5^{\text{tree}}(1,4,3,2,5). \end{aligned} \tag{6.5}$$

This representation can also be obtained directly from the KLT relations by substituting in the expressions for the gauge-theory amplitudes in terms of the same basis partial amplitudes used in Eq. (6.5).

In general, we expect the numerator relation between gravity and gauge theories to hold for an arbitrary number of external legs. Starting from the color-dressed gauge-theory amplitudes,

$$\begin{aligned} \frac{1}{g^{n-2}}\mathcal{A}_n^{\text{tree}}(1,2,3,\ldots,n) &= \sum_i \frac{n_i c_i}{(\prod_j p_j^2)_i}, \\ \frac{1}{g^{n-2}}\tilde{\mathcal{A}}_n^{\text{tree}}(1,2,3,\ldots,n) &= \sum_i \frac{\tilde{n}_i c_i}{(\prod_j p_j^2)_i}, \end{aligned} \tag{6.6}$$

where the n_i and $\tilde{n}_i$ satisfy the kinematic numerator identity (3.8), the c_i satisfy the Jacobi identity, and where the sum runs over all diagrams with only cubic vertices. We then expect gravity amplitudes to be given by

$$-iM_n^{\text{tree}}(1,2,3,\ldots,n) = \sum_i \frac{n_i\tilde{n}_i}{(\prod_j p_j^2)_i}, \tag{6.7}$$

where the sum runs over the same set of diagrams as in Eq. (6.6). We have explicitly confirmed that this is consistent with the KLT relations through eight points. We may think of formula (6.7) as a master formula for generating new representations of gravity amplitudes. As part of the conjecture, this equation has a freedom corresponding to the $2(n-3)(n-3)!$ numerators n_i and $\tilde{n}_i$ not fixed by any constraints and which drop out of the amplitudes (6.6) and (6.7). Furthermore, if we solve for the n_i, $\tilde{n}_i$ in terms of gauge-theory basis amplitudes, as done in Sec. IV B, then one has the freedom to choose these $2(n-3)!$ basis amplitudes. Every choice would result in a different KLT-like relation when fed into the master formula (6.7). The five-point representation (6.5) is one example of many possible KLT-like relations between gravity and gauge-theory amplitudes that we can construct. Note that the number of tilde and no-tilde gauge-theory amplitudes appearing in the n-point KLT relations (2.15) is $(n-3)!$ each, matching the number of independent gauge-theory amplitudes. Indeed, imposing the choice of basis amplitudes corresponding to the ones appearing in the KLT relations,

$$A_n^{\text{tree}}(1, \mathcal{P}\{2, \ldots, n-2\}, n-1, n),$$
$$\tilde{A}_n^{\text{tree}}(\mathcal{P}\{i_1, \ldots, i_j\}, 1, n-1, \mathcal{P}\{l_1, \ldots, l_{j'}\}, n), \tag{6.8}$$

where $\mathcal{P}$ signifies all permutations over the arguments, recovers the original KLT relations.

We expect that the simplified connection between gravity and gauge-theory tree amplitudes presented here should make it easier to link the ultraviolet properties of gravity theories to those of gauge theories.

VII. CONCLUSIONS

In this paper we presented a new kinematic identity for n-point tree-level color-ordered gauge-theory amplitudes. This identity is the kinematic analog of the Jacobi identity for color. By solving the constraints imposed by the identity we obtained nontrivial relations between tree-level color-ordered partial amplitudes. We derived the relevant identities at four and at five points in some detail and have confirmed it explicitly for all gluon helicity amplitudes through eight points. Beyond this it remains a conjecture, although we have performed a variety of consistency checks. A consequence of this identity is that it gives nontrivial relations between different color-ordered tree amplitudes. We conjectured an explicit all-n formula relating the different color-ordered n-point amplitudes. Under the Kleiss-Kuijf relations between color-ordered partial amplitudes [11], for a given helicity and particle configuration there are $(n-2)!$ independent partial amplitudes. The new relations reduce this number to $(n-3)!$ independent partial amplitudes.

Using generalized unitarity we demonstrated that this kinematic numerator identity implies nontrivial relations between contributions at higher loops. In this paper we applied these relations to the QCD two-loop four-point amplitude with identical helicities to explain a previously mysterious similarity between planar and nonplanar contributions. This amplitude is much simpler than for the other helicity configurations in QCD, but it does serve to illustrate the constraints imposed on higher-loop gauge-theory amplitudes by the numerator identity.

We also discussed the implication of the kinematic numerator identity for gravity tree amplitudes. We found that through at least eight points this numerator identity, together with the KLT relations, implies that gravity tree-level amplitudes can be put in a diagrammatic form where each numerator is a product of two corresponding gauge-theory numerators. We conjecture that this can be done for any number of external particles. Using the solution of color-ordered gauge-theory amplitudes in terms of a basis set of such amplitudes, we also showed how to rearrange the KLT relations into new forms.

A natural arena for applying these identities is in high-loop studies of maximally supersymmetric gauge and gravity amplitudes. As will be discussed elsewhere [27], these are very useful for obtaining four-loop four-point amplitudes in $\mathcal{N}=4$ super-Yang-Mills theory, including the subleading color contributions. The super-Yang-Mills amplitudes are useful both for studying AdS/CFT conjecture [17] and as input to the corresponding calculations in $\mathcal{N}=8$ supergravity to determine their ultraviolet properties [19,20,22], which appear better behaved than anticipated, and may even be finite. Indeed, cancellations appear to continue to all loop orders in a class of terms detectable in certain cuts [19]. These cancellations follow from the existence of novel one-loop cancellations [40]. These cancellations do not appear to be connected to supersymmetry. Instead, they appear connected to the recently uncovered behavior [41] of gravity tree amplitudes under large complex deformations [42]. String dualities have also been used to argue for improved ultraviolet behavior of $\mathcal{N}=8$ supergravity, though various difficulties with decoupling towers of massive states may spoil this conclusion [43].

It would be very interesting to explore the consequences of the kinematic numerator identity for spontaneously broken and massive theories, especially in theories of phenomenological interest. More generally, we expect the new relations discussed in this paper to be helpful for further clarifying the structure of perturbative gauge and gravity theories. In particular, we expect it to aid higher-loop investigations of gauge and gravity theories using the unitarity method.

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[1] S. J. Parke and T. R. Taylor, Phys. Rev. Lett. **56**, 2459 (1986).
[2] E. Witten, Commun. Math. Phys. **252**, 189 (2004).
[3] R. Roiban, M. Spradlin, and A. Volovich, J. High Energy Phys. 04 (2004) 012; R. Roiban, M. Spradlin, and A. Volovich, Phys. Rev. D **70**, 026009 (2004).

[4] H. Kawai, D. C. Lewellen, and S. H. H. Tye, Nucl. Phys. **B269**, 1 (1986).

[5] Z. Bern and A. K. Grant, Phys. Lett. B **457**, 23 (1999); Z. Bern, A. De Freitas, and H. L. Wong, Phys. Rev. Lett. **84**, 3531 (2000); N. E. J. Bjerrum-Bohr, Phys. Lett. B **560**, 98 (2003); Nucl. Phys. **B673**, 41 (2003); N. E. J. Bjerrum-Bohr and K. Risager, Phys. Rev. D **70**, 086011 (2004); S. Ananth and S. Theisen, Phys. Lett. B **652**, 128 (2007); M. Bianchi, H. Elvang, and D. Z. Freedman, arXiv:0805.0757.

[6] H. Elvang and D. Z. Freedman, J. High Energy Phys. 05 (2008) 096.

[7] M. L. Mangano and S. J. Parke, Phys. Rep. **200**, 301 (1991); L. J. Dixon, in *QCD & Beyond: Proceedings of TASI '95*, edited by D. E. Soper (World Scientific, Singapore, 1996).

[8] Z. Bern, L. J. Dixon, and D. A. Kosower, Annu. Rev. Nucl. Part. Sci. **46**, 109 (1996); Ann. Phys. (N.Y.) **322**, 1587 (2007).

[9] Z. Bern, Living Rev. Relativity **5**, 5 (2002).

[10] F. Cachazo and P. Svrcek, Proc. Sci., RTN2005 (2005) 004 [arXiv:hep-th/0504194].

[11] R. Kleiss and H. Kuijf, Nucl. Phys. **B312**, 616 (1989).

[12] V. Del Duca, L. J. Dixon, and F. Maltoni, Nucl. Phys. **B571**, 51 (2000).

[13] D. Zhu, Phys. Rev. D **22**, 2266 (1980); C. J. Goebel, F. Halzen, and J. P. Leveille, Phys. Rev. D **23**, 2682 (1981).

[14] Z. Bern *et al.* (NLO Multileg Working Group), arXiv:0803.0494.

[15] A. Gehrmann-De Ridder, T. Gehrmann, E. W. N. Glover, and G. Heinrich, Phys. Rev. Lett. **100**, 172001 (2008).

[16] V. A. Smirnov, Phys. Lett. B **460**, 397 (1999); S. Laporta, Int. J. Mod. Phys. A **15**, 5087 (2000); S. Moch, P. Uwer, and S. Weinzierl, J. Math. Phys. (N.Y.) **43**, 3363 (2002); T. Gehrmann and E. Remiddi, Nucl. Phys. **B601**, 248 (2001); Nucl. Phys. **B601**, 287 (2001); V. A. Smirnov, Phys. Lett. B **567**, 193 (2003); J. B. Tausk, Phys. Lett. B **469**, 225 (1999); V. A. Smirnov, Phys. Lett. B **491**, 130 (2000); **500**, 330 (2001); **524**, 129 (2002); G. Heinrich and V. A. Smirnov, Phys. Lett. B **598**, 55 (2004); M. Czakon, J. Gluza, and T. Riemann, Nucl. Phys. **B751**, 1 (2006); V. A. Smirnov, *Evaluating Feynman integrals*, Springer tracts in modern physics 211 (Springer, New York, 2004); C. Anastasiou and A. Daleo, J. High Energy Phys. 10 (2006) 031; M. Czakon, Comput. Phys. Commun. **175**, 559 (2006).

[17] C. Anastasiou, Z. Bern, L. J. Dixon, and D. A. Kosower, Phys. Rev. Lett. **91**, 251602 (2003); Z. Bern, L. J. Dixon, and V. A. Smirnov, Phys. Rev. D **72**, 085001 (2005); L. F. Alday and J. Maldacena, J. High Energy Phys. 06 (2007) 064; J. High Energy Phys. 11 (2007) 068; J. Bartels, L. N. Lipatov, and A. Sabio Vera, arXiv:0802.2065; J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, arXiv:0712.4138; arXiv:0803.1466; Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu, and A. Volovich, arXiv:0803.1465 [Phys. Rev. D (to be published)].

[18] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower, and V. A. Smirnov, Phys. Rev. D **75**, 085010 (2007).

[19] Z. Bern, L. J. Dixon, and R. Roiban, Phys. Lett. B **644**, 265 (2007).

[20] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower, and R. Roiban, Phys. Rev. Lett. **98**, 161303 (2007).

[21] Z. Bern, J. S. Rozowsky, and B. Yan, Phys. Lett. B **401**, 273 (1997).

[22] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein, and J. S. Rozowsky, Nucl. Phys. **B530**, 401 (1998).

[23] J. M. Drummond, J. Henn, V. A. Smirnov, and E. Sokatchev, J. High Energy Phys. 01 (2007) 064; J. M. Drummond, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. **B795**, 385 (2008).

[24] Z. Bern, J. J. M. Carrasco, H. Johansson, and D. A. Kosower, Phys. Rev. D **76**, 125020 (2007).

[25] F. Cachazo and D. Skinner, arXiv:0801.4574; F. Cachazo, arXiv:0803.1988.

[26] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B425**, 217 (1994); **B435**, 59 (1995).

[27] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban (unpublished).

[28] Z. Bern, L. J. Dixon, and D. A. Kosower, Nucl. Phys. **B513**, 3 (1998).

[29] Z. Bern, L. J. Dixon, and D. A. Kosower, Annu. Rev. Nucl. Part. Sci. **46**, 109 (1996).

[30] V. Del Duca, A. Frizzo, and F. Maltoni, Nucl. Phys. **B568**, 211 (2000).

[31] Z. Bern, L. J. Dixon, and D. A. Kosower, J. High Energy Phys. 01 (2000) 027.

[32] Z. Bern, L. J. Dixon, and D. A. Kosower, J. High Energy Phys. 08 (2004) 012.

[33] R. Britto, F. Cachazo, and B. Feng, Nucl. Phys. **B725**, 275 (2005); E. I. Buchbinder and F. Cachazo, J. High Energy Phys. 11 (2005) 036.

[34] J. E. Paton and H. M. Chan, Nucl. Phys. **B10**, 516 (1969); P. Cvitanović, P. G. Lauwers, and P. N. Scharbach, Nucl. Phys. **B186**, 165 (1981); D. Kosower, B. H. Lee, and V. P. Nair, Phys. Lett. B **201**, 85 (1988).

[35] Z. Bern, L. J. Dixon, M. Perelstein, and J. S. Rozowsky, Nucl. Phys. **B546**, 423 (1999).

[36] M. H. Goroff and A. Sagnotti, Phys. Lett. B **160**, 81 (1985); Nucl. Phys. **B266**, 709 (1986); R. Britto, F. Cachazo, and B. Feng, Nucl. Phys. **B725**, 275 (2005).

[37] D. A. Kosower, Nucl. Phys. **B335**, 23 (1990); F. Cachazo, P. Svrcek, and E. Witten, J. High Energy Phys. 09 (2004) 006.

[38] M. B. Green, J. H. Schwarz, and L. Brink, Nucl. Phys. **B198**, 474 (1982).

[39] Z. Bern and A. K. Grant, Phys. Lett. B **457**, 23 (1999).

[40] Z. Bern, L. J. Dixon, M. Perelstein, and J. S. Rozowsky, Nucl. Phys. **B546**, 423 (1999); Z. Bern, N. E. J. Bjerrum-Bohr, and D. C. Dunbar, J. High Energy Phys. 05 (2005) 056; N. E. J. Bjerrum-Bohr, D. C. Dunbar, and H. Ita, Phys. Lett. B **621**, 183 (2005); N. E. J. Bjerrum-Bohr, D. C. Dunbar, H. Ita, W. B. Perkins, and K. Risager, J. High Energy Phys. 12 (2006) 072; R. Kallosh, arXiv:0711.2108; N. E. J. Bjerrum-Bohr and P. Vanhove, J. High Energy Phys. 04 (2008) 065.

[41] R. Britto, F. Cachazo, B. Feng, and E. Witten, Phys. Rev. Lett. **94**, 181602 (2005); J. Bedford, A. Brandhuber, B. J.

Spence, and G. Travaglini, Nucl. Phys. **B721**, 98 (2005); F. Cachazo and P. Svrcek, arXiv:hep-th/0502160; N. E. J. Bjerrum-Bohr, D. C. Dunbar, H. Ita, W. B. Perkins, and K. Risager, J. High Energy Phys. 01 (2006) 009; P. Benincasa, C. Boucher-Veronneau, and F. Cachazo, J. High Energy Phys. 11 (2007) 057; P. Benincasa and F. Cachazo, arXiv:0705.4305; N. Arkani-Hamed and J. Kaplan, J. High Energy Phys. 04 (2008) 076; A. Hall, Phys. Rev. D **77**, 124004 (2008).

[42] Z. Bern, J. J. Carrasco, D. Forde, H. Ita, and H. Johansson, Phys. Rev. D **77**, 025010 (2008).

[43] G. Chalmers, arXiv:hep-th/0008162; M. B. Green, J. G. Russo, and P. Vanhove, J. High Energy Phys. 02 (2007) 099; M. B. Green, H. Ooguri, and J. H. Schwarz, Phys. Rev. Lett. **99**, 041601 (2007).