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Shearing expansion-free spherical anisotropic fluid evolution

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Spherically symmetric expansion-free distributions are systematically studied. The entire set of field equations and junction conditions are presented for a general distribution of dissipative anisotropic fluid (principal stresses unequal), and the expansion-free condition is integrated. In order to understand the physical meaning of expansion-free motion, two different definitions for the radial velocity of a fluid element are discussed. It is shown that the appearance of a cavity is inevitable in the expansion-free evolution. The nondissipative case is considered in detail, and the Skripkin model is recovered.

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I. INTRODUCTION

The problem of general relativistic gravitational collapse of massive stars has attracted the attention of researchers for many years, starting with the seminal paper by Oppenheimer and Snyder [1]. The motivation for such interest is easily understood: the gravitational collapse of massive stars represents one of the few observable phenomena, where general relativity is expected to play a relevant role. Ever since that work, much was written by researchers trying to provide models of evolving gravitating spheres (see [2] and references therein). However, this endeavor proved to be difficult, with different kinds of obstacles appearing, depending on the approach adopted for the modelling.

Thus, numerical methods allow for considering more realistic equations of state, but the obtained results, in general, are restricted and highly model dependent. Also, specific difficulties, associated to numerical solutions of partial differential equations in presence of shocks, complicate further the problem. Therefore, it seems useful to consider nonstatic models, which are relatively simple to analyze, but still contain some of the essential features of a realistic situation. For doing so, we need to appeal to a simple equation of state and/or to additional physically meaningful heuristic assumptions. In this work, we shall assume the fluid to be expansionfree.

As is well known, the motion of a fluid may be characterized by the four acceleration vector (a^α), the shear tensor ($\sigma_{\alpha\beta}$), the expansion scalar (Θ), and the vorticity tensor (which vanishes in the spherically symmetric case). The relevance of the shear tensor in the evolution of self-gravitating systems and the consequences emerging from its vanishing has been brought out by many authors (see [3] and references therein).

In this work, we shall study the properties of an expansion-free spherically symmetric self-gravitating fluid.

Since the expansion scalar describes the rate of change of small volumes of the fluid, it is intuitively clear that the evolution of an expansion-free spherically symmetric distribution should necessarily imply the formation of a vacuum cavity within the distribution (see a more rigorous argument on this in Sec. V). Thus, in the case of an overall expansion, the increase in volume due to the increasing area of the external boundary surface must be compensated with the increase of the area of the internal boundary surface (delimiting the cavity) in order to keep Θ vanishing. The argument in the case of collapse is similar.

For sake of generality, we shall start our discussion by considering an anisotropic dissipative viscous fluid (arguments to justify such kind of fluid distributions may found in [4–6] and references therein). For this kind of distribution, we shall write the field equations, the junction conditions, at, both, the inner and the external boundary surface (Sec. II), as well as the dynamical equations (Sec. III). Next, we shall integrate the expansion-free condition and find the general form of the metric for the anisotropic dissipative viscous fluid (Sec. IV).

In order to understand better the physical meaning of the expansion-free motion, we shall discuss two different definitions of radial velocity of a fluid element, in terms of which both the expansion and the shear can be expressed (Sec. V).

We shall next consider the nondissipative case, and we shall specialize further to the isotropic fluid. In this latter case, we shall recover as a particular example the Skripkin model [7], assuming the energy density to be constant (Sec. VI).

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Finally, a discussion on the results is presented in the last section.

II. THE ENERGY-MOMENTUM TENSOR, THE FIELD EQUATIONS, AND THE JUNCTION CONDITIONS

We consider a spherically symmetric distribution of collapsing fluid, bounded by a spherical surface $\Sigma^{(e)}$. The fluid is assumed to be locally anisotropic (principal stresses unequal) and undergoing dissipation in the form of heat flow (to model dissipation in the diffusion approximation), null radiation (to model dissipation in the free streaming approximation) and shearing viscosity.

Choosing comoving coordinates inside $\Sigma^{(e)}$, the general interior metric can be written

$$ds_-^2 = -A^2dt^2 + B^2dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (1)$$

where A, B, and R are functions of t and r and are assumed positive. We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$. Observe that A and B are dimensionless, whereas R has the same dimension as r.

The matter energy-momentum $T^-_{\alpha\beta}$ inside $\Sigma^{(e)}$ has the form

$$T^-_{\alpha\beta} = (\mu + P_\perp)V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r - P_\perp)\chi_\alpha\chi_\beta + q_\alpha V_\beta + V_\alpha q_\beta + \epsilon l_\alpha l_\beta - 2\eta\sigma_{\alpha\beta}, \qquad (2)$$

where μ is the energy density, P_r the radial pressure, $P_\perp$ the tangential pressure, q^α the heat flux, ϵ the energy density of the null fluid describing dissipation in the free streaming approximation, η the coefficient of shear viscosity, V^α the four velocity of the fluid, χ^α a unit four vector along the radial direction, and l^α a radial null four vector. These quantities satisfy

$$V^\alpha V_\alpha = -1, \qquad V^\alpha q_\alpha = 0, \qquad \chi^\alpha\chi_\alpha = 1,$$
$$\chi^\alpha V_\alpha = 0, \qquad l^\alpha V_\alpha = -1, \qquad l^\alpha l_\alpha = 0. \qquad (3)$$

The acceleration a_α and the expansion Θ of the fluid are given by

$$a_\alpha = V_{\alpha;\beta}V^\beta, \qquad \Theta = V^\alpha{}_{;\alpha}, \qquad (4)$$

and its shear $\sigma_{\alpha\beta}$ by

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \tfrac{1}{3}\Theta(g_{\alpha\beta} + V_\alpha V_\beta). \qquad (5)$$

We do not explicitly add bulk viscosity to the system because it can be absorbed into the radial and tangential pressures P_r and $P_\perp$ of the collapsing fluid [8].

Since we assumed the metric (1) comoving, then

$$V^\alpha = A^{-1}\delta^\alpha_0, \qquad q^\alpha = qB^{-1}\delta^\alpha_1,$$
$$l^\alpha = A^{-1}\delta^\alpha_0 + B^{-1}\delta^\alpha_1, \qquad \chi^\alpha = B^{-1}\delta^\alpha_1, \qquad (6)$$

where q is a function of t and r.

From (4) with (6) we have for the acceleration and its scalar a

$$a_1 = \frac{A'}{A}, \qquad a^2 = a^\alpha a_\alpha = \left(\frac{A'}{AB}\right)^2, \qquad (7)$$

and for the expansion

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right), \qquad (8)$$

where the prime stands for r differentiation and the dot stands for differentiation with respect to t. With (6) we obtain for the shear (5) its non zero components

$$\sigma_{11} = \frac{2}{3}B^2\sigma, \qquad \sigma_{22} = \frac{\sigma_{33}}{\sin^2\theta} = -\frac{1}{3}R^2\sigma, \qquad (9)$$

and its scalar

$$\sigma^{\alpha\beta}\sigma_{\alpha\beta} = \frac{2}{3}\sigma^2, \qquad (10)$$

where

$$\sigma = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right). \qquad (11)$$

A. The Einstein equations

Einstein's field equations for the interior spacetime (1) are given by

$$G^-_{\alpha\beta} = 8\pi T^-_{\alpha\beta}, \qquad (12)$$

and its non zero components with (1), (2), and (6) become

$$8\pi T^-_{00} = 8\pi(\mu + \epsilon)A^2 = \left(2\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R} - \left(\frac{A}{B}\right)^2\left[2\frac{R''}{R} + \left(\frac{R'}{R}\right)^2 - 2\frac{B'}{B}\frac{R'}{R} - \left(\frac{B}{R}\right)^2\right], \qquad (13)$$

$$8\pi T^-_{01} = -8\pi(q + \epsilon)AB = -2\left(\frac{\dot{R}'}{R} - \frac{\dot{B}}{B}\frac{R'}{R} - \frac{\dot{R}}{R}\frac{A'}{A}\right), \qquad (14)$$

$$8\pi T^-_{11} = 8\pi\left(P_r + \epsilon - \frac{4}{3}\eta\sigma\right)B^2 = -\left(\frac{B}{A}\right)^2\left[2\frac{\ddot{R}}{R} - \left(2\frac{\dot{A}}{A} - \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right] + \left(2\frac{A'}{A} + \frac{R'}{R}\right)\frac{R'}{R} - \left(\frac{B}{R}\right)^2, \qquad (15)$$

$$8\pi T_{22}^{-} = \frac{8\pi}{\sin^2\theta} T_{33}^{-} = 8\pi\left(P_\perp + \frac{2}{3}\eta\sigma\right)R^2$$
$$= -\left(\frac{R}{A}\right)^2\left[\frac{\ddot{B}}{B} + \frac{\ddot{R}}{R} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right) + \frac{\dot{B}}{B}\frac{\dot{R}}{R}\right]$$
$$+ \left(\frac{R}{B}\right)^2\left[\frac{A''}{A} + \frac{R''}{R} - \frac{A'}{A}\frac{B'}{B} + \left(\frac{A'}{A} - \frac{B'}{B}\right)\frac{R'}{R}\right]. \quad (16)$$

The component (14) can be rewritten with (8) and (10) as

$$4\pi(q + \epsilon)B = \frac{1}{3}(\Theta - \sigma)' - \sigma\frac{R'}{R}. \quad (17)$$

Next, the mass function $m(t, r)$ introduced by Misner and Sharp [9] (see also [10]) reads

$$m = \frac{R^3}{2}R_{23}^{23} = \frac{R}{2}\left[\left(\frac{\dot{R}}{A}\right)^2 - \left(\frac{R'}{B}\right)^2 + 1\right]. \quad (18)$$

Thus, in the most general case (locally anisotropic and dissipative) we have available four field Eqs. (13)–(16) for eight variables, namely A, B, R, μ, P_r, $P_\perp$, ϵ, and q. Since we are going to consider expansion-free systems, we have the additional condition $\Theta = 0$. Evidently, in order to find specific models (to close the system of equations), we need to provide additional information, which could be given in the form of constitutive equations for q and ϵ, and equations of state for both pressures.

B. The exterior spacetime and junction conditions

Outside $\Sigma^{(e)}$ we assume we have the Vaidya spacetime (i.e. we assume all outgoing radiation is massless), described by

$$ds^2 = -\left[1 - \frac{2M(v)}{r}\right]dv^2 - 2drdv + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (19)$$

where $M(v)$ denotes the total mass, and v is the retarded time.

The matching of the full nonadiabatic sphere (including viscosity) to the Vaidya spacetime, on the surface $r = r_{\Sigma^{(e)}} =$ constant, was discussed in [11]. From the continuity of the first and second differential forms it follows (see [11] for details)

$$m(t, r) \overset{\Sigma^{(e)}}{=} M(v), \quad (20)$$

and

$$2\left(\frac{\dot{R}'}{R} - \frac{\dot{B}}{B}\frac{R'}{R} - \frac{\dot{R}}{R}\frac{A'}{A}\right) \overset{\Sigma^{(e)}}{=} -\frac{B}{A}\left[2\frac{\ddot{R}}{R} - \left(2\frac{\dot{A}}{A} - \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right] + \frac{A}{B}\times\left[\left(2\frac{A'}{A} + \frac{R'}{R}\right)\frac{R'}{R} - \left(\frac{B}{R}\right)^2\right], \quad (21)$$

where $\overset{\Sigma^{(e)}}{=}$ means that both sides of the equation are evaluated on $\Sigma^{(e)}$ (observe a misprint in Eq. (40) in [11] and a slight difference in notation).

Comparing (21) with (14) and (15), one obtains

$$q \overset{\Sigma^{(e)}}{=} P_r - \frac{4}{3}\eta\sigma. \quad (22)$$

Thus, the matching of (1) and (19) on $\Sigma^{(e)}$ implies (20) and (22), which reduces to Eq. (41) in [11] with the appropriate change in notation. Observe a misprint in Eq. (27) in [5] (the σ appearing there is the one defined in [11], which is $-1/3$ of the one used here and in [5]).

As we mentioned in the introduction, the expansion-free models present an internal vacuum cavity. If we call $\Sigma^{(i)}$ the boundary surface between the cavity and the fluid, then the matching of the Minkowski spacetime within the cavity to the fluid distribution, implies

$$m(t, r) \overset{\Sigma^{(i)}}{=} 0, \quad (23)$$

$$q \overset{\Sigma^{(i)}}{=} P_r - \frac{4}{3}\eta\sigma. \quad (24)$$

III. DYNAMICAL EQUATIONS

To study the dynamical properties of the system, let us introduce, following Misner and Sharp [9], the proper time derivative D_T given by

$$D_T = \frac{1}{A}\frac{\partial}{\partial t}, \quad (25)$$

and the proper radial derivative D_R,

$$D_R = \frac{1}{R'}\frac{\partial}{\partial r}, \quad (26)$$

where R defines the areal radius of a spherical surface inside $\Sigma^{(e)}$ (as measured from its area).

Using (25), we can define the velocity U of the collapsing fluid (for another definition of velocity see Sec. VI) as the variation of the areal radius with respect to proper time, i.e.

$$U = D_T R < 0 \text{(in the case of collapse)}. \quad (27)$$

Then (18) can be rewritten as

$$E \equiv \frac{R'}{B} = \left(1 + U^2 - \frac{2m}{R}\right)^{1/2}. \quad (28)$$

With (26) we can express (17) as

$$4\pi(q + \epsilon) = E\left[\frac{1}{3}D_R(\Theta - \sigma) - \frac{\sigma}{R}\right]. \quad (29)$$

Using (13) and (14) and with (25) and (26) we obtain from (18)

$$D_T m = -4\pi\left[\left(P_r + \epsilon - \frac{4}{3}\eta\sigma\right)U + (q + \epsilon)E\right]R^2, \quad (30)$$

and

$$D_R m = 4\pi\left[\mu + \epsilon + (q+\epsilon)\frac{U}{E}\right]R^2, \quad (31)$$

which implies

$$m = 4\pi\int_0^R \left[\mu + \epsilon + (q+\epsilon)\frac{U}{E}\right]R^2 dR \quad (32)$$

(assuming a regular center to the distribution, so $m(0) = 0$).

Expression (30) describes the rate of variation of the total energy inside a surface of areal radius R. On the right-hand side of (30), $(P_r + \epsilon - 4\eta\sigma/3)U$ (in the case of collapse $U < 0$) increases the energy inside R through the rate of work being done by the "effective" radial pressure $P_r - 4\eta\sigma/3$ and the radiation pressure ϵ. Clearly, here the heat flux q does not appear, since there is no pressure associated with the diffusion process. The second term $-(q+\epsilon)E$ is the matter energy leaving the spherical surface.

Equation (31) shows how the total energy enclosed varies between neighboring spherical surfaces inside the fluid distribution. The first term on the right-hand side of (31) $\mu + \epsilon$ is due to the energy density of the fluid element plus the energy density of the null fluid describing dissipation in the free streaming approximation. The second term, $(q+\epsilon)U/E$ is negative (in the case of collapse) and measures the outflow of heat and radiation.

The nontrivial components of the Bianchi identities, $T^{-\alpha\beta}_{;\beta} = 0$, from (12) yield

$$T^{-\alpha\beta}_{;\beta}V_\alpha = -\frac{1}{A}\left[(\mu+\epsilon)^{\cdot} + \left(\mu + P_r + 2\epsilon - \frac{4}{3}\eta\sigma\right)\frac{\dot{B}}{B} + 2\left(\mu + P_\perp + \epsilon + \frac{2}{3}\eta\sigma\right)\frac{\dot{R}}{R}\right] - \frac{1}{B}\left[(q+\epsilon)' + 2(q+\epsilon)\frac{(AR)'}{AR}\right] = 0, \quad (33)$$

$$T^{-\alpha\beta}_{;\beta}\chi_\alpha = \frac{1}{A}\left[(q+\epsilon)^{\cdot} + 2(q+\epsilon)\left(\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right)\right] + \frac{1}{B}\left[\left(P_r + \epsilon - \frac{4}{3}\eta\sigma\right)' + \left(\mu + P_r + 2\epsilon - \frac{4}{3}\eta\sigma\right)\frac{A'}{A} + 2(P_r - P_\perp + \epsilon - 2\eta\sigma)\frac{R'}{R}\right] = 0, \quad (34)$$

or, by using (7), (8), (25), (26), and (28), they become, respectively,

$$D_T(\mu+\epsilon) + \frac{1}{3}(3\mu + P_r + 2P_\perp + 4\epsilon)\Theta + \frac{2}{3}(P_r - P_\perp + \epsilon - 2\eta\sigma)\sigma + ED_R(q+\epsilon) + 2(q+\epsilon)\left(a + \frac{E}{R}\right) = 0, \quad (35)$$

$$D_T(q+\epsilon) + \frac{2}{3}(q+\epsilon)(2\Theta + \sigma) + ED_R\left(P_r + \epsilon - \frac{4}{3}\eta\sigma\right) + \left(\mu + P_r + 2\epsilon - \frac{4}{3}\eta\sigma\right)a + 2(P_r - P_\perp + \epsilon - 2\eta\sigma)\frac{E}{R} = 0. \quad (36)$$

This last equation may be further tranformed as follows: The acceleration $D_T U$ of an infalling particle inside Σ can be obtained by using (7), (15), (18), and (28), producing

$$D_T U = -\frac{m}{R^2} - 4\pi\left(P_r + \epsilon - \frac{4}{3}\eta\sigma\right)R + Ea, \quad (37)$$

and then, substituting a from (37) into (36), we obtain

$$\left(\mu + P_r + 2\epsilon - \frac{4}{3}\eta\sigma\right)D_T U = -\left(\mu + P_r + 2\epsilon - \frac{4}{3}\eta\sigma\right)\times\left[\frac{m}{R^2} + 4\pi\left(P_r + \epsilon - \frac{4}{3}\eta\sigma\right)R\right] - E^2\left[D_R\left(P_r + \epsilon - \frac{4}{3}\eta\sigma\right) + 2(P_r - P_\perp + \epsilon - 2\eta\sigma)\frac{1}{R}\right] - E\left[D_T(q+\epsilon) + 2(q+\epsilon)\times\left(2\frac{U}{R} + \sigma\right)\right]. \quad (38)$$

The physical meaning of different terms in (38) has been discussed in detail in [4–6]. Suffice to say in this point that the first term on the right-hand side describes the gravitational force term.

IV. SHEARING EXPANSION-FREE MOTION

If the fluid has no expansion, i.e. $\Theta = 0$, then from (8) we have

$$\frac{\dot{B}}{B} = -2\frac{\dot{R}}{R}, \quad (39)$$

or, by integrating

$$B = \frac{g(r)}{R^2}, \quad (40)$$

where $g(r)$ is an arbitrary function of r.

Substituting (39) into (14), we obtain

$$\frac{\dot{R}'}{R} + 2\frac{\dot{R}}{R}\frac{R'}{R} - \frac{\dot{R}}{R}\frac{A'}{A} = 4\pi(q + \epsilon)AB, \tag{41}$$

which can be integrated for $\dot{R} \neq 0$ producing

$$A = \frac{R^2\dot{R}}{\tau_1}\exp\left[-4\pi\int(q + \epsilon)AB\frac{R}{\dot{R}}dr\right], \tag{42}$$

where $\tau_1(t)$ is an arbitrary function of t. With (40) and (42) then (1) becomes

$$ds^2 = -\left\{\frac{R^2\dot{R}}{\tau_1}\exp\left[-4\pi\int(q + \epsilon)AB\frac{R}{\dot{R}}dr\right]\right\}^2 dt^2 + \left(\frac{g}{R^2}\right)^2 dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{43}$$

which is the general metric for a shearing expansion-free anistropic dissipative fluid.

In the nondissipative case $q = \epsilon = 0$, we can write (43) as

$$ds^2 = -\left(\frac{R^2\dot{R}}{\tau_1}\right)^2 dt^2 + \frac{1}{R^4}dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{44}$$

where without loss of generality (by reparametrizing r) we put $g = 1$ (observe that a unit constant with dimensions $[r^4]$ is assumed to multiply dr^2). Then we have that (44) is the general metric for a spherically symmetric anisotropic perfect fluid undergoing shearing and expansion-free evolution (observe that it has the same form as for the isotropic fluid [12]).

V. ON THE PHYSICAL MEANING OF EXPANSION-FREE MOTION

We shall now analyze under which conditions a time-dependent spherically symmetric configuration may evolve without expansion. For doing so we shall try to develop more our understanding of the different speeds that involve the description of expansion as well as shear in the evolution of a self-gravitating fluid. The following discussion heavily relies on the kinematic quantities characterizing the motion of a medium presented in [13], with slight changes in notation.

In Gaussian coordinates, the position of each particle may be given as

$$x^\alpha = x^\alpha(y^a, s), \tag{45}$$

where s is the proper time along the world line of the particle, and y^a (with a running from 1 to 3) is the position of the particle on a three-dimensional hypersurface (say Σ). Then for the unit vector tangent to the world line (the four-velocity) we have

$$V^\alpha = \frac{\partial x^\alpha}{\partial s}, \tag{46}$$

and observe that

$$\frac{\partial}{\partial s} = D_T. \tag{47}$$

Next, for an infinitesimal variation of the world line we have

$$\delta x^\alpha = \frac{\partial x^\alpha}{\partial y^a}\delta y^a, \tag{48}$$

from which it follows

$$D_T(\delta x^\alpha) = V^\alpha_{;\beta}\delta x^\beta. \tag{49}$$

Introducing the projector h^α_β on Σ by

$$h^\alpha_\beta = \delta^\alpha_\beta + V^\alpha V_\beta, \tag{50}$$

we can define the position vector of the particle $y^a + \delta y^a$ relative to the particle y^a on Σ, as

$$\delta_\perp x^\alpha = h^\alpha_\beta \delta x^\beta. \tag{51}$$

Then the relative velocity between these two particles, is

$$u^\alpha = h^\alpha_\beta D_T(\delta_\perp x^\beta), \tag{52}$$

and considering (49) and (51), it follows that

$$u^\alpha = V^\alpha_{;\beta}\delta_\perp x^\beta. \tag{53}$$

Now, the infinitesimal distance between two neighboring points on Σ is

$$\delta l^2 = g_{\alpha\beta}\delta_\perp x^\beta \delta_\perp x^\alpha, \tag{54}$$

then

$$\delta l D_T(\delta l) = g_{\alpha\beta}\delta_\perp x^\beta D_T(\delta_\perp x^\alpha), \tag{55}$$

or, by using (49) and (52),

$$\delta l D_T(\delta l) = V_{\alpha;\beta}\delta_\perp x^\beta \delta_\perp x^\alpha. \tag{56}$$

Then, taking into consideration the expression for the irreducible components of a timelike vector

$$V_{\alpha;\beta} = \sigma_{\alpha\beta} - a_\alpha V_\beta + \frac{1}{3}\Theta h_{\alpha\beta}, \tag{57}$$

where we assumed zero rotation, and substituting into (56), we obtain

$$\delta l D_T(\delta l) = \delta_\perp x^\beta \delta_\perp x^\alpha\left(\sigma_{\alpha\beta} + \frac{1}{3}h_{\alpha\beta}\Theta\right), \tag{58}$$

or, introducing the spacelike unit vector

$$e^\alpha = \frac{\delta_\perp x^\alpha}{\delta l}, \tag{59}$$

it becomes

$$\frac{D_T(\delta l)}{\delta l} = e^\alpha e^\beta \sigma_{\alpha\beta} + \frac{\Theta}{3}. \tag{60}$$

Let us now consider, the spherically symmetric case, and apply (60) to two neighboring points along the radial direction. In this case, we have $e^\alpha \equiv \chi^\alpha$, and using (6), (9), and (10) in (60) we obtain

$$\frac{D_T(\delta l)}{\delta l} = \frac{2\sigma}{3} + \frac{\Theta}{3}, \tag{61}$$

or, by using (8) and (11)

$$\frac{D_T(\delta l)}{\delta l} = \frac{\dot{B}}{AB}. \tag{62}$$

Then with (27) and (62) we can write

$$\sigma = \frac{D_T(\delta l)}{\delta l} - \frac{D_T R}{R} = \frac{D_T(\delta l)}{\delta l} - \frac{U}{R}, \tag{63}$$

and

$$\Theta = \frac{D_T(\delta l)}{\delta l} + \frac{2D_T R}{R} = \frac{D_T(\delta l)}{\delta l} + \frac{2U}{R}. \tag{64}$$

Thus, we see that in general there are two different contributions to the shear (63) and to the expansion (64). One is due to the "circumferential" velocity U [14], which is related to the change of areal radius R of a layer of matter, whereas the other is related to $D_T(\delta l)$, which has also the meaning of "velocity," being the relative velocity between neighboring layers of matter, and can be in general different from U.

From (63) we see that, if the spherical distribution of matter is collapsing, $U < 0$, the shear can vanish only if the relative distance between different layers of matter diminishes, $D_T(\delta l) < 0$, and cancels the circumferential velocity.

From (64), we see that the evolution of the fluid will be expansionfree, whenever the circumferential term cancels the term related to the variation of distance of neighboring particles. Thus, the collapse will proceed expansionfree, if the decrease of the perimeter of a comoving sphere ($U < 0$) is compensated by an increase in the distance of neighboring particles (along the radial direction) according to (64). Alternatively, if the fluid is moving outward ($U > 0$) neighboring particles will get closer ($D_T(\delta l) < 0$). These observations clarify further the origin of the cavity in expansion-free models. Indeed, consider two concentric fluid shells in the neighborhood of the center. As it follows from (74), close to the center we have $U \sim R$. Now, in the process of expansion (increasing of R), the $\Theta = 0$ condition implies as mentioned before that $D_T(\delta l) < 0$, i.e. both shells become closer; however this would not be so as long as $U \sim R$, implying thereby that the $\Theta = 0$ condition requires that the innermost shell of fluid should be away from the center, initiating therefrom the formation of the cavity.

Let us see this from another perspective. Consider the infinitesimal volume of the shell between two concentric spheres of radii r and $r + \delta r$,

$$\delta V = 4\pi B R^2 \delta r, \tag{65}$$

then it follows

$$D_T(\delta V) = 4\pi (D_T B) R^2 \delta r + 8\pi B R (D_T R) \delta r, \tag{66}$$

or, dividing (66) by δV and using (62),

$$\frac{D_T(\delta V)}{\delta V} = \frac{D_T(\delta l)}{\delta l} + 2\frac{U}{R}. \tag{67}$$

Which of course coincide with (64), since we know that $D_T(\delta V)/\delta V$ is the definition of the expansion.

Thus, in the process of contraction (expansion), the elementary volume δV decreases (increases) by two factors: on the one hand, by the decreasing (increasing) of the areal radius R, and on the other hand, by the decreasing (increasing) of the proper radial distance between the two concentric surfaces. Again, we have an expansion-free evolution (the elementary volume δV remains constant) whenever the two contributions on the right hand of (67) cancel each other, in spite of the fact that neither of them vanishes.

We shall now see how these two different definitions of radial velocity considered above are related.

Let us first assume that $U = 0$. Then from (63) and (64) it follows $\Theta = \sigma$, feeding this back into (29) we get at once

$$D_T(\delta l) = -\frac{4\pi R(q + \epsilon)}{E} \delta l; \tag{68}$$

thus, $U = 0$ implies $D_T(\delta l) = 0$ only in the dissipationless case, $q = \epsilon = 0$.

Next, let us assume $D_T(\delta l) = 0$. Then it follows from (63) and (64) that $\Theta = -2\sigma$, feeding this back into (29) we get

$$D_R \sigma + \frac{\sigma}{R} = -\frac{4\pi(q + \epsilon)}{E}, \tag{69}$$

whose integration with respect to R yields

$$\sigma = \frac{\zeta}{R} - \frac{4\pi}{R} \int (q + \epsilon) \frac{R}{E} dR, \tag{70}$$

where ζ is independent of R (for any layer of fluid, characterized by $r =$ constant, ζ may depend on t. In general, however, it may depend on (t and r). On the other hand, (70) with (63) implies

$$U = -\zeta + 4\pi \int (q + \epsilon) \frac{R}{E} dR. \tag{71}$$

Since $U \to 0$ as $R \to 0$, we must put $\zeta = 0$ (if only the center of the fluid distribution is covered by the coordinate system). Thus, in the nondissipative case $U = 0$.

Therefore, only in the nondissipative case $U = 0 \leftrightarrow D_T(\delta l) = 0$ (with the condition mentioned above).

Finally, we can write (29) as

$$D_R\left(\frac{U}{R}\right) = \frac{4\pi}{E}(q+\epsilon) + \frac{\sigma}{R}, \qquad (72)$$

which after integration with respect to R becomes

$$U = \xi R + R\int_0^R \left[\frac{4\pi}{E}(q+\epsilon) + \frac{\sigma}{R}\right] dR, \qquad (73)$$

or,

$$U = \frac{U_{\Sigma^{(e)}}}{R_{\Sigma^{(e)}}} R - R\int_R^{R_{\Sigma^{(e)}}} \left[\frac{4\pi}{E}(q+\epsilon) + \frac{\sigma}{\tilde{R}}\right] d\tilde{R}. \qquad (74)$$

In the shearfree nondissipative case, we have from (74) that $U \sim R$, which is characteristic of the homologous evolution [15]. This implies that for two concentric shells of areal radii R_1 and R_2, we have in this case

$$\frac{R_1}{R_2} = \text{constant}. \qquad (75)$$

The second term on the right of (74) describes how the shear and dissipation deviate the evolution from the homologous regime.

It is worth noticing that in the shearfree nondissipative case, the sign of U for any fluid element is the same as that of $U_{\Sigma^{(e)}}$. However, if we relax any of those conditions, that might not be true. Thus, it would be possible, for example, to have an expanding outer shell with an imploding inner core. Such possibility was brought out before, but restricted to the quasistatic regime [16]. Here, we see that such an scenario is also possible in the general dynamic regime.

VI. SHEARING EXPANSION-FREE PERFECT FLUID

We shall now restrict our study to a shearing expansion-free fluid without dissipation, $q = \epsilon = 0$ and $\eta = 0$, then the metric reduces to (44) and the field Eqs. (13)–(16), using (29), become

$$8\pi\mu = -2R^3R'' - 5R^2R'^2 + \frac{1}{R^2} - 3\frac{\tau_1^2}{R^6}, \qquad (76)$$

$$\frac{1}{3}D_R\sigma + \frac{\sigma}{R} = 0, \qquad (77)$$

$$8\pi P_r = \frac{\tau_1^2}{R^5\dot{R}}\left(3\frac{\dot{R}}{R} - 2\frac{\dot{\tau}_1}{\tau_1}\right) + R^3R'\left(2\frac{\dot{R}'}{\dot{R}} + 5\frac{R'}{R}\right) - \frac{1}{R^2}, \qquad (78)$$

$$8\pi P_\perp = -\frac{\tau_1^2}{R^5\dot{R}}\left(6\frac{\dot{R}}{R} - \frac{\dot{\tau}_1}{\tau_1}\right) + R^4\left[\frac{\dot{R}''}{\dot{R}} + 7\frac{\dot{R}'}{\dot{R}}\frac{R'}{R} + 3\frac{R''}{R} + 10\left(\frac{R'}{R}\right)^2\right]; \qquad (79)$$

while the Bianchi identities (33) and (34) read

$$\dot{\mu} + 2(P_\perp - P_r)\frac{\dot{R}}{R} = 0, \qquad (80)$$

$$P_r' + (\mu + P_r)\frac{\dot{R}'}{\dot{R}} + 2(\mu + 2P_r - P_\perp)\frac{R'}{R} = 0. \qquad (81)$$

From (80) we have that if the fluid is isotropic, $P_r = P_\perp$, then the energy density μ is only r dependent.

We can now integrate (76) under the assumption $\mu = \mu(r)$, to obtain

$$R'^2 = \frac{1}{R^4} + \frac{\tau_2 - 2m}{R^5} + \frac{\tau_1^2}{R^8}, \qquad (82)$$

where $\tau_2(t)$ is an arbitrary function of t and (32) has been used.

We shall now specialize further our model to the case of constant energy density [7]

A. The Skripkin model

In [7], it is not explicitly assumed that $\Theta = 0$, instead it is assumed that the fluid is nondissipative, has its energy density $\mu = \mu_0 =$ constant and the pressure isotropic. Of course, these conditions imply, because of (35), that $\Theta = 0$. Thus, we have only one physical variable (P_r), and the system of field equations is closed and can be integrated.

From the condition $\mu = \mu_0 =$ constant, (82) becomes

$$R'^2 = -\frac{k}{R^2} + \frac{1}{R^4} + \frac{\tau_2}{R^5} + \frac{\tau_1^2}{R^8}, \qquad (83)$$

with

$$k = \frac{8\pi\mu_0}{3}. \qquad (84)$$

It should be observed that (83) imposes a maximum to the value of R (R_{max}), for which $R' = 0$. The physical origin of this maximum for the areal radius may be explained as follows:

In the Skripkin picture, the fluid is initially at rest, then there is a sudden explosion at the center producing the outward ejection of the fluid, always keeping the conditions of nondissipation, $\mu = \mu_0 =$ constant and isotropic pressure (i.e. $\Theta = 0$). Under these conditions, (38) becomes

$$(\mu + P)D_TU = -(\mu + P)\left(\frac{m}{R^2} + 4\pi PR\right) - E^2D_RP, \qquad (85)$$

with $P_r = P_\perp = P$. Now, as the fluid moves outward and R approaches R_{max}, R' tends to zero, which implies, because of (28), that E approaches zero. Furthermore, this implies that the gravitational (negative) term in (85) will prevail leading to a negative D_TU, producing a reversal of the motion at (or before) R_{max}.

As mentioned above, Skripkin assumes the pressure to be isotropic. However, if we assume the evolution to be expansionfree (and $\mu = \mu_0 =$ constant), then the isotropy of pressure follows from (80), and using (83) in (78) or (79) yields

$$8\pi P = \frac{\dot{\tau}_2}{R^2\dot{R}} - 3k, \tag{86}$$

which, of course, satisfies (81). From the matching condition (22) we have for (86)

$$8\pi P \overset{\Sigma^{(e)}}{=} \frac{\dot{\tau}_2}{R^2\dot{R}} - 3k = 0, \tag{87}$$

which gives

$$\tau_2 = kR^3_{\Sigma^{(e)}} + c_1, \tag{88}$$

where c_1 is an arbitrary constant.

The mass function (18) with (44) and (83) becomes

$$m = \frac{1}{2}(kR^3 - \tau_2) = \frac{k}{2}(R^3 - R^3_{\Sigma^{(e)}}) - \frac{c_1}{2}, \tag{89}$$

where we used (88). Measuring m on $\Sigma^{(e)}$ we obtain the total mass of the configuration M

$$m \overset{\Sigma^{(e)}}{=} M = -\frac{c_1}{2}. \tag{90}$$

Thus,

$$\tau_2 = kR^3_{\Sigma^{(e)}} - 2M, \tag{91}$$

and

$$m = \frac{k}{2}(R^3 - R^3_{\Sigma^{(e)}}) + M. \tag{92}$$

As mentioned before, it should be clear from physical considerations that the assumption of vanishing expansion (with the constant energy density condition) in the evolution of the fluid distribution, implies the formation of a vacuum cavity within the sphere.

Applying matching conditions (23) and (24) on the boundary surface $\Sigma^{(i)}$, delimiting the cavity, we obtain

$$M = \frac{k}{2}(R^3_{\Sigma^{(e)}} - R^3_{\Sigma^{(i)}}), \tag{93}$$

and using (93) in (91)

$$\tau_2 = kR^3_{\Sigma^{(i)}}. \tag{94}$$

Because of the constancy of M, we obtain from (93)

$$\dot{R}_{\Sigma^{(e)}} = \left(\frac{R_{\Sigma^{(i)}}}{R_{\Sigma^{(e)}}}\right)^2 \dot{R}_{\Sigma^{(i)}}, \tag{95}$$

and from (44) and (95)

$$A_{\Sigma^{(e)}} = A_{\Sigma^{(i)}}, \tag{96}$$

producing, because of (95),

$$U_{\Sigma^{(e)}} = \left(\frac{R_{\Sigma^{(i)}}}{R_{\Sigma^{(e)}}}\right)^2 U_{\Sigma^{(i)}}, \tag{97}$$

which implies, as expected, that the inner boundary surface $\Sigma^{(i)}$ moves faster than the outer boundary surface $\Sigma^{(e)}$. This result can also be deduced from the very definition of U.

Indeed, using (44) in (27) we obtain

$$U = \frac{\tau_1}{R^2}, \tag{98}$$

which evaluated on $\Sigma^{(i)}$ and $\Sigma^{(e)}$ produces

$$\tau_1 = U_{\Sigma^{(e)}} R^2_{\Sigma^{(e)}} = U_{\Sigma^{(i)}} R^2_{\Sigma^{(i)}}, \tag{99}$$

implying (97).

Observe that from (63), (64), and (98) it follows

$$\sigma = -\frac{3\tau_1}{R^3}, \tag{100}$$

which is the solution of (77).

It should be observed that since the pressure vanishes on $\Sigma^{(i)}$ and $\Sigma^{(e)}$, it should have a maximum somewhere between the two, i.e the pressure gradient must vanish on some spherical surface (S) within the fluid. If we denote the areal radius of such surface by $R = R_S$, then it follows from (81)

$$(R^2R')^{\cdot} \overset{S}{=} 0 \tag{101}$$

and after integration

$$R'S = \frac{c_2}{R^2}, \tag{102}$$

where c_2 is a constant. Substituting (102) into (83), we have

$$kR^6 + (c_2^2 - 1)R^4 - \tau_2 R^3 - \tau_1^2 \overset{S}{=} 0. \tag{103}$$

Thus, the surface R_S, which is the root of (103), divides the fluid into two regions. The inner one, with a positive pressure gradient, and the outer one with a negative pressure gradient.

We are now able to prescribe the strategy to determine the Skripkin models. First of all let us recall that without loss of generality Skripkin chooses $\tau_1 = R_{\Sigma^{(e)}}\dot{R}^2_{\Sigma^{(e)}}$ and, consequently, $\tau_1 = R_{\Sigma^{(i)}}\dot{R}^2_{\Sigma^{(i)}}$ and $A_{\Sigma^{(e)}} = A_{\Sigma^{(i)}} = 1$.

Then, the integration of (83), which can only be expressed in terms of elliptic functions, produces

$$R = R(r, R_{\Sigma^{(i)}}, \dot{R}_{\Sigma^{(i)}}). \tag{104}$$

Evaluating (85) on $\Sigma^{(i)}$ we obtain a differential equation for $R_{\Sigma^{(i)}}$ whose integration provides its time dependence, and therefore of $R(r, t)$.

If we deviate from the Skripkin model and relax the condition $\mu =$ constant, allowing for r dependence of μ, then we need to integrate (82) instead of (83), which of course requires the specific r dependence of μ or m.

Also, we could consider the anisotropic case, which allows for a t dependence of μ, in this case, of course, a specific equation of state for the anisotropic pressures is required (or an equivalent ansatz).

Finally, the integration in the general dissipative case would require a thorough knowledge of the energy production within the fluid (or a set of equivalent ansätze).

VII. CONCLUSIONS

We have seen so far that expansion-free condition allows for the obtention of a wide range of models for the evolution of spherically symmetric self-gravitating systems. Ranging from nondissipative spheres, with constant energy density and isotropic pressure (Skripkin model), to general dissipative anisotropic models.

Observe that even if the Skripkin model is the simplest, from the physical point of view, it might not be so from the mathematical point of view. Indeed, we could in principle choose a mass function, such that (82) could be integrated in terms of elementary functions; obviously, it remains to be seen if such models are endowed with any physical interest.

One of the most interesting features of the models, is the appearance of a vacuum cavity within the fluid distribution. It is not clear at this point if such models might be used to describe the formation of voids observed at cosmological scales (see [17] and references therein).

The two concepts of radial velocity discussed in Sec. V allows to understand the meaning of the expansion-free evolution. As a byproduct of such discussion, the shearfree flow (in the nondissipative case) appears to be equivalent to the well-known homologous evolution. Particularly remarkable is the fact that the expansion-free ejection (collapse) implying an increase (decrease) in the areal radius of a layer of matter, proceeds with a decrease (increase) in the distance of neighboring particles along the radial direction. Also, the possibility of a "splitting" of the fluid distribution (change of sign in U) due to dissipation and/or shear, as indicated by (74), deserves to be explored further.

Finally, it is worth noticing that in the locally anisotropic case, the expansion-free evolution, due to the second term on the left of Eq. (80), does not imply that energy density remains time independent. This situation becomes intelligible when it is remembered, that in the limit of hydrostatic equilibrium, when $U = q = \epsilon = 0$, we obtain from (36),

$$D_R P_r + \frac{2(P_r - P_\perp)}{R} = -\frac{\mu + P_r}{R(R - 2m)}(m + 4\pi P_r R^3), \tag{105}$$

which is just the generalization of the Tolman-Oppenheimer-Volkoff equation for anisotropic fluids, obtained in comoving coordinates [8]. Thus, the term $2(P_r - P_\perp)/R$ represents a force associated to the local anisotropy of pressure, and therefore the second term on the left of Eq. (80), is the rate of work done by that force, resulting in a change of μ.

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