First order perturbations of the Einstein-Straus and Oppenheimer-Snyder models

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We derive the linearly perturbed matching conditions between a Schwarzschild spacetime region with stationary and axially symmetric perturbations and a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime with arbitrary perturbations. The matching hypersurface is also perturbed arbitrarily and, in all cases, the perturbations are decomposed into scalars using the Hodge operator on the sphere. This allows us to write down the matching conditions in a compact way. In particular, we find that the existence of a perturbed (rotating, stationary, and vacuum) Schwarzschild cavity in a perturbed FLRW universe forces the cosmological perturbations to satisfy constraints that link rotational and gravitational wave perturbations. We also prove that if the perturbation on the FLRW side vanishes identically, then the vacuole must be perturbatively static and hence Schwarzschild. By the dual nature of the problem, the first result translates into links between rotational and gravitational wave perturbations on a perturbed Oppenheimer-Snyder model, where the perturbed FLRW dust collapses in a perturbed Schwarzschild environment which rotates in equilibrium. The second result implies, in particular, that no region described by FLRW can be a source of the Kerr metric.

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I. INTRODUCTION

A long standing question in cosmology concerns the way large scale dynamics influences the behavior on smaller scales. The most common view regarding this question is that the influence of the cosmic expansion on local physics is zero or negligible. The main argument supporting this conclusion is based on the Einstein-Straus model [1] which consists of a vacuum spherical cavity (described by the Schwarzschild metric, hence static) embedded in an expanding dust Friedmann-Lemaître-Robertson-Walker (FLRW) model. The matching between both spacetimes is performed across a timelike hypersurface using the standard matching theory in general relativity, usually known as the Darmois matching conditions (we refer to [2] for a full account on matching general hypersurfaces). In this model the local physics occurs inside the Schwarzschild vacuole which, being static, perceives no effect of the cosmological expansion.

Despite its clear physical interpretation, this model presents serious problems and involves a number of idealizations which include a spatially homogeneous and isotropic cosmological model, the assumption of spherical symmetry both for the metric inside the vacuole and for its boundary, and the assumption of an exact static vacuum in the interior. Indeed, more sophisticated models have been constructed by using Lemaître-Tolman-Bondi regions for the cosmological part (see [3,4]), and other types of cavities [5,6], but all these models are spherically symmetric. An important question is whether the Einstein-Straus conclusion is robust with respect to various plausible (nonspherically symmetric) generalizations.

A general property of the matching theory is that solving a matching problem always gives, as an immediate consequence, that the "complementary" matching also holds [6]. Consequently, the "*a priori*" interior and exterior roles assigned to the Schwarzschild and FLRW regions in the Einstein-Straus model can be interchanged, and all the previous results also apply to the Oppenheimer-Snyder model of collapse [7], in which a spherically symmetric FLRW region of dust collapses in a Schwarzschild exterior geometry.

The first attempts to provide nonspherically symmetric generalizations can be found in the works by Cocke in [8] and Shaver and Lake in [9]. The first conclusive discussion of different metrics and shapes of the static region appeared in a paper by Senovilla and Vera [10], where it was shown that a *locally* cylindrically symmetric static region cannot be matched to an expanding FLRW model across a nonspacelike hypersurface preserving the cylindrical symmetry, irrespective of the matter content in the cylindrically symmetric region. Therefore, a detailed analysis became necessary in order to decide whether the assumption of spherical symmetry of the static region was a fundamental ingredient for the models. This analysis was performed by Mars, who showed in two steps, [11,12], that a static region matched to a FRLW cosmological model is forced to be spherically symmetric (both in shape and spacetime geometry) under very weak conditions on the matter content, which include vacuum as a particular case. Therefore, *the only static vacuum region that can be matched to a nonstatic FLRW is an (either interior or exterior) spherically shaped region of Schwarzschild*, which thus leads to the Einstein-Straus or the Oppenheimer-Snyder models.

To summarize, the Einstein-Straus model, consisting of a static vacuole embedded in an exact FLRW geometry (necessarily of dust), does not allow any nonspherical generalization and, in this sense, it is unstable. The same applies to the Oppenheimer-Snyder model.

Regarding the assumption of staticity, it has recently been shown by Nolan and Vera [13] that if a stationary and axially symmetric region is to be matched to a nonstatic FLRW region across a hypersurface preserving the axial symmetry, then the stationary region must be static. Hence, the results in [11,12] can be applied to any stationary and axisymmetric region. In particular, *the only stationary and axisymmetric vacuum region that can be matched to FLRW preserving the axial symmetry is an (either interior or exterior) spherically shaped region of Schwarzschild*. This implies that the Einstein-Straus and Oppenheimer-Snyder models cannot be generalized by including stationary rotation in the non-FLRW region. Note that this shows, in particular, that no FLRW axially symmetric region can be a source of Kerr.

Now, two possibilities in trying to generalize such models can be considered: the first is to take nonspherical exact solutions which generalize the FLRW region (such as Bianchi models) and the second to consider nonspherical perturbations of FLRW. The first possibility was considered by Mena, Tavakol, and Vera in [14], where they studied a matching preserving the symmetry of a cylindrically symmetric interior spacetime with locally rotationally symmetric spatially homogeneous (but anisotropic) exteriors. This matching resulted in restrictive generalizations of the Einstein-Straus model [14], none of them physically admissible. Therefore it is interesting to consider the second possibility, i.e. to construct a perturbed model.

Perturbed matching conditions have been applied many times in the past: Hartle [15] studied first and second order stationary and axisymmetric rigidly rotating perturbations of static perfect-fluid balls in vacuum. Chamorro performed the first order matching of a Kerr cavity in an expanding perturbed FLRW model [16]. For spherical symmetry, the linearized matching conditions in an arbitrary gauge were studied by Gerlach and Sengupta [17,18] and by Martín-García and Gundlach [19]. There are also studies by Cunningham, Price, and Moncrief where axial [20] and polar [21] perturbations of the Oppenheimer-Snyder model of collapse were derived. However, a matching perturbation theory in general relativity has only recently been developed in its full generality for first order [22,23] and second order perturbations [24]. A critical review about the study of linear perturbations of matched spacetimes including gauge problems has been recently presented in [25].

Another interesting approach has been followed by Doležel, Bičák, and Deruelle [26], who have studied slowly rotating voids in cosmology with a model consisting on an interior Minkowskian void, a matter shell between the void and the cosmological model, and a FLRW universe with a particular type of perturbation describing rotation. We emphasize that, in this paper, we focus on generalizations of the Einstein-Straus and the Oppenheimer-Snyder models without surface layers of matter (i.e. such that the Darmois matching conditions are satisfied), and we do not restrict the FLRW perturbations in any way.

In this paper we consider first order (linear) perturbations of the Einstein-Straus model, as well as perturbations of the Oppenheimer-Snyder model (as the perturbative matching satisfies the same dual property as the full matching). Our background model consists then of a Schwarzschild region matched to a FLRW dust cosmological model across a spherically symmetric (timelike) hypersurface. We perturb the Schwarzschild part with vacuum stationary axially symmetric perturbations and we perturb the matching surface as well as the exterior FLRW region with arbitrary perturbations, not restrained to any material content.

Our approach to this problem consists in exploiting the underlying spherical symmetry of the background and of the matching hypersurface as much as possible. This is normally done in the literature by resorting to decompositions of all objects in terms of scalar, vector, and tensor harmonics on the sphere. Our aim is to use an alternative method based on the Hodge decomposition of all tensor objects on the sphere in terms of scalars. The two approaches are obviously related to each other. However, by working with Hodge scalars we avoid the need to deal with infinite series of objects (one for each l and m in the spherical harmonic decomposition). In particular, our set of matching conditions has a finite number of equations (involving scalars that depend on the three coordinates in the matching hypersurface) instead of an infinite collection of matching conditions for functions of only one variable (the time coordinate on the matching hypersurface). The equations are therefore much more compact.

In fact, even when working with S^2 scalars the length of the equations can grow substantially depending on how they are combined and written down. This is partly due to the level of generality we leave for the matching hypersurface and the FLRW perturbations. We have taken the effort to combine the equations and to group several terms in each equation in such a way that the set of equations becomes reasonably short and manageable. We emphasize that the gauge in the FLRW part will be left completely free, so that the readers may choose their favorite one. As an example, we rewrite the equations in the Poisson gauge in Appendix C.

Deriving and writing down the linearized matching conditions for our perturbation of the Einstein-Straus (and Oppenheimer-Snyder) model is the main result of this paper. A more detailed analysis of the resulting set is postponed to a later paper. However, in order to show the usefulness and power of the equations, we present two applications. The first one is based on the observation that the linearized matching conditions can be combined in such a way that two equations *involving only terms on the FLRW side* hold. These equations are therefore constraints on the perturbations in the FLRW part that must be necessarily satisfied if an interior stationary and axisymmetric perturbed vacuole is present (and hence the local physics can remain unaffected by the cosmic expansion, as generally believed). These two constraints link the vector and tensor (linear) FLRW perturbations on the boundary of the region, and they imply, basically, that *if the perturbed FLRW region contains vector modes (with $l \geq 2$ harmonics) on the boundary, then it must also contain tensor modes*. In other words, if a (perturbed) FLRW region contains rotational perturbations that reach the boundary of a perturbed stationary and axially symmetric vacuum region, then the cosmological model there must also carry gravitational waves. Given the estimates of cosmological rotational perturbations through observations (see e.g. [27] and references therein), the constraints due to the possible existence of stationary and axisymmetric vacuoles would provide estimates of cosmological gravitational waves.

As a second application, we consider the case when the FLRW part of the spacetime remains exact (i.e. all perturbations vanish there). As discussed above, the results in [13] combined with [12] imply that the interior has to be exactly Schwarzschild *provided the matching hypersurface is assumed to be axially symmetric*. Since we allow for nonaxially symmetric perturbations of the matching hypersurface we can address the question of whether this result generalizes to arbitrary hypersurfaces (at the linear level, of course). Our conclusion is that indeed this is the case. This result points, once again, into the fragility of the Einstein-Straus model against any reasonable generalization. From the Oppenheimer-Snyder model point of view, this also means that a body modeled by a dynamical FLRW model, irrespective of its shape and its relative rotation with the exterior, cannot be the source of any stationary (nonstatic), axially symmetric vacuum exterior, in particular, of the Kerr metric.

The paper is organized as follows. In Sec. II we give a very brief summary of the linearized matching theory, where we fix our notation. In Sec. III we summarize the standard theory of the Hodge decomposition of vectors and symmetric tensors on the sphere and apply the theory to first order perturbations of a spherical background in general. The use of Hodge decompositions introduces a so-called *kernel freedom* which is discussed and analyzed. Section IV is devoted to obtaining the explicit form of vacuum, stationary and axially symmetric perturbations of the Schwarzschild region. After perturbing the background matching hypersurface arbitrarily, we obtain, in subsection IV C, the explicit form of the Hodge scalars of the perturbed first and second fundamental forms of the matching hypersurface. The same procedure is followed in Sec. V for the FLRW side. Section VI is devoted to obtaining the linearized matching conditions for our problem in terms of S^2 scalars. As in any Hodge decomposition, the set of equations decomposes into odd and even equations, and both sets are carefully combined and rewritten to make them as compact as possible. The equations given in this section constitute the main result of this paper, so we summarize the hypotheses and conclusion in Theorem VI.1. Sections VII and VIII contain the two applications we present in this paper. Section VII is devoted to deriving the constraints in the FLRW side and Sec. VIII to the matching with an exact FLRW. The paper contains four Appendixes. Appendixes A and B contain, respectively, for the Schwarzschild and the FLRW parts, the full expressions for the linear perturbations of the first and second fundamental forms prior to their Hodge decomposition. Appendix C specifies our general matching conditions in the Poisson gauge in the FLRW part. Finally, Appendix D is devoted to the comparison between the expressions used in our formalism and the doubly gauge invariant quantities in [23,25].

Lower case Latin indices at the beginning of the alphabet $a, b, \ldots = 1, 2, 3$ refer to tensors on the constant cosmic time hypersurface in FLRW, at the middle of the alphabet $i, j, \ldots = 1, 2, 3$ are used for tensors on the matching hypersurface. The first upper case Latin indices $A, B, \ldots = 2, 3$ denote tensors on the sphere while middle indices $I, J, \ldots = 0, 1$ are used for tensors in the surfaces orthogonal to the spherical orbits. Finally, Greek indices $\alpha, \beta, \ldots = 0, 1, 2, 3$ refer to general spacetime tensors.

II. LINEARIZED PERTURBED MATCHING THEORY IN BRIEF

The linearized matching involves perturbing a background which is already constructed from the matching of two regions $(M_0^+, g^{(0)+})$ and $(M_0^-, g^{(0)-})$, with corresponding boundaries $\Sigma_0^\pm$ which are diffeomorphic to each other (then identified as the matching hypersurface). Taking local coordinates on the matching hypersurface amounts to writing down two embeddings

$$\Phi_\pm \colon \Sigma_0 \longrightarrow M_0^\pm \qquad \xi^i \mapsto x_\pm^\alpha = \Phi_\pm^\alpha(\xi^i), \tag{1}$$

such that $\Phi_\pm(\Sigma_0) = \Sigma_0^\pm$, where $x_\pm^\alpha$ are arbitrary coordinates on each of the background regions. The coordinate vectors ∂_{ξ^i} intrinsic to Σ_0 define tangent vectors to the boundaries $e_i^{\pm\alpha} = \frac{\partial \Phi_\pm^\alpha}{\partial \xi^i}$. Assuming $\Sigma_0^\pm$ not to be null at any point, there is a unique up to orientation unit normal vector $n_\pm^{(0)\alpha}$. The orientation of one of the normals can be chosen arbitrarily but the other must be chosen accordingly so that both point to the same region after the matching. The first and second fundamental forms are, respectively, $q^{(0)\pm}{}_{ij} \equiv e_i^{\pm\alpha} e_j^{\pm\beta} g^{(0)\pm}{}_{\alpha\beta}|_{\Sigma_0^\pm}$, $k^{(0)\pm}{}_{ij} = -n^{(0)}_{\pm\alpha} e_i^{\pm\beta} \nabla^\pm_\beta e_j^{\pm\alpha}|_{\Sigma_0^\pm}$ and the background matching conditions require[1]

$$q^{(0)+}{}_{ij} = q^{(0)-}{}_{ij}, \qquad k^{(0)+}{}_{ij} = k^{(0)-}{}_{ij}. \tag{2}$$

Consider now a perturbation of the background metric $g^\pm_{\text{pert}} = g^{(0)\pm} + g^{(1)\pm}$ and of the boundaries $\Sigma_0^\pm$ via the vector fields $\vec{Z}^\pm = Q^\pm \vec{n}^{(0)}_\pm + \vec{T}^\pm|_{\Sigma_0^\pm}$, where $\vec{T}^\pm$ are tangent to $\Sigma_0^\pm$. The linearized matching conditions are derived in [22,23] (see also [24] for second order matching), and read

$$q^{(1)+}{}_{ij} = q^{(1)-}{}_{ij}, \qquad k^{(1)+}{}_{ij} = k^{(1)-}{}_{ij}, \tag{3}$$

with (for a timelike matching hypersurface)

$$q^{(1)\pm}{}_{ij} = \mathcal{L}_{\vec{T}^\pm} q^{(0)\pm}{}_{ij} + 2Q^\pm k^{(0)\pm}{}_{ij} + e_i^{\pm\alpha} e_j^{\pm\beta} g^{(1)\pm}{}_{\alpha\beta}|_{\Sigma_0^\pm}, \tag{4}$$

$$\begin{aligned} k^{(1)\pm}{}_{ij} = {} & \mathcal{L}_{\vec{T}^\pm} k^{(0)\pm}{}_{ij} - D_i D_j Q^\pm \\ & + Q^\pm (n_\pm^{(0)\mu} n_\pm^{(0)\nu} R^{(0)\pm}_{\alpha\mu\beta\nu} e_i^{\pm\alpha} e_j^{\pm\beta} \\ & + k^{(0)\pm}{}_{il} k^{(0)l\pm}{}_j) + \frac{1}{2} g^{(1)\pm}{}_{\alpha\beta} n_\pm^{(0)\alpha} n_\pm^{(0)\beta} k^{(0)\pm}{}_{ij} \\ & - n^{(0)}_{\pm\mu} S^{(1)\pm\mu}_{\alpha\beta} e_i^{\pm\alpha} e_j^{\pm\beta}|_{\Sigma_0^\pm}, \end{aligned} \tag{5}$$

where D_i is the three-dimensional covariant derivative of $(\Sigma_0, q^{(0)\pm}{}_{ij})$ and

$$S^{(1)\pm\alpha}_{\beta\gamma} \equiv \frac{1}{2}(\nabla^\pm_\beta g^{(1)\pm\alpha}{}_\gamma + \nabla^\pm_\gamma g^{(1)\pm\alpha}{}_\beta - \nabla^{\pm\alpha} g^{(1)\pm}{}_{\beta\gamma}).$$

The tensors $q^{(1)\pm}$ and $k^{(1)\pm}$ are spacetime gauge invariant by construction, since they are objects intrinsically defined on $\Sigma_0^\pm$, and therefore conditions (3) are spacetime gauge invariant. Moreover, it turns out that the equations (3) are also hypersurface gauge invariant provided the background is properly matched [i.e. once (2) hold].

The quantities $Q^\pm$ and $\vec{T}^\pm$ are unknown *a priori*, and fulfilling the matching conditions requires *showing* that two vectors $\vec{Z}^\pm$ exist such that (3) are satisfied. The spacetime gauge freedom can be used to fix either or both vectors $\vec{Z}$, but these choices have to be avoided, or carefully analyzed, if additional spacetime gauge choices are made, as otherwise the possible matchings might be restricted artificially. On the other hand, the hypersurface gauge can be used in order to fix one of the vectors $\vec{T}^+$ or $\vec{T}^-$, but not both (see [25] for a full discussion).

[1]As mentioned earlier, we are not interested in a resulting spacetime with a nonvanishing energy-momentum tensor with support on the matching hypersurface (a "shell"), and therefore we do not admit jumps in the second fundamental form.

III. HODGE DECOMPOSITION OF THE LINEARIZED PERTURBED MATCHING IN SPHERICAL SYMMETRY

From now onwards we shall concentrate on spherically symmetric background configurations. In order to write the matching conditions in a way which exploits the symmetries we shall use the Hodge decomposition on the sphere. There are some good reasons to do that. As outlined in the Introduction, the equations naturally inherit the spherical symmetry of the background configuration, and thus one expects that any approach based on spherical decompositions will render the equations in a simpler form. Previously in the literature, this has been implemented by decomposing the relevant quantities into scalar, vector and tensor harmonics (see e.g. [23]). From a formal point of view, those decompositions are very useful since they provide independent sets of equations for the different spectral values, say l and m. However, in practice, the harmonic decomposition may become a problem on its own when studying explicit models. Moreover, even if the problem can be formally solved for the infinite spectrum, the sum convergence of the resulting decomposition should be eventually ensured.

Using the Hodge decomposition has the advantage that one works with all the different (l, m) harmonics of a given quantity at once. In fact, the Hodge scalars correspond in a suitable sense, to the resummation of the previous spectral decomposition. For instance instead of using all the (infinite number of) equations that correspond to each value of l and m in the matching equations involving $(\sigma_{(L)0})_{lm}$ (see [23]), only one equation for the whole sum $\sum_{lm}(\sigma_{(L)0})_{lm} Y_l^m$ ($\equiv F$ here) is needed.

It is clear that one can always go from the Hodge scalars to the spherical harmonics decomposition in a straightforward way. However, it is not always easy to rewrite the infinite number of expressions appearing in a spectral decomposition in terms of Hodge scalars. Working with Hodge scalars involves a finite number of equations and obviously there arise no convergence problems (although then one often has to deal with partial differential equations [PDEs] in 3 + 1 dimensions instead of 1 + 1 equations, which are simpler). Furthermore, their calculation entails a quite straightforward procedure, more easily implemented in algebraic computing. We devote Appendix D at the end of the paper to relate the functions of the Hodge decomposition used in the present paper and the coefficients used in [23,25] for the scalar, vector, and tensor harmonic decomposition.

A. Hodge decomposition on the sphere

We recall that the Hodge decomposition on S^2 tells us that any one-form $\boldsymbol{V}$ on S^2 can be canonically decomposed as

$$\boldsymbol{V} = dF + \star dG, \tag{6}$$

where and F and G are functions on the sphere and $(\star dG)_A = \eta^C{}_A D_C G$ is the Hodge dual with respect to the round unit metric $h_{AB}dx^A dx^B = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ on S^2. The corresponding volume form and covariant derivative are denoted, respectively, by η_{AB} and D_A. Latin indices $A, B, \ldots$ are raised and lowered with h_{AB} and the orientation is chosen so that $\eta_{\vartheta\varphi} > 0$. Furthermore, any symmetric tensor T_{AB} on the sphere can be canonically decomposed as

$$T_{AB} = D_A U_B + D_B U_A + H h_{AB},$$

for some one-form U_A on S^2, which, in turn, can be decomposed as $U_A = D_A P + (\star dR)_A$.

The Hodge decomposition on the sphere has a nontrivial kernel, i.e. the zero vector and the zero tensor on S^2 can be decomposed in terms of nonvanishing scalars, albeit of a very special form. First, we consider the kernel corresponding to the vanishing vector on the sphere:

$$D_A F + \eta^B{}_A D_B G = 0.$$

Since the only harmonic functions on the sphere are the constant functions, it follows that F and G must be independent of the angular coordinates $\{x^A\}(=\{\vartheta, \varphi\})$. Regarding the zero symmetric tensor T_{AB}, we must solve

$$D_A U_B + D_B U_A + H h_{AB} = 0, \tag{7}$$

which states that U^A is a conformal Killing vector on the sphere. There are six conformal Killing vectors on the sphere: three proper ones and three Killing vectors. They correspond to the usual longitudinal and transverse $l = 1$ vector harmonics, respectively (denoted as $V_{(L)}{}^A$ and $V_{(T)}{}^A$ with $l = 1$ in [23,25]). Explicit expressions for the conformal Killing vectors are obtained using the $l = 1$ spherical harmonics Y_1^m ($m = 1, 2, 3$)

$$Y_1^1 = \cos\vartheta, \qquad Y_1^2 = \sin\vartheta\cos\varphi, \qquad Y_1^3 = \sin\vartheta\sin\varphi.$$

The gradients $D_A Y_1^m$ correspond to three linearly independent proper conformal Killing vectors on S^2, and their Hodge duals $\eta^B{}_A D_B Y_1^m$ correspond to three linearly independent Killing vectors on S^2. Therefore, decomposing further U_A as $U_A = D_A P + \eta^B{}_A D_B R$ we obtain

$$P = P_0 + \sum_m P_m Y_1^m \qquad R = R_0 + \sum_m R_m Y_1^m$$

for some eight free coefficients P_0, P_m, R_0, R_m independent of $\{\vartheta, \varphi\}$. Substitution into (7) leads to

$$H = 2\sum_m P_m Y_1^m.$$

B. Perturbed matching conditions in terms of scalars

Our background configuration is spherically symmetric and composed of two spherically symmetric spacetimes $(M_0^\pm, g^{(0)\pm})$ matched across spherically symmetric timelike boundaries $\Sigma_0^\pm$ diffeomorphic to each other.

Let us for definiteness concentrate on the $(M_0^+, g^{(0)+})$ spacetime and drop the $+$ subindex [analogous expressions obviously hold for the $(M_0^-, g^{(0)-})$ spacetime region]. We choose coordinates adapted to the spherical symmetry, so that

$$g^{(0)}_{\alpha\beta}dx^\alpha dx^\beta = \omega_{IJ}dx^I dx^J + r^2(x^I)(d\theta^2 + \sin^2\theta d\phi^2),$$

where ω_{IJ} is a Lorentzian two-dimensional metric and $r(x^I) \geq 0$. A general spherically symmetric boundary can be described by the embedding (or parametric form)

$$\Sigma_0 := \{x^0 = \Phi^0_{(0)}(\lambda), x^1 = \Phi^1_{(0)}(\lambda), \theta = \vartheta, \phi = \varphi\}, \tag{8}$$

where $\{\xi^i\} = \{\lambda, \vartheta, \varphi\}$ is a coordinate system in Σ_0 adapted to the spherical symmetry.

The coordinate tangent vectors to Σ_0 read

$$\vec{e}_\lambda = \dot{\Phi}^0_{(0)}\partial_{x^0} + \dot{\Phi}^1_{(0)}\partial_{x^1}|_{\Sigma_0}, \qquad \vec{e}_\vartheta = \partial_\theta|_{\Sigma_0}, \qquad \vec{e}_\varphi = \partial_\phi|_{\Sigma_0}, \tag{9}$$

where the dot denotes a derivative with respect to λ. Defining $N^2 \equiv -e^I_\lambda e^J_\lambda \omega_{IJ}|_{\Sigma_0}$ the unit normal to the boundary reads

$$\boldsymbol{n}^{(0)} = \frac{\sqrt{-\det\omega}}{N}(-\dot{\Phi}^1_{(0)}dx^0 + \dot{\Phi}^0_{(0)}dx^1)|_{\Sigma_0}. \tag{10}$$

The sign of N corresponds to the choice of orientation of the normal. The first and second fundamental forms on Σ_0 read

$$q^{(0)}{}_{ij}d\xi^i d\xi^j = -N^2 d\lambda^2 + r^2|_{\Sigma_0}(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \tag{11}$$

$$k^{(0)}{}_{ij}d\xi^i d\xi^j = N^2\mathcal{K}d\lambda^2 + r^2|_{\Sigma_0}\overline{\mathcal{K}}(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \tag{12}$$

where $\mathcal{K} \equiv N^{-2}e_\lambda{}^I e_\lambda{}^J \nabla_I n^{(0)}_J|_{\Sigma_0}$, $\overline{\mathcal{K}} = n^{(0)I}\partial_{x^I}\ln r|_{\Sigma_0}$.

Applying these expressions to each spacetime region $(M_0^\pm, g^{(0)\pm})$, the background matching conditions (2) become

$$N_+^2 = N_-^2, \qquad r_+|_{\Sigma_0} = r_-|_{\Sigma_0}, \qquad \mathcal{K}_+ = \mathcal{K}_-, \qquad \overline{\mathcal{K}}_+ = \overline{\mathcal{K}}_-. \tag{13}$$

These equations involve scalars on the sphere and therefore do not require any further Hodge decomposition.

Notice that the embeddings have been implicitly chosen so that $\theta_+ = \theta_-(= \vartheta)$ and $\phi_+ = \phi_-(= \varphi)$ on the matching hypersurface. This can in principle be modified by an arbitrary rigid rotation, which is an intrinsic freedom of any matching preserving the symmetry [28]. At the background level, this rigid rotation is irrelevant and can be reabsorbed by a coordinate change. However, its effect is not so trivial at the perturbed level (see [25] for a discussion on its consequences).

Consider now an arbitrary linear perturbation and use the Hodge decomposition applied to the perturbed first and second fundamental forms. More specifically, we write

$$\begin{aligned}
q^{(1)\pm}{}_{\lambda A} &= D_A F^q_\pm + (\star dG^q_\pm)_A \\
k^{(1)\pm}{}_{\lambda A} &= D_A F^k_\pm + (\star dG^k_\pm)_A \\
q^{(1)\pm}{}_{AB} &= D_A(D_B P^q_\pm + (\star dR^q_\pm)_B) + D_B(D_A P^q_\pm \\
&\quad + (\star dR^q_\pm)_A) + H^q_\pm h_{AB} \\
k^{(1)\pm}{}_{AB} &= D_A(D_B P^k_\pm + (\star dR^k_\pm)_B) + D_B(D_A P^k_\pm \\
&\quad + (\star dR^k_\pm)_A) + H^k_\pm h_{AB},
\end{aligned} \tag{14}$$

where $F^q_\pm$, $G^q_\pm$, $P^q_\pm$, $R^q_\pm$, $H^q_\pm$, $F^k_\pm$, $G^k_\pm$, $P^k_\pm$, $R^k_\pm$, $H^k_\pm$, are scalar functions on S^2 that depend on the parameter λ. The linearized matching conditions (3) can be rewritten as conditions involving $q^{(1)\pm}{}_{\lambda\lambda}$ and $k^{(1)\pm}{}_{\lambda\lambda}$ together with the functions above. Recalling the existence of a nontrivial kernel for the Hodge decomposition, the equalities in (3) turn out to be equivalent to

$$\begin{aligned}
q^{(1)-}{}_{\lambda\lambda} &= q^{(1)+}{}_{\lambda\lambda} \\
F^q_- &= F^q_+ - N^2 F^q_0(\lambda), \\
G^q_- &= G^q_+ - N^2 G^q_0(\lambda), \\
P^q_- &= P^q_+ - N^2(P^q_0(\lambda) + P^q_m(\lambda) Y^m_1), \\
R^q_- &= R^q_+ - N^2(R^q_0(\lambda) + R^q_m(\lambda) Y^m_1), \\
H^q_- &= H^q_+ - 2N^2 P^q_m(\lambda) Y^m_1, \\
k^{(1)-}{}_{\lambda\lambda} &= k^{(1)+}{}_{\lambda\lambda} \\
F^k_- &= F^k_+ - N^2 F^\kappa_0(\lambda), \\
G^k_- &= G^k_+ - N^2 G^\kappa_0(\lambda), \\
P^k_- &= P^k_+ - N^2(P^\kappa_0(\lambda) + P^\kappa_m(\lambda) Y^m_1), \\
R^k_- &= R^k_+ - N^2(R^\kappa_0(\lambda) + R^\kappa_m(\lambda) Y^m_1), \\
H^k_- &= H^k_+ - 2N^2 P^\kappa_m(\lambda) Y^m_1,
\end{aligned} \tag{15}$$

where all the functions with a 0 or m subindex depend only on λ and correspond to the kernel freedom discussed above. They will collectively be named as *kernel functions* in what follows. The explicit factor $N^2(\lambda)(\equiv N^2_+ = N^2_-)$ in front of these functions has been added for convenience, as it simplifies some of the expressions below.

It may seem that adding these kernel functions is redundant, as they do not affect the tensors $q^{(1)\pm}{}_{ij}$ and $k^{(1)\pm}{}_{ij}$. However, it is precisely the fact that we want to impose the matching conditions at the level of S^2 scalars that forces us to include them. From a practical point of view, the explicit inclusion of the kernel functions in Eqs. (15) allows one to choose arbitrarily any particular decomposition at either ($\pm$) side. In particular, when studying existence problems for the matching of two given configurations (decomposed in terms of S^2 scalars in an explicit manner) it is important to keep the kernel functions free, as they may serve to fulfill conditions which might otherwise seem to be incompatible. For an explicit case where the kernel functions turn out to be relevant, see Sec. VIII below.

Summarizing, Eqs. (15) are the formal linearly perturbed matching conditions written in terms of S^2 scalars. The next task is to evaluate explicitly all the scalars involved in the Hodge decomposition of (4) and (5) in the cases we will be considering, namely, the matching of a Schwarzschild spacetime with a stationary and axially symmetric vacuum linear perturbation and a FLRW spacetime with a general linear perturbation.

IV. PERTURBED SCHWARZSCHILD REGION (−)

First, we describe the perturbations of the Schwarzschild region [denoted by the (−) sign] and derive the perturbed first and second fundamental forms on Σ^-_0.

A. Stationary and axially symmetric perturbations of Schwarzschild

We start by taking a stationary and axially symmetric vacuum metric which in Weyl-Papapetrou coordinates can be written as [29]

$$\begin{aligned}
g^{(0)-} = &-e^{2U}(dt + Ad\phi)^2 + e^{-2U}[e^{2k}(d\rho^2 + dz^2) \\
&+ \rho^2 d\phi^2],
\end{aligned} \tag{16}$$

where U, A, and k are functions of ρ and z. Vacuum linear perturbations can be obtained by taking derivatives of (16) with respect to a perturbation parameter. The result is

$$\begin{aligned}
g^{(1)-} = &-2e^{2U} U^{(1)} dt^2 - 4U^{(1)} A e^{2U} d\phi dt \\
&- 2U^{(1)} e^{2U} A^2 d\phi^2 - 2e^{2U} A^{(1)} dt d\phi \\
&- 2AA^{(1)} e^{2U} d\phi^2 + 2e^{-2U} e^{2k}(-U^{(1)} + k^{(1)}) \\
&\times (d\rho^2 + dz^2) - 2e^{-2U} U^{(1)} \rho^2 d\phi^2,
\end{aligned} \tag{17}$$

where the perturbation is obviously written in a specific gauge, which we shall denote by *Weyl gauge*. The functions $U^{(1)}$, $A^{(1)}$ in (17) depend on ρ and z and satisfy the perturbed vacuum equations, written explicitly below.

The Schwarzschild background is obtained from (16) by setting

$$U = \frac{1}{2}\log\left(1 - \frac{2m}{r}\right), \quad e^{2k} = \frac{r(r-2m)}{(r-m)^2 - m^2\cos^2\theta}, \quad A = 0,$$

where

$$\rho = r\sin\theta\sqrt{1-\frac{2m}{r}}, \qquad z = (r-m)\cos\theta,$$

so that the background metric reads

$$g^{(0)-} = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

In these coordinates, the first order perturbation metric (17) becomes

$$g^{(1)-} = -2\left(1-\frac{2m}{r}\right)(U^{(1)}dt^2 + A^{(1)}dtd\phi) - 2r^2\sin^2\theta U^{(1)}d\phi^2 + 2(k^{(1)} - U^{(1)}) \times\left(\frac{dr^2}{1-\frac{2m}{r}} + r^2 d\theta^2\right).$$

The perturbed vacuum equations decompose into a pair of decoupled second order PDE for $U^{(1)}$ and $A^{(1)}$

$$r(r-2m)\frac{\partial^2 U^{(1)}}{\partial r^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial U^{(1)}}{\partial\theta} + \frac{\partial^2 U^{(1)}}{\partial\theta^2} + 2(r-m)\frac{\partial U^{(1)}}{\partial r} = 0, \qquad (18)$$

$$r(r-2m)\frac{\partial^2 A^{(1)}}{\partial r^2} - \frac{\cos\theta}{\sin\theta}\frac{\partial A^{(1)}}{\partial\theta} + \frac{\partial^2 A^{(1)}}{\partial\theta^2} - 4m\frac{\partial A^{(1)}}{\partial r} = 0, \qquad (19)$$

together with a first order system for $k^{(1)}$ [which is compatible provided (18) holds]

$$\frac{\partial k^{(1)}}{\partial r} = \frac{2m\sin\theta}{(r-m)^2 - m^2\cos^2\theta}\left[(r-m)\sin\theta\frac{\partial U^{(1)}}{\partial r} + \cos\theta\frac{\partial U^{(1)}}{\partial\theta}\right],$$

$$\frac{\partial k^{(1)}}{\partial\theta} = \frac{2m\sin\theta}{(r-m)^2 - m^2\cos^2\theta}\left[-r(r-2m)\cos\theta\frac{\partial U^{(1)}}{\partial r} + (r-m)\sin\theta\frac{\partial U^{(1)}}{\partial\theta}\right].$$

B. Background matching hypersurface

The general spherically symmetric embedding (8) for Σ_0^- is given explicitly by

$$\Sigma_0^-: \{t = t_0(\lambda), r = r_0(\lambda), \theta = \vartheta, \phi = \varphi\},$$

where $t_0(\lambda)$ and $r_0(\lambda)$ are smooth functions (C^3 at least) restricted only to the condition that Σ_0^- is timelike (this implies an upper bound for $|\frac{dr_0}{dt_0}|$). The coordinate tangent vectors (9) to Σ_0^- read now

$$\vec{e}_1^{\,-} = \dot{t}_0\partial_t + \dot{r}_0\partial_r|_{\Sigma_0^-}, \qquad \vec{e}_2^{\,-} = \partial_\theta|_{\Sigma_0^-}, \qquad \vec{e}_3^{\,-} = \partial_\phi|_{\Sigma_0^-},$$

and the induced metric on Σ_0^- is

$$q^{(0)-}{}_{ij}d\xi^i d\xi^j = -N_-^2 d\lambda^2 + r_0^2(\lambda)(d\vartheta^2 + \sin^2\vartheta d\varphi^2),$$

with

$$N_-^2 = \left(1-\frac{2m}{r_0}\right)^{-1}\left[\dot{t}_0^2\left(1-\frac{2m}{r_0}\right)^2 - \dot{r}_0^2\right]. \qquad (20)$$

The sign of $N_-(\lambda)$ will be left free for the moment. The unit normal (10) to Σ_0^-

$$\vec{n}_-^{(0)} = \frac{1}{N_-}\left[\left(1-\frac{2m}{r_0}\right)\dot{t}_0\partial_r + \left(1-\frac{2m}{r_0}\right)^{-1}\dot{r}_0\partial_t\right]\Bigg|_{\Sigma_0^-},$$

$$\boldsymbol{n}_-^{(0)} = \frac{1}{N_-}(-\dot{r}_0 dt + \dot{t}_0 dr)|_{\Sigma_0^-}$$

points outwards from the interior Schwarzschild region (increasing r) whenever $\dot{t}_0 > 0$ and $N_- > 0$. The extrinsic curvature (12) relative to this normal reads

$$k^{(0)-}{}_{ij}d\xi^i d\xi^j = \frac{1}{N_-}\left[\left(-\dot{t}_0\ddot{r}_0 + \ddot{t}_0\dot{r}_0 + \frac{3m\dot{r}_0^2\dot{t}_0}{r_0(r_0-2m)} - \frac{m}{r_0^2}\times\left(1-\frac{2m}{r_0}\right)\dot{t}_0^3\right)d\lambda^2 + \dot{t}_0(r_0-2m)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right],$$

which, after comparison with (12), gives $\mathcal{K}_-$ and $\overline{\mathcal{K}}_-$ explicitly as

$$\mathcal{K}_- = \frac{1}{N_-^3}\left(-\dot{t}_0\ddot{r}_0 + \ddot{t}_0\dot{r}_0 + \frac{3m\dot{r}_0^2\dot{t}_0}{r_0(r_0-2m)} - \frac{m}{r_0^2}\left(1-\frac{2m}{r_0}\right)\dot{t}_0^3\right), \qquad (21)$$

$$\overline{\mathcal{K}}_- = \frac{1}{r_0^2 N_-}\dot{t}_0(r_0-2m). \qquad (22)$$

C. First order perturbation of the matching hypersurface

We now derive the perturbed first and second fundamental forms on Σ_0^- in terms of scalar quantities. We start by considering a general vector

$$\vec{Z}^- = Z^0(\lambda,\vartheta,\varphi)\partial_t + Z^1(\lambda,\vartheta,\varphi)\partial_r + Z^2(\lambda,\vartheta,\varphi)\partial_\theta + Z^3(\lambda,\vartheta,\varphi)\partial_\phi|_{\Sigma_0^-}, \qquad (23)$$

which describes how the matching hypersurface Σ_0^- is deformed to first order. Using (4) and (5) the perturbed first and second fundamental forms on Σ_0^- can be readily computed. The results are shown in Appendix A. In order to write down the matching conditions in terms of S^2 scalars, we need to decompose the vector $Z^2\partial_\vartheta + Z^3\partial_\varphi$ (which is tangent to the spherical orbits) according to its Hodge decomposition. Explicitly

$$Z^2\partial_\vartheta + Z^3\partial_\varphi = \left(\frac{\partial \mathcal{T}_1^-}{\partial\vartheta} - \frac{1}{\sin\vartheta}\frac{\partial \mathcal{T}_2^-}{\partial\varphi}\right)\partial_\vartheta + \left(\frac{1}{\sin^2\vartheta}\frac{\partial \mathcal{T}_1^-}{\partial\varphi} + \frac{1}{\sin\vartheta}\frac{\partial \mathcal{T}_2^-}{\partial\vartheta}\right)\partial_\varphi,$$

where $\mathcal{T}_1^-(\lambda, \vartheta, \varphi)$ and $\mathcal{T}_2^-(\lambda, \vartheta, \varphi)$ are S^2 scalars which are defined up to additive functions of λ. The radial part of $\vec{Z}^-$ can be also decomposed in the following intrinsic manner:

$$Z^0\partial_t + Z^1\partial_r|_{\Sigma_0} = Q^-\vec{n}_-^{(0)} + T^-\vec{e}_1^-, \tag{24}$$

where, again, $Q^-(\lambda, \vartheta, \varphi)$ and $T^-(\lambda, \vartheta, \varphi)$ are scalars on Σ_0^-.

When studying the Hodge decomposition of the first order perturbation tensor $g^{(1)-}$, we found it convenient to define two new scalars $\mathcal{G}(r, \theta)$ and $\mathcal{P}(r, \theta)$ by

$$A^{(1)} = \sin\theta\frac{\partial\mathcal{G}}{\partial\theta}, \qquad k^{(1)} = \frac{\partial^2\mathcal{P}}{\partial\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{\partial\mathcal{P}}{\partial\theta} \equiv \Delta_\theta\mathcal{P}.$$

The function $\mathcal{G}$ is defined up to an additive function of r while the kernel freedom in $\mathcal{P}$ corresponds to $P_0(r) + P_1(r)\cos\theta$, for arbitrary P_0 and P_1.

As a side remark, we recall that a useful function in any stationary and vacuum spacetime is the twist potential Ω, which in the axially symmetric case can be written as $dA = \rho e^{-2U} * d\Omega$ (where the Hodge dual operator $*$ refers to the $\{\rho, z\}$ plane). The background value of Ω is obviously zero. The first order perturbed twist potential $\Omega^{(1)}$ is, in fact, closely related to the function $\mathcal{G}$ above. By appropriately restricting the additive function in $\mathcal{G}$, it can be checked that $\Omega^{(1)} = (1 - \frac{2m}{r})\mathcal{G}_{,r}$ holds [the freedom left in $\mathcal{G}$ is the addition of any function $g(r)$ solving $(1 - \frac{2m}{r})g_{,r} = c$ for an arbitrary constant c].

With all the above definitions, the Hodge decomposition of the angular components of $q^{(1)-}{}_{ij}$ and $k^{(1)-}{}_{ij}$ can be computed. The corresponding scalars, following the notation in (14), are:

$$F_-^q = r_0^2\dot{\mathcal{T}}_1^- - N_-^2 T^-, \qquad G_-^q = r_0^2\dot{\mathcal{T}}_2^- - \dot{t}_0\mathcal{G}|_{\Sigma_0}\left(1 - \frac{2m}{r_0}\right) \qquad P_-^q = r_0^2(\mathcal{T}_1^- + \mathcal{P})|_{\Sigma_0}, \qquad R_-^q = r_0^2\mathcal{T}_2^-,$$

$$H_-^q = 2r_0\left(\frac{Q^-}{N_-}\left(1 - \frac{2m}{r_0}\right)\dot{t}_0 + T^-\dot{r}_0\right) - 2r_0^2U^{(1)} - 2r_0^2\frac{\cos\theta}{\sin\theta}\frac{\partial\mathcal{P}}{\partial\theta}\bigg|_{\Sigma_0}$$

$$G_-^\kappa = \frac{1}{N_-}\left[\left(1 - \frac{2m}{r_0}\right)\left(r_0\dot{t}_0\dot{\mathcal{T}}_2^- - \frac{N_-^2}{2}\frac{\partial\mathcal{G}}{\partial r}\right) - \mathcal{G}\left(\frac{\dot{r}_0^2 r_0 + mN_-^2}{r_0^2}\right)\right]\bigg|_{\Sigma_0}$$

$$F_-^\kappa = \frac{1}{N_-}\left\{\left(1 - \frac{2m}{r_0}\right)r_0\dot{t}_0\dot{\mathcal{T}}_1^- + T^-\left[\ddot{t}_0\dot{r}_0 - \dot{t}_0\ddot{r}_0 - \dot{t}_0N_-^2\frac{m}{r_0^2} + \dot{t}_0\dot{r}_0^2\frac{2m}{r_0(r_0 - 2m)}\right] + \frac{Q^-}{N_-}\left[\dot{r}_0\ddot{r}_0\frac{r_0}{r_0 - 2m} - \dot{t}_0\ddot{t}_0\frac{r_0 - 2m}{r_0} - \frac{2m\dot{r}_0^3}{(2m - r_0)^2} + \frac{N_-^2\dot{r}_0(3m - r_0)}{r_0(2m - r_0)}\right] - N_-^2\frac{d}{d\lambda}\left(\frac{Q^-}{N_-}\right) + 2\dot{r}_0\dot{t}_0U^{(1)} - \dot{r}_0\dot{t}_0\Delta_\theta\mathcal{P}\right\}\bigg|_{\Sigma_0}$$

$$P_-^\kappa = \frac{1}{2N_-}\left[(r_0 - 2m)\dot{t}_0\left(\mathcal{T}_1^- + \mathcal{P} + r_0\frac{\partial\mathcal{P}}{\partial r}\right) - N_-Q^-\right]\bigg|_{\Sigma_0}, \qquad R_-^\kappa = \frac{1}{N_-}\left((r_0 - 2m)\dot{t}_0\mathcal{T}_2^- + \frac{\dot{r}_0}{2}\mathcal{G}\right)\bigg|_{\Sigma_0},$$

$$H_-^\kappa = \frac{1}{N_-}\left\{T^-r_0\dot{r}_0\frac{1}{N_-^2}\left[\ddot{t}_0\dot{r}_0 - \dot{t}_0\ddot{r}_0 - \dot{t}_0N_-^2\frac{(m - r_0)}{r_0^2} - \dot{t}_0\dot{r}_0^2\frac{2m}{r_0(r_0 - 2m)}\right] - Q^-\frac{\dot{r}_0r_0}{N_-^3}\left(\dot{r}_0\ddot{r}_0\frac{r_0}{r_0 - 2m} - \dot{t}_0\ddot{t}_0\frac{r_0 - 2m}{r_0} - \frac{2m\dot{r}_0^3}{(2m - r_0)^2} + \frac{N_-^2\dot{r}_0(m - r_0)}{r_0(2m - r_0)}\right) + Q^-N_-\frac{r_0 - m}{r_0} + \dot{r}_0r_0\frac{d}{d\lambda}\left(\frac{Q^-}{N_-}\right) + \dot{t}_0\dot{r}_0^2r_0N_-^{-2}(\Delta_\theta\mathcal{P} - 2U^{(1)}) + \dot{t}_0(2m - r_0)\left(\Delta_{S^2}\mathcal{P} + r_0\frac{\cos\theta}{\sin\theta}\frac{\partial^2\mathcal{P}}{\partial r\partial\theta} + U^{(1)} + \frac{\partial U^{(1)}}{\partial r}\right)\right\}\bigg|_{\Sigma_0}, \tag{25}$$

where Δ_{S^2} denotes the Laplacian on (S^2, h_{AB}).

V. PERTURBED FLRW REGION (+)

In this section we describe the perturbations of the FLRW region [denoted by a (+) sign] and derive the perturbed first and second fundamental forms on the matching hypersurface Σ_0^+.

A. First order perturbations of FLRW

On a background FLRW spacetime there exists a coordinate system $\{\tau, x^a\}$ in which the metric reads

$$g^{(0)+} = a^2(\tau)(-d\tau^2 + \gamma_{ab}dx^a dx^b),$$

where $\gamma_{ab}dx^a dx^b = dR^2 + f^2(R, \epsilon)(d\theta^2 + \sin^2\theta d\phi^2)$ with $f = \sinh R$, R, $\sin R$, for $\epsilon = -1, 0, 1$, respectively. The covariant derivative associated to γ_{ab} will be denoted by ∇_a.

Let us now decompose the first order perturbation tensor $g^{(1)+}$ into scalar, vector, and tensor perturbations [30]

$$g^{(1)+}{}_{00} = -2a^2\Psi \qquad g^{(1)+}{}_{0a} = a^2 W_a$$
$$g^{(1)+}{}_{ab} = a^2(-2\Phi\gamma_{ab} + \chi_{ab})$$

with

$$\chi_{ab} = D_{ab}\chi + 2\nabla_{(a}Y_{b)} + \Pi_{ab},$$

where

$$D_{ab} \equiv \nabla_a\nabla_b - \frac{1}{3}\gamma_{ab}\nabla^2,$$

and

$$\nabla^a Y_a = \Pi_a{}^a = 0, \qquad \nabla^a \Pi_{ab} = 0. \tag{26}$$

The vector term W_a can be decomposed further into its irreducible parts

$$W_a = \partial_a W + \tilde{W}_a$$

with

$$\nabla^a \tilde{W}_a = 0. \tag{27}$$

The evolution and constraint equations for each mode, in any gauge, are given e.g. in [31] and can be written in a closed form after a gauge is specified. In this paper we intend to derive perturbed matching conditions which can be used in any (+) spacetime gauge. A gauge will only be specified in Appendix C where the Poisson gauge will be chosen as an example of how the perturbed matching equations simplify from their general expressions to a specific gauge.

Next, we introduce S^2 scalars $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{U}_1$, $\mathcal{U}_2$, and $\mathcal{H}$ according to

$$\tilde{W}_\theta d\theta + \tilde{W}_\phi d\phi = d\mathcal{W}_1 + \star d\mathcal{W}_2,$$
$$Y_\theta d\theta + Y_\phi d\phi = d\mathcal{Y}_1 + \star d\mathcal{Y}_2,$$
$$\Pi_{R\theta}d\theta + \Pi_{R\phi}d\phi = d\mathcal{Q}_1 + \star d\mathcal{Q}_2,$$
$$\Pi_{AB} = D_A(D_B\mathcal{U}_1 + (\star d\mathcal{U}_2)_B) + D_B(D_A\mathcal{U}_1 + (\star d\mathcal{U}_2)_A) + \mathcal{H}h_{AB},$$

where h_{AB}, $\star$, and D_A refer here to the coordinates $\{\theta, \phi\}$. The trace-free condition on Π_{ab} gives

$$\Pi_{RR} = -\frac{2}{f^2}(\Delta_{S^2}\mathcal{U}_1 + \mathcal{H}).$$

As before, these scalars are defined up the kernel of the Hodge operator, which in this case involves functions of τ and r. Concretely, each one of $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Q}_1$, $\mathcal{Q}_2$, admits the freedom $\mathcal{W}_1 \to \mathcal{W}_1 + w_1(\tau, R)$, etc., while $\mathcal{U}_1$ (and similarly $\mathcal{U}_2$) is defined up to $\mathcal{U}_1 \to \mathcal{U}_1 + u_1(\tau, R) + u_{1m}(\tau, R)Y_1^m$ which implies $\mathcal{H} \to \mathcal{H} + 2u_{1m}(\tau, R)Y_1^m$. Another interpretation of this freedom is that $\mathcal{W}_1, \ldots, \mathcal{Q}_2$ do not contribute to the $l = 0$ harmonic sector of the perturbations, and that $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{H}$ do not contribute to the $l = 0, 1$ sectors, since one can always choose the kernels such that $\mathcal{W}_{1(l=0)} = 0$, etc. This is made explicit in the relationship between these scalar functions and the harmonic decompositions in [23,25], as shown in Appendix D.

The constraints (26) and (27) in terms of the Hodge scalars read

$$\frac{1}{f^2}\Delta_{S^2}\mathcal{Y}_1 + \frac{2f'}{f}Y_R + Y_R' = 0,$$
$$\frac{1}{f^2}\Delta_{S^2}\mathcal{W}_1 + \frac{2f'}{f}\tilde{W}_R + \tilde{W}_R' = 0,$$
$$\Delta_{S^2}\mathcal{Q}_1 - 2\Delta_{S^2}\mathcal{U}_1' - \frac{2f'}{f}(\Delta_{S^2}\mathcal{U}_1 + \mathcal{H}) - 2\mathcal{H}' = 0,$$
$$2\Delta_{S^2}\mathcal{U}_1 + 2\mathcal{U}_1 + \mathcal{H} + f^2\mathcal{Q}_1' + 2ff'\mathcal{Q}_1 - \mathcal{B}_1(\tau, R) = 0,$$
$$\Delta_{S^2}\mathcal{U}_2 + 2\mathcal{U}_2 + f^2\mathcal{Q}_2' + 2ff'\mathcal{Q}_2 - \mathcal{B}_2(\tau, R) = 0,$$

where the prime denotes derivative with respect to R and $\mathcal{B}_1$ and $\mathcal{B}_2$ are arbitrary functions. The latter functions arise because we need to Hodge decompose the equations $\nabla^a\Pi_{aB} = 0$, from which two extra kernel functions appear.

The relationship between the Hodge scalars and the Mukohyama variables for perturbations of spherical backgrounds is summarized in Appendix D.

B. Background matching hypersurface

In the Einstein-Straus and Oppenheimer-Snyder models the matching hypersurface is comoving with respect to the FLRW flow. In fact, it is now known that this is necessary for any matching of a static, or stationary and axisymmetric, vacuum region to a FLRW spacetime [11–13]. The matching hypersurface Σ_0^+ is therefore of the form

$$\Sigma_0^+: \{\tau = \lambda, R = R_c, \theta = \vartheta, \phi = \varphi\},$$

where R_c is a constant. The tangent vectors are

$$\vec{e}_1^{\,+} = \partial_\tau|_{\Sigma_0^+}, \qquad \vec{e}_2^{\,+} = \partial_\theta|_{\Sigma_0^+}, \qquad \vec{e}_3^{\,+} = \partial_\phi|_{\Sigma_0^+},$$

and the first fundamental form is

$$q^{(0)+}{}_{ij}d\xi^i d\xi^j = a_\Sigma^2[-d\lambda^2 + f_c^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)],$$

where $f_c \equiv f(R_c, \epsilon)$ and $a_\Sigma \equiv a|_{\Sigma_0} = a(\lambda)$. Comparing this expression with (11) we have $r^+|_{\Sigma_0} = a_\Sigma f_c$ and $N_+^2 = a_\Sigma^2$. The unit normal to Σ_0^+ pointing towards the direction in which R increases reads

$$\vec{n}^+ = \frac{1}{a_\Sigma}\partial_R\bigg|_{\Sigma_0}, \qquad \boldsymbol{n}^+ = a_\Sigma dR|_{\Sigma_0},$$

and a simple calculation gives the second fundamental form on Σ_0^+ to be

$$k^{(0)+}{}_{ij}d\xi^i d\xi^j = a_\Sigma f_c f_c'(d\vartheta^2 + \sin^2\vartheta^2 d\varphi^2),$$

where $f_c' \equiv f'(R, \epsilon)|_{R=R_c}$. Comparing this expression with (12) we find

$$\mathcal{K}_+ = 0, \qquad \bar{\mathcal{K}}_+ = f_c'/(a_\Sigma f_c). \tag{28}$$

C. First order perturbation of the matching hypersurface

The first order perturbation of Σ_0^+ is defined by a vector field $\vec{Z}^+$ at points on Σ_0^+. Similarly to the case of the Schwarzschild region, we decompose $\vec{Z}^+$ as

$$\vec{Z}^+ = T^+\partial_\tau + \frac{Q^+}{a}\partial_R + \left(\frac{\partial\mathcal{T}_1^+}{\partial\vartheta} - \frac{1}{\sin\vartheta}\frac{\partial\mathcal{T}_2^+}{\partial\varphi}\right)\partial_\theta + \left(\frac{1}{\sin^2\vartheta}\frac{\partial\mathcal{T}_1^+}{\partial\varphi} + \frac{1}{\sin\vartheta}\frac{\partial\mathcal{T}_2^+}{\partial\vartheta}\right)\partial_\phi\bigg|_{\Sigma_0^+},$$

where T^+, Q^+, $\mathcal{T}_1^+$, and $\mathcal{T}_2^+$ depend on $\{\lambda, \vartheta, \varphi\}$. The Hodge decomposition of the angular parts of $q^{(1)+}{}_{ij}$ and $k^{(1)+}{}_{ij}$ (with explicit expressions given in Appendix B) in terms of the S^2 scalars introduced above can be found after a straightforward but somewhat long calculation. Recalling the notation in (14), the result is[2]

$$\begin{aligned}
F_+^q &= a_\Sigma^2(\dot{\mathcal{T}}_1^+ f_c^2 - T^+ + \mathcal{W}_1 + W)|_{\Sigma_0}, \qquad G_+^q = a_\Sigma^2(\dot{\mathcal{T}}_2^+ f_c^2 + \mathcal{W}_2)|_{\Sigma_0}\\
P_+^q &= a_\Sigma^2\left(\frac{1}{2}\chi + f_c^2\mathcal{T}_1^+ + \mathcal{U}_1 + \mathcal{Y}_1\right)\bigg|_{\Sigma_0}, \qquad R_+^q = a_\Sigma^2(f_c^2\mathcal{T}_2^+ + \mathcal{U}_2 + \mathcal{Y}_2)|_{\Sigma_0},\\
H_+^q &= a_\Sigma^2\left(-\frac{1}{3}\Delta_{S^2}\chi + \mathcal{H} + 2f_c'f_cY_R - \frac{1}{3}\chi''f_c - 2\Phi f_c^2 + \frac{1}{3}f_c'f_c\chi'\right) + 2a_\Sigma(Q^+f_c'f_c + \dot{a}_\Sigma f_c^2T^+)\bigg|_{\Sigma_0}\\
F_+^\kappa &= \frac{a_\Sigma}{2}(\mathcal{W}_1' - \tilde{W}_R - \dot{\chi}' - \dot{\mathcal{Y}}_1' - \dot{\mathcal{Q}}_1 - \dot{Y}_R) + \frac{a_\Sigma f_c'}{f_c}\left(\frac{1}{2}\dot{\chi} + \dot{\mathcal{Y}}_1 + f_c^2\dot{\mathcal{T}}_1^+\right) - \dot{Q}^+ + \frac{\dot{a}_\Sigma}{a_\Sigma}Q^+\bigg|_{\Sigma_0}\\
G_+^\kappa &= \frac{a_\Sigma}{2}(\mathcal{W}_2' - \dot{\mathcal{Y}}_2' - \dot{\mathcal{Q}}_2) + \frac{a_\Sigma f_c'}{f_c}(\dot{\mathcal{Y}}_2 + f_c^2\dot{\mathcal{T}}_2^+)\bigg|_{\Sigma_0}\\
P_+^\kappa &= \frac{a_\Sigma f_c'}{f_c}\left(\mathcal{Y}_1 + \frac{1}{2}\chi\right) + \frac{a_\Sigma}{2}\left(2f_c'f_c\mathcal{T}_1^+ + \mathcal{U}_1' - \mathcal{Q}_1 - Y_R - \frac{1}{2}\chi'\right) - \frac{Q^+}{2}\bigg|_{\Sigma_0}\\
R_+^\kappa &= a_\Sigma\left(f_c'f_c\mathcal{T}_2^+ + \frac{f_c'}{f_c}\mathcal{Y}_2 + \frac{1}{2}\mathcal{U}_2' - \frac{1}{2}\mathcal{Q}_2\right)\bigg|_{\Sigma_0}\\
H_+^\kappa &= \frac{a_\Sigma f_c'}{f_c}\left(\frac{1}{6}\Delta_{S^2}\chi + \Delta_{S^2}\mathcal{U}_1 + \mathcal{H}\right) - \frac{\dot{a}_\Sigma^2}{a_\Sigma^2}f_c^2Q^+ + \frac{\dot{a}_\Sigma}{a_\Sigma}f_c^2\dot{Q}^+ + \dot{a}_\Sigma f_c'f_cT^+ + \dot{a}_\Sigma f_c^2(\tilde{W}_R + W')\\
&\quad + a_\Sigma\left(\frac{1}{2}\chi' + \frac{1}{2}\mathcal{H}' + Y_R + \frac{Q^+}{a_\Sigma} - \frac{1}{6}\Delta_{S^2}\chi' - f_c'f_c\left(\frac{1}{2}\chi'' + \Phi\right) - f_c^2\left(\frac{2\epsilon}{3}\chi' + \Phi' + \frac{1}{6}\chi''' + 2\epsilon Y_R + 2\epsilon\frac{Q^+}{a_\Sigma}\right)\right)\bigg|_{\Sigma_0}.
\end{aligned} \tag{29}$$

This concludes the decomposition of the perturbation. Our next aim is to write down and discuss the matching conditions.

VI. MATCHING CONDITIONS

A. Background matching conditions: Einstein-Straus and Oppenheimer-Snyder models

The results in this subsection are well-known, but we reproduce their derivation for completeness. The background matching conditions are obtained simply by particularizing Eqs. (13) [which correspond to (2) in spherical symmetry] to the Schwarzschild region and the FLRW region. The second equation in (13) implies

[2]At some points we slightly abuse the notation and use dot to denote both derivative with respect to τ, and derivative with respect to λ. On the matching hypersurface they obviously coincide as $\tau = \lambda$ and R, θ, ϕ do not depend on λ there.

$$r_0 = f_c a_\Sigma. \tag{30}$$

Inserting this into (20), and using its derivative along λ, i.e. $\dot{r}_0 = f_c \dot{a}_\Sigma$, the first equation in (13) leads to a quadratic equation for $\dot{t}_0$, namely

$$\dot{t}_0^2 = \frac{f_c^2 \dot{a}_\Sigma^2 + a_\Sigma^2 - \frac{2m}{f_c} a_\Sigma}{(1 - \frac{2m}{f_c} a_\Sigma)^2}. \tag{31}$$

From $N_-^2 = a_\Sigma^2$ we write $N_- = \sigma a_\Sigma$ with $\sigma = \pm 1$ (recall that we want to keep the orientation of the normal arbitrary). The fourth equation in (13), together with expressions (22) and (28), give the following linear equation for $\dot{t}_0$:

$$\dot{t}_0 = \sigma \frac{a_\Sigma^2 f_c f_c'}{f_c a_\Sigma - 2m}, \tag{32}$$

which inserted into (31) yields

$$\dot{a}_\Sigma^2 + \epsilon a_\Sigma^2 = \frac{2m a_\Sigma}{f_c^3},$$

after using $f_c'^2 = 1 - \epsilon f_c^2$. This ordinary differential equation (ODE) for the scale factor is exactly the Friedmann equation for dust (restricted to points on Σ_0), as expected. This is just a consequence of the Israel conditions, which impose the equality of certain components of the energy-momentum tensor on both sides of the matching hypersurface.

The last matching condition, namely, the third equation in (13), is automatically fulfilled once (30) and (32) and the dust Friedmann equation hold, as a straightforward calculation shows.

Summarizing, given the necessary condition that the FLRW background is dust, the matching conditions are satisfied if and only if (30) and (32) hold. We can now study the linearly perturbed matching.

B. Linearized matching

The linearized matching conditions (3) correspond to equating the expressions for $q^{(1)\pm}{}_{ij}$ and $k^{(1)\pm}{}_{ij}$ given in Appendixes A and B. However, as discussed in Sec. III B, the angular components are much better handled if the underlying spherical symmetry is exploited through the Hodge decomposition, which allows us to work exclusively in terms of S^2 scalars. Thus, the full set of matching conditions is given by (15) after using (25) and (29), together with the nonangular expressions (A1), (B1), (A2), and (B2), given in Appendixes A and B.

The Hodge decomposition in terms of scalars involves two types of objects depending on their behavior under reflection. For instance, in the decomposition (6) the scalar F remains unchanged while G changes sign under a reflection. The former scalar is then named *even* while the latter is named *odd*. This splitting behavior occurs in any decomposition in terms of Hodge potentials. In particular, our linearized matching conditions must split into equations involving only even scalars and equations involving only odd scalars. We denote them simply as the even and odd sets of equations.

The odd set is simpler to handle. It is not difficult to see that the equations can be rewritten as the following four relations:

$$\mathcal{T}_2^+ + f_c^{-2}[\mathcal{U}_2 + \mathcal{Y}_2 - (R_0^q + R_m^q Y_1^m)] \overset{\Sigma_0}{=} \mathcal{T}_2^-, \tag{33}$$

$$\mathcal{W}_2 - G_0^q - \frac{d}{d\lambda}[\mathcal{U}_2 + \mathcal{Y}_2 - (R_0^q + R_m^q Y_1^m)|_{\Sigma_0}] \overset{\Sigma_0}{=} -\sigma \mathcal{G} f_c' a_\Sigma^{-1}, \tag{34}$$

$$\mathcal{W}_2' - 2a_\Sigma G_0^\kappa - \frac{d}{d\lambda}[\mathcal{U}_2' + \mathcal{Y}_2' - 2a_\Sigma(R_0^\kappa + R_m^\kappa Y_1^m)|_{\Sigma_0}] \overset{\Sigma_0}{=} \sigma \mathcal{G} \frac{f_c^3 a_\Sigma \epsilon - 3m}{f_c^2 a_\Sigma^2} + \sigma \frac{\partial \mathcal{G}}{\partial r}(f_c^2 \epsilon - 1), \tag{35}$$

$$\mathcal{Q}_2 - [\mathcal{U}_2' - 2a_\Sigma(R_0^\kappa + R_m^\kappa Y_1^m)] + 2f_c^{-1} f_c' [\mathcal{U}_2 - (R_0^q + R_m^q Y_1^m)] \overset{\Sigma_0}{=} -\sigma \mathcal{G} a_\Sigma^{-2} \dot{a}_\Sigma f_c. \tag{36}$$

The even set of equations requires a much more lengthy and subtle analysis. After carefully combining the equations, and defining $\delta Q \equiv Q^+ - Q^-$, it turns out that they can be written as the following eight equations:

$$\mathcal{T}_1^+ + f_c^{-2}\Big[\mathcal{U}_1 + \mathcal{Y}_1 + \frac{1}{2}\chi - (P_0^q + P_m^q Y_1^m)\Big] \overset{\Sigma_0}{=} \mathcal{T}_1^- + \mathcal{P}, \tag{37}$$

$$T^+ - \Big[\mathcal{W}_1 + W - F_0^q - \frac{d}{d\lambda}\Big[\mathcal{U}_1 + \mathcal{Y}_1 + \frac{1}{2}\chi - (P_0^q + P_m^q Y_1^m)|_{\Sigma_0}\Big]\Big] \overset{\Sigma_0}{=} f_c^3 a_\Sigma \frac{\partial \mathcal{P}}{\partial r} + T^-, \tag{38}$$

$$\mathcal{Q}_1 + Y_R + \chi' - [\mathcal{U}_1' - 2a_\Sigma(R_0^\kappa + R_m^\kappa Y_1^m)] + 2f_c^{-1} f_c' [\mathcal{U}_1 - (P_0^q + P_m^q Y_1^m)] + \frac{\delta Q}{a_\Sigma} \overset{\Sigma_0}{=} -f_c' f_c^2 a_\Sigma \frac{\partial \mathcal{P}}{\partial r}, \tag{39}$$

$$\Psi + \frac{1}{a_\Sigma}\frac{d}{d\lambda}\Big[a_\Sigma\Big(\mathcal{W}_1 + W - F_0^q - \frac{d}{d\lambda}\Big[\mathcal{U}_1 + \mathcal{Y}_1 + \frac{1}{2}\chi - (P_0^q + P_m^q Y_1^m)|_{\Sigma_0}\Big]\Big)\Big|_{\Sigma_0}\Big] \overset{\Sigma_0}{=} \frac{a_\Sigma f_c + 2m - 2a_\Sigma f_c^3 \epsilon}{a_\Sigma f_c - 2m} U^{(1)} + (2f_c^3 a_\Sigma \epsilon - 3m)\frac{\partial \mathcal{P}}{\partial r} + f_c a_\Sigma (f_c^3 a_\Sigma \epsilon - 2m)\frac{\partial^2 \mathcal{P}}{\partial r^2} + \frac{a_\Sigma f_c^3 \epsilon - 2m}{a_\Sigma f_c - 2m}\Delta_\theta \mathcal{P}, \tag{40}$$

$$\mathcal{W}_1' - \tilde{W}_R - 2a_\Sigma F_0^\kappa - \frac{d}{d\lambda}[\mathcal{U}_1' + \mathcal{Y}_1' - 2a_\Sigma(R_0^\kappa + R_m^\kappa Y_1^m)|_{\Sigma_0}] - \frac{d}{d\lambda}\left(\frac{\delta Q}{a_\Sigma}\right) \stackrel{\Sigma_0}{=} \frac{2\dot{a}_\Sigma f_c^2 f_c'}{a_\Sigma f_c - 2m}(2U^{(1)} - \Delta_\theta \mathcal{P}) - 3\dot{a}_\Sigma f_c^2 f_c' \frac{\partial \mathcal{P}}{\partial r} - \dot{a}_\Sigma a_\Sigma f_c^3 f_c' \frac{\partial^2 \mathcal{P}}{\partial r^2}, \quad (41)$$

$$\Psi' + \frac{1}{a_\Sigma}\frac{d}{d\lambda}\left(a_\Sigma\left[\tilde{W}_R + W' + \frac{d}{d\lambda}\left(\frac{\delta Q}{a_\Sigma}\right)\right]\Big|_{\Sigma_0}\right) \stackrel{\Sigma_0}{=} -\frac{3m}{a_\Sigma^2 f_c^3}Q^- + \frac{2f_c^3 a_\Sigma \epsilon + a_\Sigma f_c - 6m}{a_\Sigma f_c - 2m} a_\Sigma f_c' \frac{\partial U^{(1)}}{\partial r} - \frac{f_c^3 a_\Sigma \epsilon - 2m}{a_\Sigma f_c - 2m} a_\Sigma f_c' \frac{\partial}{\partial r}\Delta_\theta \mathcal{P} - 2\frac{f_c^2 \epsilon - 1}{(a_\Sigma f_c - 2m)^2} a_\Sigma f_c'(2U^{(1)} - \Delta_\theta \mathcal{P}), \quad (42)$$

$$\Phi' + \frac{1}{6}\chi''' + \frac{f_c'}{2f_c}(\chi'' + 2\Phi) - \frac{\dot{a}_\Sigma}{a_\Sigma}(\tilde{W}_R + W') - \frac{\dot{a}_\Sigma f_c'}{a_\Sigma f_c}\left(\mathcal{W}_1 + W - F_0^q - \frac{d}{d\lambda}\left[\mathcal{U}_1 + \mathcal{Y}_1 + \frac{1}{2}\chi - (P_0^q + P_m^q Y_1^m)|_{\Sigma_0}\right]\right) - \frac{f_c'}{f_c^3}\left[\mathcal{H} + \Delta_{S^2}\mathcal{U}_1 + \frac{1}{6}\Delta_{S^2}\chi\right] - \frac{1}{2f_c^2}[\mathcal{H}' - 4a_\Sigma P_m^\kappa Y_1^m] + \frac{1}{6f_c^2}(\Delta_{S^2}\chi' - \chi'(3 - 4\epsilon f_c^2)) - Y_R(f_c^{-2} - 2\epsilon) \stackrel{\Sigma_0}{=} \frac{1}{f_c^2}(1 - 2f_c^2\epsilon)\frac{\delta Q}{a_\Sigma} + \frac{1}{\dot{a}_\Sigma f_c^3}(2m - 2a_\Sigma f_c^3 \epsilon)\frac{d}{d\lambda}\left(\frac{\delta Q}{a_\Sigma}\right) - \frac{3m}{a_\Sigma^2 f_c^3}Q^- + \frac{f_c'(a_\Sigma f_c + 2m - 2a_\Sigma f_c^3\epsilon)}{f_c(a_\Sigma f_c - 2m)}U^{(1)} + a_\Sigma f_c' \frac{\partial U^{(1)}}{\partial r} - \frac{f_c'}{f_c}(2m - a_\Sigma f_c^3 \epsilon)\frac{\partial \mathcal{P}}{\partial r} + \frac{f_c'(a_\Sigma f_c + a_\Sigma f_c^3 \epsilon - 4m)}{f_c(a_\Sigma f_c - 2m)}\Delta_\theta \mathcal{P} + \frac{f_c'}{f_c}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\left(2\mathcal{P} + a_\Sigma f_c \frac{\partial \mathcal{P}}{\partial r}\right), \quad (43)$$

$$\frac{\delta Q}{a_\Sigma} + \frac{\dot{a}_\Sigma f_c}{a_\Sigma f_c'}\left(\mathcal{W}_1 + W - F_0^q - \frac{d}{d\lambda}\left[\mathcal{U}_1 + \mathcal{Y}_1 + \frac{1}{2}\chi - (P_0^q + P_m^q Y_1^m)|_{\Sigma_0}\right]\right) - \frac{1}{6f_c f_c'}\Delta_{S^2}\chi + \frac{1}{2f_c f_c'}(\mathcal{H} - 2P_m^q Y_1^m) + \frac{1}{6}\chi' + Y_R - \frac{f_c}{f_c'}\left(\frac{\chi''}{6} + \Phi\right) \stackrel{\Sigma_0}{=} \frac{f_c}{f_c'}\left((f_c^3 a_\Sigma \epsilon - 2m)\frac{\partial \mathcal{P}}{\partial r} - \frac{\cos\theta}{\sin\theta}\frac{\partial \mathcal{P}}{\partial \theta} - U^{(1)}\right). \quad (44)$$

This set of 12 equations represent the full set of linearized matching conditions for our problem. They are valid for any FLRW gauge and any hypersurface gauge. Moreover, they include the 20 kernel functions F_0^q, G_0^q, P_0^q, R_0^q, P_m^q, R_m^q, F_0^κ, G_0^κ, P_0^κ, R_0^κ, P_m^κ, R_m^κ in order to allow for any choice of Hodge decomposition on either ($\pm$) side. Depending on the problem, these kernel functions may play a role. For instance, if the aim is to determine perturbations in FLRW given perturbations in the Schwarzschild region, then the kernel functions can be put to zero without loss of generality since, in that problem, one is constructing the exterior data and changing the kernel functions does not affect the metric perturbations. However, in a situation when two specific perturbations are given and the problem is to determine whether they match at the linear level, then the kernel functions become relevant and cannot be dropped *a priori*.

The expressions above are written in such a way that the (+) and the (−) objects are kept on the left- and right-hand sides of the equations, respectively. The only exception being the difference δQ which we found convenient to use with a pivotal role.

The three equations (33), (37), and (38) determine the difference vector $\vec{T}^+ - \vec{T}^-$. Recall that this difference vector is tangent to the background matching hypersurface and corresponds to the freedom in perturbing points within the hypersurface without deforming it as a set of points [25]. Recall also that by choosing the appropriate hypersurface gauge one can fix either $\vec{T}^+$ or $\vec{T}^-$ arbitrarily (but not both), and that the difference $\vec{T}^+ - \vec{T}^-$ is independent of such choice. Thus, the three equations (33), (37), and (38) do not provide any essential information concerning the metric perturbations at either ($\pm$) side or the shape of the perturbed Σ_0 (defined by δQ). We have been careful in rearranging the remaining equations so that the difference $\vec{T}^+ - \vec{T}^-$ does not appear. So, this somewhat superfluous information gets, in this way, separated from the remaining (more relevant) restrictions.

We summarize the results of this section in the form of a theorem.

Theorem VI.1 *Let an Einstein-Straus or Oppenheimer-Snyder spacetime geometry be linearly perturbed in such a way that the perturbations inside the Schwarzschild region are stationary, axially symmetric and vacuum, while the* *perturbations of the matching hypersurface and of the FLRW region are arbitrary. Assume also that the Weyl gauge has been chosen for the Schwarzschild perturbation and that the Hodge decomposition has been used to write all tensors on the sphere in terms of scalars.*

Then, the linearized matching conditions are satisfied (and hence a perturbed model is obtained) if and only if the equations (34)–(36) *for the odd part, and the equations* (39)–(44) *for the even part are fulfilled.*

When a specific gauge is used on the FLRW side, the equations above simplify (sometimes notably). As an example, we present in Appendix C the linearized matching conditions for the particular case of a flat $\epsilon = 0$ FLRW region in the Poisson gauge, for which $W = \chi = Y_a = 0$.

It is also worth noticing that the Einstein equations have been used at the background level, but the linearized equations for dust at the FLRW region have not been used anywhere. Thus, the equations apply to any perturbation of FLRW *regardless of the matter content being described*. The same comment applies to the Schwarzschild side except for the fact that the perturbations have been restricted *a priori* to being stationary and axially symmetric and that the form (17) uses part of the vacuum field equations [in particular, it uses the fact that ρ is a flat harmonic function, which allows the metric to be written in the form (16)]. The reader may have also noticed that while the gauge of the FLRW is kept free, the gauge in the Schwarzschild cavity has been fixed from the very beginning. The reason for such a different treatment is that we implicitly regard the perturbed Schwarzschild metric as a *source* for the FLRW perturbations, which then become the unknowns.

Having obtained the equations in a simple and compact form (they may be compared with the equations that would result from equating all components in $q^{(1)-}{}_{ij}$ and $k^{(1)-}{}_{ij}$ in Appendix A with their pairings $q^{(1)+}{}_{ij}$ and $k^{(1)+}{}_{ij}$ in Appendix B), our aim now is to extract some of their direct consequences. A more detailed analysis of the equations is postponed to a subsequent paper.

VII. CONSTRAINTS ON THE FLRW SIDE

The first important general consequence of the linearized matching equations arises by simply considering Eqs. (34) and (36). Isolating $\mathcal{G}$ from both equations we arrive at the relation

$$\frac{\dot{a}_\Sigma f_c}{a_\Sigma f_c'}\left(\mathcal{W}_2 - G_0^q - \frac{d}{d\lambda}\left[\mathcal{U}_2 + \mathcal{Y}_2 - (R_0^q + R_m^q Y_1^m)|_{\Sigma_0}\right]\right) \stackrel{\Sigma_0}{=} \mathcal{Q}_2 - [\mathcal{U}_2{}' - 2a_\Sigma(R_0^\kappa + R_m^\kappa Y_1^m)] + 2f_c^{-1}f_c'[\mathcal{U}_2 - (R_0^q + R_m^q Y_1^m)], \tag{45}$$

which only involves objects on the FLRW side. Therefore, this equation constitutes a constraint on the FLRW perturbations on Σ_0, irrespective of the (stationary and axisymmetric) perturbations of the Schwarzschild region, and links the vector perturbations represented by the gauge invariant vector perturbation[3] $\mathcal{W}_2 - d\mathcal{Y}_2/d\tau$ and the tensor perturbations driven by $\mathcal{U}_2$ and $\mathcal{Q}_2$.

Note also that although there are kernel terms in the form of $R_0^q + R_m^q Y_1^m$ and $R_0^\kappa + R_m^\kappa Y_1^m$, which would contribute to the $l = 0, 1$ harmonics, these can be in principle absorbed into $\mathcal{U}_2$ and $\mathcal{U}_2{}'$ respectively and do not affect the value of the tensor perturbations. Thus, Eq. (45) implies that if there are no tensor perturbations, i.e. $\mathcal{U}_2 = \mathcal{Q}_2 = 0$, then, on Σ_0, $\mathcal{W}_2 - d\mathcal{Y}_2/d\tau$ cannot contain harmonics with $l \geq 2$.

In terms of the doubly gauge invariant perturbation variables of Mukohyama [23,25], defined for $l \geq 2$, the constraint (45) restricted to $l \geq 2$ is equivalent to the set of equations

$$\frac{\dot{a}_\Sigma f_c}{a_\Sigma^2 f_c'} f_0^+ \Sigma_0 = -2\kappa^+_{(LT)} \quad \text{for all } (l \geq 2, m), \tag{46}$$

as it can be easily checked by using the relations in Appendix D together with (29), and the expressions for the doubly gauge invariants in [23].

The main result of this section is then summarized in the following theorem:

Theorem VII.1 *Let a region of a general perturbed dust FLRW be (perturbatively) matched across a non-null hypersurface Σ_0 to a region of a stationary and axisymmetric, vacuum perturbation of Schwarzschild. If the perturbed FLRW contains a vector perturbation with $l \geq 2$ harmonics on Σ_0, then the FLRW region must also contain tensor perturbations on Σ_0.*

At points where $\dot{a}_\Sigma \neq 0$ there is a second constraint on the FLRW side which is obtained by differentiating (34) along λ and using (35) to isolate $\mathcal{G}_{,r}|_{\Sigma_0} = \frac{d}{d\lambda} \times (\mathcal{G}|_{\Sigma_0})/(\dot{a}_\Sigma f_c)$. This second constraint relates the values on Σ_0 of $\mathcal{W}_2 - d\mathcal{Y}_2/d\tau$ and $\mathcal{U}_2$, with their first and second derivatives, and reads

$$\begin{aligned}\dot{a}_\Sigma f_c\Big\{&(1 - a_\Sigma f_c')\Big[\mathcal{W}_2{}' - \frac{d}{d\lambda}(\mathcal{U}_2{}' + \mathcal{Y}_2{}'|_{\Sigma_0})\Big] - 2a_\Sigma G_0^\kappa \\ &+ \frac{d}{d\lambda}[2a_\Sigma(R_0^\kappa + R_m^\kappa Y_1^m)] - \Big(\mathcal{W}_2 - G_0^q - \frac{d}{d\lambda}[\mathcal{U}_2 + \mathcal{Y}_2 \\ &- (R_0^q + R_m^q Y_1^m)|_{\Sigma_0}]\Big)(3m + a_\Sigma f_c - 2\epsilon a_\Sigma f_c^3)\frac{\dot{a}_\Sigma}{a_\Sigma f_c f_c'}\Big\} \\ &- a_\Sigma f_c'\Big\{\dot{\mathcal{W}}_2 - \frac{d}{d\lambda}(\dot{\mathcal{U}}_2 + \dot{\mathcal{Y}}_2|_{\Sigma_0}) - (\mathcal{U}_2{}' + \mathcal{Y}_2{}') \\ &\times \frac{m - \epsilon a_\Sigma f_c^3}{f_c^2} - \dot{G}_0^q + \ddot{R}_0^q + \ddot{R}_m^q Y_1^m\Big\} \stackrel{\Sigma_0}{=} 0.\end{aligned} \tag{47}$$

[3] Note that $W_a - dY_a/d\tau$ is the (only) gauge invariant vector linear perturbation [30]. $\mathcal{W}_2 - d\mathcal{Y}_2/d\tau$ corresponds, then, to the divergence-free and odd part of the vector perturbation in FLRW.

In other words, Theorem VII.1 states that *if the FLRW side contains rotational perturbations that reach the matching hypersurface* Σ_0 *, then it must also contain gravitational waves there, irrespective of the matter content described by the perturbation*. How this constraint extends outside to a neighborhood of Σ_0 depends on which assumptions are made on the FLRW perturbations. Note that there is a radial derivative (transverse to Σ_0) of a tensor perturbation component in (45), and that if the expansion on the hypersurface does not vanish, there is still an additional constraint on the hypersurface given by (47), with more derivatives, both transverse and along the hypersurface.

It must be stressed that, as demonstrated in [32] (and references therein), there exist configurations of FLRW linear perturbations containing *only* vector perturbations which vanish identically inside a spherical surface. Such configurations are compatible with the results presented here, since that interior region is FLRW and the above constraints do not apply. A completely different matter is the embedding of a Schwarzschild spherical cavity (or a vacuum perturbation thereof) into any such model: the Schwarzschild cavity cannot reach the perturbed FLRW region, as otherwise the constraints above would require that tensor perturbations are also present (at least near the boundary of the Schwarzschild cavity). In fact, as we are going to see in the next section, the vacuum cavity itself must remain unperturbed because the inclusion of any slow rotation in the vacuum region necessarily induces perturbations on the FLRW region.

VIII. MATCHING PERTURBED SCHWARZSCHILD WITH EXACT FLRW

Nolan and Vera [13] have proved that in order to match stationary axially symmetric vacuum regions to FLRW regions across hypersurfaces preserving the axial symmetry, the vacuum part must be static. As an application of our formalism above, we shall show that their result can be generalized to arbitrary matching hypersurfaces (not necessarily axially symmetric) to first order in approximation theory.

Since we want to keep the FLRW exact, we set all the FLRW perturbations equal to zero. Notice that this entails a choice of Hodge scalar functions in the FLRW part, and therefore the kernel functions in the matching conditions must be kept free.

Let us start by considering Eq. (34), which differentiated with respect to ϑ and, using the fact that $\mathcal{G}$ does not depend on ϕ, leads to $\dot{R}^q_2 = \dot{R}^q_3 = 0$ plus

$$A^{(1)}|_{\Sigma_0} = -\sigma \frac{a_\Sigma \sin^2\vartheta}{f'_c} \dot{R}^q_1. \tag{48}$$

In order to determine $\dot{R}^q_1$ we use the constraints (45) and (47) derived in the previous section. Setting all FLRW perturbations equal to zero, and extracting the coefficient in Y^1_1 of Eq. (45) we get

$$R^\kappa_1 = \frac{\dot{a}_\Sigma f_c}{2a^2_\Sigma f'_c} \dot{R}^q_1 + \frac{f'_c}{a_\Sigma f_c} \dot{R}^q_1.$$

Inserting this expression into the Y^1_1 coefficient of the second constraint (47) yields a second order ODE for $R^q_1(\lambda)$, namely

$$a_\Sigma(2m - a_\Sigma f_c)\ddot{R}^q_1 + \dot{a}_\Sigma(a_\Sigma f_c - 4m)\dot{R}^q_1 = 0,$$

which can be solved to give

$$\dot{R}^q_1 = \frac{Ca^2_\Sigma}{2m - a_\Sigma f_c}, \tag{49}$$

where C is an arbitrary integration constant. Thus, (48) becomes

$$A^{(1)}|_{\Sigma_0} = \sigma C \frac{a^3_\Sigma \sin^2\vartheta}{f'_c(a_\Sigma f_c - 2m)}. \tag{50}$$

Our aim is to show that the interior source must be static in this case. Since we are working at the perturbative level, first of all we need to determine the necessary and sufficient condition that ensures that a given perturbation of a static background remains static to that order of approximation. To do that, consider an arbitrary metric $g_{\alpha\beta}$ with a static Killing vector $\vec{\xi}$ and a first order perturbation metric $g^{(1)}_{\alpha\beta}$ which admits a perturbative Killing vector, i.e. there exists a vector $\vec{\xi}^{(1)}$ such that $\vec{\xi} + \vec{\xi}^{(1)}$ is (to first order) a Killing vector of $g_{\alpha\beta} + g^{(1)}_{\alpha\beta}$. Our aim is to find the condition that has to be imposed to ensure that this vector is static (to first order).

Let us, first of all, lower its indices and define $\hat{\xi}_\alpha = (g_{\alpha\beta} + g^{(1)}_{\alpha\beta})(\xi^\beta + \xi^{(1)\beta}) = \xi_\alpha + g^{(1)}_{\alpha\beta}\xi^\beta + \xi^{(1)}_\alpha + O(2)$. The perturbed Killing vector is static (to first order) if and only if the linear term of $\hat{\xi}_{[\alpha}\partial_\beta\hat{\xi}_{\gamma]}$ vanishes, or equivalently

$$\boldsymbol{M} \wedge d\boldsymbol{\xi} + \boldsymbol{\xi} \wedge d\boldsymbol{M} = 0,$$

where $\boldsymbol{M} = \boldsymbol{\xi}^{(1)} + \boldsymbol{g}^{(1)}_t$, and $g^{(1)}_t{}_\alpha \equiv \xi^\beta g^{(1)}_{\beta\alpha}$. Now, staticity of $\boldsymbol{\xi}$ is equivalent to $\boldsymbol{\xi} \wedge \boldsymbol{w} = d\boldsymbol{\xi}$ for some one-form $\boldsymbol{w}$, which allows us to rewrite the linear staticity condition as

$$\boldsymbol{\xi} \wedge (d\boldsymbol{M} - \boldsymbol{M} \wedge \boldsymbol{w}) = 0. \tag{51}$$

By construction, this equation gives the necessary and sufficient condition for having a static first order perturbation.

If the static background is moreover axially symmetric and the perturbation is stationary and axially symmetric, with no further symmetries, the vector $\vec{\xi}^{(1)}$ is restricted to having the form $\vec{\xi}^{(1)} = a\vec{\xi} + b\vec{\eta}$, where a and b are arbitrary constants, and $\vec{\eta}$ is the axial Killing vector of the background that remains preserved in the perturbation (i.e. the one fulfilling $\mathcal{L}_{\vec{\eta}} g^{(1)}_{\alpha\beta} = 0$).

For a static background of the form (16) (with $A = 0$) we have

$$\boldsymbol{\xi} = N_\xi dt, \qquad \boldsymbol{\eta} = N_\eta d\phi,$$

where N_ξ and N_η are, respectively, the norms of $\vec{\xi}$ and $\vec{\eta}$. For a Schwarzschild background they are given by

$$N_\xi = -\left(1 - \frac{2m}{r}\right), \qquad N_\eta = r^2 \sin^2\theta \left(1 - \frac{2m}{r}\right)^{-1}. \quad (52)$$

Computing $d\boldsymbol{\xi} = N_\xi^{-1} dN_\xi \wedge \boldsymbol{\xi}$ we obtain $\boldsymbol{w} = -N_\xi^{-1} dN_\xi$. On the other hand, using (17) for the metric perturbation, we find

$$\boldsymbol{g}^{(1)}_t = A^{(1)} \frac{N_\xi}{N_\eta} \boldsymbol{\eta} + 2U^{(1)} \boldsymbol{\xi},$$

and thus

$$\boldsymbol{M} = (a + 2U^{(1)}) \boldsymbol{\xi} + \left(b + A^{(1)} \frac{N_\xi}{N_\eta}\right) \boldsymbol{\eta}.$$

It is now straightforward to rewrite (51) as

$$d\left(A^{(1)} \frac{N_\xi}{N_\eta}\right) + \left(b + A^{(1)} \frac{N_\xi}{N_\eta}\right)\left[\frac{1}{N_\eta} dN_\eta - \frac{1}{N_\xi} dN_\xi\right] = 0,$$

which simplifies to

$$dA^{(1)} = b \frac{N_\eta^2}{N_\xi^2} d\left(\frac{N_\xi}{N_\eta}\right).$$

Integrating and using (52) we find the expression

$$A^{(1)} = b \frac{r^3 \sin^2\theta}{r - 2m} + \text{const},$$

which restricted onto Σ_0 reads

$$A^{(1)}|_{\Sigma_0} = b \frac{f_c^3 a_\Sigma^3 \sin^2\vartheta}{a_\Sigma f_c - 2m} + \text{const}. \quad (53)$$

Moreover, the fact that the perturbation $A^{(1)}$ is time-independent implies that its knowledge on the matching hypersurface Σ_0 implies its knowledge on the whole region swept by the range of variation of $r_0(\lambda)$. We therefore conclude that any interior perturbation which takes the form (53) on the boundary defines a static perturbation, at least on the range of variation of $r_0(\lambda)$.

Recalling that both f_c and f'_c are constants, in view of the expressions (50) and (53) we have proven:

Theorem VIII.1 *The most general stationary and axially symmetric first order vacuum perturbation of a Schwarzschild metric, matching to an exact FLRW geometry across any linearly perturbed non-null surface, must be static on the range of variation of* $r_0(\lambda)$.

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APPENDIX A: PERTURBED (−) REGION FIRST AND SECOND FUNDAMENTAL FORMS

Using the definitions of subsections IV A, IV B, and IV C in expressions (4) and (5) we find that the first order perturbation of the induced metric on Σ_0^- has the following components:

$$q^{(1)-}{}_{\lambda\lambda} = -2\left[\frac{m}{r_0^2} Z^1 \dot{t}_0^2 + \left(1 - \frac{2m}{r_0}\right)\left(U^{(1)} \dot{t}_0 + \frac{\partial Z^0}{\partial \lambda}\right)\dot{t}_0 - \frac{r_0}{r_0 - 2m} \dot{r}_0 \frac{\partial Z^1}{\partial \lambda} + \frac{m Z^1}{(r_0 - 2m)^2} \dot{r}_0^2 + \frac{r_0}{r_0 - 2m}(U^{(1)} - k^{(1)}) \dot{r}_0^2\right]\Bigg|_{\Sigma_0},$$

$$q^{(1)-}{}_{\lambda\vartheta} = r_0^2 \frac{\partial Z^2}{\partial \lambda} - \left(1 - \frac{2m}{r_0}\right) \frac{\partial Z^0}{\partial \vartheta} \dot{t}_0 + \frac{r_0}{r_0 - 2m} \dot{r}_0 \frac{\partial Z^1}{\partial \vartheta},$$

$$q^{(1)-}{}_{\lambda\varphi} = r_0^2 \sin^2\vartheta \frac{\partial Z^3}{\partial \lambda} - \left(1 - \frac{2m}{r_0}\right)\left(\frac{\partial Z^0}{\partial \varphi} + A^{(1)}|_{\Sigma_0}\right)\dot{t}_0 + \frac{r_0}{r_0 - 2m} \dot{r}_0 \frac{\partial Z^1}{\partial \varphi},$$

$$q^{(1)-}{}_{\vartheta\vartheta} = 2r_0 Z^1 + 2r_0^2\left(\frac{\partial Z^2}{\partial \vartheta} + k^{(1)} - U^{(1)}\right)\Bigg|_{\Sigma_0}, \qquad q^{(1)-}{}_{\vartheta\varphi} = r_0^2\left(\frac{\partial Z^2}{\partial \varphi} + \sin^2\vartheta \frac{\partial Z^3}{\partial \vartheta}\right),$$

$$q^{(1)-}{}_{\varphi\varphi} = 2r_0 Z^1 \sin^2\vartheta + 2r_0^2 \sin\vartheta\left(\sin\vartheta \frac{\partial Z^3}{\partial \varphi} - U^{(1)}|_{\Sigma_0} \sin\vartheta + Z^2 \cos\vartheta\right), \quad (A1)$$

and that the first order perturbation of the extrinsic curvature has the following components:

$$
\begin{aligned}
k^{(1)-}{}_{\lambda\lambda} = \frac{1}{N_-}\Bigg(& -\frac{(r_0-2m)^2}{r_0^2}\dot{t}_0^3\frac{\partial U^{(1)}}{\partial r} + \frac{m(r_0-2m)}{r_0^3}\dot{t}_0^3(k^{(1)}-3U^{(1)}) - \frac{2m(r_0-2m)}{r_0^3}\dot{t}_0^2\frac{\partial Z^0}{\partial\lambda} - \dot{t}_0\frac{\partial^2 Z^1}{\partial\lambda^2} + \dot{t}_0^3\frac{m(2r_0-5m)}{r_0^4}Z^1 \\
& + \dot{t}_0\dot{r}_0^2\Big(3\frac{\partial U^{(1)}}{\partial r} - \frac{\partial k^{(1)}}{\partial r} + \dot{t}_0^2\frac{2m}{N_-^2 r_0^2}k^{(1)}\Big) + \frac{\dot{t}_0\ddot{r}_0}{N_-^2}\Big(\frac{r_0\dot{r}_0^2}{r_0-2m}\frac{\partial Z^0}{\partial\lambda} - \frac{r_0\dot{r}_0}{r_0-2m}\frac{\partial Z^1}{\partial\lambda}\dot{t}_0 - \dot{t}_0^3\Big(1-\frac{2m}{r_0}\Big)k^{(1)}\Big) + \dot{r}_0\frac{\partial^2 Z^0}{\partial\lambda^2} \\
& + \dot{t}_0\ddot{r}_0\frac{(r_0-2m)^2\dot{t}_0^2 + r_0^2\dot{r}_0^2}{(r_0-2m)^2\dot{t}_0^2 - r_0^2\dot{r}_0^2}\Big(U^{(1)} + \frac{mZ^1}{r_0(r_0-2m)}\Big) + \frac{m\dot{r}_0^2}{(r_0-2m)^2\dot{t}_0^2 - r_0^2\dot{r}_0^2}\Big(\dot{t}_0 Z^1\Big(\frac{5m-6r_0}{r_0^2}\dot{t}_0^2 + 3\dot{r}_0^2\frac{2r_0-3m}{(r_0-2m)^2}\Big) \\
& + \frac{\dot{t}_0^2(r_0-2m)^2 - 3r_0^2\dot{r}_0^2}{r_0(r_0-2m)}\frac{\partial Z^0}{\partial\lambda} - \dot{t}_0\frac{(r_0-2m)^2\dot{t}_0^2 + 3r_0^2\dot{r}_0^2}{r_0(r_0-2m)}U^{(1)}\Big) + \frac{m\dot{t}_0\dot{r}_0}{\dot{t}_0^2(r_0-2m)^2 - r_0^2\dot{r}_0^2}\frac{5\dot{t}_0^2(r_0-2m)^2 - 3r_0^2\dot{r}_0^2}{r_0(r_0-2m)}\frac{\partial Z^1}{\partial\lambda}\Bigg)\Bigg|_{\Sigma_0}, \\
k^{(1)-}{}_{\lambda\vartheta} = \frac{1}{N_-}\Bigg(& -\dot{t}_0\frac{\partial^2 Z^1}{\partial\lambda\partial\vartheta} + \dot{t}_0(r_0-2m)\Big(\frac{\partial Z^2}{\partial\lambda} - \frac{m}{r_0^3}\frac{\partial Z^0}{\partial\vartheta}\dot{t}_0\Big) + \dot{t}_0\dot{r}_0\frac{1}{r_0-2m}\frac{\partial Z^1}{\partial\vartheta} + \dot{r}_0\Big(\frac{\partial^2 Z^0}{\partial\lambda\partial\vartheta} + 2\dot{t}_0\frac{\partial U^{(1)}}{\partial\theta} - \dot{t}_0\frac{\partial k^{(1)}}{\partial\theta}\Big) \\
& - \frac{(r_0-3m)\dot{r}_0^2}{r_0(r_0-2m)}\frac{\partial Z^0}{\partial\vartheta}\Bigg)\Bigg|_{\Sigma_0}, \\
k^{(1)-}{}_{\lambda\varphi} = \frac{1}{N_-}\Bigg(& -\dot{t}_0\frac{\partial^2 Z^1}{\partial\lambda\partial\varphi} + (r_0-2m)\Big(\dot{t}_0\sin^2\vartheta\frac{\partial Z^3}{\partial\lambda} - \frac{m}{r_0^3}\dot{t}_0^2\frac{\partial Z^0}{\partial\varphi}\Big) - \frac{(r_0-2m)^2}{2r_0^2}\dot{t}_0^2\frac{\partial A^{(1)}}{\partial r} + \dot{r}_0\Big(\frac{\partial Z^0}{\partial\lambda\partial\varphi} + \frac{1}{r_0-2m}\dot{t}_0\frac{\partial Z^1}{\partial\varphi} \\
& - \frac{(r_0-3m)\dot{r}_0}{r_0(r_0-2m)}\frac{\partial Z^0}{\partial\varphi}\Big) - \frac{m(r_0-2m)}{r_0^3}\dot{t}_0^2 A^{(1)} + \dot{r}_0\Big(\frac{\dot{r}_0}{2}\frac{\partial A^{(1)}}{\partial r} - \frac{(r_0-3m)\dot{r}_0}{r_0(r_0-2m)}A^{(1)}\Big)\Bigg)\Bigg|_{\Sigma_0}, \\
k^{(1)-}{}_{\vartheta\vartheta} = \frac{1}{N_-}\Bigg(& -\dot{t}_0\frac{\partial^2 Z^1}{\partial\vartheta^2} + \dot{t}_0\frac{r_0-m}{r_0}Z^1 + \dot{t}_0(r_0-2m)\Big(2\frac{\partial Z^2}{\partial\vartheta} + r_0\frac{\partial k^{(1)}}{\partial r} - r_0\frac{\partial U^{(1)}}{\partial r} + k^{(1)} - U^{(1)}\Big) + \dot{r}_0\Big(\frac{\partial^2 Z^0}{\partial\vartheta^2}\Big) \\
& + \frac{\dot{t}_0 r_0\dot{r}_0^2}{N_-^2}\Big(-\frac{\partial Z^0}{\partial\lambda}\frac{1}{\dot{t}_0} + k^{(1)} - 2U^{(1)}\Big) - \dot{t}_0\frac{2m\dot{r}_0^2}{N_-^2(r_0-2m)}Z^1 + \frac{\dot{t}_0 r_0\dot{r}_0}{N_-^2}\frac{\partial Z^1}{\partial\lambda}\Bigg)\Bigg|_{\Sigma_0}, \\
k^{(1)-}{}_{\vartheta\varphi} = \frac{1}{N_-}\Bigg(& -\dot{t}_0\frac{\partial^2 Z^1}{\partial\vartheta\partial\varphi} + \dot{t}_0\frac{\cos\vartheta}{\sin\vartheta}\frac{\partial Z^1}{\partial\varphi} + \dot{t}_0(r_0-2m)\Big(\frac{\partial Z^2}{\partial\varphi} + \sin^2\vartheta\frac{\partial Z^3}{\partial\vartheta}\Big) + \dot{r}_0\Big(\frac{\partial Z^0}{\partial\varphi\partial\vartheta} + \frac{1}{2}\frac{\partial A^{(1)}}{\partial\theta} - \frac{\cos\vartheta}{\sin\vartheta}\Big(\frac{\partial Z^0}{\partial\varphi} + A^{(1)}\Big)\Big)\Bigg)\Bigg|_{\Sigma_0}, \\
k^{(1)-}{}_{\varphi\varphi} = \frac{1}{N_-}\dot{t}_0\Bigg(& -\frac{\partial^2 Z^1}{\partial\varphi^2} - \sin\vartheta\cos\vartheta\frac{\partial Z^1}{\partial\vartheta} + \frac{(r_0-m)\sin^2\vartheta}{r_0}Z^1 - \frac{2m\dot{r}_0^2\sin^2\vartheta Z^1}{N_-^2(r_0-2m)} + (r_0-2m)\Big(2\sin^2\vartheta\frac{\partial Z^3}{\partial\varphi} + 2\cos\vartheta\sin\vartheta Z^2 - \sin^2\vartheta \\
& \times\Big(r_0\frac{\partial U^{(1)}}{\partial r} + U^{(1)} + k^{(1)}\Big)\Big)\Big) + \frac{r_0\dot{r}_0}{N_-^2}\sin^2\vartheta\frac{\partial Z^1}{\partial\lambda} + \frac{r_0\dot{r}_0^2}{N_-^2}\sin^2\vartheta\Big(k^{(1)} - 2U^{(1)} - \frac{1}{\dot{t}_0}\frac{\partial Z^0}{\partial\lambda}\Big) + \frac{\dot{r}_0}{\dot{t}_0}\Big(\frac{\partial^2 Z^0}{\partial\varphi^2} + \cos\vartheta\sin\vartheta\frac{\partial Z^0}{\partial\vartheta}\Big)\Bigg)\Bigg|_{\Sigma_0}.
\end{aligned}
\tag{A2}
$$

APPENDIX B: PERTURBED (+)–REGION FIRST AND SECOND FUNDAMENTAL FORMS

Using the definitions of subsections V A, V B, and V C in expressions (4) and (5) we find that the first order perturbation of the induced metric on Σ_0^+ has the following components:

$$
\begin{aligned}
& q^{(1)+}{}_{\lambda\lambda} = g^{(1)+}{}_{\tau\tau}|_{\Sigma_0} - 2a_\Sigma^2\frac{\partial T^+}{\partial\lambda} - 2a_\Sigma\dot{a}_\Sigma T^+, \qquad q^{(1)+}{}_{\lambda\vartheta} = g^{(1)+}{}_{\tau\theta}|_{\Sigma_0} - a_\Sigma^2\frac{\partial T^+}{\partial\vartheta} + a_\Sigma^2 f_c^2\Big(\frac{\partial^2\mathcal{T}_1^+}{\partial\lambda\partial\vartheta} - \frac{1}{\sin\vartheta}\frac{\partial^2\mathcal{T}_2^+}{\partial\lambda\partial\varphi}\Big), \\
& q^{(1)+}{}_{\lambda\varphi} = g^{(1)+}{}_{\tau\phi}|_{\Sigma_0} - a_\Sigma^2\frac{\partial T^+}{\partial\varphi} + a_\Sigma^2 f_c^2\Big(\frac{\partial^2\mathcal{T}_1^+}{\partial\lambda\partial\varphi} + \sin\vartheta\frac{\partial^2\mathcal{T}_2^+}{\partial\lambda\partial\vartheta}\Big), \\
& q^{(1)+}{}_{\vartheta\vartheta} = g^{(1)+}{}_{\theta\theta}|_{\Sigma_0} + 2a_\Sigma f_c f_c' Q^+ + 2a_\Sigma^2 f_c^2\Big(\frac{\dot{a}_\Sigma}{a_\Sigma}T^+ + \frac{\partial^2\mathcal{T}_1^+}{\partial\vartheta^2} - \frac{1}{\sin\vartheta}\frac{\partial^2\mathcal{T}_2^+}{\partial\vartheta\partial\varphi} + \frac{\cos\vartheta}{\sin^2\vartheta}\frac{\partial\mathcal{T}_2^+}{\partial\varphi}\Big), \\
& q^{(1)+}{}_{\vartheta\varphi} = g^{(1)+}{}_{\theta\phi}|_{\Sigma_0} + a_\Sigma^2 f_c^2\sin\vartheta\Big(\frac{\partial^2\mathcal{T}_2^+}{\partial\vartheta^2} - \frac{1}{\sin^2\vartheta}\frac{\partial^2\mathcal{T}_2^+}{\partial\varphi^2} - \frac{\cos\vartheta}{\sin\vartheta}\frac{\partial\mathcal{T}_2^+}{\partial\vartheta} + \frac{2}{\sin\vartheta}\frac{\partial^2\mathcal{T}_1^+}{\partial\vartheta\partial\varphi} - \frac{2\cos\vartheta}{\sin^2\vartheta}\frac{\partial\mathcal{T}_1^+}{\partial\varphi}\Big), \\
& q^{(1)+}{}_{\varphi\varphi} = g^{(1)+}_{\phi\phi}|_{\Sigma_0} + 2a_\Sigma f_c f_c' Q^+\sin^2\vartheta + 2a_\Sigma f_c^2\dot{a}_\Sigma T^+\sin^2\vartheta \\
& \qquad + 2a_\Sigma^2 f_c^2\Big(\frac{\partial^2\mathcal{T}_1^+}{\partial\varphi^2} + \cos\vartheta\sin\vartheta\frac{\partial\mathcal{T}_1^+}{\partial\vartheta} + \sin\vartheta\frac{\partial^2\mathcal{T}_2^+}{\partial\vartheta\partial\varphi} - \cos\vartheta\frac{\partial\mathcal{T}_2^+}{\partial\varphi}\Big),
\end{aligned}
\tag{B1}
$$

and that the first order perturbation of the extrinsic curvature has the following components:

$$k^{(1)+}{}_{\lambda\lambda} = -\frac{1}{a_\Sigma}\frac{\partial g^{(1)+}{}_{\tau R}}{\partial\tau} + \frac{\dot a_\Sigma}{a_\Sigma^2}g^{(1)+}{}_{\tau R} + \frac{1}{2a_\Sigma}\frac{\partial g^{(1)+}{}_{\tau\tau}}{\partial R} - \frac{\partial^2 Q^+}{\partial\lambda^2} + \frac{\dot a_\Sigma}{a_\Sigma}\frac{\partial Q^+}{\partial\lambda} + \left(\frac{\ddot a_\Sigma}{a_\Sigma} - \frac{\dot a_\Sigma^2}{a_\Sigma^2}\right)Q^+ \Big|_{\Sigma_0},$$

$$k^{(1)+}{}_{\lambda\vartheta} = \frac{1}{2a_\Sigma}\frac{g^{(1)+}{}_{\tau\theta}}{\partial R} - \frac{1}{2a_\Sigma}\frac{g^{(1)+}{}_{\tau R}}{\partial\theta} + \frac{\dot a_\Sigma}{a_\Sigma^2}g^{(1)+}{}_{R\theta} - \frac{1}{2a_\Sigma}\frac{\partial g^{(1)+}{}_{R\theta}}{\partial\tau} - \frac{\partial^2 Q^+}{\partial\lambda\partial\vartheta} + \frac{\dot a_\Sigma}{a_\Sigma}\frac{\partial Q^+}{\partial\vartheta} + a_\Sigma f_c f_c'\left(\frac{\partial^2\mathcal{T}_1^+}{\partial\lambda\partial\vartheta} - \frac{1}{\sin\vartheta}\frac{\partial^2\mathcal{T}_2^+}{\partial\lambda\partial\varphi}\right)\Big|_{\Sigma_0},$$

$$k^{(1)+}{}_{\lambda\varphi} = \frac{1}{2a_\Sigma}\frac{g^{(1)+}{}_{\tau\phi}}{\partial R} - \frac{1}{2a_\Sigma}\frac{g^{(1)+}{}_{\tau R}}{\partial\phi} + \frac{\dot a_\Sigma}{a_\Sigma^2}g^{(1)+}{}_{R\phi} - \frac{1}{2a_\Sigma}\frac{\partial g^{(1)+}{}_{R\phi}}{\partial\tau} - \frac{\partial^2 Q^+}{\partial\lambda\partial\varphi} + \frac{\dot a_\Sigma}{a_\Sigma}\frac{\partial Q^+}{\partial\varphi} + a_\Sigma f_c f_c'\left(\frac{\partial^2\mathcal{T}_1^+}{\partial\lambda\partial\varphi} + \sin\vartheta\frac{\partial^2\mathcal{T}_2^+}{\partial\lambda\partial\vartheta}\right)\Big|_{\Sigma_0},$$

$$k^{(1)+}{}_{\vartheta\vartheta} = -\frac{f_c f_c'}{2a_\Sigma}g^{(1)+}{}_{RR} + \frac{\dot a_\Sigma f_c^2}{a_\Sigma^2}g^{(1)+}{}_{\tau R} + \frac{1}{2a_\Sigma}\frac{\partial g^{(1)+}{}_{\theta\theta}}{\partial R} - \frac{1}{a_\Sigma}\frac{\partial g^{(1)+}{}_{R\theta}}{\partial\theta}$$
$$+ f_c f_c' a_\Sigma\left(\frac{\dot a_\Sigma}{a_\Sigma}T^+ + 2\frac{\partial^2\mathcal{T}_1^+}{\partial\vartheta^2} - \frac{2}{\sin\vartheta}\frac{\partial^2\mathcal{T}_2^+}{\partial\vartheta\partial\varphi} + \frac{2\cos\vartheta}{\sin^2\vartheta}\frac{\partial\mathcal{T}_2^+}{\partial\varphi}\right) - \frac{\partial^2 Q^+}{\partial\vartheta^2} + \frac{\dot a_\Sigma f_c^2}{a_\Sigma}\frac{\partial Q^+}{\partial\lambda} + f_c^2 Q^+\left(-\epsilon + \frac{f_c'^2}{f_c^2} - \frac{\dot a_\Sigma^2}{a_\Sigma^2}\right)\Big|_{\Sigma_0},$$

$$k^{(1)+}{}_{\vartheta\varphi} = \frac{1}{2a_\Sigma}\left(\frac{\partial g^{(1)+}{}_{\theta\phi}}{\partial R} - \frac{\partial g^{(1)+}{}_{R\theta}}{\partial\phi} - \frac{\partial g^{(1)+}{}_{R\phi}}{\partial\theta}\right) + \frac{\cos\vartheta}{a_\Sigma\sin\vartheta}g^{(1)+}{}_{R\phi} - \frac{\partial^2 Q^+}{\partial\vartheta\partial\varphi} + \frac{\cos\vartheta}{\sin\vartheta}\frac{\partial Q^+}{\partial\varphi}$$
$$+ f_c f_c' a_\Sigma\left(2\frac{\partial^2\mathcal{T}_1^+}{\partial\vartheta\partial\varphi} - 2\frac{\cos\vartheta}{\sin\vartheta}\frac{\partial\mathcal{T}_1^+}{\partial\varphi} + \sin\vartheta\frac{\partial^2\mathcal{T}_2^+}{\partial\vartheta^2} - \cos\vartheta\frac{\partial\mathcal{T}_2^+}{\partial\vartheta} - \frac{1}{\sin\vartheta}\frac{\partial^2\mathcal{T}_2^+}{\partial\varphi^2}\right)\Big|_{\Sigma_0},$$

$$k^{(1)+}{}_{\varphi\varphi} = -\frac{\sin^2\vartheta f_c f_c'}{2a_\Sigma}g^{(1)+}{}_{RR} + \frac{\dot a_\Sigma\sin^2\vartheta f_c^2}{a_\Sigma^2}g^{(1)+}{}_{\tau R} + \frac{1}{2a_\Sigma}\frac{\partial g^{(1)+}{}_{\phi\phi}}{\partial R} - \frac{1}{a_\Sigma}\frac{\partial g^{(1)+}{}_{R\phi}}{\partial\phi} - \frac{\cos\vartheta\sin\vartheta}{a_\Sigma}g^{(1)+}{}_{R\theta} - \frac{\partial^2 Q^+}{\partial\varphi^2}$$
$$- \frac{\cos\vartheta}{\sin\vartheta}\frac{\partial Q^+}{\partial\vartheta} + \sin^2\vartheta\frac{\dot a_\Sigma f_c^2}{a_\Sigma}\frac{\partial Q^+}{\partial\lambda} + \sin^2\vartheta f_c^2 Q^+\left(-\epsilon + \frac{f_c'^2}{f_c^2} - \frac{\dot a_\Sigma^2}{a_\Sigma^2}\right)$$
$$+ \sin^2\vartheta f_c f_c' a_\Sigma\left(\frac{\dot a_\Sigma}{a_\Sigma}T^+ + \frac{2}{\sin^2\vartheta}\frac{\partial^2\mathcal{T}_1^+}{\partial\varphi^2} + \frac{2\cos\vartheta}{\sin\vartheta}\frac{\partial\mathcal{T}_1^+}{\partial\vartheta} + \frac{2}{\sin\vartheta}\left(\frac{\partial^2\mathcal{T}_2^+}{\partial\vartheta\partial\varphi} - \frac{\cos\vartheta}{\sin\vartheta}\frac{\partial\mathcal{T}_2^+}{\partial\varphi}\right)\right)\Big|_{\Sigma_0}. \quad \text{(B2)}$$

APPENDIX C: LINEARIZED MATCHING IN POISSON GAUGE FOR $\epsilon = 0$

Here, as an example, we shall write the linearized matching conditions for the particular case of a flat $\epsilon = 0$ FLRW region in the Poisson spacetime gauge in FLRW, for which $W = \chi = Y_a = 0$.

The odd part equations are given by

$$\mathcal{T}_2^+ + f_c^{-2}[\mathcal{U}_2 - (R_0^q + R_m^q Y_1^m)] \overset{\Sigma_0}{=} \mathcal{T}_2^-, \quad \text{(C1)}$$

$$\mathcal{W}_2 - G_0^q - \frac{d}{d\lambda}[\mathcal{U}_2 - (R_0^q + R_m^q Y_1^m)|_{\Sigma_0}] \overset{\Sigma_0}{=} -\sigma\mathcal{G}a_\Sigma^{-1}, \quad \text{(C2)}$$

$$\mathcal{W}_2' - 2a_\Sigma G_0^\kappa - \frac{d}{d\lambda}[\mathcal{U}_2' - 2a_\Sigma(R_0^\kappa + R_m^\kappa Y_1^m)|_{\Sigma_0}] \overset{\Sigma_0}{=} -\sigma\mathcal{G}\frac{3m}{f_c^2 a_\Sigma^2} - \sigma\frac{\partial\mathcal{G}}{\partial r}, \quad \text{(C3)}$$

$$\mathcal{Q}_2 - [\mathcal{U}_2' - 2a_\Sigma(R_0^\kappa + R_m^\kappa Y_1^m)] + 2f_c^{-1}[\mathcal{U}_2 - (R_0^q + R_m^q Y_1^m)] \overset{\Sigma_0}{=} -\sigma\mathcal{G}a_\Sigma^{-2}\dot a_\Sigma f_c. \quad \text{(C4)}$$

The even part equations read

$$\mathcal{T}_1^+ + f_c^{-2}[\mathcal{U}_1 - (P_0^q + P_m^q Y_1^m)] \overset{\Sigma_0}{=} \mathcal{T}_1^- + \mathcal{P}, \quad \text{(C5)}$$

$$T^+ - \left[\mathcal{W}_1 - F_0^q - \frac{d}{d\lambda}[\mathcal{U}_1 - (P_0^q + P_m^q Y_1^m)|_{\Sigma_0}]\right] \overset{\Sigma_0}{=} f_c^3 a_\Sigma\frac{\partial\mathcal{P}}{\partial r} + T^-, \quad \text{(C6)}$$

$$\mathcal{Q}_1 + Y_R - [\mathcal{U}_1' - 2a_\Sigma(R_0^\kappa + R_m^\kappa Y_1^m)] + 2f_c^{-1}[\mathcal{U}_1 - (P_0^q + P_m^q Y_1^m)] + \frac{\delta Q}{a_\Sigma} \overset{\Sigma_0}{=} -f_c^2 a_\Sigma\frac{\partial\mathcal{P}}{\partial r}, \quad \text{(C7)}$$

$$\Psi + \frac{1}{a_\Sigma}\frac{d}{d\lambda}\left[a_\Sigma\left(\mathcal{W}_1 - F_0^q - \frac{d}{d\lambda}[\mathcal{U}_1 - (P_0^q + P_m^q Y_1^m)|_{\Sigma_0}]\right)\Big|_{\Sigma_0}\right] \overset{\Sigma_0}{=} \frac{a_\Sigma f_c + 2m}{a_\Sigma f_c - 2m}U^{(1)} - 3m\frac{\partial\mathcal{P}}{\partial r} - 2m\frac{\partial^2\mathcal{P}}{\partial r^2} - \frac{2m}{a_\Sigma f_c - 2m}\Delta_\theta\mathcal{P}, \quad \text{(C8)}$$

$$\mathcal{W}_1' - \tilde{W}_R - 2a_\Sigma F_0^\kappa - \frac{d}{d\lambda}[\mathcal{U}_1' - 2a_\Sigma(R_0^\kappa + R_m^\kappa Y_1^m)|_{\Sigma_0}] - \frac{d}{d\lambda}\left(\frac{\delta Q}{a_\Sigma}\right) \overset{\Sigma_0}{=} \frac{2\dot{a}_\Sigma f_c^2}{a_\Sigma f_c - 2m}(2U^{(1)} - \Delta_\theta \mathcal{P}) - 3\dot{a}_\Sigma f_c^2 \frac{\partial \mathcal{P}}{\partial r} - \dot{a}_\Sigma a_\Sigma f_c^3 \frac{\partial^2 \mathcal{P}}{\partial r^2}, \tag{C9}$$

$$\Psi' + \frac{1}{a_\Sigma}\frac{d}{d\lambda}\left(a_\Sigma\left[\tilde{W}_R + \frac{d}{d\lambda}\left(\frac{\delta Q}{a_\Sigma}\right)\right]\Big|_{\Sigma_0}\right) \overset{\Sigma_0}{=} -\frac{3m}{a_\Sigma^2 f_c^3}Q^- + \frac{a_\Sigma f_c - 6m}{a_\Sigma f_c - 2m}a_\Sigma \frac{\partial U^{(1)}}{\partial r} + \frac{2m}{a_\Sigma f_c - 2m} \times a_\Sigma \frac{\partial}{\partial r}\Delta_\theta \mathcal{P} + \frac{2a_\Sigma}{(a_\Sigma f_c - 2m)^2}(2U^{(1)} - \Delta_\theta \mathcal{P}), \tag{C10}$$

$$\Phi' + \frac{1}{f_c}\Phi - \frac{\dot{a}_\Sigma}{a_\Sigma}\tilde{W}_R - \frac{\dot{a}_\Sigma}{a_\Sigma f_c}\left(\mathcal{W}_1 - F_0^q - \frac{d}{d\lambda}[\mathcal{U}_1 - (P_0^q + P_m^q Y_1^m)|_{\Sigma_0}]\right) - \frac{1}{f_c^3}[\mathcal{H} + \Delta_{S^2}\mathcal{U}_1] - \frac{1}{2f_c^2}[\mathcal{H}' - 4a_\Sigma P_m^\kappa Y_1^m] - Y_R f_c^{-2} \overset{\Sigma_0}{=} \frac{1}{f_c^2}\frac{\delta Q}{a_\Sigma} + \frac{2m}{\dot{a}_\Sigma f_c^3}\frac{d}{d\lambda}\left(\frac{\delta Q}{a_\Sigma}\right) - \frac{3m}{a_\Sigma^2 f_c^3}Q^- + \frac{a_\Sigma f_c + 2m}{f_c(a_\Sigma f_c - 2m)}U^{(1)} + a_\Sigma \frac{\partial U^{(1)}}{\partial r} - \frac{1}{f_c}2m\frac{\partial \mathcal{P}}{\partial r} + \frac{a_\Sigma f_c - 4m}{f_c(a_\Sigma f_c - 2m)}\Delta_\theta \mathcal{P} + \frac{1}{f_c}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\left(2\mathcal{P} + a_\Sigma f_c \frac{\partial \mathcal{P}}{\partial r}\right), \tag{C11}$$

$$\frac{\delta Q}{a_\Sigma} + \frac{\dot{a}_\Sigma f_c}{a_\Sigma}\left(\mathcal{W}_1 - F_0^q - \frac{d}{d\lambda}[\mathcal{U}_1 - (P_0^q + P_m^q Y_1^m)|_{\Sigma_0}]\right) + \frac{1}{2f_c}(\mathcal{H} - 2P_m^q Y_1^m) + Y_R - f_c\Phi \overset{\Sigma_0}{=} f_c\left(-2m\frac{\partial \mathcal{P}}{\partial r} - \frac{\cos\theta}{\sin\theta}\frac{\partial \mathcal{P}}{\partial\theta} - U^{(1)}\right). \tag{C12}$$

APPENDIX D: IDENTIFICATION WITH MUKOHYAMA'S PERTURBATION VARIABLES

The variables N, $\mathcal{K}$, and $\bar{\mathcal{K}}$ here (see subsection III B) have been chosen so that they correspond to those of [23,25], while the function r there corresponds to the Schwarzschild radius here, and takes the value $a_\Sigma f_c$ on the FLRW side of Σ_0. In the FLRW region, the S^2 scalar variables used here are related to the variables in [23,25] in the following way:

$$\sum_{l=0}^{\infty} h_{00}^+ Y = -2a^2\Psi \qquad \sum_{l=0}^{\infty} h_{01}^+ Y = a^2(\tilde{W}_R + W) \qquad \sum_{l=0}^{\infty} h_{11}^+ Y = a^2(-2\Phi + \chi)$$

$$\sum_{l=0}^{\infty} h_{(Y)}^+ Y = a^2\left[-2\Phi f^2 + \frac{1}{3}ff'\chi' + \frac{1}{6}\Delta_{S^2}\chi - \frac{1}{3}f^2\chi'' + 2ff'Y_R + \Delta_{S^2}(\mathcal{U}_1 + \mathcal{Y}_1) + \mathcal{H}\right]$$

$$\sum_{l=1}^{\infty} h_{(L)_0}^+ Y = a^2(W + \mathcal{W}_1)|_{l\geq 1} \qquad \sum_{l=1}^{\infty} h_{(T)_0}^+ Y = a^2\mathcal{W}_2|_{l\geq 1} \qquad \sum_{l=1}^{\infty} h_{(T)_1}^+ Y = a^2\left(\mathcal{Q}_2 + \mathcal{Y}_2' - 2\frac{f'}{f}\mathcal{Y}_2\right)\Big|_{l\geq 1}$$

$$\sum_{l=1}^{\infty} h_{(L)_1}^+ Y = a^2\left(\chi' + \mathcal{Q}_1 + Y_R + \mathcal{Y}_1' - 2\frac{f'}{f}\mathcal{Y}_1 - \frac{f'}{f}\chi\right)\Big|_{l\geq 1} \qquad \sum_{l=2}^{\infty} h_{(LT)}^+ Y = a^2(\mathcal{U}_2 + \mathcal{Y}_2)|_{l\geq 2}$$

$$\sum_{l=2}^{\infty} h_{(LL)}^+ Y = a^2\left(\mathcal{U}_1 + \mathcal{Y}_1 + \frac{1}{2}\chi\right)\Big|_{l\geq 2},$$

where Y stands for the corresponding spherical harmonic for any given pair (l, m), and where the sum over the values of m is to be understood. Also note that, as explicitly indicated, the expressions on the right-hand side must be restricted to the corresponding values of l.

At either $(\pm)$ side on Σ_0, the decomposition of the first fundamental form perturbation tensor $q^{(1)}$ in [23,25], is related to the Hodge scalars decomposition used here by

$$\sum_{l=0}^{\infty} \sigma_{00} Y = q^{(1)}{}_{\lambda\lambda} \qquad \sum_{l=1}^{\infty} \sigma_{(T)0} Y = G^q|_{l\geq 1}$$
$$\sum_{l=1}^{\infty} \sigma_{(L)0} Y = F^q|_{l\geq 1} \qquad \sum_{l=2}^{\infty} \sigma_{(LT)} Y = R^q|_{l\geq 2}$$
$$\sum_{l=0}^{\infty} \sigma_{(Y)} Y = H^q + \Delta_{S^2}P^q \qquad \sum_{l=2}^{\infty} \sigma_{(LL)} Y = P^q|_{l\geq 2}. \tag{D1}$$

Analogously, the relations regarding the second fundamental form perturbation $k^{(1)}$ follow from the above replacing σ by κ on the left-hand side, and q by k on the superscripts on the right-hand side quantities.

Finally, the perturbation of the vector $\vec{Z}$ at either side is represented in [25] by

$$\sum_{l=0}^{\infty} \tilde{Q}\, Y = Q, \qquad \sum_{l=0}^{\infty} z_{\lambda} Y = N^2 T,$$
$$\sum_{l=1}^{\infty} z_{(T)} Y = \mathcal{T}_1|_{l\geq 1}, \qquad \sum_{l=1}^{\infty} z_{(L)} Y = \mathcal{T}_2|_{l\geq 1},$$

where $\tilde{Q}$ stands for the Q used in [25].

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