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Estimating total momentum at finite distances

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We study the difficulties associated with the evaluation of the total Bondi momentum at finite distances around the central source of a general (asymptotically flat) spacetime. Since the total momentum is only rigorously defined at future null infinity, both finite distance and gauge effects must be taken into account for a correct computation of this quantity. Our discussion is applicable in general contexts but is particularly relevant in numerically constructed spacetimes for both extracting important physical information and assessing the accuracy of additional quantities.

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I. INTRODUCTION

The possibility of studying highly dynamical, compact systems by the observation of gravitational waves with the new generation of gravitational wave detectors is about to become a reality. Efforts at the experimental front have allowed the coming on-line of several highly sensitive detectors which will be followed by even more refined ones [1–3]. Confronting the output with theoretical models will shed light to a number of spectacular events which have so far remained out of reach. In fact, the analysis of the detectors' output has already provided upper bounds to several systems, giving insight into the possible source or location of a gamma ray burst event [4], and constraining important physical parameters in the crab pulsar [5]. A detection and subsequent analysis will go further than this by providing key information of the system that produced it, if accurate models are available to tie theory with observation. For binary black hole systems, concentrated efforts are now focusing on providing effective templates covering from early orbiting stages through the merger and post-merger stages and assessing their accuracy [6–9]. Intimately related to this latter issue is the suitable identification of physical parameters at each of the different stages (treated by different techniques) and the extraction of sought after information. When studying an isolated system in general relativity, there are several quantities that have strict physical meaning only in the asymptotic region; for example: gravitational radiation fields, total momentum, total angular momentum, etc.; which are defined at null infinity. Numerical approximations however, often are not able to reach the asymptotic region and so the relevant quantities must either be computed at finite distances or propagated somehow to infinity for its unambiguous computation. In a previous article [10], we have analyzed the difficulties that appear when estimating gravitational radiation at finite distances and how to remove them. In this article, we discuss the subject of estimating total momentum also at finite distances from the sources. This is useful to extract further physical information and to check the numerical solution itself. In particular we concentrate on the total Bondi momentum which is an important physical quantity. For example, it is directly related to the computation of velocity recoils produced by kicks in collisions of compact body systems, or in the gravitational radiation recoils produced in anisotropic collapse of stars, etc. [11–20]. The prevalent approach to compute these quantities within a simulation is to do so through a temporal integration of the Bondi momentum flux. The Bondi momentum on the other hand needs only be evaluated at the initial and final times for this purpose. Furthermore a balance law relates the difference between the Bondi momentum with the integral of the momentum flux, which can be employed as a further check of the implementation.

In what follows we review the Bondi momentum calculation at infinity. This will serve as a motivation for the rest of the work, by addressing the limitations that finite-distance calculations must address to reduce the associated errors when attempting such a task. The notion of total momentum is normally associated to an integration on an asymptotic sphere at future null infinity of an asymptotically flat spacetime. The integration employs the so called four-translations of the Bondi–Metzner-Sachs group (BMS), and an appropriate integrand. A typical expression in terms of standard notation is

$$P^a = -\frac{1}{4\pi}\int_S \hat{l}^a(\Psi_2^0 + \sigma^0\dot{\bar{\sigma}}^0)dS^2; \qquad (1)$$

where $\hat{l}^a = (1, \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ when expressed in standard angular coordinates (θ, ϕ) and all these quantities are calculated in terms of a Bondi coordinate and tetrad system [21]. In particular, recall that the Bondi mass M is the timelike component of this vector; namely

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$$M = -\frac{1}{4\pi}\int_S (\Psi_2^0 + \sigma^0\dot{\bar{\sigma}}^0) dS^2. \quad (2)$$

The Bondi momentum can also be expressed in terms of the Psi supermomentum [23] $\Psi \equiv \Psi_2^0 + \sigma^0\dot{\bar{\sigma}}^0 + \eth^2\bar{\sigma}^0$ by

$$P^a = -\frac{1}{4\pi}\int_S \hat{l}^a \Psi dS^2. \quad (3)$$

This supermomentum has a couple of useful properties; namely, it is real, $\bar{\Psi} = \Psi$, and its time derivative is simply

$$\dot{\Psi} = \dot{\sigma}^0\dot{\bar{\sigma}}^0. \quad (4)$$

Thus, one can see that the time variation of the Bondi momentum, in terms of time Bondi coordinate, is just given by

$$\dot{P}^a = -\frac{1}{4\pi}\int_S \hat{l}^a \dot{\sigma}^0\dot{\bar{\sigma}}^0 dS^2. \quad (5)$$

As we discussed in [10], unless further structure is added to 3 + 1 implementations, these cannot access future null infinity to calculate useful quantities and must resort to do so at finite distances. This might not only bring about finite size effects but also difficulties in adopting the appropriate frame—analogous to a Bondi one—since the advantageous coordinates at the evolution level need not agree with Bondi-type ones. In [10] we examined how these issues influence the calculation of the radiation and a way to address them. We next carry out a similar analysis for the momentum. Certainly, as the underlying issues are the same, we rely on our discussion and refer the reader to that work for further details. For the sake of completeness however, we include here briefly the list of basic assumptions required in the calculation of relevant quantities at future null infinity (further details are presented in [10]):

(i) Peeling is assumed.

(ii) Outgoing null hypersurfaces, parameterized by u intersect $\mathcal{I}^+$ (future null infinity) defining a sequence of S^2 surfaces.

(iii) Each of these surfaces is conformal to a unit sphere metric; off this surface (into the spacetime) the departure from it is of lower order. Namely the angular metric in a neighborhood of $\mathcal{I}^+$ can be expressed, as $g_{AB} = r^2 h_{AB} = r^2(q_{AB}/V^2 + c_{AB}/r + O(r^{-2}))$; with V a conformal factor, q_{AB} a unit sphere metric, and r a suitably defined radial distance.

(iv) A null-tetrad $\{\ell^a, n^a, m^a, \bar{m}^a\}$ satisfying $\ell^a n_a = -m^a\bar{m}_a = 1$ (with all other products being 0).

(v) Using standard [24,25] conventions for the Riemann tensor and spinor dyad, particularly useful scalars obtained from them are

$$\Psi_4 = C_{abcd} n^a \bar{m}^b n^c \bar{m}^d, \quad (6)$$

$$\Psi_3 = C_{abcd} \ell^a n^b \bar{m}^c \bar{n}^d, \quad (7)$$

$$\Psi_2 = C_{abcd} \ell^a m^b \bar{m}^c n^d, \quad (8)$$

$$\sigma = m^a m^b \nabla_a l_b. \quad (9)$$

(vi) A suitable (Bondi type) expansion in terms of $1/r$ with coordinates chosen such that $V = 1$, $g_{ur}^0 = 1$ and $g_{uA}^0 = 0$ (x^A labeling angular coordinates at $u = \text{const}$, $r \to \infty$) gives rise, in particular, to [26,27]

$$\Psi_4^0 = -\ddot{\bar{\sigma}}^0, \quad (10)$$

$$\Psi_3^0 = -\eth\dot{\bar{\sigma}}^0, \quad (11)$$

$$\dot{\Psi}_2^0 = \eth\Psi_3^0 + \sigma^0\Psi_4^0; \quad (12)$$

where the supraindex "0" indicates leading order in an expansion in the radial coordinate and $\eth$ is the edth operator [24] of the unit sphere. However, if one does not adopt a Bondi frame, the previous equations become much more complicated.

Working with finite size regions one would like to estimate the asymptotic fields in terms of null tetrads based on a choice of coordinate system. The nature of the numerical work will suggest the choice of coordinates—adopted to simplify the simulation—and consequently some natural tetrads that can be defined. These null tetrads however, will not be in general Bondi tetrads. In order to make these estimations, an extraction world tube is assumed to be far enough to calculate the asymptotic quantities. See Fig. 1.

The metric is expressed in terms of a null tetrad through

$$g_{ab} = \ell_a n_b + n_a \ell_b - m_a \bar{m}_b - \bar{m}_a m_b. \quad (13)$$

The null tetrad can be related to a dyad of spinors (o^A, ι^A). The relation among the null-tetrad vectors and a spinor

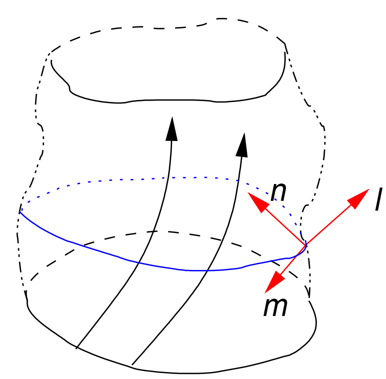

FIG. 1 (color online). The extraction world-tube Γ and the associated null tetrad and coordinates. The nearly vertical curves denote the time evolution of the angular coordinates on the world tube. The tetrad null vector ℓ points outwards, while n points inwards. Complex null vectors $(m, \bar{m})$ are tangent to the spheres contained in the world tube.

dyad is given by $\ell^a \Leftrightarrow o^A o^{A'}$, $m^a \Leftrightarrow o^A \iota^{A'}$, $\bar{m}^a \Leftrightarrow \iota^A o^{A'}$, and $n^a \Leftrightarrow \iota^A \iota^{A'}$.

Therefore, the transformation of the tetrad frame can be obtained, as we do here, from the transformation of the dyad frame [28]. We will also make use of a null polar coordinate system $(x^0, x^1, x^2, x^3) = (u, r, (\zeta + \bar{\zeta}), \frac{1}{i} \times (\zeta - \bar{\zeta}))$; where r has the meaning of a radial coordinate and $(\zeta, \bar{\zeta})$ angular coordinates.

Using the numerical time coordinate t, we construct the coordinate system $(u, r, \zeta, \bar{\zeta})$ in the following way. We define Γ to be the timelike surface given by $(r = R = \text{const})$. On Γ, a null tetrad $\{\ell, n, m, \bar{m}\}$ can be defined in the following way. Let the null function u be such that on Γ one has $u = t$; and $\ell = du$ everywhere. The function u is chosen such that the future directed vector ℓ points outwards with respect to the 2-surfaces $(t = \text{const}, r = R)$; which are, topologically, two-dimensional spheres.

Then, the complex vectors m and $\bar{m}$ are defined to be tangent to the spheres $(u = \text{const}, r = \text{const})$. Furthermore, one can choose m to be proportional to $\frac{\partial}{\partial \zeta}$; and $\bar{m}$ to be proportional to $\frac{\partial}{\partial \bar{\zeta}}$ in the asymptotic region for large r. The remaining null vector n^a is set by requiring the standard normalization conditions. This is the way in which the coordinate system $(u, r, \zeta, \bar{\zeta})$ is constructed.

Now, in order to compute the total momentum one must be able to express the different Bondi quantities appearing in Eq. (1), in terms of analogous quantities constructed from the tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ and coordinate system $(u, r, \zeta, \bar{\zeta})$. In the next sections we discuss how to do so.

II. BASIC STRUCTURE OF ASYMPTOTICALLY FLAT SPACETIMES

A. Asymptotic behavior of the tetrad's components

Let us use the coordinate system $(u, r, \zeta, \bar{\zeta})$. Keeping $(\zeta = \text{const})$ and $(\bar{\zeta} = \text{const})$ on the null hypersurface $(u = \text{const})$, one can see that incrementing r one moves along a null direction. Since ℓ is contained on the hypersurface $(u = \text{const})$, one deduces that ℓ is proportional to $\frac{\partial}{\partial r}$. We will prove below that

$$(\ell^a) = \left(\frac{1}{g_{ur}} \frac{\partial}{\partial r}\right)^a. \tag{14}$$

A natural null tetrad is then completed with,

$$\ell_a = (du)_a, \tag{15}$$

$$m^a = \xi^i \left(\frac{\partial}{\partial x^i}\right)^a, \tag{16}$$

$$\bar{m}^a = \bar{\xi}^i \left(\frac{\partial}{\partial x^i}\right)^a; \tag{17}$$

$$n^a = \left(\frac{\partial}{\partial u}\right)^a + U\left(\frac{\partial}{\partial r}\right)^a + X^i \left(\frac{\partial}{\partial x^i}\right)^a, \tag{18}$$

with $i = 2, 3$ and components ξ^i, U and X^i are:

$$\xi^2 = \frac{\xi_0^2}{r} + O\left(\frac{1}{r^2}\right), \qquad \xi^3 = \frac{\xi_0^3}{r} + O\left(\frac{1}{r^2}\right), \tag{19}$$

with

$$\xi_0^2 = \sqrt{2} P_0 V, \qquad \xi_0^3 = -i \xi_0^2; \tag{20}$$

where $V = V(u, \zeta, \bar{\zeta})$ and the square of $P_0 = \frac{(1+\zeta\bar{\zeta})}{2}$ is the conformal factor of the unit sphere;

$$U = rU_{00} + U_0 + \frac{U_1}{r} + O\left(\frac{1}{r^2}\right); \tag{21}$$

and the other components of the vector n^a have the asymptotic form

$$X^2 = X_0^2 + O\left(\frac{1}{r}\right), \qquad X^3 = X_0^3 + O\left(\frac{1}{r}\right). \tag{22}$$

B. Coordinate and tetrad transformations

With the objective of computing the total momentum, one must transform to an asymptotic Bondi coordinate $(\tilde{u}, \tilde{r}, \tilde{\zeta}, \bar{\tilde{\zeta}})$ and tetrad frame $(\tilde{\ell}, \tilde{n}, \tilde{m}, \bar{\tilde{m}})$.

The transformation is of the form

$$\tilde{u} = \alpha(u, \zeta, \bar{\zeta}) + \frac{\tilde{u}_1(u, \zeta, \bar{\zeta})}{r} + O\left(\frac{1}{r^2}\right), \tag{23}$$

$$\tilde{r} = \frac{r}{w(u, \zeta, \bar{\zeta})} + O(r^0), \tag{24}$$

$$\tilde{\zeta} = \tilde{\zeta}_0(u, \zeta, \bar{\zeta}) + O\left(\frac{1}{r}\right). \tag{25}$$

with $\dot{\alpha} > 0$.

Let us note that in a Bondi system, the angular coordinates $(\tilde{\zeta}, \bar{\tilde{\zeta}})$ can be chosen as the stereographic coordinates of asymptotic spheres. If one further assumes that ζ is a stereographic coordinate of the spheres $(t = \text{const}, r = \text{const})$; which are conformally related to the Bondi coordinates, it is only necessary to consider an angular transformation of the form [10]

$$\tilde{\zeta} = \tilde{\zeta}_0(u, \zeta) + O\left(\frac{1}{r}\right). \tag{26}$$

The contravariant metric components for the standard Bondi-like coordinate system are given by equations (3.13–18) of [26]; whose inverse is given by equations (3.19–24) of the same reference.

For the general tetrad the only difference is that now

$$g^{ur} = \frac{1}{g_{ur}} \tag{27}$$

has not necessarily a unit value.

In order to deduce the relations among the dyads let us start from the null tetrad defined by (14),

$$(\ell^a) = \left(A\frac{\partial}{\partial r}\right)^a, \tag{28}$$

together with Eqs. (16)–(18). Then the inverse metric is given by

$$g^{uu} = 0, \tag{29}$$

$$g^{ur} = A, \tag{30}$$

$$g^{ui} = 0, \tag{31}$$

$$g^{rr} = 2AU, \tag{32}$$

$$g^{ri} = AX^i, \tag{33}$$

$$g^{ij} = -(\xi^i\bar{\xi}^j + \bar{\xi}^i\xi^j). \tag{34}$$

While the metric is given by

$$g_{ur} = \frac{1}{A}, \tag{35}$$

$$g_{rr} = 0, \tag{36}$$

$$g_{ri} = 0, \tag{37}$$

$$g_{uu} = -2\frac{U}{A} + X^iX^jg_{ij}, \tag{38}$$

$$g_{ui} = -g_{ij}X^j, \tag{39}$$

$$g_{ij} = (g^{ij})^{-1} = -d\epsilon_{ik}\epsilon_{jl}(\xi^k\bar{\xi}^l + \bar{\xi}^k\xi^l); \tag{40}$$

with $i, j, k, l = 2, 3$, $d = \det(g_{ij})$, $\epsilon_{ij} = -\epsilon_{ji}$ and $\epsilon_{23} = 1$. In particular, defining the quantity

$$\lambda = \epsilon_{ij}\xi^i\bar{\xi}^j; \tag{41}$$

one has that

$$d = \frac{1}{|\lambda|^2}. \tag{42}$$

The coordinate transformation induces a tetrad transformation as described in [10]. The main resulting equations are:

$$\dot{\alpha}\frac{1}{w} = g^0_{ur} = \frac{1}{A^0}; \tag{43}$$

which determines a relation among w, α and g^0_{ur}, where a superscript ^{0} means leading behavior for large r.

The function $\tilde{\zeta}$ must be chosen [10] such that

$$\dot{\tilde{\zeta}} + X^{0\bar{\zeta}}\tilde{\zeta}_{\bar{\zeta}} = 0; \tag{44}$$

so that angular coordinates remain constant along the generators of future null infinity. In particular if to leading order $g_{ui} = 0$ ($X^i = 0$), one could adopt initial conditions so that $\tilde{\zeta} = \zeta$. In the general case however, the coordinates will "shift" around the world tube and the explicit transformation must be taken into account as described in [29].

The transformation of the vector n is given by

$$\tilde{n} = \frac{\partial}{\partial\tilde{u}} + O\left(\frac{1}{r}\right) = \frac{1}{\dot{\alpha} - \alpha_\zeta\frac{\dot{\tilde{\zeta}}_0}{\tilde{\zeta}_{0\zeta}} - \alpha_{\bar{\zeta}}\frac{\dot{\bar{\tilde{\zeta}}}_0}{\bar{\tilde{\zeta}}_{0\bar{\zeta}}}}n + O\left(\frac{1}{r}\right). \tag{45}$$

Similarly, the asymptotic behavior of the vector $\tilde{m}$ is given by

$$\tilde{m} = \frac{\sqrt{2}\tilde{P}}{\tilde{r}}\frac{\partial}{\partial\bar{\tilde{\zeta}}} + O\left(\frac{1}{r^2}\right) = -\frac{\sqrt{2}\tilde{P}_0\tilde{V}w}{r}\frac{\alpha_\zeta}{n^0(\alpha)\tilde{\zeta}_\zeta}n^0 + \frac{\tilde{P}_0\tilde{V}w}{P_0V\tilde{\zeta}_\zeta}m + O\left(\frac{1}{r^2}\right); \tag{46}$$

where n^0 is the vector n evaluated at future null infinity;

$$n^0 = \frac{\partial}{\partial u} + X^{0\zeta}\frac{\partial}{\partial\zeta} + X^{0\bar{\zeta}}\frac{\partial}{\partial\bar{\zeta}}; \tag{47}$$

and therefore an operator acting on functions.

Since the angular part of the metric expressed in terms of the new null tetrad must coincide with the angular part of the metric expressed in terms of the original null tetrad, it follows that

$$\frac{P_0^2V^2}{w^2\tilde{P}_0^2\tilde{V}^2}\tilde{\zeta}_\zeta\bar{\tilde{\zeta}}_{\bar{\zeta}} = 1. \tag{48}$$

So that

$$\frac{w\tilde{P}_0\tilde{V}}{P_0V} = \sqrt{\tilde{\zeta}_\zeta\bar{\tilde{\zeta}}_{\bar{\zeta}}}; \tag{49}$$

and therefore the factor of m is

$$\frac{w\tilde{P}_0\tilde{V}}{P_0V\tilde{\zeta}_\zeta} = \sqrt{\frac{\bar{\tilde{\zeta}}_{\bar{\zeta}}}{\tilde{\zeta}_\zeta}}. \tag{50}$$

Then, in particular, for a Bondi system one has

$$1 = \tilde{V} = \frac{P_0V\sqrt{\tilde{\zeta}_\zeta\bar{\tilde{\zeta}}_{\bar{\zeta}}}}{\tilde{P}_0w} = \frac{P_0Vg^0_{ur}\sqrt{\tilde{\zeta}_\zeta\bar{\tilde{\zeta}}_{\bar{\zeta}}}}{\tilde{P}_0\dot{\alpha}}; \tag{51}$$

where we are using that the tilde system is of Bondi type. Equation (51) determines α.

From the transformation expressed in Eqs. (45) and (46) and the requirement that the spinor metric ϵ_{AB} be invariant, one can deduce the transformation for the spinor dyad

$$\tilde{\iota}^A = \frac{1}{\sqrt{n^0(\alpha)}}\left(\frac{\tilde{\zeta}_\zeta}{\bar{\tilde{\zeta}}_{\bar{\zeta}}}\right)^{1/4}\iota^A; \tag{52}$$

and

$$\begin{aligned}\tilde{o}^A &= \sqrt{n^0(\alpha)}\left(\frac{\tilde{\zeta}_\zeta}{\bar{\tilde{\zeta}}_{\bar{\zeta}}}\right)^{1/4}\left[\left(\frac{\tilde{P}_0\tilde{V}w}{P_0V\tilde{\zeta}_\zeta}\right)o^A \right.\\ &\quad \left. - \left(\frac{\sqrt{2}\tilde{P}_0\tilde{V}w}{r}\frac{\alpha_\zeta}{n^0(\alpha)\tilde{\zeta}_\zeta}\right)\iota^A\right]\\ &= \sqrt{n^0(\alpha)}\left(\frac{\tilde{\zeta}_\zeta}{\bar{\tilde{\zeta}}_{\bar{\zeta}}}\right)^{1/4}\frac{\tilde{P}_0\tilde{V}w}{P_0V\tilde{\zeta}_\zeta}\left(o^A - \frac{1}{r}\frac{\eth_V\alpha}{n^0(\alpha)}\iota^A\right)\\ &= \sqrt{n^0(\alpha)}\left(\frac{\bar{\tilde{\zeta}}_{\bar{\zeta}}}{\tilde{\zeta}_\zeta}\right)^{1/4}\left(o^A - \frac{1}{r}\frac{\eth_V\alpha}{n^0(\alpha)}\iota^A\right). \end{aligned} \tag{53}$$

The regular dyad at future null infinity is then given by

$$\hat{\tilde{\iota}}^A = \tilde{\iota}^A = \frac{1}{\sqrt{n^0(\alpha)}}\left(\frac{\tilde{\zeta}_\zeta}{\bar{\tilde{\zeta}}_{\bar{\zeta}}}\right)^{1/4}\hat{\iota}^A; \tag{54}$$

and

$$\begin{aligned}\hat{\tilde{o}}^A &= \tilde{\Omega}^{-1}\tilde{o}^A = \frac{r}{w}\sqrt{n^0(\alpha)}\left(\frac{\bar{\tilde{\zeta}}_{\bar{\zeta}}}{\tilde{\zeta}_\zeta}\right)^{1/4}\left(o^A - \frac{1}{r}\frac{\eth_V\alpha}{n^0(\alpha)}\iota^A\right)\\ &= \frac{1}{w}\sqrt{n^0(\alpha)}\left(\frac{\bar{\tilde{\zeta}}_{\bar{\zeta}}}{\tilde{\zeta}_\zeta}\right)^{1/4}\left(\hat{o}^A - \frac{\eth_V\alpha}{n^0(\alpha)}\hat{\iota}^A\right) = B(\hat{o}^A + C\hat{\iota}^A);\end{aligned} \tag{55}$$

where

$$B = \frac{1}{w}\sqrt{n^0(\alpha)}\left(\frac{\bar{\tilde{\zeta}}_{\bar{\zeta}}}{\tilde{\zeta}_\zeta}\right)^{1/4} = \frac{g^0_{ur}}{\dot{\alpha}}\sqrt{n^0(\alpha)}\left(\frac{\bar{\tilde{\zeta}}_{\bar{\zeta}}}{\tilde{\zeta}_\zeta}\right)^{1/4} \tag{56}$$

and

$$C = -\frac{\eth_V\alpha}{n^0(\alpha)}. \tag{57}$$

Notice that one could generalize (52) and (53) to include higher order terms in an asymptotic expansion of the form

$$\tilde{\iota}^A = \frac{1}{\sqrt{n^0(\alpha)}}\left(\frac{\tilde{\zeta}_\zeta}{\bar{\tilde{\zeta}}_{\bar{\zeta}}}\right)^{1/4}(\iota^A + ho^A); \tag{58}$$

and

$$\tilde{o}^A = wB(o^A(1+\delta) + \Omega C(1+\gamma)\iota^A); \tag{59}$$

however its contribution to the (spinorial) metric is of higher order as shown by the following expression

$$\tilde{\epsilon}^{AB} = [(1+\delta) - \Omega h C(1+\gamma)]\epsilon^{AB}; \tag{60}$$

which in turn implies,

$$\delta = \Omega h C(1+\gamma). \tag{61}$$

III. CALCULATION OF THE BONDI MOMENTUM

In order to compute the Bondi momentum, as expressed in Eq. (1), one needs to evaluate the Bondi quantities $\tilde{\Psi}^0_2$, $\tilde{\sigma}^0$ and $\dot{\bar{\tilde{\sigma}}}^0$ in terms of the quantities obtained in the more readily accessible quantities to numerical implementations.

A. Transformation of Ψ^0_2

We can now easily calculate the component Ψ_2 of the Weyl tensor, in leading orders, with respect to the new null tetrad, obtaining

$$\begin{aligned}\tilde{\Psi}^0_2 &= \tilde{\Omega}^{-1}\Psi_{ABCD}\hat{\tilde{o}}^A\hat{\tilde{o}}^B\hat{\tilde{\iota}}^C\hat{\tilde{\iota}}^D\\ &= \frac{B^2}{wn^0(\alpha)}\left(\frac{\tilde{\zeta}_\zeta}{\bar{\tilde{\zeta}}_{\bar{\zeta}}}\right)^{1/2}\left(\Psi^0_2 - 2\frac{\eth_V\alpha}{n^0(\alpha)}\Psi^0_3 + \frac{(\eth_V\alpha)^2}{n^0(\alpha)^2}\Psi^0_4\right)\\ &= \left(\frac{g^0_{ur}}{\dot{\alpha}}\right)^3\left(\Psi^0_2 - 2\frac{\eth_V\alpha}{n^0(\alpha)}\Psi^0_3 + \frac{(\eth_V\alpha)^2}{n^0(\alpha)^2}\Psi^0_4\right).\end{aligned} \tag{62}$$

Note that the asymptotic transformation of the dyad $\{o^A, \iota^A\}$ can be reinterpreted as a combination of the null rotations of types II (rotation around n^a) and III (boost/spin) in the standard Newman-Penrose formalism (see Appendix A). If one makes use of these rotations to compute the transformation of the complete scalar Ψ_2 an expression similar to Eq. (62) is obtained but *without* the factor $(g^0_{ur}\dot{\alpha}^{-1})^3$. This factor is present in Eq. (62) since we are computing the leading order behaviors, which due to the assumed peeling property obey,

$$\Psi_2 = \frac{\Psi^0_2}{r^3} + O\left(\frac{1}{r^4}\right); \tag{63}$$

$$\tilde{\Psi}_2 = \frac{\tilde{\Psi}^0_2}{\tilde{r}^3} + O\left(\frac{1}{\tilde{r}^4}\right). \tag{64}$$

Then, taking into account the relation

$$\tilde{r} = \left(\frac{g^0_{ur}}{\dot{\alpha}}\right)r + O(r^0); \tag{65}$$

if follows that the aforementioned factor will be present in the transformation of Ψ^0_2.

B. Transformation of $\tilde{\sigma}^0$ and $\dot{\bar{\tilde{\sigma}}}^0$

Let us recall that the shear is defined by

$$\sigma = o^A\bar{\iota}^{A'}o^B\nabla_{AA'}o_B = \frac{\sigma_0}{r^2} + O\left(\frac{1}{r^3}\right). \tag{66}$$

Only the leading order transformation of the dyad is needed to calculate the transformation of the leading order behavior of the shear.

Then, one has

$$\tilde{\sigma} = (wB)^2\left(\frac{\bar{\tilde{\zeta}}_{\bar{\zeta}}}{\tilde{\zeta}_{\zeta}}\right)^{1/2}(o^A + C\Omega\iota^A)\iota^{A'}(o^B + C\Omega\iota^B)\nabla_{AA'}(o_B + C\Omega\iota_B)$$
$$= (wB)^2\left(\frac{\bar{\tilde{\zeta}}_{\bar{\zeta}}}{\tilde{\zeta}_{\zeta}}\right)^{1/2}[\sigma - \Omega\hat{m}(\Omega C) - \Omega C\hat{n}(\Omega C)\Omega C\tau + 2\Omega C(\beta - \Omega C\epsilon') + \Omega^2C^2(-\rho' - \Omega C\kappa') + O(\Omega^3)]; \quad (67)$$

where $\kappa = 0$.

The details of the calculations leading to $\tilde{\sigma}_0$ are given in the Appendix B. One finds

$$\tilde{\sigma}_0 = B^2\left(\frac{\bar{\tilde{\zeta}}_{\bar{\zeta}}}{\tilde{\zeta}_{\zeta}}\right)^{1/2}\left[\sigma_0 - \eth_V C + 2\frac{C}{A}\eth_V A - Cn^0(C)\right]; \quad (68)$$

and where C is recognized as a quantity of boost-spin weight $\{p, q\}$ type $\{2, 0\}$ [24].

In the calculation of the Bondi momentum, one also needs the time derivative of the shear that is given by

$$\dot{\tilde{\sigma}}_0 \equiv \frac{\partial\tilde{\sigma}_0}{\partial\tilde{u}} = \frac{1}{n^0(\alpha)}n^0\left(B^2\left(\frac{\bar{\tilde{\zeta}}_{\bar{\zeta}}}{\tilde{\zeta}_{\zeta}}\right)^{1/2}\left[\sigma_0 - \eth_V C + 2\frac{C}{A}\eth_V A - Cn^0(C)\right]\right). \quad (69)$$

Then, we see that by expressing the Bondi scalars $\tilde{\Psi}_2^0$, $\tilde{\sigma}_0$ and $\dot{\tilde{\sigma}}_0$ in terms of Ψ_2^0, σ_0 and $\dot{\sigma}_0$, we get nontrivial factors and terms containing Ψ_3^0, Ψ_4^0, g_{ur}^0, $\tilde{\zeta}$, α, V and some of their derivatives. This is an analogous situation as that observed in [10] for the radiation field Ψ_4^0 where these correcting terms will have in general nontrivial angular dependence which would affect relevant quantities. These would be important, for example, in computations of kicks observed in collisions of different masses or spinning compact objects where the merged black hole moves through the computational grid and, consequently, the extraction sphere will be noncentered introducing extra multipolar structure.

For completeness of the discussion we point out how to calculate the auxiliary functions: the conformal function V, was explained in our previous article [10], the angular transformation $\tilde{\zeta}$ is determined by the angular components of the vector n, while α is determined from the knowledge of g_{ur}, $\tilde{\zeta}$ and V.

IV. EXAMPLES

In what follows we discuss two examples to illustrate the effects brought by nonadapted tetrads in the calculation of the momentum. The first example shows how the computation can be affected already at future null infinity and without even introducing a 3 + 1 description of the spacetime. The second example illustrates how coordinates not adapted to a Bondi system in a 3 + 1 description can give rise to results with spurious gauge dependence.

A. Robinson-Trautman metrics

As a first example of applications of our expression for the Bondi momentum, let us study the Robinson-Trautman metrics. These metrics are vacuum solutions of Einstein's equations containing a congruence of diverging null geodesics, with vanishing shear and twist [30]. They can be written as [31]

$$ds^2 = \left(-2Hr + K - 2\frac{M(u)}{r}\right)du^2 + 2dudr - \frac{r^2}{P^2}d\zeta d\bar{\zeta}; \quad (70)$$

where $P = P(u, \zeta, \bar{\zeta})$, $H = \frac{\dot{P}}{P}$, $K = \Delta \ln P$, a doted quantity denotes its time derivative and Δ is the two-dimensional Laplacian for the two-surfaces $u =$ constant, $r =$ constant with line element

$$dS^2 = \frac{1}{P^2}d\zeta d\bar{\zeta}.$$

We can describe this line element in terms of the line element of the unit sphere; this is done by expressing P in terms of $P = V(u, \zeta, \bar{\zeta})P_0(\zeta, \bar{\zeta})$, with P_0 the value of P for the unit sphere. The function $V(u, \zeta, \bar{\zeta})$, must satisfy a fourth order parabolic differential equation

$$-3M\dot{V} = V^4\eth^2\bar{\eth}^2V - V^3\eth^2V\bar{\eth}^2V; \quad (71)$$

where we fixed the freedom of redefining u in such a way that $M(u)$ is actually a constant. This equation is known as the Robinson-Trautman (RT) equation. If l denotes the vector field that generates the null congruence, then $l_a = du$, and $l^a = (\frac{\partial}{\partial r})^a$. By completing the tetrad in the usual way with the three null vectors,

$$n^a = \left(\frac{\partial}{\partial u}\right)^a + \left(Hr - \frac{K}{2} + \frac{M}{r}\right)\left(\frac{\partial}{\partial r}\right)^a, \quad (72)$$

$$m^a = \frac{\sqrt{2}P}{r}\left(\frac{\partial}{\partial\zeta}\right)^a, \quad (73)$$

$$\bar{m}^a = \frac{\sqrt{2}P}{r}\left(\frac{\partial}{\partial\bar{\zeta}}\right)^a, \quad (74)$$

we have a frame adapted to the geometry. We call this coordinate/tetrad system RT frame.

The relation between the RT coordinate system and a Bondi system $\{\tilde{u}, \tilde{r}, \tilde{\zeta}, \bar{\tilde{\zeta}}\}$ will be of the form given by Eqs. (23)–(25).

Now, let us compute the Bondi linear momentum

$$P^a = -\frac{1}{4\pi}\int\left(\tilde{\Psi}_2^0 + \tilde{\sigma}^0\frac{\partial\bar{\tilde{\sigma}}^0}{\partial\tilde{u}}\right)\hat{l}^a d\tilde{S}^2; \qquad (75)$$

in a given $u = u_o =$ const, RT section, and let us write this P^a in terms of an RT frame. The relation between Robinson-Trautman and Bondi quantities are given by Eqs. (62), (68), and (69). But in order to simplify the expressions we will use the freedom of BMS group in such a way that the "origin" of the new Bondi system $\tilde{u} = 0$ coincides with the RT section u_0, where we wish to compute the momentum. That is, the relation between the Bondi and RT coordinates is such that $\alpha(u_0, \zeta, \bar{\zeta}) = 0$. This implies, in particular, that $\eth_V\alpha(u_0, \zeta, \bar{\zeta}) = \bar{\eth}_V\alpha(u_0, \zeta, \bar{\zeta}) = 0$.

For Robinson-Trautman metrics we have:

$$\Psi^0_{2RT} = -M, \qquad (76)$$

$$\sigma_{RT} = \sigma^0 = 0, \qquad (77)$$

$$g_{ur} = 1, \qquad (78)$$

$$\dot{\alpha} = V. \qquad (79)$$

then in terms of these quantities, and using the Eqs. (62), (68), and (69), we have that the Bondi momentum reads,

$$P^a = \frac{1}{4\pi}\int\frac{M}{V^3}\hat{l}^a dS^2. \qquad (80)$$

Here it is important to emphasize that the absence of the V^{-3} correcting factor would imply an erroneous constant mass result.

Note additionally, that if we wish to compute the Bondi momentum at two different RT times using Eq. (80), we will obtain two Bondi momenta calculated with respect to two different Bondi systems which are related by some BMS transformation. In order to compare the second momentum with the first, one should express it with respect to the first Bondi system.

B. Vaidya solution

As a second example let us study the Vaidya metric which represents a spherically-symmetric solution radiating a null fluid. The usual representation of this metric is

$$ds^2 = \left(1 - 2\frac{M(\tilde{u})}{\tilde{r}}\right)d\tilde{u}^2 + 2d\tilde{u}d\tilde{r} - \tilde{r}^2 d\tilde{\Omega}^2 \qquad (81)$$

where

$$d\tilde{\Omega}^2 = d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2, \qquad (82)$$

and $\{\tilde{u}, \tilde{r}, \tilde{\theta}, \tilde{\phi}\}$ define a null polar coordinate system. The mass of this spacetime at a given time $\tilde{u}$ is given by $M(\tilde{u})$ which decreases with time. Notice that the coordinate system used to express the line element is naturally of Bondi type with an associated tetrad given by

$$\tilde{l}^a = \partial^a_{\tilde{r}}, \qquad (83)$$

$$\tilde{n}^a = \partial^a_{\tilde{u}} - \frac{1}{2}\left(1 - \frac{2M(\tilde{u})}{\tilde{r}}\right)\partial^a_{\tilde{r}}, \qquad (84)$$

$$\tilde{m}^a = \frac{1}{\sqrt{2}\tilde{r}}\left(\partial^a_{\tilde{\theta}} + \frac{i}{\sin\tilde{\theta}}\partial^a_{\tilde{\phi}}\right), \qquad (85)$$

$$\bar{\tilde{m}}^a = \frac{1}{\sqrt{2}\tilde{r}}\left(\partial^a_{\tilde{\theta}} - \frac{i}{\sin\tilde{\theta}}\partial^a_{\tilde{\phi}}\right). \qquad (86)$$

Thus leading order behavior of $\tilde{n}^a$ is

$$\tilde{n}^a = \partial^a_{\tilde{u}} + O\left(\frac{1}{\tilde{r}^2}\right); \qquad (87)$$

in terms of an asymptotic regular frame at future null infinity.

We can now compute some geometrical quantities pertinent to the present discussion with this tetrad. In particular we obtain for the Weyl's scalars and the shear are

$$\tilde{\psi}_1 = \tilde{\psi}_3 = \tilde{\psi}_4 = \tilde{\sigma} = 0, \qquad (88)$$

$$\tilde{\psi}_2 = -\frac{M(\tilde{u})}{\tilde{r}^3}; \qquad (89)$$

thus, in particular, the only nonvanishing component of the leading order behavior of Weyl's scalar is $\tilde{\psi}_2^0 = -M(\tilde{u})$, and the null congruence determined by $\tilde{l}^a$ is a free shear congruence ($\tilde{\sigma} = 0$). If we compute the Bondi momentum to this metric we obtain the well know result

$$P^{\underline{a}} = -\frac{1}{4\pi}\int(\tilde{\psi}_2^0 + \tilde{\sigma}^0\dot{\bar{\tilde{\sigma}}}^0)\hat{l}^{\underline{a}} d\tilde{S}^2 = (M(\tilde{u}), 0, 0, 0); \qquad (90)$$

where $\underline{a}$ is a numerical index. This relation shows that $M(\tilde{u})$ is the Bondi mass, and that this particular Bondi system is at rest with the source (the spatial part of the Bondi momentum is zero).

So far the discussion has been centered around a null coordinate system. However, most numerical simulations adopt a natural $3 + 1$ coordinate system and it is with respect to this system that relevant quantities are calculated. In order to make contact with such task, we must reexpress the line element in terms of a $3 + 1$ coordinate system through some given transformation $(\tilde{u}, \tilde{r}, \tilde{\theta}, \tilde{\phi}) \rightarrow (T, R, \Theta, \Phi)$. Naturally different transformations can be envisaged and, in particular, ones that will be consistent with a Bondi system. However, our goal here is to illustrate the issues discussed in the previous sections in a simple scenario where the coordinates do not necessarily conform to Bondi ones. We therefore introduce the following transformation,

$$T = \Delta(\tilde{u}) + \frac{\tilde{r}}{\Delta^{(1)}(\tilde{u})}, \tag{91}$$

$$R = \frac{\tilde{r}}{\Delta^{(1)}(\tilde{u})}, \tag{92}$$

$$\Theta = \tilde{\theta}, \tag{93}$$

$$\Phi = \tilde{\phi}, \tag{94}$$

where $\Delta^{(1)} = \frac{d\Delta}{d\tilde{u}}$ and in general we will have $\Delta^{(n)} = \frac{d^n\Delta}{d\tilde{u}^n}$. Incidentally, let us note that Δ is playing the role of the inverse of the transformation α presented previously. Therefore we have,

$$d\tilde{u} = \frac{1}{\Delta^{(1)}}(dT - dR), \tag{95}$$

$$d\tilde{r} = \frac{\Delta^{(2)}R}{\Delta^{(1)}}dT + \left(\Delta^{(1)} - \frac{\Delta^{(2)}R}{\Delta^{(1)}}\right)dR, \tag{96}$$

$$d\tilde{\theta} = d\Theta, \tag{97}$$

$$d\tilde{\phi} = d\Phi. \tag{98}$$

With this transformation, we can construct a "standard" null tetrad in the usual way as in numerical efforts, i.e., from a unit vector N^a orthogonal to the surface $T =$ const, and three vectors $\{R^a, \partial^a_\Theta, \partial^a_\Phi\}$ obtained on the sphere defined by the $T =$ const, $R =$ const by Gram-Schmidt procedure. The result is,

$$l'^a = \frac{1}{\sqrt{2}}(N^a + R^a) = \frac{1}{\sqrt{2}N}[\partial_T + \partial_R], \tag{99}$$

$$n'^a = \frac{1}{\sqrt{2}}(N^a - R^a) = \frac{1}{\sqrt{2}N}[\partial_T - (2N^2 - 1)\partial_R], \tag{100}$$

$$m'^a = \frac{1}{\sqrt{2}\Delta^{(1)}R}\left(\partial^a_\Theta + \frac{i}{\sin\Theta}\partial^a_\Phi\right), \tag{101}$$

$$\bar{m}'^a = \frac{1}{\sqrt{2}\Delta^{(1)}R}\left(\partial^a_\Theta - \frac{i}{\sin\Theta}\partial^a_\Phi\right); \tag{102}$$

where

$$N = \sqrt{2 - \frac{1}{(\Delta^{(1)})^2}\left(1 - 2\frac{M(\tilde{u})}{\tilde{r}} + \frac{2\Delta^{(2)}}{\Delta^{(1)}}\tilde{r}\right)}; \tag{103}$$

and where $\tilde{r}$ and $\tilde{u}$ must be thought of as functions of $\{T, R\}$. If we compute relevant scalar quantities with this tetrad, the only nonvanishing scalar is ψ'^0_2 which is,

$$\psi'^0_2 = -\frac{M(\tilde{u})}{(\Delta^{(1)})^3}; \tag{104}$$

where one finds an additional nontrivial factor which is completely coordinate dependent. In order to obtain the correct expression we need to take into account the corrective factors that were discussed in Sec. III. For this, assuming one starts with the 3 + 1 version of the Vaidya metric induced by the coordinates (T, R, Θ, Φ), and following our discussion we need first a null coordinate/tetrad system naturally induced by these coordinates, namely

$$u = T - R, \tag{105}$$

$$r = R, \tag{106}$$

$$\theta = \Theta, \tag{107}$$

$$\phi = \Phi. \tag{108}$$

We can now construct the "naturally" induced tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ to leading order and compare this with the Bondi tetrad $\{\tilde{l}^a, \tilde{n}^a, \tilde{m}^a, \bar{\tilde{m}}^a\}$:

$$l_a = (du)_a, \qquad l^a = \partial^a_r = \Delta^{(1)}\tilde{l}^a, \tag{109}$$

$$n^a = \partial^a_u + O\left(\frac{1}{r}\right) = \frac{1}{\Delta^{(1)}}\tilde{n}^a + O\left(\frac{1}{r}\right), \tag{110}$$

$$m^a = \frac{1}{\sqrt{2}\Delta^{(1)}r}\left(\partial^a_\theta + \frac{i}{\sin\theta}\partial^a_\phi\right) = \tilde{m}^a, \tag{111}$$

$$\bar{m}^a = \frac{1}{\sqrt{2}\Delta^{(1)}r}\left(\partial^a_\theta - \frac{i}{\sin\theta}\partial^a_\phi\right) = \bar{\tilde{m}}^a. \tag{112}$$

Then, from Eqs. (109) and (110), we see that

$$g_{ur} = 1, \qquad \dot{\alpha} = \frac{1}{\Delta^{(1)}}. \tag{113}$$

If we compute ψ^0_2 with this tetrad we obtain,

$$\psi^0_2 = \psi'^0_2 = -\frac{M(\tilde{u})}{(\Delta^{(1)})^3}. \tag{114}$$

Then, employing Eq. (62), we get

$$\tilde{\psi}^0_2 = \left(\frac{1}{\dot{\alpha}}\right)^3\psi^0_2 = -M(\tilde{u}); \tag{115}$$

which yields the correct result.

As a final comment, let us note that $\psi_2 = \psi'_2$. This is due to the fact that the relation between the two tetrads is

$$l'^a = \frac{1}{\sqrt{2}N}l^a, \tag{116}$$

$$n'^a = \sqrt{2}Nn^a, \tag{117}$$

$$m'^a = m^a, \tag{118}$$

$$\bar{m}'^a = \bar{m}^a; \tag{119}$$

and therefore the $\sqrt{2}N$ factors cancel.

V. FINAL COMMENTS

The calculation of total momentum at finite distances can be employed to extract valuable information from a numerically obtained spacetime both to obtain relevant physical quantities and check related results obtained in the simulation. In particular, the final black hole recoil velocity can be readily obtained and the total energy radiated during the length of the simulation. Standard calculations of kicks in numerical relativity have relied on integrating the flux of energy over a period of interest; alternatively, it could also be computed simply by the difference between initial and final spacetime momenta; provided that gauge issues are addressed so that they do not negatively influence either the calculation or the comparison between the results of the two methods. Our work provides a way of dealing with possible problems and sets up a common framework, consistent with Bondi's construction, to remove ambiguities from the calculation and facilitate the comparison of results obtained by different numerical implementations.

It is probably worthwhile to note that we have been concerned with the notion of total momentum at future null infinity, as one is interested in the relation between total momentum and outgoing radiation which can carry momentum. For this reason our approach has been to imagine that topological closed 2-surfaces at finite but large distances can be understood as limiting spheres in the vicinity of future null infinity. However, once one has a finite closed 2-surface, one can wonder whether to regard it as a limiting sphere in a vicinity of past null infinity instead. With this point of view, one should relate the difference in the total momentum at two different finite spheres with the incoming radiation; which in this case will be characterized by the leading order behavior, at constant advanced time, of Ψ_0. Consequently the notion of total momentum at finite spheres will give different quantities whether one regards it as limiting spheres to future null infinity or to past null infinity. This brings the issue of whether one should consider both contributions when dealing with finite spheres. The answer is negative if one wants to describe what is observable at very long distances from the sources; since, for all practical purposes, an observer will be situated at future null infinity. Therefore, the notion of total momentum and its flux should be those naturally appearing at future null infinity.

To be more concrete on this point, let us consider a sphere S at a finite distance belonging to a timelike hypersurface Γ; and which can be regarded as one of a sequence of limiting spheres to future null infinity. Then in a numerical calculation one constructs these spheres far enough and aims to have enough control on boundary conditions so that incoming radiation through Γ is negligible. In this scenario, the estimation of total momentum by using the limiting behavior of fields in the interior, gives accurate information of the total momentum of a limiting sphere at future null infinity after gauge issues are appropriately handled.

Last, we find it worthwhile to stress that the calculation of momentum changes from the asymptotic fields at future null infinity does take into account all contributions in the interior of the spacetime, including incoming radiation that ultimately might cross the horizon of a final black hole. The role of this radiation in the computed kick velocity is accounted by the measured variation of spatial momentum and black hole mass which changes as energy flows into the horizon. Thus, the expression $v^i_{\rm kick} = \Delta P^i/P^0$ accounts for all relevant contributions.

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APPENDIX A

The transformation between the dyad $\{o^A, \iota^A\}$ and $\{\tilde{o}^A, \tilde{\iota}^A\}$ induces a transformation on the corresponding null tetrads. If we work with regular quantities at future null infinity we have,

$$\hat{\ell}^a = \Omega^{-2}\ell^a, \tag{A1}$$

$$\hat{m}^a = \Omega^{-1}m^a, \tag{A2}$$

$$\hat{n}^a = n^a; \tag{A3}$$

and similar expression for its "tilde" version. Using the relations (54) and (55), and having into account that the conformal factors are related as

$$\tilde{\Omega} = w\Omega; \tag{A4}$$

we find

$$\hat{\tilde{\ell}}^a = \frac{n^0(\alpha)}{w^2}(\hat{\ell}^a + \bar{C}\hat{m}^a + C\hat{\bar{m}}^a + C\bar{C}\hat{n}^a), \tag{A5}$$

$$\hat{\tilde{m}}^a = \frac{e^{i\lambda}}{w}(\hat{m}^a + C\hat{n}^a), \tag{A6}$$

$$\hat{\tilde{n}}^a = \frac{1}{n^0(\alpha)}\hat{n}^a; \tag{A7}$$

where in order to simplify the notation we have introduced

$$e^{i\lambda} \equiv \left(\frac{\bar{\tilde{\zeta}}_{\bar{\zeta}}}{\tilde{\zeta}_{\zeta}}\right)^{1/2}. \tag{A8}$$

From this transformation of the regular null tetrad we can see that it is originated as a combination of transformations of types II and III. The factor $n^0(\alpha)$ represents locally the action of a boost, similarly, C the action of a null rotation around $\hat{n}$ and $e^{i\lambda}$ a spatial rotation. The factor w comes as a result of both null tetrads belonging to the same conformal metric class (see Eq. (A4)).

APPENDIX B

The calculation of $\tilde{\sigma}_0$ is conveniently done referring to a "starred" tetrad * which is defined in the following way. Let us first define a * coordinate system such that

$$\ell = \frac{\partial}{\partial r^*}; \tag{B1}$$

in other words, r^* is an affine parameter of the null geodesic vector defined by

$$\ell = du. \tag{B2}$$

We complete the coordinate system by

$$u^* = u, \tag{B3}$$

$$r^* = \frac{r}{A} + r_0, \tag{B4}$$

$$\zeta^* = \zeta, \tag{B5}$$

$$\bar{\zeta}^* = \bar{\zeta}. \tag{B6}$$

The * null tetrad related to this coordinate system can be given in terms of the original tetrad by a transformation of the form

$$\ell^* = \ell, \tag{B7}$$

$$m^* = \gamma(m + \Gamma\ell), \tag{B8}$$

$$n^* = n + \Gamma\bar{m} + \bar{\Gamma}m + \Gamma\bar{\Gamma}\ell. \tag{B9}$$

One can show that in our case $\gamma = 1$ and

$$\Gamma = \frac{1}{A^2}\eth_V A. \tag{B10}$$

The null tetrad $(\ell^*, n^*, m^*, \bar{m}^*)$ have the characteristic that is associated to a system $(u^*, r^*, \zeta^*, \bar{\zeta}^*)$, were r^* an affine radial coordinate. Then we can use the known behavior of the spin coefficients in this tetrad-coordinate system [26] in order to relate the leading order behavior of this quantities with those obtained in the more general tetrad $(l, n, m, \bar{m})$.

In order to compute $\tilde{\sigma}_0$, we must know the leading order behavior of τ, β, ϵ', ρ' and κ'.

The spin coefficients transform according to

$$\rho^* = \rho, \tag{B11}$$

$$\sigma^* = \sigma, \tag{B12}$$

$$\tau^* = \tau + \Gamma\rho + \bar{\Gamma}\sigma; \tag{B13}$$

which implies

$$\tau = \tau^* - \Gamma\rho^* - \bar{\Gamma}\sigma^*. \tag{B14}$$

From this one can deduce that the leading order behavior of the original τ has a $1/r$ term, namely

$$\tau_{-1} = \frac{1}{A}\eth_V A. \tag{B15}$$

Similarly one has

$$\beta^* = \beta + \bar{\Gamma}\sigma + \Gamma\epsilon; \tag{B16}$$

with

$$\epsilon^* = \epsilon; \tag{B17}$$

but our choice of tetrad is such that

$$\epsilon = 0. \tag{B18}$$

Then in leading order the betas coincide, that is

$$\frac{\beta_0}{r} = \frac{\beta_0^*}{r^*} = A\frac{\beta_0^*}{r}. \tag{B19}$$

But one has

$$\beta_0^* = -\frac{V^*}{\sqrt{2}}\frac{\partial P_0^*}{\partial \zeta^*} - \frac{1}{2V^*}\eth_{V^*}V^*; \tag{B20}$$

and using that

$$V^* = \frac{V}{A}; \tag{B21}$$

one obtains

$$\beta_0^* = \frac{\beta_{0V}}{A} + \frac{1}{2A^2}\eth_V A; \tag{B22}$$

with

$$\beta_{0V} = -\frac{V}{\sqrt{2}}\frac{\partial P_0}{\partial \zeta} - \frac{1}{2V}\eth_V V. \tag{B23}$$

Then one has

$$\beta_0 = \beta_{0V} + \frac{1}{2A}\eth_V A. \tag{B24}$$

Let us also note that

$$-\epsilon'^* = -\epsilon' - \Gamma\beta' + \bar{\Gamma}(\tau + \beta) + \bar{\Gamma}^2\sigma + \Gamma\bar{\Gamma}\rho; \tag{B25}$$

and

$$-\beta'^{*} = -\beta' + \bar{\Gamma}\rho. \tag{B26}$$

Using that in the first two orders in the expansion of ϵ' one has

$$\epsilon' = \frac{1}{2}\frac{\partial U}{\partial r} + O\left(\frac{1}{r}\right); \tag{B27}$$

one can deduce that

$$\hat{n}(\Omega C) + 2\Omega C\epsilon' = \Omega n^0(C); \tag{B28}$$

which was used above.

Finally one can see that the other spin coefficients needed in order to compute σ have the following behavior

$$\rho'^{*} = \rho + 2\Gamma\bar{\Gamma}\epsilon - \Gamma\tau' + \bar{\Gamma}^2\sigma - \bar{\Gamma}(\beta' - \beta) + \Gamma\bar{\Gamma}^2\kappa + \eth\bar{\Gamma} + \Gamma\ell(\bar{\Gamma}), \tag{B29}$$

and

$$\kappa' = \Omega\hat{m}(U); \tag{B30}$$

then one can see that they do not contribute to $\tilde{\sigma}_0$.

Collecting all these quantities in the expression for $\tilde{\sigma}$, one has to leading order

$$\frac{\tilde{\sigma}_0}{\tilde{r}^2} = \frac{(wB)^2}{r^2}\left(\frac{\bar{\tilde{\zeta}}_{\bar{\zeta}}}{\tilde{\zeta}_{\zeta}}\right)^{1/2}\left[\sigma_0 - \eth_V C + 2\frac{C}{A}\eth_V A - Cn^0(C)\right]; \tag{B31}$$

which implies Eq. (68).

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