

Slow relaxation of rapidly rotating black holes

Shahar Hod

The Ruppin Academic Center, Emeq Hefer 40250, Israel

and The Hadassah Institute, Jerusalem 91010, Israel

(Received 15 June 2008; published 27 October 2008)

We study *analytically* the relaxation phase of perturbed, rapidly rotating black holes. In particular, we derive a simple formula for the fundamental quasinormal resonances of near-extremal Kerr black holes. The formula is expressed in terms of the black hole physical parameters: $\omega = m\Omega - i2\pi T_{\text{BH}}(n + \frac{1}{2})$, where T_{BH} and Ω are the temperature and angular velocity of the black hole, and m is the azimuthal harmonic index of a corotating equatorial mode. This formula implies that the relaxation period $\tau \sim 1/\Im\omega$ of the black hole becomes extremely long as the extremal limit $T_{\text{BH}} \rightarrow 0$ is approached. The analytically derived formula is shown to agree with direct numerical computations of the black hole resonances. We use our results to demonstrate analytically the fact that near-extremal Kerr black holes saturate the recently proposed universal relaxation bound.

DOI: [10.1103/PhysRevD.78.084035](https://doi.org/10.1103/PhysRevD.78.084035)

PACS numbers: 04.70.Bw

The radiative perturbations of a complete gravitational collapse decay with time leaving behind a “bald” black hole. This is the essence of the no-hair conjecture introduced by Ruffini and Wheeler [1] more than 30 years ago. It asserts that perturbation fields left outside the collapsing star would either be radiated away to infinity, or be swallowed by the newly born black hole.

According to the uniqueness theorems [2–6], the metric outside the black hole should relax into a Kerr-Newman spacetime, characterized solely by the black hole mass, charge, and angular momentum. This relaxation phase in the dynamics of perturbed black holes is characterized by “quasinormal ringing,” damped oscillations with a discrete spectrum (see e.g. [7] for a detailed review). At late times, all perturbations are radiated away in a manner reminiscent of the last pure dying tones of a ringing bell [8–11].

The black hole quasinormal modes (QNMs) correspond to solutions of the perturbations equations with the physical boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves crossing the event horizon [12]. Such boundary conditions single out a *discrete* set of black hole resonances $\{\omega_n\}$ (assuming a time dependence of the form $e^{-i\omega t}$). In analogy with standard scattering theory, the QNMs can be regarded as the scattering resonances of the black hole spacetime. They thus correspond to poles of the transmission and reflection amplitudes of a standard scattering problem in a black hole spacetime.

The characteristic quasinormal frequencies are *complex*. This reflects the fact that the perturbations fields decay with time, in accord with the spirit of the no-hair conjecture. It turns out that there exist an infinite number of quasinormal modes, characterizing oscillations with decreasing relaxation times (increasing imaginary part) [13]. The mode with the smallest imaginary part (known as the *fundamental* mode) determines the characteristic dynamical time scale τ for generic perturbations to decay.

Quasinormal resonances are expected to play a prominent role in gravitational radiation emitted by a variety of astrophysical scenarios involving black holes. Being the characteristic “sound” of the black hole itself, these free oscillations are of great importance from the astrophysical point of view. They allow a direct way of identifying the spacetime parameters, especially the mass and angular momentum of the black hole. This has motivated a flurry of research during the last four decades aiming to compute the quasinormal mode spectrum of various types of black hole spacetimes [7].

It is well known that realistic stellar objects generally rotate about their axis, and are therefore not spherical. Thus, an astrophysically realistic model of wave dynamics in black hole spacetimes must involve a nonspherical background geometry with angular momentum. In this work, we determine analytically the fundamental (least-damped) resonant frequencies of such rapidly rotating Kerr black holes. (For a recent progress in the study of the highly damped resonances, see [14,15].) The spectrum of quasinormal resonances can be studied analytically in the near-extremal limit $(M^2 - a^2)^{1/2} \ll a \lesssim M$, where M and a are the mass and angular momentum per unit mass of the black hole, respectively.

Before going on we would like to summarize what is already known about these fundamental, slowly damped Kerr QNMs:

- (i) In the extremal limit ($a \rightarrow M$) one finds $\Re\omega \simeq m/2M$, where $m > 0$ is the azimuthal harmonic index of a perturbation field corotating with the black hole. This fact has been found in many numerical computations (see e.g. [13,16–18]) and is well understood analytically [19–21].
- (ii) Numerical computations [13,16–18] have indicated that, for corotating modes, $\Im\omega \rightarrow 0$ in the extremal limit. This implies that these modes are long lived. Detweiler [19] presented a semianalytical formula for the long-lived black hole resonances in the ex-

TABLE I. Quasinormal resonances of a near-extremal Kerr black hole with $a/M = 0.9999$. The data shown is for the equatorial mode $l = m = 2$; see also [17]. The proposed analytical formula is $\omega = m\Omega - i2\pi T_{\text{BH}}(n + 1/2)$. (Here $m\Omega \approx 0.9859$.) The agreement between the numerical data and the proposed formula is of $\sim 2\%$.

n	$\Re\omega$	$\Im\omega$ (numerical)	$\Im\omega$ (analytical)
0	0.993 24	0.003 41	0.003 48
1	0.993 22	0.010 20	0.010 45
2	0.993 21	0.016 99	0.017 43
3	0.993 20	0.023 85	0.024 40
4	0.993 17	0.030 67	0.031 37
5	0.993 13	0.037 49	0.038 34

tremal limit. It is important to emphasize that Detweiler's result assumes that $\Im\omega/T_{\text{BH}} \gg 1$ and is not appropriate for the most long-lived modes in the nonextremal case, where it might be that $\Im\omega \sim T_{\text{BH}}$, where $T_{\text{BH}} = \frac{(M^2 - a^2)^{1/2}}{4\pi M[M + (M^2 - a^2)^{1/2}]}$ is the Bekenstein-Hawking temperature of the black hole.

- (iii) Motivated by the recently proposed time-temperature universal relaxation bound (TTT) [22], we have recently reexamined the role played by the imaginary parts of the black hole quasinormal resonances. It has been observed [22,23] that the *numerically* computed equatorial resonances of near-extremal Kerr black holes are well approximated by the simple *analytical* relation $\omega = m\Omega - i2\pi T_{\text{BH}}(n + 1/2)$, where $n = 0, 1, 2, \dots$. The successfulness of this conjectured formula is demonstrated in Table I. The numerical results therefore imply $\Im\omega = O(T_{\text{BH}})$ in the near-extremal limit. A similar relation which is valid only for the fundamental mode was obtained in [24] for equatorial modes. Table II demonstrates the fact that the predictions of the formula improve as the extremal limit is approached.

In order to determine the black hole resonances, we shall analyze the scattering of massless waves in the Kerr spacetime. The dynamics of a perturbation field Ψ in the rotating Kerr spacetime is governed by the Teukolsky equation

TABLE II. The ratio between the relation $\omega = m\Omega - i\pi T_{\text{BH}}$ for the fundamental equatorial black hole resonance, and the numerically computed value. The data shown is for the mode $l = m = 2$. The agreement between the numerical data and the analytical formula improves as the black hole approaches its extremal limit.

a/M	$\Re\omega_{\text{ana}}/\Re\omega_{\text{num}}$	$\Im\omega_{\text{ana}}/\Im\omega_{\text{num}}$
0.9	0.933	1.170
0.96	0.977	1.106
0.9999	0.993	1.022

[25]. One may decompose the field as (we use natural units in which $G = c = \hbar = 1$)

$$\Psi_{slm}(t, r, \theta, \phi) = e^{im\phi} S_{slm}(\theta; a\omega) \psi_{slm}(r) e^{-i\omega t}, \quad (1)$$

where (t, r, θ, ϕ) are the Boyer-Lindquist coordinates, ω is the (conserved) frequency of the mode, l is the spheroidal harmonic index, and m is the azimuthal harmonic index with $-l \leq m \leq l$. The parameter s is called the spin weight of the field, and is given by $s = \pm 2$ for gravitational perturbations, $s = \pm 1$ for electromagnetic perturbations, $s = \pm \frac{1}{2}$ for massless neutrino perturbations, and $s = 0$ for scalar perturbations. (We shall henceforth omit the indices s, l, m for brevity.) With the decomposition (1), ψ and S obey radial and angular equations, both of confluent Heun type [26,27], coupled by a separation constant $A(a\omega)$.

The angular functions $S(\theta; a\omega)$ are the spin-weighted spheroidal harmonics which are solutions of the angular equation [25,27]

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial S}{\partial\theta} \right) + \left[a^2 \omega^2 \cos^2\theta - 2a\omega s \cos\theta - \frac{(m + s \cos\theta)^2}{\sin^2\theta} + s + A \right] S = 0. \quad (2)$$

The angular functions are required to be regular at the poles $\theta = 0$ and $\theta = \pi$. These boundary conditions pick out a discrete set of eigenvalues A_l labeled by an integer l . [In the $a\omega \ll 1$ limit, these angular functions become the familiar spin-weighted spherical harmonics with the corresponding angular eigenvalues $A = l(l + 1) - s(s + 1) + O(a\omega)$.]

The radial Teukolsky equation is given by

$$\Delta^{-s} \frac{d}{dr} \left(\Delta^{s+1} \frac{d\psi}{dr} \right) + \left[\frac{K^2 - 2is(r - M)K}{\Delta} - a^2 \omega^2 + 2ma\omega - A + 4is\omega r \right] \psi = 0, \quad (3)$$

where $\Delta \equiv r^2 - 2Mr + a^2$ and $K \equiv (r^2 + a^2)\omega - am$. The zeroes of Δ , $r_{\pm} = M \pm (M^2 - a^2)^{1/2}$, are the black hole (event and inner) horizons.

For the scattering problem one should impose physical boundary conditions of purely ingoing waves at the black hole horizon and a mixture of both ingoing and outgoing waves at infinity (these correspond to incident and scattered waves, respectively). That is,

$$\psi \sim \begin{cases} e^{-i\omega y} + \mathcal{R}(\omega) e^{i\omega y} & \text{as } r \rightarrow \infty (y \rightarrow \infty); \\ \mathcal{T}(\omega) e^{-i(\omega - m\Omega)y} & \text{as } r \rightarrow r_+ (y \rightarrow -\infty), \end{cases} \quad (4)$$

where the ‘‘tortoise’’ radial coordinate y is defined by $dy = [(r^2 + a^2)/\Delta] dr$. Here $\Omega \equiv \frac{a}{2Mr_+}$ is the angular velocity of the black hole horizon. The coefficients $\mathcal{T}(\omega)$ and $\mathcal{R}(\omega)$ are the transmission and reflection amplitudes for a wave incident from infinity. The discrete quasinormal frequencies are the scattering resonances of the black hole space-

time. They thus correspond to poles of the transmission and reflection amplitudes. (The pole structure reflects the fact that the QNMs correspond to purely outgoing waves at spatial infinity.) These resonances determine the ringdown response of a black hole to outside perturbations.

The transmission and reflection amplitudes satisfy the usual probability conservation equation $|\mathcal{T}(\omega)|^2 + |\mathcal{R}(\omega)|^2 = 1$. Teukolsky and Press [28] and also Starobinsky and Churilov [29] have analyzed the black hole scattering problem in the double limit $a \rightarrow M$ and

$\omega \rightarrow m\Omega$. Detweiler [19] then used that solution to determine the long-lived black hole resonances in the extremal limit. Define

$$\sigma \equiv \frac{r_+ - r_-}{r_+}; \quad \tau \equiv M(\omega - m\Omega); \quad \hat{\omega} \equiv \omega r_+. \quad (5)$$

Then the resonance condition obtained in [19] for $\sigma \ll 1$ and $\tau \ll 1$ is

$$-\frac{\Gamma(2i\delta)\Gamma(1+2i\delta)\Gamma(1/2+s-2i\hat{\omega}-i\delta)\Gamma(1/2-s-2i\hat{\omega}-i\delta)}{\Gamma(-2i\delta)\Gamma(1-2i\delta)\Gamma(1/2+s-2i\hat{\omega}+i\delta)\Gamma(1/2-s-2i\hat{\omega}+i\delta)} = (-2i\hat{\omega}\sigma)^{2i\delta} \frac{\Gamma(1/2+2i\hat{\omega}+i\delta-4i\tau/\sigma)}{\Gamma(1/2+2i\hat{\omega}-i\delta-4i\tau/\sigma)}, \quad (6)$$

where $\delta^2 \equiv 4\hat{\omega}^2 - 1/4 - A - a^2\omega^2 + 2ma\omega$. In the extremal limit, Detweiler solved this equation with the assumption that $\tau/\sigma \rightarrow \infty$. For near-extremal black holes, it should be emphasized that the numerical data presented above is consistent with the limit $\tau/\sigma \rightarrow \text{const}$ as $\sigma \rightarrow 0$. Thus, the quasinormal frequencies obtained in [19] do not include the most long-lived, fundamental resonances of near-extremal black holes. We now derive analytically the fundamental resonances of rapidly rotating, near-extremal Kerr black holes.

The left-hand side of Eq. (6) has a well-defined limit as $a \rightarrow M$ and $\omega \rightarrow m\Omega$. We denote that limit by \mathcal{L} . In the limit $\omega \rightarrow m\Omega$, where ω is almost purely real, one finds from Eq. (2) that the separation constants $\{A\}$ are also almost purely real. This in turn implies that the δ^2 's are almost purely real. For some modes, including most of the equatorial $l = m$ modes (and also other modes with m close enough to l), δ^2 is found to be positive, which implies that in these cases δ is almost purely real and positive [28,30]. For the rest of the modes one finds $\delta^2 < 0$, which implies that δ is almost purely imaginary with positive imaginary part [28,30].

If δ is almost purely real and positive then one has $(-i)^{-2i\delta} = e^{-2i\delta \ln(-i)} = e^{-2i\delta \ln e^{-i\pi/2}} = e^{-2i\delta(-i\pi/2)} = e^{-\pi\delta} \ll 1$. Here we have used the fact that $\delta > 2$ for all gravitational equatorial modes [28,30]. (In fact, this is also true for many other modes for which $l \sim m \gg 1$.) If δ is almost purely imaginary with a positive imaginary part then one has $\sigma^{-2i\delta} \rightarrow 0$ in the near-extremal limit $\sigma \rightarrow 0$. In both cases, one therefore finds $\epsilon \equiv (-2i\hat{\omega}\sigma)^{-2i\delta} \ll 1$. Thus, a consistent solution of the resonance condition, Eq. (6), may be obtained if $1/\Gamma(1/2+2i\hat{\omega}-i\delta-4i\tau/\sigma) = O(\epsilon)$ [31]. Suppose

$$1/2 + 2i\hat{\omega} - i\delta - 4i\tau/\sigma = -n + \eta\epsilon + O(\epsilon^2), \quad (7)$$

where $n \geq 0$ is a non-negative integer and η is an unknown constant to be determined below. Then one has

$$\begin{aligned} \Gamma(1/2 + 2i\hat{\omega} - i\delta - 4i\tau/\sigma) &\simeq \Gamma(-n + \eta\epsilon) \\ &\simeq (-n)^{-1} \Gamma(-n + 1 + \eta\epsilon) \\ &\simeq \dots \simeq [(-1)^n n!]^{-1} \Gamma(\eta\epsilon), \end{aligned} \quad (8)$$

where we have used the relation $\Gamma(z+1) = z\Gamma(z)$ [32]. Next, using the series expansion $1/\Gamma(z) = \sum_{k=1}^{\infty} c_k z^k$ with $c_1 = 1$ [see Eq. (6.1.34) of [32]], one obtains

$$1/\Gamma(1/2 + 2i\hat{\omega} - i\delta - 4i\tau/\sigma) = (-1)^n n! \eta\epsilon + O(\epsilon^2). \quad (9)$$

Substituting this into Eq. (6) one finds $\eta = \mathcal{L}/[(-1)^n n! \Gamma(-n + 2i\delta)]$.

Finally, recalling that $4\tau/\sigma = (\omega - m\Omega)/2\pi T_{\text{BH}}$, we obtain from Eq. (7) the resonance condition

$$(\omega - m\Omega)/2\pi T_{\text{BH}} = i[-n + \eta\epsilon - 1/2 + i(\delta - m)], \quad (10)$$

where we have substituted $2i\hat{\omega} \simeq im$ for $\omega \simeq m\Omega \simeq m/2M$. The black hole quasinormal resonances of equatorial $l = m \geq 0$ modes (and, in general, corotating modes for which $\delta^2 > 0$) are therefore given by the leading-order formula

$$\omega = m\Omega - i2\pi T_{\text{BH}}(n + 1/2) + O(T_{\text{BH}}), \quad (11)$$

where $n = 0, 1, 2, \dots$. One thus finds that $\Re\omega \rightarrow m\Omega$ and $\Im\omega \rightarrow 0$ in the near extremal. These analytical findings are in accord with direct numerical computations; see Tables I and II [33].

The black hole quasinormal resonances of nonequatorial $l \neq m \geq 0$ modes (and, in general, corotating modes for which $\delta^2 < 0$) are given by the leading-order formula [see Eq. (10)]

$$\omega = m\Omega - i2\pi T_{\text{BH}}(n + 1/2 - i\delta) + O(T_{\text{BH}}), \quad (12)$$

where $n = 0, 1, 2, \dots$. It is worth pointing out that, in this case, $\Im\omega > 2\pi T_{\text{BH}}(n + 1/2)$ (recall that $\Im\delta > 0$). This

implies that for these modes $\Im\omega$ approaches zero slower as compared to the equatorial ones for which $\Im\omega = 2\pi T_{\text{BH}}(n + 1/2)$. We have therefore established the fact that nonequatorial modes decay faster than the equatorial ones.

In summary, we have studied analytically the quasinormal spectrum of rapidly rotating Kerr black holes. The main results and their physical implications are as follows:

- (1) We have shown that the fundamental resonances can be expressed in terms of the black hole physical parameters: the temperature T_{BH} , and the angular velocity Ω of the horizon.
- (2) It was found that for all corotating modes (modes having $m > 0$) $\Re\omega \rightarrow m\Omega$ in the near-extremal limit. This conclusion is in agreement with, and generalizes, the $l = m$ result obtained in [19].
- (3) We find that, in the near-extremal limit $\Im\omega$ approaches zero linearly with the black hole temperature. Namely, $\Im\omega = O(T_{\text{BH}})$. This conclusion holds true for all modes corotating with the black hole. Thus, all corotating modes become long lived as the black hole spins up. Moreover, it is realized that equatorial $l = m$ modes (and in general, modes for which the quantity δ is real) decay slower than other nonequatorial perturbations.
- (4) It is worth mentioning that a fundamental bound on the relaxation time τ of a perturbed thermodynam-

ical system has recently been suggested [22], $\tau \geq \hbar/\pi T$, where T is the system's temperature. Taking cognizance of this relaxation bound, one deduces an upper bound on the black hole fundamental (slowest damped) frequency

$$\min\{\Im\omega\} \leq \pi T_{\text{BH}}. \quad (13)$$

Thus the relaxation bound implies that a black hole must have (at least) one quasinormal resonance whose imaginary part conform to the upper bound (13). This mode would dominate the relaxation dynamics of the perturbed black hole and will determine its characteristic relaxation time scale. Taking cognizance of Eq. (11) for the equatorial modes, and substituting $n = 0$ for the fundamental resonance, one obtains $\min\{\Im\omega\} = \pi T_{\text{BH}} + O(T_{\text{BH}}^2)$ [22–24,34,35]. One therefore concludes that rapidly rotating Kerr black holes actually *saturate* the universal relaxation bound.

This research is supported by the Meltzer Science Foundation. I thank Uri Keshet, Yael Oren, Liran Shimshi, and Clovis Hoppman for helpful discussions. I also thank Andrei Gruzinov for interesting correspondence. It is my pleasure to thank Steve Detweiler for valuable comments.

[1] R. Ruffini and J. A. Wheeler, *Phys. Today* **24**, 30 (1971).
 [2] W. Israel, *Phys. Rev.* **164**, 1776 (1967); *Commun. Math. Phys.* **8**, 245 (1968).
 [3] B. Carter, *Phys. Rev. Lett.* **26**, 331 (1971).
 [4] S. W. Hawking, *Commun. Math. Phys.* **25**, 152 (1972).
 [5] D. C. Robinson, *Phys. Rev. D* **10**, 458 (1974); *Phys. Rev. Lett.* **34**, 905 (1975).
 [6] J. Ispert, *Phys. Rev. Lett.* **27**, 529 (1971).
 [7] For an excellent review and a detailed list of references see H. P. Nollert, *Classical Quantum Gravity* **16**, R159 (1999).
 [8] W. H. Press, *Astrophys. J.* **170**, L105 (1971).
 [9] V. de la Cruz, J. E. Chase, and W. Israel, *Phys. Rev. Lett.* **24**, 423 (1970).
 [10] C. V. Vishveshwara, *Nature (London)* **227**, 936 (1970).
 [11] M. Davis, R. Ruffini, W. H. Press, and R. H. Price, *Phys. Rev. Lett.* **27**, 1466 (1971).
 [12] S. L. Detweiler, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979).
 [13] E. W. Leaver, *Proc. R. Soc. A* **402**, 285 (1985).
 [14] U. Keshet and S. Hod, *Phys. Rev. D* **76**, 061501(R) (2007).
 [15] E. Berti, V. Cardoso, and S. Yoshida, *Phys. Rev. D* **69**, 124018 (2004).
 [16] H. Onozawa, *Phys. Rev. D* **55**, 3593 (1997).
 [17] See Table I of K. Glampedakis and N. Andersson, arXiv:gr-qc/0103054.
 [18] K. Glampedakis and N. Andersson, *Classical Quantum Gravity* **20**, 3441 (2003).
 [19] S. Detweiler, *Astrophys. J.* **239**, 292 (1980).
 [20] K. Glampedakis and N. Andersson, *Phys. Rev. D* **64**, 104021 (2001).
 [21] V. Cardoso, *Phys. Rev. D* **70**, 127502 (2004).
 [22] S. Hod, *Phys. Rev. D* **75**, 064013 (2007).
 [23] S. Hod, *Classical Quantum Gravity* **24**, 4235 (2007).
 [24] A. Gruzinov, arXiv:gr-qc/0705.1725.
 [25] S. A. Teukolsky, *Phys. Rev. Lett.* **29**, 1114 (1972); *Astrophys. J.* **185**, 635 (1973).
 [26] A. Ronveaux, *Heun's Differential Equations* (Oxford University Press, Oxford, UK, 1995).
 [27] C. Flammer, *Spheroidal Wave Functions* (Stanford University Press, Stanford, 1957).
 [28] S. A. Teukolsky and W. H. Press, *Astrophys. J.* **193**, 443 (1974).
 [29] A. A. Starobinsky, *Zh. Eksp. Teor. Fiz.* **64**, 48 (1973) [*Sov. Phys. JETP* **37**, 28 (1973)]; A. A. Starobinsky and S. M. Churilov, *Zh. Eksp. Teor. Fiz.* **65**, 3 (1973) [*Sov. Phys. JETP* **38**, 1 (1973)].
 [30] W. H. Press and S. A. Teukolsky, *Astrophys. J.* **185**, 649 (1973).

- [31] Had we taken $\Re\delta < 0$ or $\Im\delta < 0$, we would have found that $\epsilon \gg 1$. In this case, a consistent solution of the resonance condition, Eq. (6), would require $1/\Gamma(1/2 + 2i\hat{\omega} + i\delta - 4i\tau/\sigma) = O(\epsilon^{-1})$. It is straightforward to show that the black hole resonances would still be given by the same analytical expressions; see Eqs. (11) and (12) below.
- [32] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1970).
- [33] It is worth emphasizing again that this simple relation is valid in the $\Im\omega \ll \Re\omega$ regime.
- [34] S. Hod, Phys. Lett. B **666**, 483 (2008).
- [35] We note that a spherically symmetric Schwarzschild black hole has only one time/length scale—its horizon radius,

r_+ (or equivalently, its mass M). One therefore expects to find $\tau \sim r_+$ (and $\omega_I \sim r_+^{-1}$) on dimensional grounds. On the other hand, rotating Kerr black holes have an additional length scale—the black hole inverse temperature T_{BH}^{-1} . Here we have established that the relevant relaxation time scale of a perturbed black hole is determined by its inverse temperature T_{BH}^{-1} , and not by its horizon radius r_+ . We emphasize that T_{BH}^{-1} is much larger than r_+ in the extremal limit, $T_{\text{BH}} \rightarrow 0$. Thus, our result $\min\{\Im\omega\} = \pi T_{\text{BH}} + O(T_{\text{BH}}^2)$ is *stronger* than a relation of the form $\min\{\Im\omega\} \sim r_+^{-1}$, which one could have anticipated from some naive dimensionality considerations. In particular, the present analytical results along with numerical computations [13,16–18] imply that extremal Kerr black holes have *infinitely* long relaxation times.