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Nonrotating black hole in a post-Newtonian tidal environment

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We examine the motion and tidal dynamics of a nonrotating black hole placed within a post-Newtonian external spacetime. The black hole's gravity is described accurately to all orders in Gm/c^2r, where m is the black-hole mass and r is the distance to the black hole. The tidal perturbation created by the external environment is treated as a small perturbation. At a large distance from the black hole, the gravitational field of the external distribution of matter is assumed to be sufficiently weak to be adequately described by the (first) post-Newtonian approximation to general relativity. There, the black hole is treated as a monopole contribution to the total gravitational field. There exists an overlap in the domains of validity of each description, and the black-hole and post-Newtonian metrics are matched in the overlap. The matching procedure produces (i) a justification of the statement that a nonrotating black hole is a post-Newtonian monopole; (ii) a complete characterization of the coordinate transformation between the inertial, barycentric frame and the accelerated, black-hole frame; (iii) the equations of motion for the black hole; and (iv) the gravito-electric and gravito-magnetic tidal fields acting on the black hole. We first calculate the equations of motion and tidal fields by making no assumptions regarding the nature of the post-Newtonian environment; this could contain a continuous distribution of matter (so as to model a galactic core) or any number of condensed bodies. We next specialize our discussion to a situation in which the black hole is a member of a post-Newtonian two-body system. As an application of our results, we examine the geometry of the deformed event horizon and calculate the tidal heating of the black hole, the rate at which it acquires mass as a result of its tidal interaction with the companion body.

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I. INTRODUCTION AND SUMMARY

A. This work and its context

How does a black hole move in an external spacetime, and what effects do the tidal fields created in the external spacetime have on the black hole? These are the questions that are investigated in this work, in a context in which the black hole is nonrotating and the gravity of the external universe is sufficiently weak to be adequately described by the post-Newtonian approximation to general relativity. This work is a continuation of a line of inquiry that was initiated by Manasse [1] in the early nineteen sixties, and that has been pursued to the present day.

To pose our questions more precisely, and to better discuss the place of this work in the context of what was achieved previously, we introduce two length scales that are relevant to this problem. The first is set by m, the mass of the black hole, which gives rise to an associated length scale $M := Gm/c^2$, the gravitational radius of the black hole. The second is $\mathcal{R}$, the radius of curvature of the external spacetime, evaluated at the black hole's position. Our work, and all others that preceded it, is carried out in a context in which $M/\mathcal{R} \ll 1$, so that there is a clean separation between these scales. Only in this context can one meaningfully speak of a black hole moving in an external spacetime; when M is comparable to $\mathcal{R}$, no distinction can be made between the "black hole" and the "external spacetime."

As a concrete example we may consider a situation in which the black hole is a member of a binary system. Then $\mathcal{R} \sim \sqrt{b^3/M_{\rm tot}}$, where b is the separation between the bodies and $M_{\rm tot} := G(m + m')/c^2$ is a measure of the total mass within the system (m' is the external mass). In this case we have

$$\frac{M}{\mathcal{R}} \sim \frac{M}{M_{\rm tot}}\left(\frac{M_{\rm tot}}{b}\right)^{3/2},$$

and for this work, this is required to be small. There are two particular ways to achieve this. In the *small-hole approximation* the black-hole mass is assumed to be much smaller than the external mass, so that $M/M_{\rm tot} \sim m/m' \ll 1$; then $M/\mathcal{R}$ is small irrespective of the size of $M_{\rm tot}/b$, and the binary system can be strongly relativistic. In the *weak-field approximation* it is $M_{\rm tot}/b$ that is assumed to be small, while the mass ratio is left unconstrained.

Our work is concerned with the weak-field approximation. The black hole is placed within a post-Newtonian external spacetime, and the external gravitational potentials determine its motion as well as the tidal gravity acting upon it. We determine the motion of the black hole, the tidal fields, and the effects of the tidal fields on the structure of spacetime around the black hole, all within the post-Newtonian approximation to general relativity. At first, we do not specify the nature of the post-Newtonian environment. We leave it completely general; the black hole might be immersed within a smooth distribution of matter (a model for a galactic core, for example), or it might be part of an N-body system (with the number, nature, and state of motion of the bodies left arbitrary). As our work progresses, we specialize our results to a two-body system undergoing generic orbital motion, and finally we examine the special case of circular orbits.

The motion of a black hole in an arbitrary external spacetime was first investigated by D'Eath [2–4] and Kates [5], who showed that in the limit $M/\mathcal{R} \to 0$, the black hole moves on a geodesic of the external spacetime. In this limit the black hole behaves as a test mass, in spite of the fact that the self-gravity of the black hole never ceases to be strong. The corrections to geodesic motion produced by the coupling of the black-hole spin with the curvature of the external spacetime were worked out by Thorne and Hartle [6], who also obtained precession equations for the spin vector. These authors exploited the power of matched asymptotic expansions in their derivation of the equations of motion. In their approach, the metric of the black hole (deformed by the conditions in the external spacetime) is matched to the metric of the external spacetime (perturbed by the moving black hole). The matching is carried out in a region in which both descriptions are valid, and it produces both the equations of motion and the tidal fields, with only the Einstein field equations as additional input. Our work is a continuation of this program.

These investigations were next specialized to systems for which the gravity of the external spacetime is weak; this is the context that interests us in this paper. Demianski and Grishchuk [7] showed that to the leading order in a post-Newtonian expansion of the external gravity, the black hole moves according to the Newtonian equations of motion. Their results were generalized to the first post-Newtonian order by D'Eath [3] and Damour [8], who found agreement between the equations of motion for black holes in binary systems and the standard (Einstein-Infeld-Hoffman) equations of motion of post-Newtonian theory. Our work is a continuation of this effort, and our results are slightly more general than theirs: While our black hole is still immersed within a post-Newtonian environment, this environment is completely general, and the black hole is not required to be a member of a binary system. When, however, we specialize our results to this particular case, we recover the results of D'Eath and Damour.

The motion of a black hole in a post-Newtonian external spacetime is well understood, and our contribution to this understanding is a relatively minor one. The same cannot be said, however, of the effects of the external tidal gravity on the black hole, which have not been much discussed in the literature. This is the true focus of this work, and our main goal in this paper is to calculate the post-Newtonian tidal fields acting on the black hole, and to explore the physical consequences of the tidal interaction.

We are not claiming that ours is the first calculation of post-Newtonian tidal fields acting on a self-gravitating body. It is not. In their pioneering work on relativistic celestial mechanics, Damour, Soffel, and Xu [9–11] calculated the post-Newtonian tidal fields acting on an arbitrarily-structured body with weak internal gravity. This work was recently generalized to arbitrarily-structured, strongly-gravitating bodies by Racine and Flanagan [12]. Our work is concerned instead with a very specific type of strongly self-gravitating body: a nonrotating black hole. We calculate the post-Newtonian tidal fields acting on this black hole, and observe that they are the same as those obtained by Damour, Soffel, and Xu in the case of weakly self-gravitating monopoles. We confirm, therefore, the general expectation (known as the "effacement principle") that the post-Newtonian tidal fields must depend on the body's multipole moments only (in addition to the conditions in the external spacetime), and not on additional details concerning its internal structure.

The effects of tidal fields on the structure of spacetime around a black hole were first investigated by Manasse [1], who provided an essential input to the work later carried out by D'Eath, Kates, Thorne, and Hartle. Adopting the small-hole approximation defined previously, Manasse calculated the metric around a small black hole that falls radially toward a much larger black hole. Each black hole was taken to be nonrotating, and the small hole was taken to move on a geodesic of the (unperturbed) Schwarzschild spacetime of the large hole. The tidal gravity exerted by the large black hole creates a perturbation in the Schwarzschild metric of the small hole, and employing the techniques of Regge and Wheeler [13], Manasse was able to calculate this perturbation in a local neighborhood of the black hole. His metric is expressed as an expansion in powers of $r/\mathcal{R}$, where r is the distance to the small hole; it is accurate through the second order in $r/\mathcal{R}$, and it is valid to all orders in M/r, which measures the strength of the small hole's self-gravity. The case of circular motion around a large Schwarzschild black hole was treated much later by Poisson [14].

The methods used by Manasse are not necessarily restricted to the small-hole approximation. Alvi [15,16] realized that these methods could be seamlessly extended to the general setting defined by the requirement $M/\mathcal{R} \ll 1$, which includes both the small-hole and weak-field approximations as special cases. Alvi exploited this insight to calculate the tidal fields acting on a black hole in a post-Newtonian binary system (perhaps with another black hole). In Alvi's work, the two bodies have comparable masses and the black hole has a significant influence on the geometry of the external spacetime. Alvi calculated the tidal fields to the leading (Newtonian) order in the post-Newtonian approximation to general relativity, for circular orbits. The metric of the distorted black hole was next joined to the post-Newtonian two-body metric, and the global metric was presented in a single coordinate system

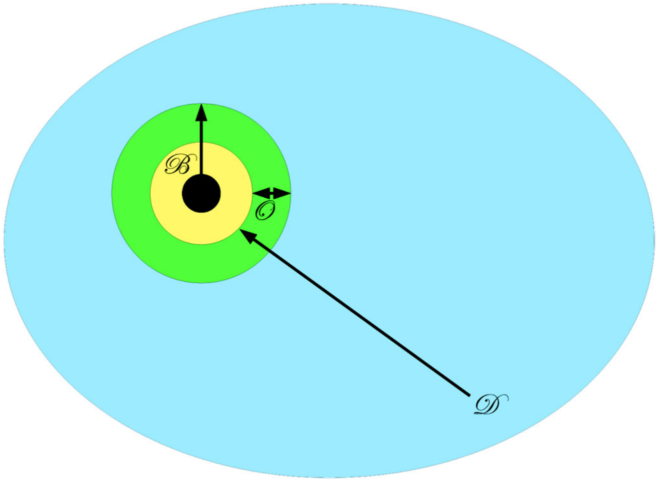

FIG. 1 (color online). The post-Newtonian domain $\mathcal{D}$, the black-hole neighborhood $\mathcal{B}$, and the overlap region $\mathcal{O}$. The post-Newtonian domain is depicted in blue (light gray), and it includes the green (dark gray) annulus that surrounds the black hole. The black-hole neighborhood is drawn as a yellow (white) disk around the black hole, and it also includes the green (dark gray) annulus. The black hole is represented as a black disk. The overlap region is the union of $\mathcal{B}$ and $\mathcal{D}$ that is shown in green (dark gray), the union of the blue and yellow colors.

that corotates with the system. (In Alvi's original work there is a discontinuity in the metric at the common boundary between the two descriptions. The joint was made continuous in a follow-up paper by Yunes *et al.* [17]).

Alvi's insight was exploited by Poisson [18] in a calculation of the metric of a nonrotating black hole placed within an arbitrary tidal environment (still restricted by $M/\mathcal{R} \ll 1$). Working, like Manasse [1], in a local neighborhood of the black hole, Poisson was able to calculate the metric through the third order in $r/\mathcal{R}$, while keeping the expressions accurate to all orders in M/r. Poisson's metric is parametrized by a number of *tidal moments*, freely-specifiable tensorial functions of time that characterize the black hole's tidal environment. This metric gives a general description of the spacetime around a black hole in *any* tidal environment, but a more complete description requires the determination of the tidal moments. This is our task in this paper: We calculate the tidal moments for the specific case described above, in which the black hole is immersed within a post-Newtonian external spacetime. This is a generalization of Alvi's work [15–17]; we appeal to the weak-field approximation, we calculate the tidal fields created by an arbitrary post-Newtonian spacetime, and we do so to a higher order of accuracy than what was achieved by Alvi.

B. Our results

The metric of a nonrotating black hole immersed in a tidal environment is expressed as a perturbation of the Schwarzschild metric. We take the black hole to have a mass m when it is in complete isolation (unperturbed), and we denote its gravitational radius by $M := Gm/c^2$. The strength of the tidal perturbation is measured by the inverse length scale $\mathcal{R}^{-1}$, and we assume that $M/\mathcal{R} \ll 1$; the tidal perturbation is weak. In addition, we assume that the black hole moves in an empty region of spacetime, so that in the hole's neighborhood $\mathcal{B}$, the perturbed metric satisfies the vacuum field equations linearized about the exact Schwarzschild solution. We present the metric in the comoving reference frame of the black hole, in a quasi-Cartesian system of coordinates $\bar{x}^\alpha = (\bar{x}^0, \bar{x}^a) = (c\bar{t}, \bar{x}, \bar{y}, \bar{z})$ that enforce the harmonic conditions $\partial_\beta(\sqrt{-g}g^{\alpha\beta}) = 0$. Our convention is that Greek indices run from 0 to 3, while Latin indices cover the spatial coordinates and run from 1 to 3. We raise and lower Latin indices with the Euclidean metric δ_{ab}, and we let ϵ_{abc} denote the permutation symbol of ordinary vector calculus (with $\epsilon_{123} = \epsilon_{xyz} = 1$).

The time-time and time-space components of the black-hole metric are (Sec. III)

$$g_{\bar{0}\bar{0}} = -\frac{1 - M/\bar{r}}{1 + M/\bar{r}} - \frac{1}{c^2}(1 - M/\bar{r})^2 \bar{\mathcal{E}}_{ab}(\bar{t})\bar{x}^a\bar{x}^b + O(\bar{r}^3/\mathcal{R}^3), \tag{1.1}$$

$$g_{\bar{0}\bar{a}} = \frac{2}{3c^3}(1 - M/\bar{r})(1 + M/\bar{r})^2 \epsilon_{abp}\bar{\mathcal{B}}^p{}_c(\bar{t})\bar{x}^b\bar{x}^c + O(\bar{r}^3/\mathcal{R}^3); \tag{1.2}$$

for our purposes here we shall not need an expression for $g_{\bar{a}\bar{b}}$, the space-space components of the metric. The metric is expressed as an expansion in powers of $\bar{r}/\mathcal{R}$, the ratio of $\bar{r} := \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}$, the distance from the black hole, to $\mathcal{R}$, the local radius of curvature of the external spacetime. The metric is valid in the black hole's local neighborhood $\mathcal{B}$, which is defined by $\bar{r} < \bar{r}_{\max}$ (see Fig. 1); we demand that $\bar{r}_{\max}/\mathcal{R}$ be small, but within $\mathcal{B}$ the ratio $M/\bar{r}$ is allowed to be arbitrarily large.

The first term on the right-hand side of Eq. (1.1) is the Schwarzschild piece of the metric; it is expressed in harmonic coordinates, and in these coordinates the (unperturbed) event horizon is situated at $\bar{r} = M$. The second term and the right-hand side of Eq. (1.2) represent the tidal perturbation. The tensorial functions $\bar{\mathcal{E}}_{ab}(\bar{t})$ and $\bar{\mathcal{B}}_{ab}(\bar{t})$ are the *tidal moments* (Sec. II), and it is these tensors that characterize the black hole's tidal environment. The tidal moments are symmetric and tracefree (STF) tensors, in the sense that $\bar{\mathcal{E}}_{ba} = \bar{\mathcal{E}}_{ab}$ and $\delta^{ab}\bar{\mathcal{E}}_{ab} = 0$, with similar relations holding for $\bar{\mathcal{B}}_{ab}$. The tensors $c^{-2}\bar{\mathcal{E}}_{ab}$ and $c^{-3}\bar{\mathcal{B}}_{ab}$ have a dimension of squared inverse length, and their scale *defines* $\mathcal{R}$, the local radius of curvature; we have $c^{-2}\bar{\mathcal{E}}_{ab} \sim \mathcal{R}^{-2}$ and $c^{-3}\bar{\mathcal{B}}_{ab} \sim \mathcal{R}^{-2}$. The tidal moments are not determined by solving the Einstein field equations in $\mathcal{B}$. They are *a priori* arbitrary, and their determination is accomplished by matching the black-hole metric to a global metric defined in a domain that is much larger than $\mathcal{B}$. In this work the tidal moments are determined by placing the black hole within a post-Newtonian environment; the global metric is obtained by solving the Einstein field equations in the first post-Newtonian approximation.

We next describe this post-Newtonian environment. We consider a spatial domain $\mathcal{D}$ that is much larger than $\mathcal{B}$, the black-hole neighborhood (see Fig. 1). This domain contains an arbitrary distribution of matter,[1] and it is assumed that everywhere within $\mathcal{D}$ gravity is sufficiently weak to be adequately described by the post-Newtonian approximation to general relativity. The domain is spatially limited by a sphere of radius $r_{\rm near}$ centered on the post-Newtonian barycenter. This sphere marks the boundary of the near zone: If $\mathcal{T}$ is a typical time scale for processes taking place within $\mathcal{D}$, and if $\lambda_c = c\mathcal{T}$ is a typical wavelength of the gravitational waves escaping the domain, then $r_{\rm near} < \lambda_c$. The domain also excludes a sphere of radius $\bar{r}_{\rm min}$ centered on the black hole, inside which gravity is too strong to be adequately described by post-Newtonian theory. We demand both that $M/\bar{r}_{\rm min} \ll 1$ and $\bar{r}_{\rm min}/\mathcal{R} \ll 1$, which is possible when $M \ll \mathcal{R}$. There exists an overlap region $\mathcal{O}$ between the black-hole neighborhood $\mathcal{B}$ and the post-Newtonian domain $\mathcal{D}$. This region is described by $\bar{r}_{\rm min} < \bar{r} < \bar{r}_{\rm max}$, and we assume that there is no matter in $\mathcal{O}$. So while matter is present somewhere within $\mathcal{D}$, we assume that the black hole is moving in an empty region of spacetime.

The metric in $\mathcal{D}$ can be expressed as a post-Newtonian expansion of the form (Sec. IV)

$$g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}(\Psi - U^2) + O(c^{-6}), \tag{1.3}$$

$$g_{0a} = -\frac{4}{c^3}U_a + O(c^{-5}), \tag{1.4}$$

$$g_{ab} = \left(1 + \frac{2}{c^2}U\right)\delta_{ab} + O(c^{-4}), \tag{1.5}$$

in which U is a Newtonian potential, U_a a vector potential, and Ψ a post-Newtonian potential; the metric is presented in harmonic coordinates $x^\alpha = (x^0, x^a) = (ct, x, y, z)$. These barycentric coordinates differ from the black-hole coordinates $(c\bar{t}, \bar{x}, \bar{y}, \bar{z})$ introduced previously; the black-hole and post-Newtonian metrics are presented in different coordinate systems. In spite of the fact that each system is harmonic, the coordinates are indeed distinct: They are defined in different domains (x^α in $\mathcal{D}$, $\bar{x}^\alpha$ in $\mathcal{B}$), and they have a different spatial origin (x^α is centered on the post-Newtonian barycenter, whose position is fixed in the global reference frame, while $\bar{x}^\alpha$ is centered on the moving black hole).

In the overlap region $\mathcal{O}$, the spacetime is empty of matter, and the potentials U, U_a, and Ψ satisfy the vacuum field equations of post-Newtonian theory. The Newtonian potential, for example, must satisfy Laplace's equation in flat space, $\nabla^2 U = 0$. The solution must account for the presence of a black hole, and it must also account for the presence of matter outside $\mathcal{O}$. We treat the black hole as a post-Newtonian monopole, and we write

$$U(t, \boldsymbol{x}) = \frac{Gm}{|\boldsymbol{x} - \boldsymbol{z}(t)|} + U_{\rm ext}(t, \boldsymbol{x}), \tag{1.6}$$

in which the three-dimensional vector $\boldsymbol{z}(t)$ denotes the position of the black hole in the barycentric coordinates. The external potential $U_{\rm ext}$ is created by the matter outside $\mathcal{O}$, and within the overlap region we have $\nabla^2 U_{\rm ext} = 0$. The potentials U_a and Ψ are handled in a similar fashion (Sec. IV), and in this way we construct the post-Newtonian metric in $\mathcal{O}$.

The post-Newtonian metric of Eqs. (1.3), (1.4), and (1.5) and the black-hole metric of Eqs. (1.1) and (1.2) both give a valid description of the gravitational field in $\mathcal{O}$. The metrics must agree in the overlap region, and matching them determines the equations of motion for $\boldsymbol{z}(t)$ as well as the tidal moments $\bar{\mathcal{E}}_{ab}(\bar{t})$ and $\bar{\mathcal{B}}_{ab}(\bar{t})$. This matching, however, can only be done after the post-Newtonian metric is transformed from the barycentric coordinates x^α to the black-hole coordinates $\bar{x}^\alpha$. This transformation, between two systems of harmonic coordinates, can be fully worked out (Sec. V), relying on previous work by Kopeikin [19], Brumberg and Kopeikin [20], Damour, Soffel, and Xu [9], Kopeikin and Vlasov [21], and Racine and Flanagan [12].

The matching procedure determines the coordinate transformation completely (Sec. VI C), and it produces a justification of the earlier statement that the black hole can be treated as a post-Newtonian monopole. (A fuller discussion of this point is presented at the end of Sec. IV). This statement, therefore, is a strict consequence of the Einstein field equations, rather than an artificial assumption. At the first post-Newtonian order, the gravitational field of a black hole is that of a pure monopole, and it would be inconsistent to endow the black hole with an additional multipole structure.

The matching procedure produces also an equation of motion for the moving black hole. It reads (Sec. VI E)

[1]Here and below, the word "matter" describes a number of different situations. The matter could be a continuous fluid, so as to model an accretion disk or a galactic core. It could also correspond to a collection of $N - 1$ bodies with weak self-gravity, making the black hole a member of an N-body system. Or else the domain $\mathcal{D}$ could exclude a number $N - 1$ of small regions that would each contain a condensed body such as a neutron star or a black hole. In this last case, the post-Newtonian domain would contain no matter at all, but we will nevertheless refer to the $N - 1$ excluded regions as "matter."

$$a^a = \partial^a U_{\rm ext} + \frac{1}{c^2}[\partial^a \Psi_{\rm ext} - 4(\partial^a U^b_{\rm ext} - \partial^b U^a_{\rm ext})v_b + 4\partial_t U^a_{\rm ext} + (v^2 - 4U_{\rm ext})\partial^a U_{\rm ext} - v^a(4v^b\partial_b U_{\rm ext} + 3\partial_t U_{\rm ext})] + O(c^{-4}), \tag{1.7}$$

in which $\boldsymbol{v} = d\boldsymbol{z}/dt$ is the black hole's velocity vector in the barycentric frame, and $\boldsymbol{a} = d\boldsymbol{v}/dt$ is its acceleration. The external potentials $U_{\rm ext}$, $U^a_{\rm ext}$, and $\Psi_{\rm ext}$ are defined as in Eq. (1.6), and they are evaluated at $\boldsymbol{x} = \boldsymbol{z}(t)$ after differentiation. Equation (1.7) applies to a black hole moving in any post-Newtonian environment. When this environment consists of $(N-1)$ external bodies, so that the black hole is a member of an N-body system, Eq. (1.7) reduces to the standard (Einstein-Infeld-Hoffman) post-Newtonian equations of motion. These are listed, for example, in Exercise 39.15 of Misner, Thorne, and Wheeler [22]. In effect, Eq. (1.7) states that the black hole moves on a geodesic of the metric of Eqs. (1.3), (1.4), and (1.5), in which the (singular) potentials U, U_a, and Ψ are replaced by the (smooth) external potentials created by the distribution of matter outside the black-hole neighborhood $\mathcal{B}$.

It is interesting to compare the differences between our derivation of Eq. (1.7) and the approach followed by Racine and Flanagan (RF) [12]. First, the work of RF is concerned with arbitrarily structured bodies (with weak or strong internal gravity), while our own work is concerned specifically with a nonrotating black hole, which is necessarily treated as a post-Newtonian monopole. Our work, therefore, is a specialization of theirs. Second, RF define the frame $(\bar{t}, \bar{x}^a)$, which they call the body-adapted frame, by (essentially) setting the body's intrinsic mass dipole moment to zero; this is (essentially) the piece of $g_{\bar{0}\bar{0}}$ that behaves as $\bar{x}^a/\bar{r}^3$. This coordinate choice does not, in general, constrain the tidal dipole moment; this is (essentially) the piece of $g_{\bar{0}\bar{0}}$ that grows as $\bar{x}^a$. In our work, the coordinates $(\bar{t}, \bar{x}^a)$ are defined so as to eliminate all mass dipole moments (both intrinsic and tidal) from the metric. This is made possible by the fact that we are dealing here with a specific type of body—a nonrotating black hole—instead of a general body whose nature is characterized only by an infinite set of multipole moments. Indeed, the work of Zerilli [23] shows that in vacuum, an even-parity dipole perturbation of the Schwarzschild metric can always be removed by a gauge transformation; it is this gauge choice that defines our own version of the body-adapted frame, and the metric of Eqs. (1.1) and (1.2) reflects the complete absence of dipole terms. Third, in RF, the equations of motion are obtained by exploiting the integral form of the momentum-conservation identities that come as a consequence of the Landau-Lifshitz formulation of the Einstein field equations [24]. In our approach, the equations of motion are obtained directly by matching the black-hole and post-Newtonian metrics, and the computations are considerably simpler. This advantage is intimately tied to our complete control over the dipole terms; a derivation of the equations of motion involving matching only would not be possible without the ability to set both the intrinsic and tidal mass dipole moments to zero.

Finally, the matching procedure produces expressions for the tidal moments (Secs. VI D and VI E). In the barycentric frame they are given by

$$\mathcal{E}_{ab} = -\partial_{ab}U_{\rm ext} + \frac{1}{c^2} - (\partial_{\langle ab\rangle}\Psi_{\rm ext} + 4v^c(\partial_{ab}U^{\rm ext}_c - \partial_{c\langle a}U^{\rm ext}_{b\rangle}) - 4\partial_{t\langle a}U^{\rm ext}_{b\rangle} - 2(v^2 - U_{\rm ext})\partial_{ab}U_{\rm ext} + 3v^c v_{\langle a}\partial_{b\rangle c}U_{\rm ext} + 2v_{\langle a}\partial_{b\rangle t}U_{\rm ext} + 3\partial_{\langle a}U_{\rm ext}\partial_{b\rangle}U_{\rm ext}) + O(c^{-4}), \tag{1.8}$$

and

$$\mathcal{B}_{ab} = 2\epsilon_{pq(a}\partial^p{}_{b)}(U^q_{\rm ext} - v^q U_{\rm ext}) + O(c^{-2}), \tag{1.9}$$

in which the external potentials are evaluated at $\boldsymbol{x} = \boldsymbol{z}(t)$ after differentiation. The brackets around indices indicate symmetrization, while angular brackets indicate an STF operation: For any tensor A_{ab} we have $A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba})$ and $A_{\langle ab\rangle} := A_{(ab)} - \frac{1}{3}\delta_{ab}A$, where $A := \delta^{ab}A_{ab}$. The expressions of Eqs. (1.8) and (1.9) are valid for any post-Newtonian environment.

Given the vast difference in notations and ways of expressing our results, we have not attempted to compare Eqs. (1.8) and (1.9) to the results obtained by Racine and Flanagan [12], nor to those of Damour, Soffel, and Xu [9–11]. We can state, however, that the specialization of Eqs. (1.8) and (1.9) to an N-body system is in perfect agreement with the corresponding results of Damour, Soffel, and Xu—see, in particular, Eqs. (4.29)–(4.31) of Ref. [11]. We shall provide evidence for this statement in Sec. VII A.

When the black hole is part of a binary system in circular motion, the nonvanishing components of the tidal moments are given by (Sec. VII C)

$$\mathcal{E}_{11} + \mathcal{E}_{22} = -\frac{Gm'}{b^3}\left[1 + \frac{m}{2(m+m')}(v_{\rm rel}/c)^2 + O(c^{-4})\right], \tag{1.10}$$

$$\mathcal{E}_{11} - \mathcal{E}_{22} = -\frac{3Gm'}{b^3}\left[1 - \frac{3m+4m'}{2(m+m')}(v_{\rm rel}/c)^2 + O(c^{-4})\right]\cos 2\omega t, \tag{1.11}$$

$$\mathcal{E}_{12} = -\frac{3Gm'}{2b^3}\left[1 - \frac{3m+4m'}{2(m+m')}(v_{\rm rel}/c)^2 + O(c^{-4})\right] \times \sin 2\omega t, \tag{1.12}$$

$$\mathcal{B}_{13} = -\frac{3Gm'}{b^3}v_{\rm rel}\cos\omega t + O(c^{-2}), \tag{1.13}$$

$$\mathcal{B}_{23} = -\frac{3Gm'}{b^3} v_{\rm rel} \sin\omega t + O(c^{-2}), \qquad (1.14)$$

where m is the mass of the black hole, m' the mass of the companion, and $b := |z - z'|$ the orbital separation between the two bodies (in barycentric harmonic coordinates); the components $\mathcal{E}_{13}$, $\mathcal{E}_{23}$, $\mathcal{B}_{11}$, $\mathcal{B}_{22}$, and $\mathcal{B}_{12}$ all vanish for circular orbits, and $\mathcal{E}_{33} = -(\mathcal{E}_{11} + \mathcal{E}_{22})$. In these equations $v_{\rm rel} := |\boldsymbol{v} - \boldsymbol{v}'| = \sqrt{G(m + m')/b}$ stands for the (Newtonian) orbital velocity of the relative orbit, and

$$\omega = \sqrt{\frac{G(m + m')}{b^3}}\bigg[1 - \frac{3m^2 + 5mm' + 3m'^2}{2(m + m')^2}(v_{\rm rel}/c)^2 + O(c^{-4})\bigg] \qquad (1.15)$$

is the orbit's post-Newtonian angular velocity. Equations (1.8), (1.9), (1.10), (1.11), (1.12), (1.13), (1.14), and (1.15) are expressed in the barycentric coordinates x^α. In the main text (see Sec. VII D) we also list expressions that are valid in the black hole's moving frame. In the comoving coordinates $\bar{x}^\alpha$ we find that Eqs. (1.10), (1.11), (1.12), (1.13), and (1.14) are unchanged, except for the fact that the tidal moments must now be expressed in terms of the transformed phase variable $\bar{\omega}\,\bar{t}$. The transformed angular velocity $\bar{\omega}$, given by

$$\bar{\omega} = \sqrt{\frac{G(m + m')}{b^3}}\bigg[1 - \frac{3m^2 + 7mm' + 3m'^2}{2(m + m')^2}(v_{\rm rel}/c)^2 + O(c^{-4})\bigg], \qquad (1.16)$$

accounts for the change in time coordinate as well as the geodetic precession of the moving frame relative to the barycentric frame. Notice the different coefficient in front of mm' in the post-Newtonian term.

Equations (1.7), (1.8), (1.9), (1.10), (1.11), (1.12), (1.13), (1.14), (1.15), and (1.16) are the main results of this work. As an application we examine (Sec. VIII), in a suitable choice of gauge, the intrinsic geometry of the event horizon of a tidally-deformed black hole. We also calculate (Sec. IX A) the rate at which the black hole acquires mass as a result of its tidal interaction with the companion body. We find that this rate of tidal heating is given by

$$G\dot{m} = \frac{32}{5c^{15}}\frac{m^6 m'^2}{(m + m')^8}\bigg(\frac{G(m + m')}{b}\bigg)^9 \times \bigg[1 - \frac{5m^2 + 12mm' + 6m'^2}{(m + m')^2} \times (v_{\rm rel}/c)^2 + O(c^{-4})\bigg]. \qquad (1.17)$$

The rate at which the tidal coupling increases the black hole's angular momentum J can next be obtained from the rigid-rotation relation $\dot{m}c^2 = \bar{\omega}\,\dot{J}$.

C. Relevance of this work

Our interest in this paper is mostly in the exploration of the tidal dynamics of black holes, in a weak-field context in which the tidal fields can be determined explicitly. These dynamics produce the tidal heating of the black hole, an increase in mass (and angular momentum and surface area) that is produced entirely by an influx of gravitational energy across the horizon. This fascinating phenomenon was studied before, most notably by Poisson and Sasaki [25], Alvi [26], Price and Whelan [27], Hughes [28], Martel [29], and Poisson [18,30]. We provide here some additional insights.

The tidal heating of a nonrotating black hole is generally very small. Relative to the energy radiated away by gravitational waves, the effect is of order $(v/c)^8$. In practical terms, the effect is likely to be too small to be observed in a gravitational-wave signal that would be measured by ground-based detectors such as LIGO, VIRGO, and GEO600. For example, Alvi [26] calculated that for binary systems involving black holes with masses ranging from 5 to 50 solar masses, tidal heating is negligible: It contributes only a small fraction of a wave cycle during the signal's sweep through the detector's frequency band.

In some circumstances, however, the tidal heating is a significant effect that should not be neglected [27]. In particular, it is likely to be observed in gravitational-wave signals that would be measured by a space-based detector such as LISA. For example, Martel [29] showed that during a close encounter between a massive black hole and a compact body of much smaller mass, up to approximately 5% of the total radiated energy is absorbed by the black hole, the rest making its way out to infinity. Hughes [28] calculated that when the massive black hole is rapidly rotating, tidal heating slows down the inspiral of the orbiting body, thereby increasing the duration of the gravitational-wave signal. For example, a $1M_\odot$ compact body on a slightly inclined, circular orbit around a $10^6 M_\odot$ black hole of near-maximum spin will spend approximately two years in the LISA frequency band before its final plunge into the black hole; Hughes shows that tidal heating contributes approximately 20 days (and 10^4 wave cycles) to these two years.

Another situation in which the tidal heating might be "measured" is in the numerical simulations of black-hole mergers. To reveal this small effect the simulations will need to be performed with very high accuracy and resolution, but this should become possible within the next few years. The simulations would reveal a steady growth in the irreducible mass of each black hole at a rate that should be compatible with Eq. (1.17) if the holes are nonrotating. A recent paper by Boyle *et al.* [31] suggests that the tidal heating of black holes might already have been seen in numerical simulations—witness their Fig. 4.

As an additional application of our work we mention another connection with numerical relativity. If the simulations of black-hole mergers are to describe realistic situations of astrophysical interest, it is imperative that they proceed from data that correctly describe the initial state of the system (the correct state of motion, and the correct amount of initial radiation). The idea that well-controlled initial-data sets could be constructed with the help of post-Newtonian theory is an old one [32–37], but its initial formulation did not account for the strong internal gravity of each black hole, which cannot be approximated by a post-Newtonian series. Alvi [15,16] was the first to remedy this situation by matching the post-Newtonian metric to black-hole metrics, in the way that was reviewed in Secs. I A and I B. Alvi's work was improved upon by Yunes *et al.* [17] (see Sec. I A), and our work contributes an additional improvement. While Alvi and Yunes calculated the black-hole tidal moments for circular motion only, and only to the leading order in a post-Newtonian expansion, here we calculate $\mathcal{E}_{ab}$ to the first post-Newtonian order, and $\mathcal{B}_{ab}$ to the leading order, for more general situations. Our results could be used to generate an improved version of the Alvi-Yunes metric, which could then be used to construct improved initial-data sets for numerical relativity.

D. Organization of this paper

The technical portion of the paper begins in Sec. II with an introduction to the description of tidal environments in terms of STF moments $\bar{\mathcal{E}}_{ab}$ and $\bar{\mathcal{B}}_{ab}$. In Sec. III we present the black-hole metric of Eqs. (1.1) and (1.2). In Sec. IV we introduce the post-Newtonian metric of Eqs. (1.3), (1.4), and (1.5), and discuss the decomposition of the gravitational potentials into a black-hole piece and an external piece, as in Eq. (1.6). In Sec. V we review the coordinate transformation between the barycentric frame (t, x^a) and the black-hole frame $(\bar{t}, \bar{x}^a)$, and we calculate how the post-Newtonian potentials change under this transformation. The end result of this computation is a post-Newtonian metric expressed in the same coordinates as the black-hole metric of Sec. III. In Sec. VI we match the black-hole and post-Newtonian metrics and derive our expression of Eq. (1.7) for the black hole's acceleration vector, as well as Eqs. (1.8) and (1.9) for the tidal moments. In Sec. VII we specialize our results to the specific case of a two-body system. We first calculate the tidal moments for generic orbital motion, and we next specialize these results to circular motion; this gives rise to Eqs. (1.10), (1.11), (1.12), (1.13), (1.14), (1.15), and (1.16) above. In Sec. VIII we present an application of our results: We examine the intrinsic geometry of the event horizon of a tidally-deformed black hole in a suitable choice of gauge. And finally, in Sec. IX we apply our results to a calculation of the tidal heating of a black hole by an external body on a circular orbit; this is a gauge-independent effect. We first consider the case of a nonrotating black hole and obtain Eq. (1.17). We next consider the case of a rotating black hole; the result was not displayed above, but it can be found in Eq. (8.7) below.

Throughout the paper (except in Secs. III B and VIII) we work in quasi-Lorentzian coordinates $x^\alpha = (ct, x^a)$ or $\bar{x}^\alpha = (c\bar{t}, \bar{x}^a)$, and we adopt a standard three-dimensional notation when we deal with spatial components. For example, we use $\boldsymbol{v} = (v^x, v^y, v^z)$ to denote a Cartesian vector with components v^a in a flat, three-dimensional space. Indices on v^a are manipulated with the Euclidean metric δ_{ab}, and ϵ_{abc} is the familiar permutation symbol. The Euclidean norm of $\boldsymbol{v}$ is $|\boldsymbol{v}| := \sqrt{\delta_{ab} v^a v^b}$, and we let $r := |\boldsymbol{x}|$ and $\bar{r} := |\bar{\boldsymbol{x}}|$. Because the paper is devoted to a post-Newtonian treatment of tidal gravity, we find it useful to use conventional units in which G and c are not set equal to 1 (Sec. III B is again an exception in this regard); as in much of the literature on post-Newtonian theory, we use c^{-2} as a formal expansion parameter.

II. TIDAL SCALES AND TIDAL MOMENTS

As shown in Sec. I B, the black hole's tidal environment is described by the STF tensors $\bar{\mathcal{E}}_{ab}(\bar{t})$ and $\bar{\mathcal{B}}_{ab}(\bar{t})$, and it is characterized by the length scale $\mathcal{R}$. Our purpose in this section is to formally introduce these quantities, and to set the stage toward the computation of the black-hole metric in Sec. III. The gravito-electric tidal moments $\bar{\mathcal{E}}_{ab}$ can be introduced most simply in the context of Newtonian gravity; we shall do this first. The gravito-magnetic moments $\bar{\mathcal{B}}_{ab}$ do not exist in the Newtonian theory, but they appear in a relativistic description of tidal fields. (The overbar, we recall, indicates that the tidal moments are evaluated in a reference frame that moves with the black hole).

In Newtonian gravity the total potential U can be expressed as in Eq. (1.6), with the first term $Gm/\bar{r}$ describing the black hole (here $\bar{r}^2 := \delta_{ab}\bar{x}^a\bar{x}^b$), and the second term $U_{\rm ext}$ describing the gravitational influence of the external matter. Assuming that the external potential varies slowly in the black-hole neighborhood $\mathcal{B}$, we express it as a Taylor expansion in powers of $\bar{r}$,

$$U_{\rm ext}(\bar{t}, \bar{x}^a) = U_{\rm ext}(\bar{t}, 0) + \bar{g}_a(\bar{t})\bar{x}^a + \tfrac{1}{2}\bar{\mathcal{E}}_{ab}(\bar{t})\bar{x}^a\bar{x}^b + \tfrac{1}{6}\bar{\mathcal{E}}_{abc}(\bar{t})\bar{x}^a\bar{x}^b\bar{x}^c + O(\bar{r}^4/\mathcal{R}^4).$$

Here $\bar{g}_a := \partial_{\bar{a}} U_{\rm ext}(\bar{t}, 0)$ is the gravitational force (per unit mass) acting on the black hole, $\bar{\mathcal{E}}_{ab} := \partial_{\bar{a}\bar{b}} U_{\rm ext}(\bar{t}, 0)$ is the quadrupole moment of the external potential, and $\bar{\mathcal{E}}_{abc} := \partial_{\bar{a}\bar{b}\bar{c}} U_{\rm ext}(\bar{t}, 0)$ is the octupole moment. The first term in the expansion does not depend on the spatial coordinates and plays no role in the gravitational interaction of the black hole with the external distribution of matter. The second term $\bar{g}_a\bar{x}^a$ also plays no role because we are working in a noninertial frame attached to the moving black hole. What remains is the pure tidal potential, $U_{\rm tidal} = \frac{1}{2}\bar{\mathcal{E}}_{ab}\bar{x}^a\bar{x}^b + \frac{1}{6}\bar{\mathcal{E}}_{abc}\bar{x}^a\bar{x}^b\bar{x}^c + O(\bar{r}^4/\mathcal{R}^4)$, expressed in terms of the spatial coordinates and the tidal moments (which depend on time). Notice that the tidal moments are defined as fully symmetric tensors. And because $U_{\rm ext}$ satisfies Laplace's equation in $\mathcal{B}$ (recall that the black-hole neighborhood is assumed to be empty of matter), the tidal moments are also tracefree. The quantities $\bar{\mathcal{E}}_{ab}$ and $\bar{\mathcal{E}}_{abc}$ (and all higher-order moments) are therefore STF tensors: $\bar{\mathcal{E}}_{ab} = \bar{\mathcal{E}}_{\langle ab\rangle}$ and $\bar{\mathcal{E}}_{abc} = \bar{\mathcal{E}}_{\langle abc\rangle}$. In addition, because $c^{-2}U_{\rm ext}$ is dimensionless, $c^{-2}\bar{\mathcal{E}}_{ab}$ has a dimension of inverse length squared, and the tidal length scale $\mathcal{R}$ is defined such that the components of $c^{-2}\bar{\mathcal{E}}_{ab}$ are typically of order $\mathcal{R}^{-2}$. In the preceding expression for $U_{\rm ext}$ we have assumed that the components of $c^{-2}\bar{\mathcal{E}}_{abc}$ are of order $\mathcal{R}^{-3}$, and that each additional term in the expansion comes with an additional power of $\bar{r}/\mathcal{R}$.

In general relativity the definition of the tidal moments $\bar{\mathcal{E}}_{ab}$, $\bar{\mathcal{E}}_{abc}$, $\bar{\mathcal{B}}_{ab}$, $\bar{\mathcal{B}}_{abc}$, (and higher-order moments) requires more refinement. The relevant tools were introduced by Thorne and Hartle [6] and Zhang [38]. We consider a neighborhood of a geodesic world line γ in an arbitrary spacetime. In this neighborhood the metric is assumed to satisfy the vacuum field equations. (In Sec. III a black hole will be placed on this geodesic, and our spacetime will become the "external spacetime" of Sec. I A. For the time being, however, the black hole is absent.) We install a normal coordinate system $(\bar{t}, \bar{x}^a)$ in the neighborhood. It possesses the following properties: (1) the spatial coordinates $\bar{x}^a$ vanish on γ, and $\bar{t}$ is proper time on the geodesic; (2) the metric takes on Minkowski values on γ; (3) all Christoffel symbols vanish on γ; and (4) the coordinates are harmonic, in the sense that the metric satisfies the conditions $\partial_\beta(\sqrt{-g}g^{\alpha\beta}) = 0$ everywhere in the neighborhood.

The metric near γ admits an expansion in powers of $\bar{r}$. By virtue of property (2) the zeroth-order terms are constant, and by virtue of property (3) there are no terms at the first order. The terms at the second and higher orders contain information about the curvature of spacetime near the geodesic, and it is those terms that describe the tidal environment around γ. This environment is characterized by the scaling quantities $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{T}$, with $\mathcal{R}$ denoting the radius of curvature on γ; $\mathcal{L}$, the scale of spatial inhomogeneity; and $\mathcal{T}$, the time scale over which changes occur in the environment. The tidal environment is described precisely by the tidal moments $\bar{\mathcal{E}}_{ab}(\bar{t}), \bar{\mathcal{E}}_{abc}(\bar{t}), \cdots$ and $\bar{\mathcal{B}}_{ab}(\bar{t}), \bar{\mathcal{B}}_{abc}(\bar{t}), \cdots$, which are STF tensors that depend on $\bar{t}$ only. They are related to components of the Riemann tensor and its derivatives evaluated on γ—see Zhang's Eqs. (1.3) for definitions [38]. The gravito-electric moments scale as

$$c^{-2}\bar{\mathcal{E}}_{ab} \sim \mathcal{R}^{-2}, \qquad c^{-2}\bar{\mathcal{E}}_{abc} \sim \mathcal{R}^{-2}\mathcal{L}^{-1}, \qquad c^{-2}\dot{\bar{\mathcal{E}}}_{ab} \sim \mathcal{R}^{-2}\mathcal{T}^{-1}, \tag{2.1}$$

in which an overdot indicates differentiation with respect to $\bar{t}$. It is the relations of Eq. (2.1) that define the scales $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{T}$. The gravito-magnetic moments scale as

$$c^{-3}\bar{\mathcal{B}}_{ab} \sim (v/c)\mathcal{R}^{-2}, \qquad c^{-3}\bar{\mathcal{B}}_{abc} \sim (v/c)\mathcal{R}^{-2}\mathcal{L}^{-1}, \qquad c^{-3}\dot{\bar{\mathcal{B}}}_{ab} \sim (v/c)\mathcal{R}^{-2}\mathcal{T}^{-1}, \tag{2.2}$$

in which $v \sim \mathcal{L}/\mathcal{T}$ is a velocity scale. In a slow-motion context we have that $v/c \ll 1$, and the gravito-magnetic moments are smaller than the gravito-electric moments by a small factor of order v/c.

To illustrate the meaning of these tidal scales we return to the example presented in Sec. I A, in which the tidal environment is provided by an external body of mass m'. We suppose that the geodesic is at a distance b from the external body. In this situation $v/c \sim \sqrt{M'/b}$, where $M' := Gm'/c^2$ is the characteristic gravitational radius of the external body. The tidal scales are then given by

$$\mathcal{R} \sim \sqrt{b^3/M'}, \qquad \mathcal{L} \sim b, \qquad \mathcal{T} \sim \sqrt{b^3/Gm'}.$$

We notice that $\mathcal{L} \sim (v/c)\mathcal{R}$ and $\mathcal{T} \sim \mathcal{R}/c$; in a slow-motion situation we have that $\mathcal{L} \ll \mathcal{R}$.

The metric near γ takes the form derived by Zhang [38]

$$g_{\bar{0}\bar{0}} = -1 - \frac{1}{c^2}\bar{\mathcal{E}}_{ab}\bar{x}^a\bar{x}^b - \frac{1}{3c^2}\bar{\mathcal{E}}_{abc}\bar{x}^a\bar{x}^b\bar{x}^c + \cdots, \tag{2.3}$$

$$g_{\bar{0}\bar{a}} = \frac{2}{3c^3}\epsilon_{abp}\bar{\mathcal{B}}^p{}_c\bar{x}^b\bar{x}^c + \frac{1}{3c^3}\epsilon_{abp}\bar{\mathcal{B}}^p{}_{cd}\bar{x}^b\bar{x}^c\bar{x}^d - \frac{10}{21c^3}\left(\bar{x}_a\dot{\bar{\mathcal{E}}}_{bc}\bar{x}^b\bar{x}^c - \frac{2}{5}\bar{r}^2\dot{\bar{\mathcal{E}}}_{ab}\bar{x}^b\right) + \cdots, \tag{2.4}$$

$$g_{\bar{a}\bar{b}} = \delta_{ab}\left(1 + \frac{1}{c^2}\bar{\mathcal{E}}_{cd}\bar{x}^c\bar{x}^d + \frac{1}{3c^2}\bar{\mathcal{E}}_{cde}\bar{x}^c\bar{x}^d\bar{x}^e\right) + \frac{5}{21c^4} \times\left(\bar{x}_{(a}\epsilon_{b)cp}\dot{\bar{\mathcal{B}}}^p{}_d\bar{x}^c\bar{x}^d - \frac{1}{5}\bar{r}^2\epsilon_{cp(a}\dot{\bar{\mathcal{B}}}^p{}_{b)}\bar{x}^c\right) + \cdots, \tag{2.5}$$

where ϵ_{abc} is the permutation symbol. The neglected terms involve higher powers of $\bar{r}$, and higher-order tidal moments.

The first tidal term on the right-hand side of Eq. (2.3) is of order $(\bar{r}/\mathcal{R})^2$, and the second term is smaller than this by a factor of order $\bar{r}/\mathcal{L}$; the neglected terms are smaller still, by additional factors of order $\bar{r}/\mathcal{L}$. In Eq. (2.4) the first term is of order $(v/c)(\bar{r}/\mathcal{R})^2$, and the second term is smaller than this by a factor of order $\bar{r}/\mathcal{L}$; taking into account the scalings $\mathcal{L} \sim (v/c)\mathcal{R}$ and $\mathcal{T} \sim \mathcal{R}/c$, the same is true of the third and fourth terms. In Eq. (2.5) the first tidal term is of order $(\bar{r}/\mathcal{R})^2$, and the second term is smaller by a factor of order $\bar{r}/\mathcal{L}$; the third and fourth terms are smaller than this by another factor of order $\bar{r}/\mathcal{L}$, and they come also with an additional factor of order $(v/c)^2$.

These considerations lead us to the following conclusion: If we restrict the neighborhood of γ to be such that $\bar{r}$ is everywhere much smaller than $\mathcal{L}$, then Zhang's metric can be simplified to

$$g_{\bar{0}\bar{0}} = -1 - \frac{1}{c^2}\bar{\mathcal{E}}_{ab}\bar{x}^a\bar{x}^b + O\left(\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}\right), \tag{2.6}$$

$$g_{\bar{0}\bar{a}} = \frac{2}{3c^3}\epsilon_{abp}\bar{\mathcal{B}}^p{}_c\bar{x}^b\bar{x}^c + O\left(\frac{v}{c}\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}\right), \tag{2.7}$$

$$g_{\bar{a}\bar{b}} = \delta_{ab}\left(1 + \frac{1}{c^2}\bar{\mathcal{E}}_{cd}\bar{x}^c\bar{x}^d\right) + O\left(\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}\right). \tag{2.8}$$

This simplified form, which involves the lowest-order tidal moments only, and which neglects their time derivatives, shall be sufficient for our purposes below.

III. TIDALLY DEFORMED BLACK HOLE

A. Metric of a deformed black hole

The metric of Eqs. (2.6), (2.7), and (2.8) describes the tidal environment around a geodesic γ in an arbitrary spacetime, with the only restriction that the geodesic's neighborhood must be empty of matter. We now place a nonrotating black hole of mass m on this geodesic, and modify the metric to account for its gravitational effects. As we shall show below (in Sec. III B), the metric of the tidally deformed black hole is given by

$$g_{\bar{0}\bar{0}} = -\frac{1 - M/\bar{r}}{1 + M/\bar{r}} - \bar{r}^2(1 - M/\bar{r})^2\mathcal{E}^{\mathsf{q}} + O\left(\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}\right), \tag{3.1}$$

$$g_{\bar{0}\bar{a}} = \frac{2}{3}\bar{r}^2(1 - M/\bar{r})(1 + M/\bar{r})^2\mathcal{B}^{\mathsf{q}}_a + O\left(\frac{v}{c}\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}\right), \tag{3.2}$$

$$\begin{aligned} g_{\bar{a}\bar{b}} = {} & \frac{1 + M/\bar{r}}{1 - M/\bar{r}}\Omega_a\Omega_b + (1 + M/\bar{r})^2\gamma_{ab} \\ & - \bar{r}^2(1 + M/\bar{r})^2\mathcal{E}^{\mathsf{q}}\Omega_a\Omega_b - M\bar{r}(1 + M/\bar{r})^2 \\ & \times (1 + M^2/3\bar{r}^2)(\Omega_a\mathcal{E}^{\mathsf{q}}_b + \mathcal{E}^{\mathsf{q}}_a\Omega_b) - \bar{r}^2(1 - M/\bar{r})^2 \\ & \times (1 + M/\bar{r})^3\gamma_{ab}\mathcal{E}^{\mathsf{q}} - M\bar{r}(1 + M/\bar{r})^2(1 - M^2/3\bar{r}^2) \\ & \times \mathcal{E}^{\mathsf{q}}_{ab} + O\left(\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}\right) \end{aligned} \tag{3.3}$$

in the hole's neighborhood $\mathcal{B}$, which is formally defined by $\bar{r} < \bar{r}_{\max}$ with $\bar{r}_{\max} \ll \mathcal{L}$ (this is a refinement of the definition presented in Sec. I B). Here $M := Gm/c^2$ is the black hole's gravitational radius, $\Omega^a := \bar{x}^a/\bar{r}$ is a unit radial vector, and $\gamma_{ab} := \delta_{ab} - \Omega_a\Omega_b$. The tidal potentials are given by

$$\mathcal{E}^{\mathsf{q}} = \frac{1}{c^2}\bar{\mathcal{E}}_{cd}\Omega^c\Omega^d, \tag{3.4}$$

$$\mathcal{E}^{\mathsf{q}}_a = \frac{1}{c^2}\gamma_a{}^c\bar{\mathcal{E}}_{cd}\Omega^d, \tag{3.5}$$

$$\mathcal{E}^{\mathsf{q}}_{ab} = \frac{1}{c^2}(2\gamma_a{}^c\gamma_b{}^d\bar{\mathcal{E}}_{cd} + \gamma_{ab}\mathcal{E}^{\mathsf{q}}), \tag{3.6}$$

$$\mathcal{B}^{\mathsf{q}}_a = \frac{1}{c^3}\epsilon_{apq}\Omega^p\bar{\mathcal{B}}^q{}_c\Omega^c; \tag{3.7}$$

the label q stands for "quadrupole." The metric of Eqs. (3.1), (3.2), and (3.3) is presented in harmonic coordinates, and it is an approximate solution to the vacuum field equations linearized about the Schwarzschild metric. Indeed, setting $\bar{\mathcal{E}}_{ab} = \bar{\mathcal{B}}_{ab} = 0$ in Eqs. (3.1), (3.2), and (3.3) returns the Schwarzschild metric in harmonic coordinates, and the tidal potentials represent a pure-quadrupole metric perturbation. It is easy to see that when $\bar{r}$ is much larger than M (but still much smaller than $\mathcal{L}$), Eqs. (3.1), (3.2), and (3.3) reduce to Eqs. (2.6), (2.7), and (2.8). This shows that Zhang's metric provides the appropriate asymptotic conditions for the metric of a tidally deformed black hole; these replace the asymptotically-flat conditions that would be appropriate for an isolated black hole. Like the metric of Eqs. (2.6), (2.7), and (2.8), the black-hole metric neglects tidal terms that are smaller than the dominant ones by additional factors of order $\bar{r}/\mathcal{L} \ll 1$; within this approximation the metric is accurate to all orders in $M/\bar{r}$.

The method by which the metric of Eqs. (3.1), (3.2), and (3.3) is obtained is explained in the next subsection. The reader not interested in those details can immediately proceed to Sec. IV, in which we present the post-Newtonian metric to which the black-hole metric will be matched. Before we proceed, however, it is useful to note that the components of the black-hole metric that are required for matching are

$$g_{\bar{0}\bar{0}} = -\frac{1 - M/\bar{r}}{1 + M/\bar{r}} - \frac{1}{c^2}(1 - M/\bar{r})^2\bar{\mathcal{E}}_{ab}(\bar{t})\bar{x}^a\bar{x}^b + O\left(\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}\right), \tag{3.8}$$

and

$$g_{\bar{0}\bar{a}} = \frac{2}{3c^3}(1 - M/\bar{r})(1 + M/\bar{r})^2\epsilon_{abp}\bar{\mathcal{B}}^p{}_c(\bar{t})\bar{x}^b\bar{x}^c + O\left(\frac{v}{c}\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}\right). \tag{3.9}$$

We recall that $M = Gm/c^2$ is the black hole's gravitational radius.

B. Derivation

We begin this subsection with a warning on a change of notation; this change applies to this subsection only. To simplify the notation we shall refrain from displaying the overbar, in spite of the fact that we continue to work in a reference frame that moves with the black hole. In addition, we shall use relativist's units, in which c and G are both set equal to unity. And finally, in this section we use $g_{\alpha\beta}$ to denote the Schwarzschild metric (as opposed to the perturbed metric displayed in Sec. III A), and we let $h_{\alpha\beta}$ denote the tidal perturbation; the perturbed metric is therefore $g_{\alpha\beta} + h_{\alpha\beta}$.

To obtain Eqs. (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), and (3.7) we rely on Ref. [18], in which the metric of a tidally deformed, nonrotating black hole is presented in light-cone coordinates (v, ρ, θ, ϕ). The advanced-time coordinate v is constant on past light cones that converge toward $\rho = 0$, ρ is the usual Schwarzschild radial coordinate that measures the area of closed surfaces of constant (v, ρ), and $\theta^A = (\theta, \phi)$ are angular coordinates on these surfaces. In the light-cone coordinates the Schwarzschild metric takes the form

$$g_{\alpha\beta}dx^\alpha dx^\beta = -fdv^2 + 2dvd\rho + \rho^2\Omega_{AB}d\theta^A d\theta^B, \tag{3.10}$$

with $f := 1 - 2M/\rho$ and $\Omega_{AB}d\theta^A d\theta^B := d\theta^2 + \sin^2\theta d\phi^2$ denoting the metric on the unit two-sphere. The nonvanishing components of the tidal perturbation are

$$h^{\text{light}}_{vv} = -\rho^2 f^2 \mathcal{E}^{\mathsf{q}}, \tag{3.11}$$

$$h^{\text{light}}_{vA} = -\tfrac{2}{3}\rho^3 f(\mathcal{E}^{\mathsf{q}}_A - \mathcal{B}^{\mathsf{q}}_A), \tag{3.12}$$

$$h^{\text{light}}_{AB} = -\tfrac{1}{3}\rho^4(1 - 2M^2/\rho^2)\mathcal{E}^{\mathsf{q}}_{AB} + \tfrac{1}{3}\rho^4(1 - 6M^2/\rho^2)\mathcal{B}^{\mathsf{q}}_{AB}, \tag{3.13}$$

in which the label "light" indicates that the perturbation is presented in the light-cone gauge. The tidal potentials are given in terms of the tidal moments $\mathcal{E}_{ab}(v)$ and $\mathcal{B}_{ab}(v)$ by the relations

$$\mathcal{E}^{\mathsf{q}} = \mathcal{E}_{ab}\Omega^a\Omega^b, \tag{3.14}$$

$$\mathcal{E}^{\mathsf{q}}_A = \Omega^a_A \mathcal{E}_{ab}\Omega^b, \tag{3.15}$$

$$\mathcal{E}^{\mathsf{q}}_{AB} = 2\Omega^a_A\Omega^b_B\mathcal{E}_{ab} + \Omega_{AB}\mathcal{E}^{\mathsf{q}}, \tag{3.16}$$

$$\mathcal{B}^{\mathsf{q}}_A = \Omega^a_A \epsilon_{apq}\Omega^p\mathcal{B}^q{}_b\Omega^b, \tag{3.17}$$

$$\mathcal{B}^{\mathsf{q}}_{AB} = \Omega^a_A\epsilon_{apq}\Omega^p\mathcal{B}^q{}_b\Omega^b_B + \Omega^b_B\epsilon_{bpq}\Omega^p\mathcal{B}^q{}_a\Omega^a_A, \tag{3.18}$$

where $\Omega^a := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $\Omega^a_A := \partial\Omega^a/\partial\theta^A$. The perturbed metric is $g_{\alpha\beta} + h_{\alpha\beta}$, and it is straightforward to show that this metric satisfies the vacuum field equations linearized about the Schwarzschild solution. The metric is accurate up to terms of order $(r^3/\mathcal{R}^2\mathcal{L})$ that come from higher-order tidal moments, and terms of order $(r^3/\mathcal{R}^2\mathcal{T})$ that come from the time derivative of the quadrupole moments. For the purposes of substitution into the field equations, $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ can be considered to be time-independent.

The black-hole metric is expressed in the light-cone coordinates (v, ρ, θ^A), and we wish to perform a transformation to harmonic coordinates (t, x^a). We accomplish this in two steps. First, we perform a *gauge transformation* to change the perturbation $h_{\alpha\beta}$ from its current light-cone gauge to a harmonic gauge. Second, we perform a *background coordinate transformation* from the (background) light-cone coordinates to the (background) harmonic coordinates. This strategy, and its implementation detailed below, was suggested to us by Detweiler [39].

Let a set of four scalar fields be defined by

$$T := v - \rho - 2M\log(\rho/2M - 1), \tag{3.19}$$

$$X := (\rho - M)\sin\theta\cos\phi, \tag{3.20}$$

$$Y := (\rho - M)\sin\theta\sin\phi, \tag{3.21}$$

$$Z := (\rho - M)\cos\theta, \tag{3.22}$$

and let us collectively denote the members of this set by $X^{(\mu)}$. It is straightforward to show that each one of the scalar fields $X^{(\mu)}$ satisfies the wave equation

$$\Box X^{(\mu)} := g^{\alpha\beta}\nabla_\alpha\nabla_\beta X^{(\mu)} = \frac{1}{\sqrt{-g}}\partial_\beta(g^{\alpha\beta}\partial_\alpha X^{(\mu)}) = 0 \tag{3.23}$$

in the Schwarzschild spacetime. Here ∇_α is the covariant-derivative operator compatible with the Schwarzschild metric $g_{\alpha\beta}$, and g is the metric determinant. The statement of Eq. (3.23) is coordinate independent. When, however, we choose $t = T$, $x = X$, $y = Y$, and $z = Z$ as coordinates, then Eq. (3.23) becomes $\partial_\beta(\sqrt{-g}g^{\mu\beta}) = 0$, the familiar statement of the harmonic coordinate condition. Equation (3.23) therefore provides a coordinate-invariant way of stating that the scalar fields $X^{(\mu)}$ form a set of harmonic coordinates for the Schwarzschild spacetime.

We now demand that $X^{(\mu)}$ be harmonic coordinates for the perturbed spacetime, *in addition to* being harmonic coordinates for the Schwarzschild spacetime. To achieve this we rewrite Eq. (3.23) in terms of the full metric $g_{\alpha\beta} + h_{\alpha\beta}$ and its associated covariant-derivative operator. We write $g^{\alpha\beta} - h^{\alpha\beta}$ for the inverse metric and $\sqrt{-g}(1 + \frac{1}{2}h)$ for the metric determinant, and we manipulate the indices with the background metric; for example, $h^{\alpha\beta} = g^{\alpha\gamma}g^{\beta\delta}h_{\gamma\delta}$ and $h = g^{\alpha\beta}h_{\alpha\beta}$. After some straightforward manipulations we find that in addition to Eq. (3.23), the scalar fields must also satisfy the set of equations

$$\nabla_\alpha(\psi^{\alpha\beta}\nabla_\beta X^{(\mu)}) = 0, \tag{3.24}$$

where $\psi_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}h$ is the "trace-reversed" metric perturbation. This equation can be interpreted as a gauge condition on $h_{\alpha\beta}$: The perturbation will be in the *harmonic gauge* if it satisfies the conditions of Eq. (3.24). This equation is coordinate invariant, and the harmonic-gauge condition can be imposed even when the coordinates x^α do not coincide with the scalar fields $X^{(\mu)}$.

In Eq. (3.11), (3.12), and (3.13) the metric perturbation is presented in the light-cone gauge. A transformation to the harmonic gauge will be generated by the vector field ξ_α, so that

$$h^{\rm harm}_{\alpha\beta} = h^{\rm light}_{\alpha\beta} - \nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha. \qquad (3.25)$$

It is straightforward to show that if $h^{\rm harm}_{\alpha\beta}$ is to satisfy Eq. (3.24), then the vector field must satisfy the four equations

$$\nabla_\alpha(\psi^{\alpha\beta}_{\rm light}\nabla_\beta X^{(\mu)}) = (\Box\xi^\alpha)\nabla_\alpha X^{(\mu)} + 2(\nabla^\alpha\xi^\beta)\nabla_\alpha\nabla_\beta X^{(\mu)}; \qquad (3.26)$$

in our case the perturbation is traceless in the light-cone gauge, so $\psi^{\rm light}_{\alpha\beta} = h^{\rm light}_{\alpha\beta}$. It may be verified that the vector

$$\xi_v = -\tfrac{1}{3}\rho^3 f\mathcal{E}^{\sf q}, \qquad (3.27)$$

$$\xi_\rho = \tfrac{1}{3}\rho^3\mathcal{E}^{\sf q}, \qquad (3.28)$$

$$\xi_A = -\frac{\rho^5 f^2}{3(\rho - M)}\mathcal{E}^{\sf q}_A + \tfrac{1}{3}\rho^2(\rho^2 - 6M^2)\mathcal{B}^{\sf q}_A \qquad (3.29)$$

is a solution to Eqs. (3.26). In this computation the tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ can be considered to be time-independent, because the time derivative of ξ_α is smaller than its spatial derivatives by a small factor of order $\rho/\mathcal{T}$.

Substitution of Eqs. (3.11), (3.12), (3.13), (3.27), (3.28), and (3.29) into Eq. (3.25) returns the tidal perturbation in the desired harmonic gauge. We obtain

$$h^{\rm harm}_{vv} = -\rho^2 f^2\mathcal{E}^{\sf q}, \qquad (3.30)$$

$$h^{\rm harm}_{v\rho} = \rho^2 f\mathcal{E}^{\sf q}, \qquad (3.31)$$

$$h^{\rm harm}_{\rho\rho} = -2\rho^2\mathcal{E}^{\sf q}, \qquad (3.32)$$

$$h^{\rm harm}_{vA} = \tfrac{1}{3}\rho^3 f\mathcal{B}^{\sf q}_A, \qquad (3.33)$$

$$h^{\rm harm}_{\rho A} = -\frac{M\rho^2}{3(\rho-M)^2}(3\rho^2 - 6M\rho + 4M^2)\mathcal{E}^{\sf q}_A - \frac{2}{3}\rho^3\mathcal{B}^{\sf q}_A, \qquad (3.34)$$

$$h^{\rm harm}_{AB} = -\frac{\rho^5 f^2}{\rho - M}\Omega_{AB}\mathcal{E}^{\sf q} - \frac{M\rho^2}{3(\rho - M)}(3\rho^2 - 6M\rho + 2M^2)\mathcal{E}^{\sf q}_{AB}. \qquad (3.35)$$

The full metric is next obtained by adding $h_{\alpha\beta}$ to $g_{\alpha\beta}$ as given by Eq. (3.10).

The tidal perturbation is now correctly expressed in the harmonic gauge, but the metric is still written in terms of the original coordinates (v, ρ, θ, ϕ). Our final step is therefore to perform a coordinate transformation from these coordinates to the harmonic coordinates ($t = T$, $x = X$, $y = Y$, $z = Z$). We carry this out in two stages. First, we effect a transformation from (v, ρ) to (t, r), leaving the angular coordinates alone; here $t = v - \rho - 2M\log(\rho/2M - 1)$ is harmonic time and $r = \rho - M$ is the harmonic radial coordinate. This coordinate transformation brings the Schwarzschild metric to the new form

$$g_{\alpha\beta}dx^\alpha dx^\beta = -\frac{1 - M/r}{1 + M/r}dt^2 + \frac{1 + M/r}{1 - M/r}dr^2 + (r + M)^2\Omega_{AB}d\theta^A d\theta^B, \qquad (3.36)$$

and the tidal perturbation becomes

$$h^{\rm light}_{tt} = -r^2(1 - M/r)^2\mathcal{E}^{\sf q}, \qquad (3.37)$$

$$h^{\rm light}_{rr} = -r^2(1 + M/r)^2\mathcal{E}^{\sf q}, \qquad (3.38)$$

$$h^{\rm light}_{tA} = \tfrac{2}{3}r^3(1 - M/r)(1 + M/r)^2\mathcal{B}^{\sf q}_A, \qquad (3.39)$$

$$h^{\rm light}_{rA} = -Mr^2(1 + M/r)^2(1 + M^2/3r^2)\mathcal{E}^{\sf q}_A, \qquad (3.40)$$

$$h^{\rm light}_{AB} = -r^4(1 - M/r)^2(1 + M/r)^3\Omega_{AB}\mathcal{E}^{\sf q} - Mr^3(1 + M/r)^2(1 - M^2/3r^2)\mathcal{E}^{\sf q}_{AB}. \qquad (3.41)$$

In the second stage we go from the spherical coordinates (r, θ^A) to the associated Cartesian coordinates $x^a = r\Omega^a$, that is, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$. The transformation matrix is $\partial r/\partial x^a = \Omega_a$ and $\partial\theta^A/\partial x^a = r^{-1}\Omega^A_a$, in which $\Omega_a := \delta_{ab}\Omega^b$ and $\Omega^A_a := \Omega^{AB}\delta_{ab}\Omega^b_B$. The transformation of $g_{\alpha\beta} + h_{\alpha\beta}$ under this change of coordinates is easy to carry out, and the end result is the metric of Eqs. (3.1), (3.2), and (3.3).

IV. POST-NEWTONIAN METRIC

In this section we take our attention away from the black-hole neighborhood $\mathcal{B}$, and we take the global view of the post-Newtonian domain $\mathcal{D}$, which was introduced in Sec. I B. Let us recall that $\mathcal{D}$ is spatially limited by a sphere of radius $r_{\rm near}$ centered on the post-Newtonian barycenter, and that this sphere marks the boundary of the near zone. The domain excludes a sphere of radius $\bar{r}_{\rm min}$ centered on the black hole, inside which the hole's gravity is too strong to be adequately described by post-Newtonian theory. We recall also that there exists an overlap $\mathcal{O}$ between the black-hole neighborhood $\mathcal{B}$ and the post-Newtonian domain $\mathcal{D}$. This is described by $\bar{r}_{\rm min} < \bar{r} < \bar{r}_{\rm max}$, in which $\bar{r}_{\rm max} \ll \mathcal{L}$ is the boundary of $\mathcal{B}$. We assume that there is no matter in $\mathcal{O}$.

The metric in $\mathcal{D}$ is expressed in the post-Newtonian form

$$g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}(\Psi - U^2) + O(c^{-6}), \qquad (4.1)$$

$$g_{0a} = -\frac{4}{c^3}U_a + O(c^{-5}), \tag{4.2}$$

$$g_{ab} = \left(1 + \frac{2}{c^2}U\right)\delta_{ab} + O(c^{-4}), \tag{4.3}$$

in which U is a Newtonian potential, U_a a vector potential, and Ψ a post-Newtonian potential. The metric is presented in harmonic coordinates $x^\alpha = (ct, x^a)$. These coordinates are centered on the post-Newtonian barycenter, which defines the origin of an inertial reference frame; we recall that they are distinct from the harmonic coordinates $\bar{x}^\alpha = (c\bar{t}, \bar{x}^a)$ used in $\mathcal{B}$.

We assume that there is a distribution of matter within $\mathcal{D}$, and we make only two assumptions regarding its nature. First, we assume that its gravity is sufficiently weak to be adequately described by the post-Newtonian metric of Eqs. (4.1), (4.2), and (4.3). Second, we assume that there is no matter in $\mathcal{O}$, so that the black hole moves in an empty region of spacetime. Otherwise the distribution of matter within $\mathcal{D}$ is completely arbitrary. It could be a continuous fluid, so as to model an accretion disk or a galactic core; or it could correspond to a collection of $N-1$ bodies with weak self-gravity, making the black hole a member of an N-body system; or else the domain $\mathcal{D}$ could exclude a number $N-1$ of small regions that would each contain a condensed body such as a neutron star or a black hole. Our considerations allow for this degree of generality.

We wish to solve the Einstein field equations in the vacuum domain $\mathcal{O}$. To write them down it is convenient to express the post-Newtonian potential as

$$\Psi = \psi + \frac{1}{2}\frac{\partial^2 X}{\partial t^2}, \tag{4.4}$$

in terms of two new potentials ψ and X. The degeneracy is broken by the field equations, which are given by (see, for example, Ref. [12])

$$\nabla^2 U = 0, \tag{4.5}$$

$$\nabla^2 U^a = 0, \tag{4.6}$$

$$\nabla^2 \psi = 0, \tag{4.7}$$

$$\nabla^2 X = 2U, \tag{4.8}$$

where ∇^2 is the usual Laplacian operator of three-dimensional flat space. The potential X is commonly referred to as the post-Newtonian superpotential. The vacuum field equations are augmented by the harmonic coordinate condition

$$\partial_t U + \partial_a U^a = 0, \tag{4.9}$$

which takes the form of a gauge condition on the gravitational potentials.

Each one of the field equations is linear, and a solution representing a black hole and an external distribution of matter can be obtained by linear superposition. We treat the black hole as a post-Newtonian monopole of mass m moving on a trajectory described by the position vector $\boldsymbol{x} = \boldsymbol{z}(t)$; we let $\boldsymbol{v} := d\boldsymbol{z}/dt$ be the black hole's velocity vector, and $\boldsymbol{a} := d\boldsymbol{v}/dt$ is the acceleration vector. In this treatment the solutions to the field equations are expressed as

$$U(t, \boldsymbol{x}) = \frac{Gm}{|\boldsymbol{x} - \boldsymbol{z}|} + U_{\rm ext}(t, \boldsymbol{x}), \tag{4.10}$$

$$U^a(t, \boldsymbol{x}) = \frac{Gmv^a}{|\boldsymbol{x} - \boldsymbol{z}|} + U^a_{\rm ext}(t, \boldsymbol{x}), \tag{4.11}$$

$$\psi(t, \boldsymbol{x}) = \frac{Gm\mu}{|\boldsymbol{x} - \boldsymbol{z}|} + \psi_{\rm ext}(t, \boldsymbol{x}), \tag{4.12}$$

$$X(t, \boldsymbol{x}) = Gm|\boldsymbol{x} - \boldsymbol{z}| + X_{\rm ext}(t, \boldsymbol{x}), \tag{4.13}$$

where $|\boldsymbol{x} - \boldsymbol{z}(t)|$ is the Euclidean distance between the field point $\boldsymbol{x}$ and the black hole.

The first term on the right-hand side of each equation represents the black hole, and we see that indeed, each black-hole potential has a monopole structure. The fact that the vector $\boldsymbol{v}$ appears in the vector potential U^a is a consequence of the gauge condition of Eq. (4.9). The quantity $\mu(t)$ that appears in ψ is a post-Newtonian correction to the mass parameter m; this cannot be determined directly from the vacuum field equations.[2]

The second terms on the right-hand sides of Eqs. (4.10), (4.11), (4.12), and (4.13) are the potentials created by the matter distribution external to $\mathcal{O}$. In $\mathcal{O}$ they separately satisfy the vacuum field equations of Eqs. (4.5), (4.6), (4.7), and (4.8), and they separately satisfy the harmonic-gauge condition of Eq. (4.9)

$$\partial_t U_{\rm ext} + \partial_a U^a_{\rm ext} = 0. \tag{4.14}$$

We assume that the external potentials are smooth functions of the coordinates in a neighborhood of $\boldsymbol{x} = \boldsymbol{z}(t)$.

From Eq. (4.4) we find that the post-Newtonian potential Ψ is given by

[2] If the black hole were treated as a point particle, solving the field equations in the presence of a distributional energy-momentum tensor would reveal that $\mu = \frac{3}{2}v^2 - U_{\rm ext}$ $(t, \boldsymbol{x} = \boldsymbol{z})$, where $v^2 = |\boldsymbol{v}|^2$. This is indeed the correct expression, but in our approach the determination of μ will come at a later stage, from a careful matching of the post-Newtonian metric with the black-hole metric of Sec. III. For the time being $\mu(t)$ will remain undetermined.

$$\Psi = -\frac{Gm}{2s^3}(\boldsymbol{v}\cdot\boldsymbol{s})^2 + \frac{Gm}{s}\left(\mu + \frac{1}{2}v^2\right) - \frac{Gm}{2s}\boldsymbol{a}\cdot\boldsymbol{s} + \Psi_{\rm ext}, \tag{4.15}$$

where $\boldsymbol{s} := \boldsymbol{x} - \boldsymbol{z}(t)$, $s := |\boldsymbol{s}|$, and $\Psi_{\rm ext} = \psi_{\rm ext} + \frac{1}{2}\partial_t^2 X_{\rm ext}$.

The foregoing equations provide necessary and sufficient information regarding the post-Newtonian environment of the black hole. A remarkable and important aspect of our discussion is that the black hole is treated as a post-Newtonian monopole. This feature requires a justification. At the level of Eqs. (4.10), (4.11), (4.12), and (4.13) the monopole nature of the black hole is introduced as an assumption. As we proceed with the matching of the post-Newtonian metric with the black-hole metric of Sec. III, however, we shall see that our assumption is the only one that is consistent with the given structure of the black-hole metric. If we had instead given an arbitrary multipole structure to our black hole, the matching procedure would eventually force us to set all higher multipole moments to zero; the only surviving moment is the black-hole mass m. As we shall see, therefore, *the statement that the black hole is a post-Newtonian monopole is a strict consequence of the field equations of general relativity; no other multipole structure is possible.* The monopole structure is not assumption introduced for its simplicity; it is a direct outcome of the matching procedure.

V. TRANSFORMATION FROM BARYCENTER FRAME TO BLACK-HOLE FRAME

The post-Newtonian metric of Eqs. (4.1), (4.2), and (4.3), with the potentials of Eqs. (4.10), (4.11), and (4.15), is not yet ready to be matched to the black-hole metric of Eqs. (3.1), (3.2), and (3.3). While both metrics are defined in the overlap region $\mathcal{O}$ and describe the same physical situation, they are expressed in different coordinate systems: The post-Newtonian coordinates (t, x^a) are defined everywhere in $\mathcal{D}$, and they are attached to the post-Newtonian barycenter; the black-hole coordinates $(\bar{t}, \bar{x}^a)$ are defined only in $\mathcal{B}$, and they are attached to the moving black hole. Each coordinate system is harmonic, and since they overlap in $\mathcal{O}$ there exists a coordinate transformation between them. This transformation, from harmonic coordinates to harmonic coordinates, is worked out in Sec. V A, following closely the treatment of Racine and Flanagan [12]. Their work was built on the post-Newtonian theory of reference frames developed by Kopeikin, [19], Brumberg and Kopeikin [20], Damour, Soffel, and Xu [9], and Kopeikin and Vlasov [21]. In Sec. V B we calculate how the potentials of Eqs. (4.10), (4.11), and (4.15) change under the transformation from the barycenter frame to the black-hole frame. And finally, in Sec. V C we express the transformed potentials in terms of irreducible quantities that will facilitate the matching with the black-hole metric, to be carried out in Sec. VI.

A. Coordinate transformation

The most general coordinate transformation that preserves the post-Newtonian expansion of the metric is given by [12]

$$t = \bar{t} + \frac{1}{c^2}\alpha(\bar{t}, \bar{x}^a) + \frac{1}{c^4}\beta(\bar{t}, \bar{x}^a) + O(c^{-6}), \tag{5.1}$$

$$x^a = \bar{x}^a + z^a(\bar{t}) + \frac{1}{c^2}h^a(\bar{t}, \bar{x}^a) + O(c^{-4}), \tag{5.2}$$

where

$$\alpha(\bar{t}, \bar{x}^a) = A(\bar{t}) + v_a\bar{x}^a, \tag{5.3}$$

$$h^a(\bar{t}, \bar{x}^a) = H^a(\bar{t}) + H^a{}_b(\bar{t})\bar{x}^b + \tfrac{1}{2}H^a{}_{bc}(\bar{t})\bar{x}^b\bar{x}^c, \tag{5.4}$$

with

$$H_{ab}(\bar{t}) = \epsilon_{abc}R^c(\bar{t}) + \tfrac{1}{2}v_a v_b - \delta_{ab}(\dot{A} - \tfrac{1}{2}v^2), \tag{5.5}$$

$$H_{abc}(\bar{t}) = -\delta_{ab}a_c - \delta_{ac}a_b + \delta_{bc}a_a. \tag{5.6}$$

The functions A, z^a, H^a, and R^a are freely specifiable functions of time $\bar{t}$, while β is a free function of all the coordinates; these functions characterize the coordinate transformation. An overdot indicates differentiation with respect to $\bar{t}$, and we have introduced

$$v^a := \dot{z}^a, \qquad a^a := \dot{v}^a = \ddot{z}^a. \tag{5.7}$$

As before indices are raised and lowered with the Euclidean metric δ_{ab}, and we let $v^2 = \delta_{ab}v^a v^b$.

The transformation of Eqs. (5.1) and (5.2) preserves the post-Newtonian expansion of the metric, but it does not necessarily keep the coordinates harmonic. To preserve this also we must set

$$\beta(\bar{t}, \bar{x}^a) = \tfrac{1}{6}\ddot{A}\delta_{ab}\bar{x}^a\bar{x}^b + \tfrac{1}{30}(\delta_{ab}\dot{a}_c + \delta_{ac}\dot{a}_b + \delta_{bc}\dot{a}_a)\bar{x}^a\bar{x}^b\bar{x}^c + \gamma(\bar{t}, \bar{x}^a), \tag{5.8}$$

and γ is required to satisfy Laplace's equation: $\bar{\nabla}^2\gamma = 0$, with $\bar{\nabla}^2$ denoting the Laplacian operator in the coordinates $\bar{x}^a$.

Under the coordinate transformation the potentials become

$$\bar{U}(\bar{t}, \bar{x}^a) = \hat{U} - \dot{A} + \tfrac{1}{2}v^2 - a_a\bar{x}^a, \tag{5.9}$$

$$\bar{U}^a(\bar{t}, \bar{x}^a) = \hat{U}^a - \hat{U}v^a + \tfrac{1}{4}(V^a + V^a{}_b\bar{x}^b + \tfrac{1}{2}V^a{}_{bc}\bar{x}^b\bar{x}^c + \partial_{\bar{a}}\gamma), \tag{5.10}$$

$$\bar{\Psi}(\bar{t}, \bar{x}^a) = \hat{\Psi} - 4\hat{U}^a v_a + 2v^2\hat{U} + (A + v_b\bar{x}^b)\partial_{\bar{t}}\hat{U} + (F^a + F^a{}_b\bar{x}^b + \tfrac{1}{2}F^a{}_{bc}\bar{x}^b\bar{x}^c)\partial_{\bar{a}}\hat{U} + G + G_a\bar{x}^a + \tfrac{1}{2}G_{ab}\bar{x}^a\bar{x}^b + \tfrac{1}{6}G_{abc}\bar{x}^a\bar{x}^b\bar{x}^c - \partial_{\bar{t}}\gamma, \tag{5.11}$$

where

$$V^a = (2\dot{A} - v^2)v^a - \dot{H}^a + \epsilon^a{}_{bc}v^b R^c, \tag{5.12}$$

$$V^a{}_b = \tfrac{3}{2}v^a a_b + \tfrac{1}{2}a^a v_b + \delta^a{}_b(\tfrac{4}{3}\ddot{A} - 2v^c a_c) - \epsilon^a{}_{bc}\dot{R}^c, \tag{5.13}$$

$$V^a{}_{bc} = \tfrac{6}{5}(\delta^a{}_b \dot{a}_c + \delta^a{}_c \dot{a}_b) - \tfrac{4}{5}\delta_{bc}\dot{a}^a, \tag{5.14}$$

$$F^a = H^a - Av^a, \tag{5.15}$$

$$F^a{}_b = -\delta^a{}_b(\dot{A} - \tfrac{1}{2}v^2) - \tfrac{1}{2}v^a v_b + \epsilon^a{}_{bc}R^c, \tag{5.16}$$

$$F^a{}_{bc} = -(\delta^a{}_b a_c + \delta^a{}_c a_b) + a^a \delta_{bc}, \tag{5.17}$$

$$G = \tfrac{1}{2}\dot{A}^2 - \dot{A}v^2 + \tfrac{1}{4}v^4 + \dot{H}^a v_a, \tag{5.18}$$

$$G_a = (\dot{A} - \tfrac{1}{2}v^2)a_a - (\ddot{A} - \tfrac{3}{2}v^c a_c)v_a - \epsilon_{abc}v^b\dot{R}^c, \tag{5.19}$$

$$G_{ab} = a_a a_b - v_a \dot{a}_b - \dot{a}_a v_b + \delta_{ab}(v_c \dot{a}^c) - \tfrac{1}{3}\delta_{ab}A^{(3)}, \tag{5.20}$$

$$G_{abc} = -\tfrac{1}{5}(\delta_{ab}\ddot{a}_c + \delta_{ac}\ddot{a}_b + \delta_{bc}\ddot{a}_a). \tag{5.21}$$

Here $A^{(3)}$ stands for $d^3A/d\bar{t}^3$.

The "hatted" potentials are equal to the original potentials evaluated at time $t = \bar{t}$ and position $x^a = \bar{x}^a + z^a(\bar{t})$. For example,

$$\hat{U}(\bar{t}, \bar{x}^a) := U(t = \bar{t}, x^a = \bar{x}^a + z^a(\bar{t})). \tag{5.22}$$

Because U now possesses, in addition to its original explicit time dependence, an implicit time dependence contained in $z^a(\bar{t})$, some care must be exercised when taking time derivatives of $\hat{U}$. We have, for example,

$$\frac{\partial \hat{U}}{\partial \bar{t}} = \left(\frac{\partial U}{\partial t} + v^a \frac{\partial U}{\partial x^a}\right)_{t=\bar{t}, x=\bar{x}+z}, \qquad \frac{\partial \hat{U}}{\partial \bar{x}^a} = \left(\frac{\partial U}{\partial x^a}\right)_{t=\bar{t}, x=\bar{x}+z}. \tag{5.23}$$

The harmonic coordinate condition reads

$$\partial_{\bar{t}}\hat{U} - v^a\partial_{\bar{a}}\hat{U} + \partial_{\bar{a}}\hat{U}^a = 0 \tag{5.24}$$

when it is expressed in terms of the hatted potentials.

B. Post-Newtonian potentials in the black-hole frame

Our task in this subsection is to transform the potentials of Eqs. (4.10), (4.11), and (4.15) from the barycenter coordinates (t, x^a) to coordinates $(\bar{t}, \bar{x}^a)$ that are centered on the black hole. Each coordinate system is harmonic, and the transformation was described in the preceding subsection. The most important pieces of the coordinate transformation are the functions $z^a(\bar{t})$, and for these we choose

$$z^a(\bar{t}) = z^a(t = \bar{t}), \tag{5.25}$$

where $\boldsymbol{z}(t)$ is the black hole's position vector in the barycentric frame. In words, the coordinate displacements $z^a(\bar{t})$ are given by the black-hole position in the barycentric system, evaluated at the time $t = \bar{t}$. With this choice it follows that the quantities v^a and a^a that appear in the coordinate transformation are the same as those contained in the post-Newtonian potentials. The other freely specifiable pieces of the coordinate transformation will be determined in due course.

The transformed potentials $\bar{U}$, $\bar{U}^a$, and $\bar{\Psi}$ are expressed partly in terms of the hatted potentials. We have, for example, $\hat{U}(\bar{t}, \bar{x}^a) = U(\bar{t}, \bar{x}^a + z^a)$. According to Eqs. (4.10), (4.11), and (4.15), the hatted potentials are

$$\hat{U} = \frac{Gm}{\bar{r}} + \hat{U}_{\rm ext}, \tag{5.26}$$

$$\hat{U}^a = \frac{Gmv^a}{\bar{r}} + \hat{U}^a_{\rm ext}, \tag{5.27}$$

$$\hat{\Psi} = -\frac{Gm}{2\bar{r}^3}v_a v_b \bar{x}^a\bar{x}^b + \frac{Gm}{\bar{r}}\left(\mu + \frac{1}{2}v^2\right) - \frac{Gm}{2\bar{r}}a_a\bar{x}^a + \hat{\Psi}_{\rm ext}, \tag{5.28}$$

in which the vector $\boldsymbol{s} := \boldsymbol{x} - \boldsymbol{z}(t)$ has been identified with $\bar{\boldsymbol{x}}$; we continue to use the notation $\bar{r} := |\bar{\boldsymbol{x}}| = (\delta_{ab}\bar{x}^a\bar{x}^b)^{1/2}$.

The external potentials are given, for example, by

$$\hat{U}_{\rm ext}(\bar{t}, \bar{x}^a) := U_{\rm ext}(t = \bar{t}, x^a = \bar{x}^a + z^a(\bar{t})). \tag{5.29}$$

They satisfy the harmonic coordinate condition

$$\partial_{\bar{t}}\hat{U}_{\rm ext} - v^a\partial_{\bar{a}}\hat{U}_{\rm ext} + \partial_{\bar{a}}\hat{U}^a_{\rm ext} = 0. \tag{5.30}$$

Because it is well behaved near $\bar{x}^a$, each external potential can be expressed as a Taylor expansion about $\bar{\boldsymbol{x}} = 0$. For example,

$$\hat{U}_{\rm ext}(\bar{t}, \bar{\boldsymbol{x}}) = \hat{U}_{\rm ext}(\bar{t}, \boldsymbol{0}) + \partial_{\bar{a}}\hat{U}_{\rm ext}(\bar{t}, \boldsymbol{0}) + \tfrac{1}{2}\partial_{\bar{a}\bar{b}}\hat{U}_{\rm ext}(\bar{t}, \boldsymbol{0}) + \cdots. \tag{5.31}$$

This defines the strategy behind our calculation of the transformed potentials: Each quantity that is smooth at $\bar{\boldsymbol{x}} = \boldsymbol{0}$ will be expressed as a Taylor expansion. The potentials, therefore, will contain a piece that is singular at $\bar{\boldsymbol{x}} = \boldsymbol{0}$, and a smooth piece that will be expressed as a Taylor series. All Taylor expansions will be truncated at the quadratic order, and they will involve derivatives of the external potentials evaluated at $\bar{\boldsymbol{x}} = \boldsymbol{0}$.

The harmonic function $\gamma(\bar{t}, \bar{x}^a)$, introduced in Eq. (5.8), is smooth at $\bar{\boldsymbol{x}} = \boldsymbol{0}$, and it also can be expressed as a Taylor expansion. We write

$$\gamma(\bar{t}, \bar{x}^a) = C(\bar{t}) + \gamma_a(\bar{t})\bar{x}^a + \tfrac{1}{2}\gamma_{ab}(\bar{t})\bar{x}^a\bar{x}^b + \tfrac{1}{6}\gamma_{abc}(\bar{t})\bar{x}^a\bar{x}^b\bar{x}^c + \cdots. \tag{5.32}$$

To ensure that this is a solution to Laplace's equation, we must choose the expansion coefficients to be STF tensors. We express this property as $\gamma_{ab} = \gamma_{\langle ab\rangle}$ and $\gamma_{abc} = \gamma_{\langle abc\rangle}$. The expansion coefficients are otherwise arbitrary, and they will be determined in due course.

After a lengthy computation we obtain

$$\bar{U} = \frac{Gm}{\bar{r}} + {}_0U + {}_1U_a\bar{x}^a + {}_2U_{ab}\bar{x}^a\bar{x}^b + \cdots, \tag{5.33}$$

$$\bar{U}^a = {}_0U^a + {}_1U^a{}_b\bar{x}^b + {}_2U^a{}_{bc}\bar{x}^b\bar{x}^c + \cdots, \tag{5.34}$$

$$\bar{\Psi} = -\frac{Gm}{\bar{r}^3}(H_a - Av_a)\bar{x}^a + \frac{Gm}{\bar{r}}(\mu + \dot{A} - 2v^2) + {}_0\Psi + {}_1\Psi_a\bar{x}^a + {}_2\Psi_{ab}\bar{x}^a\bar{x}^b + \cdots, \tag{5.35}$$

with

$${}_0U = \tfrac{1}{2}v^2 - \dot{A} + \hat{U}_{\rm ext}, \tag{5.36}$$

$${}_1U_a = -a_a + \partial_{\bar{a}}\hat{U}_{\rm ext}, \tag{5.37}$$

$${}_2U_{ab} = \tfrac{1}{2}\partial_{\bar{a}\bar{b}}\hat{U}_{\rm ext}, \tag{5.38}$$

$${}_0U^a = \hat{U}_{\rm ext} - v^a\hat{U}_{\rm ext} + \tfrac{1}{4}(2\dot{A} - v^2) - \tfrac{1}{4}\dot{H}^a + \tfrac{1}{4}\epsilon^a{}_{bc}v^bR^c + \tfrac{1}{4}\gamma^a, \tag{5.39}$$

$${}_1U^a{}_b = \partial_{\bar{b}}\hat{U}^a_{\rm ext} - v^a\partial_{\bar{b}}\hat{U}_{\rm ext} + \tfrac{3}{8}v^aa_b + \tfrac{1}{8}a^av_b + \tfrac{1}{4}\delta^a{}_b(\tfrac{4}{3}\ddot{A} - 2v^ca_c) - \tfrac{1}{4}\epsilon^a{}_{bc}\dot{R}^c + \tfrac{1}{4}\gamma^a{}_b, \tag{5.40}$$

$${}_2U^a{}_{bc} = \tfrac{1}{2}\partial_{\bar{b}\bar{c}}\hat{U}^a_{\rm ext} - \tfrac{1}{2}v^a\partial_{\bar{b}\bar{c}}\hat{U}_{\rm ext} + \tfrac{3}{20}(\delta^a{}_b\dot{a}_c + \delta^a{}_c\dot{a}_b) - \tfrac{1}{10}\dot{a}^a\delta_{bc} + \tfrac{1}{8}\gamma^a{}_{bc}, \tag{5.41}$$

$${}_0\Psi = \hat{\Psi}_{\rm ext} - 4v_a\hat{U}^a_{\rm ext} + 2v^2\hat{U}_{\rm ext} + A\partial_{\bar{t}}\hat{U}_{\rm ext} + (H^a - Av^a)\partial_{\bar{a}}\hat{U}_{\rm ext} + \tfrac{1}{2}\dot{A}^2 - \dot{A}v^2 + \tfrac{1}{4}v^4 + \dot{H}^av_a - \dot{C}, \tag{5.42}$$

$$\begin{aligned}{}_1\Psi_a = {}& \partial_{\bar{a}}\hat{\Psi}_{\rm ext} - 4v_b\partial_{\bar{a}}\hat{U}^b_{\rm ext} + (\tfrac{5}{2}v^2 - \dot{A})\partial_{\bar{a}}\hat{U}_{\rm ext} - \tfrac{1}{2}v_av^b\partial_{\bar{b}}\hat{U}_{\rm ext} + v_a\partial_{\bar{t}}\hat{U}_{\rm ext} + A\partial_{\bar{t}\bar{a}}\hat{U}_{\rm ext} \\ & + (H^b - Av^b)\partial_{\bar{a}\bar{b}}\hat{U}_{\rm ext} + (\dot{A} - \tfrac{1}{2}v^2)a_a - (\ddot{A} - \tfrac{3}{2}v^ca_c)v_a - \epsilon_{abc}(\partial_{\bar{b}}\hat{U}_{\rm ext}R^c + v^b\dot{R}^c) - \dot{\gamma}_a,\end{aligned} \tag{5.43}$$

$$\begin{aligned}{}_2\Psi_{ab} = {}& \tfrac{1}{2}\partial_{\bar{a}\bar{b}}\hat{\Psi}_{\rm ext} - 2v_c\partial_{\bar{a}\bar{b}}\hat{U}^c_{\rm ext} + (\tfrac{3}{2}v^2 - \dot{A})\partial_{\bar{a}\bar{b}}\hat{U}_{\rm ext} - \tfrac{1}{2}v^cv_{(a}\partial_{\bar{b})\bar{c}}\hat{U}_{\rm ext} + v_{(a}\partial_{\bar{b})\bar{t}}\hat{U}_{\rm ext} + \tfrac{1}{2}A\partial_{\bar{t}\bar{a}\bar{b}}\hat{U}_{\rm ext} \\ & - a_{(a}\partial_{\bar{b})}\hat{U}_{\rm ext} + \tfrac{1}{2}\delta_{ab}a^c\partial_{\bar{c}}\hat{U}_{\rm ext} - \epsilon^c{}_{p(a}R^p\partial_{\bar{b})\bar{c}}\hat{U}_{\rm ext} + \tfrac{1}{2}(H^c - Av^c)\partial_{\bar{c}\bar{a}\bar{b}}\hat{U}_{\rm ext} + \tfrac{1}{2}a_aa_b - v_{(a}\dot{a}_{b)} \\ & + \tfrac{1}{2}\delta_{ab}(v^c\dot{a}_c) - \tfrac{1}{6}\delta_{ab}A^{(3)} - \tfrac{1}{2}\dot{\gamma}_{ab}.\end{aligned} \tag{5.44}$$

It is understood that in these expressions, the external potentials are evaluated at $\bar{\boldsymbol{x}} = \boldsymbol{0}$ after differentiation. This notational convention will be retained below.

C. Decomposition into irreducible pieces

To facilitate the matching procedure it is useful to decompose the tensors ${}_2U_{ab}$, ${}_1U_{ab}$, ${}_2U_{abc}$, and ${}_2\Psi_{ab}$ into their irreducible components. Equation (5.38) reveals that ${}_2U_{ab}$ is already a pure STF tensor, because $\bar{\nabla}^2\hat{U}_{\rm ext} = 0$ everywhere near the black hole.

We write

$${}_1U_{ab} = \tfrac{1}{3}\delta_{ab}{}_1U + {}_1U_{\langle ab\rangle} + {}_1U_{[ab]}, \tag{5.45}$$

which is a decomposition of ${}_1U_{ab}$ into a trace part, an STF part, and an antisymmetric part. Equation (5.40) implies

$${}_1U = \ddot{A} - \partial_{\bar{t}}\hat{U}_{\rm ext} - v^ca_c, \tag{5.46}$$

$${}_1U_{\langle ab\rangle} = \partial_{\langle\bar{a}}\hat{U}^{\rm ext}_{b\rangle} - v_{\langle a}\partial_{\bar{b}\rangle}\hat{U}_{\rm ext} + \tfrac{1}{2}v_{\langle a}a_{b\rangle} + \tfrac{1}{4}\gamma_{ab}, \tag{5.47}$$

$${}_1U_{[ab]} = -\partial_{[\bar{a}}\hat{U}^{\rm ext}_{b]} - v_{[a}\partial_{\bar{b}]}\hat{U}_{\rm ext} + \tfrac{1}{4}v_{[a}a_{b]} - \tfrac{1}{4}\epsilon_{abc}\dot{R}^c. \tag{5.48}$$

To decompose ${}_2U_{abc}$ we first isolate its completely symmetric part, and we write $U_{abc} = U_{(abc)} + V_{abc}$, where $V_{abc} = \frac{2}{3}(U_{[ab]c} + U_{[ac]b})$ is what is left over of U_{abc} after complete symmetrization. The first term is decomposed into trace and STF parts. For the second term we note that $\frac{2}{3}U_{[ab]c}$ possesses $3 \times 3 = 9$ independent components, so that it can be expressed as $\epsilon_{abp}X^p{}_c$, in terms of a general 3×3 matrix X_{ab}. This, in turn, can be decomposed as $X_{ab} = \frac{1}{3}\delta_{ab}X + V_{ab} + X_{[ab]}$, in terms of trace, STF, and antisymmetric components. Finally, we write $X_{[ab]} = \epsilon_{abp}V^p$, which relates the 3 independent components of $X_{[ab]}$ to those of a vector V^a. Altogether, we have $\frac{2}{3}U_{[ab]c} = \frac{1}{3}\epsilon_{abc}X + \delta_{ac}V_b - \delta_{bc}V_a + \epsilon_{abp}V^p{}_c$. This produces $V_{abc} = \delta_{ab}V_c + \delta_{ac}V_b - 2\delta_{bc}V_a + \epsilon_{abp}V^p{}_c + \epsilon_{acp}V^p{}_b$, and we obtain the decomposition

$${}_2U_{abc} = {}_2U_{\langle abc\rangle} + \tfrac{1}{5}(\delta_{ab2}U_c + \delta_{ac2}U_b + \delta_{bc2}U_a) + \delta_{ab2}V_c + \delta_{ac2}V_b - 2\delta_{bc2}V_a + \epsilon_{abp2}V^p{}_c + \epsilon_{acp2}V^p{}_b,$$

where ${}_2U_a := \delta^{bc}{}_2U_{(abc)}$. The 18 independent components of ${}_2U_{abc}$ have been packaged into the 7 components of

${}_2U_{\langle abc\rangle}$, the 3 components of ${}_2U_a$, the 3 components of ${}_2V_a$, and the 5 components of ${}_2V_{ab}$. Calculation shows that ${}_2V_a = \frac{1}{4}{}_2U_a$ and we obtain, finally,

$$ {}_2U_{abc} = {}_2U_{\langle abc\rangle} + \tfrac{9}{20}(\delta_{ab2}U_c + \delta_{ac2}U_b) - \tfrac{3}{10}\delta_{bc2}U_a + \epsilon_{abp2}V^p{}_c + \epsilon_{acp2}V^p{}_b, \tag{5.49} $$

with

$$ {}_2U_a = \tfrac{1}{3}(\dot{a}_a - \partial_{\bar{t}\bar{a}}\hat{U}_{\rm ext}), \tag{5.50} $$

$$ {}_2V_{ab} = -\tfrac{1}{6}\epsilon_{(a}{}^{pq}\partial_{\bar{b})\bar{p}}(\hat{U}_q^{\rm ext} - v_q\hat{U}_{\rm ext}), \tag{5.51} $$

$$ \begin{aligned} {}_2U_{\langle abc\rangle} = {} & \tfrac{1}{6}(\partial_{\bar{a}\bar{b}}\hat{U}_c^{\rm ext} + \partial_{\bar{a}\bar{c}}\hat{U}_b^{\rm ext} + \partial_{\bar{b}\bar{c}}\hat{U}_a^{\rm ext}) \\ & - \tfrac{1}{6}(v_c\partial_{\bar{a}\bar{b}}\hat{U}_{\rm ext} + v_b\partial_{\bar{a}\bar{c}}\hat{U}_{\rm ext} + v_a\partial_{\bar{b}\bar{c}}\hat{U}_{\rm ext}) \\ & + \tfrac{1}{15}\partial_{\bar{t}}(\delta_{ab}\partial_{\bar{c}}\hat{U}_{\rm ext} + \delta_{ac}\partial_{\bar{b}}\hat{U}_{\rm ext} + \delta_{bc}\partial_{\bar{a}}\hat{U}_{\rm ext}) \\ & + \tfrac{1}{8}\gamma_{abc}. \end{aligned} \tag{5.52} $$

That the right-hand side of Eq. (5.52) is STF follows from the facts that: (i) the potentials $\hat{U}_{\rm ext}$ and $\hat{U}^a_{\rm ext}$ satisfy Laplace's equation; (ii) they obey the harmonic condition of Eq. (5.30); and (iii) γ_{abc} is itself a STF tensor.

The decomposition of ${}_2\Psi_{ab}$ is

$$ {}_2\Psi_{ab} = {}_2\Psi_{\langle ab\rangle} + \tfrac{1}{3}\delta_{ab2}\Psi, \tag{5.53} $$

with

$$ {}_2\Psi = \tfrac{1}{2}(\partial_{\bar{t}\bar{t}}\hat{U}_{\rm ext} + a^2 + v^a\dot{a}_a - A^{(3)}), \tag{5.54} $$

$$ \begin{aligned} {}_2\Psi_{\langle ab\rangle} = {} & \tfrac{1}{2}\partial_{\bar{a}\bar{b}}\hat{\Psi}_{\rm ext} - 2v_c\partial_{\bar{a}\bar{b}}\hat{U}^c_{\rm ext} + (\tfrac{3}{2}v^2 - \dot{A})\partial_{\bar{a}\bar{b}}\hat{U}_{\rm ext} \\ & - \tfrac{1}{2}v^c v_{(a}\partial_{\bar{b})\bar{c}}\hat{U}_{\rm ext} + v_{(a}\partial_{\bar{b})\bar{t}}\hat{U}_{\rm ext} + \tfrac{1}{2}A\partial_{\bar{t}\bar{a}\bar{b}}\hat{U}_{\rm ext} \\ & - a_{(a}\partial_{\bar{b})}\hat{U}_{\rm ext} + \tfrac{1}{2}\delta_{ab}a^c\partial_{\bar{c}}\hat{U}_{\rm ext} \\ & - \epsilon^c{}_{p(a}R^p\partial_{\bar{b})\bar{c}}\hat{U}_{\rm ext} + \tfrac{1}{2}(H^c - Av^c)\partial_{\bar{c}\bar{a}\bar{b}}\hat{U}_{\rm ext} \\ & - \tfrac{1}{6}\delta_{ab}\partial_{\bar{t}\bar{t}}\hat{U}_{\rm ext} + \tfrac{1}{2}a_{\langle a}a_{b\rangle} - v_{\langle a}\dot{a}_{b\rangle} - \tfrac{1}{2}\dot{\gamma}_{ab}. \end{aligned} \tag{5.55} $$

The right-hand side of Eq. (5.55) is STF by virtue of the field equation satisfied by $\hat{\Psi}_{\rm ext}$. In the barycentric frame we have $\nabla^2\Psi_{\rm ext} = 2\partial_{tt}U_{\rm ext}$; transforming to the black hole's moving frame gives instead

$$ \bar{\nabla}^2\hat{\Psi}_{\rm ext} = \partial_{\bar{t}\bar{t}}\hat{U}_{\rm ext} - a^a\partial_{\bar{a}}\hat{U}_{\rm ext} - 2v^a\partial_{\bar{t}\bar{a}}\hat{U}_{\rm ext} + v^a v^b\partial_{\bar{a}\bar{b}}\hat{U}_{\rm ext}. \tag{5.56} $$

VI. MATCHING THE BLACK-HOLE AND POST-NEWTONIAN METRICS

Because the black-hole and post-Newtonian metrics are now expressed in the same coordinate system $(\bar{t}, \bar{x}^a)$ in the overlap region $\mathcal{O}$, we are finally ready to compare their expressions. We recall that the black-hole metric is valid when $\bar{r} \ll \mathcal{L}$, while the global post-Newtonian metric is valid when $\bar{r} \gg M$; the expansions provided in Eqs. (5.33), (5.34), and (5.35) are also restricted to the domain $\bar{r} \ll \mathcal{L}$. The metrics can be compared directly when $\bar{r}$ is such that $M \ll \bar{r} \ll \mathcal{L}$. A typical value of $\bar{r}$ in $\mathcal{O}$ might be $\bar{r}_c \sim \sqrt{M\mathcal{L}}$, and we have that $\bar{r}_c/\mathcal{L} \sim M/\bar{r}_c \sim v/c$; each quantity is indeed small. We write the metrics in Sec. VI A in a form that is ready for matching, and the matching conditions are extracted in Sec. VI B. In Sec. VI C we use them to determine the free functions associated with the coordinate transformation; this procedure reveals, in particular, the equations of motion for the black hole. In Sec. VI D we determine the free functions that appear in the black-hole and post-Newtonian metrics; it is here that the tidal moments $\bar{\mathcal{E}}_{ab}$ and $\bar{\mathcal{B}}_{ab}$ are finally obtained. In our last subsection, Sec. VI E, we transform the equations of motion and tidal moments from the black-hole frame back to the barycenter frame, in which they are most easily interpreted.

A. Metrics

In the overlap region $\mathcal{O}$ the black-hole metric can be expressed as a post-Newtonian expansion. Writing

$$ \bar{\mathcal{E}}_{ab} = \bar{\mathcal{E}}^{\rm N}_{ab} + \frac{1}{c^2}\bar{\mathcal{E}}^{\rm PN}_{ab} + O(c^{-4}), \tag{6.1} $$

we get

$$ g_{\bar{0}\bar{0}} = -1 + \frac{2Gm}{c^2\bar{r}} - \frac{2G^2m^2}{c^4\bar{r}^2} - \frac{1}{c^2}\bar{\mathcal{E}}^{\rm N}_{ab}\bar{x}^a\bar{x}^b - \frac{1}{c^4}\bar{\mathcal{E}}^{\rm PN}_{ab}\bar{x}^a\bar{x}^b + \frac{2Gm}{c^4\bar{r}}\bar{\mathcal{E}}^{\rm N}_{ab}\bar{x}^a\bar{x}^b + O(c^{-6}) + O\left(\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}\right), \tag{6.2} $$

and

$$ g_{\bar{0}\bar{a}} = \frac{2}{3c^3}\epsilon_{abp}\bar{\mathcal{B}}^p{}_c(\bar{t})\bar{x}^b\bar{x}^c + O(c^{-5}) + O\left(\frac{v}{c}\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}\right). \tag{6.3} $$

The post-Newtonian metric is obtained by inserting Eqs. (5.33), (5.34), and (5.35) within Eqs. (4.1), (4.2), and (4.3). The result is

$$ \begin{aligned} g_{\bar{0}\bar{0}} = {} & -1 + \frac{2Gm}{c^2\bar{r}} - \frac{2Gm}{c^4\bar{r}^3}(H_a - Av_a)\bar{x}^a + \frac{2Gm}{c^4\bar{r}}(\mu + \dot{A} - 2v^2) - \frac{2G^2m^2}{c^4\bar{r}^2} - \frac{4Gm}{c^4\bar{r}}({}_0U + {}_1U_a\bar{x}^a + {}_2U_{ab}\bar{x}^a\bar{x}^b) \\ & + \frac{2}{c^2}\left({}_0U + \frac{1}{c^2}{}_0\Psi - \frac{1}{c^2}({}_0U)^2\right) + \frac{2}{c^2}\left({}_1U_a + \frac{1}{c^2}{}_1\Psi_a - \frac{2}{c^2}{}_0U_1U_a\right)\bar{x}^a + \frac{2}{c^2}\left({}_2U_{ab} + \frac{1}{c^2}{}_2\Psi_{ab} - \frac{2}{c^2}{}_0U_2U_{ab}\right. \\ & \left. - \frac{1}{c^2}{}_1U_{a1}U_b\right)\bar{x}^a\bar{x}^b + O(c^{-6}) + O\left(\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}\right) \end{aligned} \tag{6.4} $$

and

$$g_{\bar{0}\bar{a}} = -\frac{4}{c^3}({}_0U_a + {}_1U_{ab}\bar{x}^b + {}_2U_{abc}\bar{x}^b\bar{x}^c) + O(c^{-5}) + O\left(\frac{v}{c}\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}\right). \quad (6.5)$$

Comparing Eq. (6.4) to Eq. (6.2), and Eq. (6.5) to Eq. (6.3) produces a complete set of matching conditions.

B. Matching conditions

From the absence of $\bar{x}^a$-independent terms in Eq. (6.2) we get ${}_0U + c^{-2}[{}_0\Psi - ({}_0U)^2] = O(c^{-4})$, which implies that ${}_0U = O(c^{-2})$. The equation simplifies to

$${}_0U + \frac{1}{c^2}{}_0\Psi = O(c^{-4}). \quad (6.6)$$

From the absence of terms linear in $\bar{x}^a$ we get ${}_1U_a + c^{-2}[{}_1\Psi_a - 2{}_0U\,{}_1U_a] = O(c^{-4})$, which implies

$${}_1U_a + \frac{1}{c^2}{}_1\Psi_a = O(c^{-4}). \quad (6.7)$$

Comparing the singular terms in $g_{\bar{0}\bar{0}}$ and taking into account the facts that ${}_0U = O(c^{-2})$ and ${}_1U = O(c^{-2})$, we obtain

$$H^a - Av^a = O(c^{-2}), \quad (6.8)$$

$$\mu + \dot{A} - 2v^2 = O(c^{-2}), \quad (6.9)$$

and

$$\bar{\mathcal{E}}^{\rm N}_{ab} = -2{}_2U_{ab}; \quad (6.10)$$

the last equation can be valid if and only if ${}_2U_{ab}$ is an STF tensor, a property that was already established in Sec. V C. Equation (6.10) is recovered by matching the terms that are quadratic in $\bar{x}^a$ in $g_{\bar{0}\bar{0}}$, and this also reveals that

$$\bar{\mathcal{E}}^{\rm PN}_{ab} = -2{}_2\Psi_{ab}. \quad (6.11)$$

We observe that unlike all preceding equations, Eqs. (6.10) and (6.11) do not include an error term $O(c^{-2})$; this is because each member of these equations is defined so as to possess a specific post-Newtonian order.

From the absence of $\bar{x}^a$-independent terms in Eq. (6.3) we get

$${}_0U_a = O(c^{-2}), \quad (6.12)$$

and from the absence of linear terms we obtain

$${}_1U_{ab} = O(c^{-2}). \quad (6.13)$$

Finally, matching the quadratic terms in $g_{\bar{0}\bar{a}}$ produces

$$\epsilon_{pa(b}\bar{\mathcal{B}}^p{}_{c)} = -6{}_2U_{abc} + O(c^{-2}). \quad (6.14)$$

The matching conditions of Eqs. (6.6), (6.7), (6.8), (6.9), (6.10), (6.11), (6.12), (6.13), and (6.14) allow the determination of the quantities

$$A, \quad z^a, \quad H^a, \quad R^a, \quad C, \quad \gamma_a, \quad \gamma_{ab}, \quad \gamma_{abc},$$

that appear in the transformation from the barycentric frame (t, x^a) to the black hole's moving frame $(\bar{t}, \bar{x}^a)$. They allow also the determination of the functions

$$\mu, \quad \bar{\mathcal{E}}^{\rm N}_{ab}, \quad \bar{\mathcal{E}}^{\rm PN}_{ab}, \quad \bar{\mathcal{B}}_{ab},$$

that appear in the post-Newtonian and black-hole metrics.

C. Determination of the coordinate transformation

From Eq. (6.6) we learn that ${}_0U = O(c^{-2})$, and with the expression for ${}_0U$ given in Eq. (5.36), we find that $A(\bar{t})$ is determined by the differential equation

$$\dot{A} = \tfrac{1}{2}v^2 + \hat{U}_{\rm ext}, \quad (6.15)$$

where $\hat{U}_{\rm ext} \equiv \hat{U}_{\rm ext}(\bar{t}, \mathbf{0})$ is the external Newtonian potential evaluated at $\bar{x}^a = 0$. With A determined, Eq. (6.8) implies

$$H^a = Av^a + O(c^{-2}), \quad (6.16)$$

and this determines $H^a(\bar{t})$.

From Eq. (6.7) we learn that ${}_1U_a = O(c^{-2})$, and with the expression for ${}_1U_a$ given in Eq. (5.37), we obtain an expression for the black hole's acceleration vector

$$a_a = \partial_{\bar{a}}\hat{U}_{\rm ext} + O(c^{-2}), \quad (6.17)$$

where the external potential is evaluated at $\bar{x}^a = 0$ after differentiation. This is a Newtonian approximation to the acceleration vector, and the post-Newtonian corrections will be determined below.

With ${}_0U = O(c^{-2})$ Eq. (6.6) implies ${}_0\Psi = O(c^{-2})$, and taking into account Eqs. (6.15), (6.16), and (6.17), Eq. (5.42) reveals that

$$\dot{C} = \hat{\Psi}_{\rm ext} - 4v_a\hat{U}^a_{\rm ext} + \tfrac{5}{2}v^2\hat{U}_{\rm ext} + \tfrac{1}{2}\hat{U}^2_{\rm ext} + A(\partial_{\bar{t}}\hat{U}_{\rm ext} + v^a\partial_{\bar{a}}\hat{U}_{\rm ext}) + \tfrac{3}{8}v^4 + O(c^{-2}), \quad (6.18)$$

in which each external potential is evaluated at $\bar{x}^a = 0$ after differentiation. This equation determines $C(\bar{t})$.

Equation (6.13) implies that each irreducible piece of ${}_1U_{ab}$ must vanish to order c^0. According to Eq. (5.45) we must have ${}_1U = 0 = {}_1U_{\langle ab\rangle} = {}_1U_{[ab]}$. With Eq. (5.46) we reproduce Eq. (6.15). From Eqs. (5.47) and (6.17) we get

$$\gamma_{ab} = -4\partial_{\langle\bar{a}}\hat{U}^{\rm ext}_{b\rangle} + 2v_{\langle a}\partial_{\bar{b}\rangle}\hat{U}_{\rm ext}. \quad (6.19)$$

And from Eq. (5.48) we find

$$\epsilon_{abc}\dot{R}^c = -4\partial_{[\bar{a}}\hat{U}^{\rm ext}_{b]} - 3v_{[a}\partial_{\bar{b}]}\hat{U}_{\rm ext}, \quad (6.20)$$

an equation that determines $R^a(\bar{t})$.

Taking into account Eqs. (6.15), (6.16), and (6.17), Eqs. (5.39) and (6.12) imply

$$\gamma_a = -4\hat{U}_a^{\rm ext} + (\tfrac{1}{2}v^2 + 3\hat{U}_{\rm ext})v_a + A\partial_{\bar{a}}\hat{U}_{\rm ext} - \epsilon_{abc}v^b R^c + O(c^{-2}). \tag{6.21}$$

We may now determine the post-Newtonian corrections to the acceleration vector. We return to Eq. (6.7), in which we insert Eqs. (5.37) and (5.43). We next incorporate Eqs. (6.15), (6.16), and (6.17), as well as Eqs. (6.20) and (6.21). After simplification we obtain

$$a^a = \partial^{\bar{a}}\hat{U}^{\rm ext} + \frac{1}{c^2}[\partial^{\bar{a}}\hat{\Psi}_{\rm ext} - 4v_b\partial^{\bar{a}}\hat{U}^b_{\rm ext} + 4\partial_{\bar{t}}\hat{U}^a_{\rm ext} + (v^2 - 4\hat{U}_{\rm ext})\partial^{\bar{a}}\hat{U}_{\rm ext} - (v^c\partial_{\bar{c}}\hat{U}_{\rm ext} + 3\partial_{\bar{t}}\hat{U}_{\rm ext})v^a] + O(c^{-4}), \tag{6.22}$$

where (as always) the external potentials are evaluated at $\bar{x}^a = 0$ after differentiation. Equation (6.22) is a system of second-order differential equations for the functions $z^a(\bar{t})$; they represent *equations of motion* for the black hole.

The last piece of the coordinate transformation that must be determined is γ_{abc}. The information comes from Eq. (6.14) and the decomposition of Eq. (5.49). Comparing the equations reveals that ${}_2U_a$ and ${}_2U_{\langle abc\rangle}$ must both vanish. The first statement reproduces Eq. (6.17), while the second implies

$$\gamma_{abc} = -\tfrac{4}{3}(\partial_{\bar{a}\bar{b}}\hat{U}^{\rm ext}_c + \partial_{\bar{a}\bar{c}}\hat{U}^{\rm ext}_b + \partial_{\bar{b}\bar{c}}\hat{U}^{\rm ext}_a) + \tfrac{4}{3}(v_c\partial_{\bar{a}\bar{b}}\hat{U}^{\rm ext} + v_b\partial_{\bar{a}\bar{c}}\hat{U}^{\rm ext} + v_a\partial_{\bar{b}\bar{c}}\hat{U}^{\rm ext}) - \tfrac{8}{15}\partial_{\bar{t}}(\delta_{ab}\partial_{\bar{c}}\hat{U}_{\rm ext} + \delta_{ac}\partial_{\bar{b}}\hat{U}_{\rm ext} + \delta_{bc}\partial_{\bar{a}}\hat{U}_{\rm ext}).$$

This can be expressed more compactly as

$$\gamma_{abc} = -4(\partial_{\langle\bar{a}\bar{b}}\hat{U}^{\rm ext}_{c\rangle} - v_{\langle a}\partial_{\bar{b}\bar{c}\rangle}\hat{U}^{\rm ext}). \tag{6.23}$$

The coordinate transformation is now completely determined by the matching conditions.

D. Determination of the metric functions

Equations (6.9) and (6.15) imply

$$\mu = \tfrac{3}{2}v^2 - \hat{U}_{\rm ext}, \tag{6.24}$$

in which the external potential is evaluated at $\bar{x}^a = 0$. As was first pointed out in the footnote that follows Eq. (4.13), this assignment can also be obtained by calculating the post-Newtonian potential ψ—see Eq. (4.4)—for a point particle of the same mass as the black hole. Our matching procedure shows that μ keeps the same value when the particle is replaced by the black hole.

The electric components of the tidal fields are determined by Eqs. (6.10) and (6.11). After inserting Eqs. (5.38) and (5.53) and invoking Eq. (6.15) to eliminate the trace part of ${}_2\Psi_{ab}$, we obtain

$$\bar{\mathcal{E}}^{\rm N}_{ab} = -\partial_{\bar{a}\bar{b}}\hat{U}_{\rm ext}, \tag{6.25}$$

and $\bar{\mathcal{E}}^{\rm PN}_{ab} = -2{}_2\Psi_{\langle ab\rangle}$. An explicit evaluation of this yields

$$\bar{\mathcal{E}}^{\rm PN}_{ab} = -\partial_{\langle\bar{a}\bar{b}\rangle}\hat{\Psi}_{\rm ext} + 4v_c\partial_{\bar{a}\bar{b}}\hat{U}^c_{\rm ext} - 4\partial_{\bar{t}\langle\bar{a}}\hat{U}^{\rm ext}_{b\rangle} - 2(v^2 - \hat{U}_{\rm ext})\partial_{\bar{a}\bar{b}}\hat{U}_{\rm ext} + v^c v_{\langle a}\partial_{\bar{b}\rangle\bar{c}}\hat{U}_{\rm ext} + 2v_{\langle a}\partial_{\bar{b}\rangle\bar{t}}\hat{U}_{\rm ext} + 3\partial_{\langle\bar{a}}\hat{U}_{\rm ext}\partial_{\bar{b}\rangle}\hat{U}_{\rm ext} - A\partial_{\bar{t}\bar{a}\bar{b}}\hat{U}_{\rm ext} + 2\epsilon_{cp(a}R^p\partial^{\bar{c}}{}_{\bar{b})}\hat{U}_{\rm ext}, \tag{6.26}$$

where, as usual, the external potentials are evaluated at $\bar{x}^a = 0$ after differentiation. The complete tidal potentials are $\bar{\mathcal{E}}_{ab} = \bar{\mathcal{E}}^{\rm N}_{ab} + c^{-2}\bar{\mathcal{E}}^{\rm PN}_{ab} + O(c^{-4})$, as was expressed in Eq. (6.1).

The magnetic components of the tidal fields are determined by Eq. (6.14) and the decomposition of Eq. (5.49). Taking into account the facts that ${}_2U_a$ and ${}_2U_{\langle abc\rangle}$ must both vanish (as was noted previously), comparing the equations reveals that $\bar{\mathcal{B}}_{ab} = -12{}_2V_{ab} + O(c^{-2})$. With Eq. (5.51), this is

$$\bar{\mathcal{B}}_{ab} = 2\epsilon_{(a}{}^{pq}\partial_{\bar{b})\bar{p}}(\hat{U}^{\rm ext}_q - v_q\hat{U}_{\rm ext}) + O(c^{-2}). \tag{6.27}$$

The metric functions are now fully determined by the matching conditions.

E. Transformation to the barycentric frame

In the moving frame $(\bar{t}, \bar{x}^a)$ the black hole is situated at $\bar{x}^a = 0$. According to Eqs. (5.1), (5.2), (5.3), and (5.4), the position of the black hole in the barycentric frame is described by the parametric equations

$$t_{\rm bh} = \bar{t} + \frac{1}{c^2}A(\bar{t}) + O(c^{-4}),\qquad x^a_{\rm bh} = z^a(\bar{t}) + \frac{1}{c^2}H^a(\bar{t}) + O(c^{-4}). \tag{6.28}$$

The first equation can be approximately inverted as $\bar{t} = t_{\rm bh} - c^{-2}A(t_{\rm bh}) + O(c^{-4})$, and substitution into the second equation yields $x^a_{\rm bh} = z^a(t_{\rm bh}) + c^{-2}[H^a(t_{\rm bh}) - A(t_{\rm bh})v^a(t_{\rm bh})] + O(c^{-4})$. With Eq. (6.16) this becomes

$$x^a_{\rm bh} = z^a(t_{\rm bh}) + O(c^{-4}), \tag{6.29}$$

which is the same statement as Eq. (5.25). The position of the black hole in the barycenter frame is therefore obtained simply by evaluating the functions $z^a(\bar{t})$ at the time $\bar{t} = t_{\rm bh}$. From this observation it follows that the black hole's barycentric velocity is $v^a(t_{\rm bh})$, and its acceleration is $a^a(t_{\rm bh})$. Henceforth we shall omit the label "bh" on the barycentric time coordinate.

The equations of motion for the black hole, expressed in the barycentric frame, are obtained from Eq. (6.22) by replacing the hatted potentials ($\hat{U}_{\rm ext}$ and so on) with the original potentials ($U_{\rm ext}$ and so on) using the correspondence of Eqs. (5.22) and (5.23). Noting that Eq. (6.22) is to be evaluated at $\bar{t} = t_{\rm bh} \equiv t$, we get

$$
\begin{aligned}
a^a = \partial^a U^{\rm ext} + \frac{1}{c^2}[\partial^a \Psi_{\rm ext} - 4(\partial^a U^b_{\rm ext} - \partial^b U^a_{\rm ext})v_b \\
+ 4\partial_t U^a_{\rm ext} + (v^2 - 4U_{\rm ext})\partial^a U_{\rm ext} \\
- v^a(4v^b\partial_b U_{\rm ext} + 3\partial_t U_{\rm ext})] + O(c^{-4}).
\end{aligned}
\tag{6.30}
$$

The external potentials were introduced in Eqs. (4.10), (4.11), and (4.15), and here they are evaluated at $\boldsymbol{x} = \boldsymbol{z}(t)$ after differentiation. Equation (6.30) applies to a black hole moving in any post-Newtonian environment. When this environment consists of $(N - 1)$ external bodies, so that the black hole is a member of an N-body system, Eq. (6.30) reduces to the standard (Einstein-Infeld-Hoffman) post-Newtonian equations of motion. Because this connection is well understood, we shall not provide here a derivation of this well-known fact; the Einstein-Infeld-Hoffman equations are listed, for example, in Exercise 39.15 of Misner, Thorne, and Wheeler [22].

Following Racine and Flanagan [12] we define barycentric tidal moments $\mathcal{E}_{ab}(t)$ and $\mathcal{B}_{ab}(t)$ that are related to those of the black-hole frame by the transformation

$$
\begin{aligned}
\mathcal{E}_{ab}(t) := \mathcal{M}_a{}^c(\bar{t})\mathcal{M}_b{}^d(\bar{t})\bar{\mathcal{E}}_{cd}(\bar{t}), \\
\mathcal{B}_{ab}(t) := \mathcal{M}_a{}^c(\bar{t})\mathcal{M}_b{}^d(\bar{t})\bar{\mathcal{B}}_{cd}(\bar{t}),
\end{aligned}
\tag{6.31}
$$

where

$$
\mathcal{M}_{ab}(\bar{t}) := \delta_{ab} + \frac{1}{c^2}\epsilon_{abc}R^c(\bar{t}) + O(c^{-4}) \tag{6.32}
$$

is a post-Newtonian rotation matrix that accounts for the precession of the moving frame relative to the barycentric frame. We recall that the time coordinates are related by $t = \bar{t} + c^{-2}A(\bar{t}) + O(c^{-4})$. The quantities $A(\bar{t})$ and $R^a(\bar{t})$ that appear in the transformation are determined by Eqs. (6.15) and (6.20), respectively. The inverse transformation is

$$
\begin{aligned}
\bar{\mathcal{E}}_{ab}(\bar{t}) = \mathcal{N}_a{}^c(t)\mathcal{N}_b{}^d(t)\mathcal{E}_{cd}(t), \\
\bar{\mathcal{B}}_{ab}(\bar{t}) = \mathcal{N}_a{}^c(t)\mathcal{N}_b{}^d(t)\mathcal{B}_{cd}(t),
\end{aligned}
\tag{6.33}
$$

where

$$
\mathcal{N}_{ab}(t) := \delta_{ab} - \frac{1}{c^2}\epsilon_{abc}R^c(t) + O(c^{-4}) \tag{6.34}
$$

is the inverse to $\mathcal{M}_{ab}$. In these equations we have $\bar{t} = t - c^{-2}A(t) + O(c^{-4})$, and the quantities $A(t)$ and $R^a(t)$ are determined by

$$
\begin{aligned}
\frac{dA}{dt} = \frac{1}{2}v^2 + U_{\rm ext}, \\
\epsilon_{abc}\frac{dR^c}{dt} = -4\partial_{[a}U^{\rm ext}_{b]} - 3v_{[a}\partial_{b]}U_{\rm ext}.
\end{aligned}
\tag{6.35}
$$

Expanding Eqs. (6.32) in powers of c^{-2} produces $\mathcal{E}_{ab} = \bar{\mathcal{E}}_{ab} + c^{-2}[-A\partial_{\bar{t}}\bar{\mathcal{E}}_{ab} + 2\epsilon_{cp(a}R^p\bar{\mathcal{E}}^c{}_{b)}] + O(c^{-4})$ and $\mathcal{B}_{ab} = \bar{\mathcal{B}}_{ab} + O(c^{-2})$. In this we substitute Eqs. (6.25), (6.26), and (6.27), and we replace the hatted potentials by the original potentials using the correspondence of Eqs. (5.22) and (5.23). After simplification we obtain

$$
\begin{aligned}
\mathcal{E}_{ab} = -\partial_{ab}U_{\rm ext} + \frac{1}{c^2} - (\partial_{\langle ab\rangle}\Psi_{\rm ext} + 4v^c(\partial_{ab}U^{\rm ext}_c \\
- \partial_{c\langle a}U^{\rm ext}_{b\rangle}) - 4\partial_{t\langle a}U^{\rm ext}_{b\rangle} - 2(v^2 - U_{\rm ext})\partial_{ab}U_{\rm ext} \\
+ 3v^c v_{\langle a}\partial_{b\rangle c}U_{\rm ext} + 2v_{\langle a}\partial_{b\rangle t}U_{\rm ext} \\
+ 3\partial_{\langle a}U_{\rm ext}\partial_{b\rangle}U_{\rm ext}) + O(c^{-4})
\end{aligned}
\tag{6.36}
$$

and

$$
\mathcal{B}_{ab} = 2\epsilon_{pq(a}\partial^p{}_{b)}(U^q_{\rm ext} - v^q U_{\rm ext}) + O(c^{-2}). \tag{6.37}
$$

In these equations the external potentials are evaluated at $\boldsymbol{x} = \boldsymbol{z}(t)$ after differentiation. Notice that unlike $\bar{\mathcal{E}}_{ab}$, the barycentric tidal moment $\mathcal{E}_{ab}$ does not involve the functions A and R^a that must be obtained by integrating first-order differential equations; this was the reason for introducing the transformation of Eq. (6.32). Notice also that since $\bar{\mathcal{B}}_{ab}$ has been worked out to leading-order only, its transformation to the barycentric frame is trivial.

VII. TIDAL MOMENTS FOR A TWO-BODY SYSTEM

The results obtained in the preceding section apply to any post-Newtonian environment described by the external potentials $U_{\rm ext}$, $U^a_{\rm ext}$, and $\Psi_{\rm ext}$. Given these potentials, the motion of the black hole in the barycentric frame (t, x^a) is determined by Eq. (6.30), and the barycentric tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ are obtained by evaluating Eqs. (6.36) and (6.37), respectively. The tidal moments perceived by the black hole are $\bar{\mathcal{E}}_{ab}$ and $\bar{\mathcal{B}}_{ab}$, and these are calculated using the transformation of Eqs. (6.34), (6.35), and (6.36).

In this section we specialize our discussion to a specific post-Newtonian environment that consists of a single external body (perhaps another black hole), so that the black hole is a member of a post-Newtonian two-body system. Adapting our notation to this specific situation, we let our original black hole have a mass m_1, position $\boldsymbol{z}_1(t)$, velocity $\boldsymbol{v}_1(t)$, acceleration $\boldsymbol{a}_1(t)$, and so on. (These quantities were previously denoted m, $\boldsymbol{z}$, $\boldsymbol{v}$, and $\boldsymbol{a}$, respectively. Below we will redefine m to be the system's total mass $m_1 + m_2$, and $\boldsymbol{v}$ to be the system's relative velocity $\boldsymbol{v}_1 - \boldsymbol{v}_2$.) The second body also is modeled as a post-Newtonian monopole, and it has a mass m_2, position $\boldsymbol{z}_2(t)$, velocity $\boldsymbol{v}_2(t)$, acceleration $\boldsymbol{a}_2(t)$, and so on.

In Sec. VII A we list the potentials associated with the external body, evaluate their derivatives, and calculate the barycentric tidal moments. In Sec. VII B we simplify our expressions by writing them in terms of $\boldsymbol{r} := \boldsymbol{z}_1 - \boldsymbol{z}_2$, the relative separation between the two bodies, from which the individual trajectories can be recovered. In Sec. VII C we restrict our attention to a binary system in circular motion, and in Sec. VII D we compute the tidal moments as viewed in the moving frame of the black hole. Finally, in Sec. VII E we compare our post-Newtonian answers to those obtained by Poisson [14] in the context of the small-hole approximation (see Sec. I A).

Our calculations in this section rely on well-known results from post-Newtonian theory. These can be found, for example, in Blanchet's review article [40].

A. Two-body potentials and tidal moments

Assuming that the external body is a post-Newtonian monopole of mass m_2, the external potentials can be expressed in a form that is directly analogous to that of the black-hole potentials of Eqs. (4.10), (4.11), (4.12), and (4.13). We have $U_{\rm ext} = Gm_2/s$, $U^a_{\rm ext} = Gm_2 v_2^a/s$, $\psi_{\rm ext} = Gm_2\mu_2/s$, $X_{\rm ext} = Gm_2 s$, and $\Psi_{\rm ext} = \psi_{\rm ext} + \frac{1}{2}\partial_t^2 X_{\rm ext}$, where s now stands for $|\boldsymbol{x} - \boldsymbol{z}_2(t)|$ and $\mu_2 := \frac{3}{2}v_2^2 - Gm_1/|\boldsymbol{z}_1 - \boldsymbol{z}_2|$.

These potentials are easily differentiated, and after evaluation at $\boldsymbol{x} = \boldsymbol{z}_1(t)$ we obtain

$$U_{\rm ext} = \frac{Gm_2}{r}, \tag{7.1}$$

$$\partial_a U_{\rm ext} = -\frac{Gm_2}{r^2} n_a, \tag{7.2}$$

$$\partial_{ab} U_{\rm ext} = \frac{Gm_2}{r^3}(3n_a n_b - \delta_{ab}), \tag{7.3}$$

$$\partial_{ta} U_{\rm ext} = -\frac{Gm_2}{r^3}(3n_a n_b - \delta_{ab})v_2^b, \tag{7.4}$$

$$\partial_b U^a_{\rm ext} = -\frac{Gm_2 v_2^a}{r^2} n_b, \tag{7.5}$$

$$\partial_{bc} U^a_{\rm ext} = \frac{Gm_2 v_2^a}{r^3}(3n_b n_c - \delta_{bc}), \tag{7.6}$$

$$\partial_{tb} U^a_{\rm ext} = -\frac{G^2 m_1 m_2}{r^4} n^a n_b - \frac{Gm_2 v_2^a}{r^3}(3n_b n_c - \delta_{bc})v_2^c, \tag{7.7}$$

$$\begin{aligned}\partial_{ab}\Psi_{\rm ext} = {}& \frac{Gm_2}{r^3}\left(2v_2^2 - \frac{Gm_1}{r}\right)(3n_a n_b - \delta_{ab}) \\ &+ \frac{Gm_2}{2r^3}[3(\boldsymbol{n}\cdot\boldsymbol{v}_2)^2(\delta_{ab} - 5n_a n_b) \\ &+ 12(\boldsymbol{n}\cdot\boldsymbol{v}_2)v_{2(a}n_{b)} - 2v_{2a}v_{2b}] \\ &+ \frac{G^2 m_1 m_2}{2r^4}(\delta_{ab} - n_a n_b).\end{aligned} \tag{7.8}$$

We have introduced the new quantities

$$\boldsymbol{r} := \boldsymbol{z}_1 - \boldsymbol{z}_2, \qquad r := |\boldsymbol{z}_1 - \boldsymbol{z}_2|, \qquad \boldsymbol{n} := \boldsymbol{r}/r. \tag{7.9}$$

To arrive at Eqs. (7.7) and (7.8) we used the equations of motion $\boldsymbol{a}_2 = Gm_1\boldsymbol{n}/r^2 + O(c^{-2})$ to replace the acceleration vector of the second body by its Newtonian expression.

Making the substitutions into Eqs. (6.36) and (6.37) gives

$$\begin{aligned}\mathcal{E}_{ab} = {}& -\frac{3Gm_2}{r^3}n_{\langle ab\rangle} - \frac{3Gm_2}{c^2 r^3}\bigg(\bigg[2v_1^2 - 4(\boldsymbol{v}_1\cdot\boldsymbol{v}_2) + 2v_2^2 \\ &- \frac{5}{2}(\boldsymbol{n}\cdot\boldsymbol{v}_2)^2 - \frac{5}{2}\frac{Gm_1}{r} - 3\frac{Gm_2}{r}\bigg]n_{\langle ab\rangle} \\ &- [3(\boldsymbol{n}\cdot\boldsymbol{v}_1) - 2(\boldsymbol{n}\cdot\boldsymbol{v}_2)]n_{\langle a}v_{1b\rangle} + [4(\boldsymbol{n}\cdot\boldsymbol{v}_1) \\ &- 2(\boldsymbol{n}\cdot\boldsymbol{v}_2)]n_{\langle a}v_{2b\rangle} + v_{1\langle a}v_{1b\rangle} - 2v_{1\langle a}v_{2b\rangle} \\ &+ v_{2\langle a}v_{2b\rangle}\bigg) + O(c^{-4}),\end{aligned} \tag{7.10}$$

and

$$\mathcal{B}_{ab} = -\frac{6Gm_2}{r^3}[\boldsymbol{n}\times(\boldsymbol{v}_1 - \boldsymbol{v}_2)]_{(a}n_{b)} + O(c^{-2}), \tag{7.11}$$

where $n_{\langle ab\rangle} := n_a n_b - \frac{1}{3}\delta_{ab}$. The quantities $A(t)$ and $R^a(t)$ that appear in the transformation from the barycentric frame to the black-hole frame are determined by the equations

$$\begin{aligned}\dot{A} &= \frac{1}{2}v_1^2 + \frac{Gm_2}{r} + O(c^{-2}), \\ \dot{\boldsymbol{R}} &= \frac{Gm_2}{2r^2}\boldsymbol{n}\times(4\boldsymbol{v}_2 - 3\boldsymbol{v}_1) + O(c^{-2});\end{aligned} \tag{7.12}$$

these are obtained from Eqs. (6.36).

It is straightforward to generalize Eqs. (7.10), (7.11), and (7.12) from a two-body system to an N-body system by simply writing the external potentials as a sum of single-body terms. The generalized expressions can then be compared with the corresponding results of Damour, Soffel, and Xu—see, in particular, Eqs. (4.29)–(4.31) of Ref. [11]. As was already stated in Sec. I B, we find that our expressions agree with theirs.

B. Generic orbital motion

To simplify the foregoing results we incorporate the fact that the motion of each body in a post-Newtonian two-body system can be related to the motion of the *relative orbit*, which is described by the separation vector $\boldsymbol{r} = \boldsymbol{z}_1 - \boldsymbol{z}_2$ and the relative velocity vector $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$. The post-Newtonian dynamics implies that if the system's barycenter is placed at the origin of the coordinate system, then $\boldsymbol{z}_1 = (m_2/m)\boldsymbol{r} + O(c^{-2})$ and $\boldsymbol{z}_2 = -(m_1/m)\boldsymbol{r} + O(c^{-2})$, where $m := m_1 + m_2$ is the total mass of the two-body system. As a consequence we also have $\boldsymbol{v}_1 = (m_2/m)\boldsymbol{v} + O(c^{-2})$ and $\boldsymbol{v}_2 = -(m_1/m)\boldsymbol{v} + O(c^{-2})$, and we make these substitutions in Eqs. (7.10), (7.11), and (7.12).

In addition we incorporate the fact that the post-Newtonian motion of a two-body system takes place in a fixed orbital plane. We take this plane to be the x-y plane, and we use polar coordinates r and ϕ to describe the orbital motion. We write $\boldsymbol{r} = (r\cos\phi, r\sin\phi, 0)$, and we resolve all vectors in the basis $\boldsymbol{n} = (\cos\phi, \sin\phi, 0)$, $\boldsymbol{\phi} = (-\sin\phi, \cos\phi, 0)$, and $\boldsymbol{l} = (0, 0, 1)$ associated with the polar coordinates; the vector $\boldsymbol{l}$ is normal to the plane, and it is aligned with the system's total angular momentum. We have

$$\boldsymbol{r} = r\boldsymbol{n}, \qquad \boldsymbol{v} = \dot{r}\boldsymbol{n} + (r\dot{\phi})\boldsymbol{\phi}, \tag{7.13}$$

and we also make these substitutions in Eqs. (7.10), (7.11), and (7.12).

After simplification our results become

$$\begin{aligned}\mathcal{E}_{ab} = &-\frac{3Gm_2}{r^3}n_{\langle ab\rangle} - \frac{3Gm_2}{c^2r^3}\bigg(\bigg[-\frac{3m_1^2}{2m^2}\dot{r}^2 + 2(r\dot{\phi})^2 \\ &- \frac{5Gm_1 + 6Gm_2}{2r}\bigg]n_{\langle ab\rangle} \\ &- \frac{(2m_1 + m_2)m_2}{m^2}\dot{r}(r\dot{\phi})n_{(a}\phi_{b)} + (r\dot{\phi})^2\phi_{\langle ab\rangle}\bigg) \\ &+ O(c^{-4}),\end{aligned} \tag{7.14}$$

$$\mathcal{B}_{ab} = -\frac{6Gm_2}{r^3}(r\dot{\phi})l_{(a}n_{b)} + O(c^{-2}), \tag{7.15}$$

$$\dot{A} = \frac{m_2^2}{2m^2}[\dot{r}^2 + (r\dot{\phi})^2] + \frac{Gm_2}{r} + O(c^{-2}), \tag{7.16}$$

$$\dot{R}^a = -\frac{Gm_2}{2r^2}\frac{4m_1 + 3m_2}{m}(r\dot{\phi})l^a + O(c^{-2}). \tag{7.17}$$

We recall that $m = m_1 + m_2$ is the total mass of the system, $r := |z_1 - z_2|$ is the interbody distance, ϕ is the angular position of the relative orbit in the orbital plane, $\boldsymbol{n}$ is a unit vector that points from body 2 to body 1, $\boldsymbol{\phi}$ is a unit vector that points in the direction of increasing ϕ, and finally, $\boldsymbol{l}$ is the unit normal to the orbital plane. We use the notation $n_{\langle ab\rangle} = n_a n_b - \frac{1}{3}\delta_{ab}$ and $\phi_{\langle ab\rangle} = \phi_a\phi_b - \frac{1}{3}\delta_{ab}$.

C. Circular motion

To specialize to circular orbits we set $\dot{r} = 0$ and $r\dot{\phi} = v$ in the preceding results. The post-Newtonian equations of motion imply that

$$\omega := \dot{\phi} = \sqrt{\frac{Gm}{r^3}}\bigg[1 - \frac{1}{2}(3 - \eta)(v/c)^2 + O(c^{-4})\bigg], \tag{7.18}$$

where $m = m_1 + m_2$ is total mass and $\eta := m_1m_2/m^2$ is a dimensionless reduced mass. As a consequence of Eq. (7.18) we find that $v = \sqrt{Gm/r} + O(c^{-2})$. Making these substitutions in Eqs. (7.14), (7.15), (7.16), and (7.17) gives

$$\begin{aligned}\mathcal{E}_{ab} = &-\frac{3Gm_2}{r^3}\bigg(\bigg[1 - \frac{m_1 + 2m_2}{2m}(v/c)^2\bigg]n_{\langle ab\rangle} \\ &+ (v/c)^2\phi_{\langle ab\rangle}\bigg) + O(c^{-4}),\end{aligned} \tag{7.19}$$

$$\mathcal{B}_{ab} = -\frac{6Gm_2}{r^3}vl_{(a}n_{b)} + O(c^{-2}), \tag{7.20}$$

$$\dot{A} = \frac{(2m_1 + 3m_2)m_2}{2m^2}v^2 + O(c^{-2}), \tag{7.21}$$

$$\dot{R}^a = -\frac{Gm_2}{2r^2}\frac{4m_1 + 3m_2}{m}vl^a + O(c^{-2}). \tag{7.22}$$

For circular orbits we also have $\phi = \omega t$, with ω given by Eq. (7.18).

To list the components of the tidal moments it is useful to define

$$\begin{aligned}&\mathcal{E}_0 := \tfrac{1}{2}(\mathcal{E}_{11} + \mathcal{E}_{22}), \qquad \mathcal{E}_{1c} := \mathcal{E}_{13}, \qquad \mathcal{E}_{1s} := \mathcal{E}_{23}, \\ &\mathcal{E}_{2c} := \tfrac{1}{2}(\mathcal{E}_{11} - \mathcal{E}_{22}), \qquad \mathcal{E}_{2s} := \mathcal{E}_{12},\end{aligned} \tag{7.23}$$

and

$$\begin{aligned}&\mathcal{B}_0 := \tfrac{1}{2}(\mathcal{B}_{11} + \mathcal{B}_{22}), \qquad \mathcal{B}_{1c} := \mathcal{B}_{13}, \qquad \mathcal{B}_{1s} := \mathcal{B}_{23}, \\ &\mathcal{B}_{2c} := \tfrac{1}{2}(\mathcal{B}_{11} - \mathcal{B}_{22}), \qquad \mathcal{B}_{2s} := \mathcal{B}_{12}.\end{aligned} \tag{7.24}$$

With the vectorial basis $\boldsymbol{n} = (\cos\phi, \sin\phi, 0)$, $\boldsymbol{\phi} = (-\sin\phi, \cos\phi, 0)$, and $\boldsymbol{l} = (0, 0, 1)$ we find that the nonvanishing components of the tidal moments are

$$\mathcal{E}_0 = -\frac{Gm_2}{2r^3}\bigg[1 + \frac{m_1}{2m}(v/c)^2 + O(c^{-4})\bigg], \tag{7.25}$$

$$\begin{aligned}\mathcal{E}_{2c} = &-\frac{3Gm_2}{2r^3}\bigg[1 - \frac{3m_1 + 4m_2}{2m}(v/c)^2 + O(c^{-4})\bigg] \\ &\times \cos 2\phi,\end{aligned} \tag{7.26}$$

$$\begin{aligned}\mathcal{E}_{2s} = &-\frac{3Gm_2}{2r^3}\bigg[1 - \frac{3m_1 + 4m_2}{2m}(v/c)^2 + O(c^{-4})\bigg] \\ &\times \sin 2\phi,\end{aligned} \tag{7.27}$$

$$\mathcal{B}_{1c} = -\frac{3Gm_2}{r^3}v\cos\phi + O(c^{-2}), \tag{7.28}$$

$$\mathcal{B}_{1s} = -\frac{3Gm_2}{r^3}v\sin\phi + O(c^{-2}). \tag{7.29}$$

The components $\mathcal{E}_{1c}$, $\mathcal{E}_{1s}$, $\mathcal{B}_0$, $\mathcal{B}_{2c}$, and $\mathcal{B}_{2s}$ all vanish for circular orbits. These results were already displayed in Sec. I A; in Eqs. (1.10), (1.11), (1.12), (1.13), and (1.14) we used the symbol b (instead of r) for the interbody distance, $v_{\rm rel}$ (instead of v) for the relative orbital velocity, m (instead of m_1) for the black-hole mass, and m' (instead of m_2) for the mass of the external body.

D. Tidal moments in the black-hole frame

The tidal moments of Eqs. (7.25), (7.26), (7.27), (7.28), and (7.29) refer to the barycentric frame. We may express them in the moving frame of the black hole by invoking the transformation of Eqs. (6.34), (3.35), and (3.36). The transformation involves a switch from global time t to local time $\bar{t}$, and a rotation of the Cartesian axes mediated by the vector $\boldsymbol{R}(t)$. According to Eqs. (7.21) the transformation of the time coordinate is given by

$$t = \left[1 + \frac{(2m_1 + 3m_2)m_2}{2m^2}(v/c)^2 + O(c^{-4})\right]\bar{t}. \quad (7.30)$$

And according to Eq. (7.22) we have $c^{-2}R^a = -(\Omega\bar{t})l^a$, with

$$\Omega := \sqrt{\frac{Gm}{r^3}}\frac{(4m_1 + 3m_2)m_2}{2m^2}(v/c)^2 + O(c^{-4}) \quad (7.31)$$

denoting the precessional angular velocity of the moving frame relative to the barycentric frame. (This is the rotation of the coordinate axes, not the rotational motion of the black hole on its orbit.) The rotation takes place around the z axis, and it is easy to show that it is effected by the transformation $\phi \to \bar{\phi} = \phi - \Omega\bar{t}$.

Altogether we find that the tidal moments are given by

$$\bar{\mathcal{E}}_0 = -\frac{Gm_2}{2r^3}\left[1 + \frac{m_1}{2m}(v/c)^2 + O(c^{-4})\right], \quad (7.32)$$

$$\bar{\mathcal{E}}_{2c} = -\frac{3Gm_2}{2r^3}\left[1 - \frac{3m_1 + 4m_2}{2m}(v/c)^2 + O(c^{-4})\right] \times \cos 2\bar{\phi}, \quad (7.33)$$

$$\bar{\mathcal{E}}_{2s} = -\frac{3Gm_2}{2r^3}\left[1 - \frac{3m_1 + 4m_2}{2m}(v/c)^2 + O(c^{-4})\right] \times \sin 2\bar{\phi}, \quad (7.34)$$

$$\bar{\mathcal{B}}_{1c} = -\frac{3Gm_2}{r^3}v\cos\bar{\phi} + O(c^{-2}), \quad (7.35)$$

$$\bar{\mathcal{B}}_{1s} = -\frac{3Gm_2}{r^3}v\sin\bar{\phi} + O(c^{-2}), \quad (7.36)$$

in the moving frame. Tensorial expressions for $\bar{\mathcal{E}}_{ab}$ and $\bar{\mathcal{B}}_{ab}$ can be obtained directly from Eqs. (7.19) and (7.20) by making the substitution $\phi \to \bar{\phi}$. After involvement of Eqs. (7.18), (7.30), and (7.31) we find that $\bar{\phi} = \bar{\omega}\,\bar{t}$, with

$$\bar{\omega} = \sqrt{\frac{Gm}{r^3}}\left[1 - \frac{1}{2}(3 + \eta)(v/c)^2 + O(c^{-4})\right]. \quad (7.37)$$

This is the angular frequency of the tidal moments as measured in the moving frame of the black hole. The transformation from ω to $\bar{\omega}$ involves a switch from barycenter time to local proper time, and a rotation of the local accelerated frame relative to the global inertial frame. Notice the change in sign in front of $\eta := m_1 m_2/m^2$ between Eqs. (7.18) and (7.37).

E. Comparison with Schwarzschild tidal fields

The tidal moments of a black hole of small mass m_1 moving in the gravitational field of another black hole of large mass m_2 can be obtained simply by evaluating the components of the Riemann tensor for the large black hole; the Riemann tensor is evaluated in the moving frame of the small black hole. The details of such a computation are presented in Poisson [14], and in this subsection we compare our results. Poisson uses different definitions for the harmonic components of the tidal moments, and his results are presented in Schwarzschild coordinates. With the conventions adopted here, in relativist's units, and in harmonic coordinates, Poisson's results are

$$\bar{\mathcal{E}}_0 = -\frac{m_2}{2(r + m_2)^2(r - 2m_2)},$$

$$\bar{\mathcal{E}}_{2c} = -\frac{3m_2}{2(r + m_2)^3}\frac{r - m_2}{r - 2m_2}\cos 2\bar{\phi},$$

$$\bar{\mathcal{E}}_{2s} = -\frac{3m_2}{2(r + m_2)^3}\frac{r - m_2}{r - 2m_2}\sin 2\bar{\phi},$$

$$\bar{\mathcal{B}}_{1c} = -\frac{3m_2^{3/2}}{(r + m_2)^3}\frac{\sqrt{r - m_2}}{r - 2m_2}\cos\bar{\phi},$$

$$\bar{\mathcal{B}}_{1s} = -\frac{3m_2^{3/2}}{(r + m_2)^3}\frac{\sqrt{r - m_2}}{r - 2m_2}\sin\bar{\phi},$$

where $\bar{\phi} = \bar{\omega}\,\bar{t}$, with

$$\bar{\omega} = \frac{m_2}{(r + m_2)^3}.$$

Here, r is the orbital radius of the small black hole (in harmonic coordinates), and $\bar{t}$ is proper time on the circular orbit.

It is easy to check that when $(v/c)^2 := m_2/r \ll 1$, the Schwarzschild expressions reduce to Eqs. (7.32), (7.33), (7.34), (7.35), (7.36), and (7.37) when $m_1 \ll m_2$; in this limit $m \simeq m_2$ and $\eta \simeq 0$. In their common domain of validity, our results agree with those of Poisson.

VIII. GEOMETRY OF THE EVENT HORIZON

In this section we present an application of the results obtained in Sec. VII. We examine, in a particular gauge, the intrinsic geometry of the event horizon of a tidally-deformed black hole. We emphasize that the discussion presented here is tied to a specific choice of gauge; a different slicing of the event horizon would produce a different intrinsic geometry.

The harmonic coordinates $(\bar{t}, \bar{x}^a)$ are singular on the black-hole horizon, and an examination of its geometry requires a change of coordinates. For this purpose we return to the light-cone coordinates (v, ρ, θ^A) of

Sec. III B. It is known [18] that in the light-cone gauge, the coordinate description of the event horizon is $\rho = 2M_1 = 2Gm_1/c^2$, the same as in the unperturbed Schwarzschild geometry. Equations (3.10), (3.11), (3.12), and (3.13) then imply that the induced metric on the event horizon is given by $g_{AB} = (2M_1)^2\Omega_{AB} + h_{AB} + O(M_1^5\mathcal{R}^{-2}\mathcal{L}^{-1})$, where Ω_{AB} is the metric on the unit two-sphere, and

$$h_{AB} = -\tfrac{1}{6}(2M_1)^4(\mathcal{E}^{\rm q}_{AB} + \mathcal{B}^{\rm q}_{AB}) \tag{8.1}$$

is the tidal perturbation. Here $\mathcal{E}^{\rm q}_{AB}$ and $\mathcal{B}^{\rm q}_{AB}$ are the tidal potentials defined in Eqs. (3.16) and (3.18), respectively. As in Sec. III B, we (momentarily) refrain from displaying the factors of c as well as the overbar.

To simplify the horizon metric we implement a gauge transformation generated by the vector field

$$\xi_A = -\tfrac{1}{6}(2M_1)^4(\mathcal{E}^{\rm q}_{A} + \mathcal{B}^{\rm q}_{A}), \tag{8.2}$$

where $\mathcal{E}^{\rm q}_A$ and $\mathcal{B}^{\rm q}_A$ are introduced in Eqs. (3.15) and (3.17), respectively. The transformation changes the metric perturbation according to $h_{AB} \to h'_{AB} = h_{AB} - D_A\xi_B - D_B\xi_A$, where D_A is the covariant-derivative operator compatible with Ω_{AB}. Using the relations $D_A\mathcal{E}^{\rm q}_B = D_B\mathcal{E}^{\rm q}_A = \frac{1}{2}\mathcal{E}^{\rm q}_{AB} - \frac{3}{2}\Omega_{AB}\mathcal{E}^{\rm q}$ and $D_A\mathcal{B}^{\rm q}_B + D_B\mathcal{B}^{\rm q}_A = \mathcal{B}^{\rm q}_{AB}$, we find that the new perturbation is given by

$$h'_{AB} = -\tfrac{1}{2}(2M_1)^4\Omega_{AB}\mathcal{E}^{\rm q}, \tag{8.3}$$

where $\mathcal{E}^{\rm q}$ is defined by Eq. (3.14).

Reinstating the factors of c and the overbar on the tidal moments (to emphasize that we are working in the black hole's comoving frame), we find that the induced metric on the black-hole horizon is given by

$$g_{AB} = (2M_1)^2\left[1 - \frac{(2M_1)^2}{2c^2}\bar{\mathcal{E}}_{ab}\Omega^a\Omega^b\right]\Omega_{AB} + O\left(\frac{M_1^5}{\mathcal{R}^2\mathcal{L}}\right) \tag{8.4}$$

in this choice of gauge. The line element on the horizon is

$$ds^2 = (2M_1)^2\left[1 - \frac{(2M_1)^2}{2c^2}\bar{\mathcal{E}}_{ab}\Omega^a\Omega^b\right](d\theta^2 + \sin^2\theta d\phi^2) + O\left(\frac{M_1^5}{\mathcal{R}^2\mathcal{L}}\right), \tag{8.5}$$

and $\Omega^a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is the unit radial vector. According to these equations, the area of each cross section $v = \text{const}$ of the event horizon is given by $A = 4\pi(2M_1)^2$, so that $2M_1$ is the radius averaged over each cross section.

The metric of Eqs. (8.4) and (8.5) can be reproduced by embedding a closed two-surface described by

$$r = 2M_1[1 + \varepsilon(\theta, \phi)] \tag{8.6}$$

in a flat, three-dimensional space charted by spherical coordinates (r, θ, ϕ). Working consistently to the first order in ε, we find that the metric on this two-surface is given by $ds^2 = (2M_1)^2(1 + 2\varepsilon)(d\theta^2 + \sin^2\theta d\phi^2)$, and this agrees with Eq. (8.5) when

$$\varepsilon(\theta, \phi) = -\frac{M_1^2}{c^2}\bar{\mathcal{E}}_{ab}\Omega^a\Omega^b. \tag{8.7}$$

This equation evidently describes a quadrupole deformation of a round two-sphere of radius $2M_1$. To the first order in ε, the deformation produces no change in area.

So far our considerations have been limited only by the restriction $M_1 \ll \mathcal{R}$, which ensures that the tidal perturbation is small. As was discussed in Sec. I A, this restriction includes both the small-hole and weak-field approximations as limiting cases. If we now restrict our attention to the weak-field approximation and place the black hole within a post-Newtonian environment, then the tidal moments $\bar{\mathcal{E}}_{ab}$ can be imported from Sec. VI D and substituted within Eq. (8.7). To illustrate this, we restrict our attention further to the situation examined in Sec. VII C, in which the black hole is a member of a two-body system in circular motion. The relevant tidal moments are listed in Eqs. (7.32), (7.33), and (7.34), and when these are inserted within Eq. (8.7), we obtain

$$\begin{aligned}\varepsilon = {}& \frac{M_1^2M_2}{2b^3}\left[1 + \frac{M_1}{2M}(v_{\rm rel}/c)^2 + O(c^{-4})\right](1 - 3\cos^2\theta) \\ & + \frac{3M_1^2M_2}{2b^3}\left[1 - \frac{3M_1 + 4M_2}{2M}(v_{\rm rel}/c)^2 + O(c^{-4})\right]\sin^2\theta\cos2\psi.\end{aligned} \tag{8.8}$$

Here $M_1 := Gm_1/c^2$ is the black hole's gravitational radius, and $M_2 := Gm_2/c^2$ measures the mass of the companion body. We use the same notation as in Sec. I B: b is the separation between the two bodies (in harmonic coordinates), $M := M_1 + M_2$ is a measure of the total mass within the system, and $v_{\rm rel} = c\sqrt{M/b}$ is the relative orbital velocity. The symbol ψ stands for $\phi - \bar{\omega}v$, where v is the advanced-time coordinate on the event horizon and $\bar{\omega}$ is the orbital frequency of Eq. (7.37).

Equation (8.8) implies that the event horizon is bulging along an axis directed toward the orbiting body. To see this clearly, we calculate from the metric the circumference of a line of longitude $\psi = \text{constant}$, and we obtain

$$\begin{aligned}C_l = 2\pi(2M_1)\Bigg\{1 &- \frac{M_1^2M_2}{4b^3}\left[1 + \frac{M_1}{2M}(v_{\rm rel}/c)^2 + O(c^{-4})\right] \\ &+ \frac{3M_1^2M_2}{4b^3}\left[1 - \frac{3M_1 + 4M_2}{2M}(v_{\rm rel}/c)^2 + O(c^{-4})\right] \\ &\times \cos2\psi\Bigg\}.\end{aligned} \tag{8.9}$$

This equation reveals that the circumference is largest (stretched) when $\psi = \{0, \pi\}$ and smallest (squeezed) when $\psi = \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. We also calculate the circumference of the equator (at $\theta = \frac{\pi}{2}$) and obtain

$$C_e = 2\pi(2M_1)\left\{1 + \frac{M_1^2 M_2}{2b^3}\left[1 + \frac{M_1}{2M}(v_{\rm rel}/c)^2 + O(c^{-4})\right]\right\}. \tag{8.10}$$

This equation also reveals a bulging of the horizon at the equator.

IX. TIDAL HEATING

In this final section, we present another application of the results obtained in Sec. VII. We calculate the tidal heating of a black hole of mass m_1 placed in a post-Newtonian tidal environment created by an external body of mass m_2. For simplicity, we restrict our attention to circular motion. These results, unlike those presented in Sec. VIII, are gauge-invariant. In Sec. IX A we calculate the tidal heating of a nonrotating black hole, and in Sec. IX B we examine the case of a rotating black hole. The foundations for this calculation are given in Ref. [30].

A. Nonrotating black hole

The rate at which a black hole of mass m_1 acquires mass by tidal heating is given by [30]

$$G\dot{m}_1 = \frac{16}{45}\frac{(Gm_1)^6}{c^{15}}\left[\dot{\bar{\mathcal{E}}}_{ab}\dot{\bar{\mathcal{E}}}^{ab} + \frac{1}{c^2}\dot{\bar{\mathcal{B}}}_{ab}\dot{\bar{\mathcal{B}}}^{ab} + O(c^{-4})\right], \tag{9.1}$$

in which an overdot indicates differentiation with respect to $\bar{t}$. This equation excludes contributions from octupole and higher-order tidal moments—see Ref. [18]. It is easy to show, however, that these contributions occur at order c^{-4} (and smaller) relative to the dominant, quadrupole term; they are therefore included in the neglected terms of Eq. (9.1).

According to our results in Sec. VII D, the tidal moments are given by

$$\bar{\mathcal{E}}_{ab} = -\frac{3Gm_2}{r^3}\left(\left[1 - \frac{m_1 + 2m_2}{2m}(v/c)^2\right]\bar{n}_{\langle ab\rangle} + (v/c)^2\bar{\phi}_{\langle ab\rangle}\right) + O(c^{-4}), \tag{9.2}$$

$$\bar{\mathcal{B}}_{ab} = -\frac{6Gm_2}{r^3}v\bar{l}_{(a}\bar{n}_{b)} + O(c^{-2}) \tag{9.3}$$

in the black-hole frame. Here $\bar{n}^a = (\cos\bar{\phi}, \sin\bar{\phi}, 0)$, $\bar{\phi}^a = (-\sin\bar{\phi}, \cos\bar{\phi}, 0)$, and $\bar{l}^a = (0, 0, 1)$, with $\bar{\phi} = \bar{\omega}\,\bar{t}$; the angular frequency $\bar{\omega}$ is displayed in Eq. (7.37). The time derivatives of the basis vectors are given by $\dot{\bar{n}}_a = \bar{\omega}\bar{\phi}_a$, $\dot{\bar{\phi}}_a = -\bar{\omega}\bar{n}_a$, and $\dot{\bar{l}}_a = 0$; this gives rise to $\dot{\bar{n}}_{\langle ab\rangle} = 2\bar{\omega}\bar{n}_{(a}\bar{\phi}_{b)}$ and $\dot{\bar{\phi}}_{\langle ab\rangle} = -2\bar{\omega}\bar{n}_{(a}\bar{\phi}_{b)}$.

Evaluating Eq. (9.1) from Eqs. (9.2) and (9.3) produces

$$G\dot{m}_1 = \frac{32}{5c^{15}}\frac{m_1^6 m_2^2}{m^8}\left(\frac{Gm}{r}\right)^9\left[1 - \frac{5m_1^2 + 12m_1m_2 + 6m_2^2}{m^2} \times (v/c)^2 + O(c^{-4})\right], \tag{9.4}$$

where $m = m_1 + m_2$ is the total mass, r is the orbital separation (in harmonic coordinates), and $v = \sqrt{Gm/r}$ is the relative orbital velocity. The rate at which the tidal coupling increases the black hole's angular momentum can next be obtained from the rigid-rotation relation $\dot{m}_1c^2 = \bar{\omega}\dot{J}_1$. Equation (9.4) was already displayed in Sec. I B; in Eq. (1.17) we used the symbol b (instead of r) for the orbital separation, $v_{\rm rel}$ (instead of v) for the relative orbital velocity, m (instead of m_1) for the black-hole mass, and m' (instead of m_2) for the mass of the external body.

Equation (9.4) can be compared with the result obtained by Poisson [30] for a black hole of small mass m_1 moving in the field of another black hole of large mass m_2. In geometrized units, and in harmonic coordinates, Poisson's result is

$$\dot{m}_1 = \frac{32}{5}\left(\frac{m_1}{m_2}\right)^6\left(\frac{m_2}{r}\right)^9\frac{1 - m_2/r}{(1 + m_2/r)^9(1 - 2m_2/r)^2}.$$

When $m_2/r = (v/c)^2$ is small the relativistic factor becomes $1 - 6(v/c)^2 + O(c^{-4})$, and this expression agrees with Eq. (9.4) when $m_1 \ll m_2$.

B. Rotating black hole

We next calculate the tidal heating of a rotating black hole, assuming that the tidal fields are not affected (at the first post-Newtonian order) when the nonrotating black hole is replaced by a rapidly rotating hole. The rate at which the hole's angular momentum is increased by the tidal coupling is given by [30]

$$G\dot{J}_1 = -\frac{2}{45}\frac{(Gm_1)^5}{c^{10}}\chi[8(1 + 3\chi^2)(E_1 + c^{-2}B_1) - 3(4 + 17\chi^2)(E_2 + c^{-2}B_2) + 15\chi^2(E_3 + c^{-2}B_3) + O(c^{-4})], \tag{9.5}$$

where $\chi := cJ_1/(Gm_1^2)$ is a dimensionless angular-momentum parameter that ranges between 0 and 1, and $E_1 := \bar{\mathcal{E}}_{ab}\bar{\mathcal{E}}^{ab}$, $E_2 := (\bar{\mathcal{E}}_{ab}s^b)(\bar{\mathcal{E}}^a{}_c s^c)$, $E_3 := (\bar{\mathcal{E}}_{ab}s^a s^b)^2$, $B_1 := \bar{\mathcal{B}}_{ab}\bar{\mathcal{B}}^{ab}$, $B_2 := (\bar{\mathcal{B}}_{ab}s^b)(\bar{\mathcal{B}}^a{}_c s^c)$, $B_3 := (\bar{\mathcal{B}}_{ab}s^a s^b)^2$. Here, the unit vector s^a points in the direction of the hole's spin angular-momentum vector, so that $\boldsymbol{J}_1 = J_1\boldsymbol{s}$. In this application, the spin and orbital angular momenta are aligned or antialigned, so that $s^a = \epsilon l^a$ with $\epsilon = \pm 1$.

Evaluation of Eq. (9.5) produces

$$G\dot{J}_1 = -\frac{8}{5}\chi(1+3\chi^2)\frac{m_1^5 m_2^2}{m^7}\frac{(Gm)^7}{c^{10}r^6}\bigg[1-\bigg(\frac{8+39\chi^2}{4+12\chi^2}\frac{m_1}{m} + \frac{12+51\chi^2}{4+12\chi^2}\frac{m_2}{m}\bigg)(v/c)^2 + O(c^{-4})\bigg], \tag{9.6}$$

where $m = m_1 + m_2$ is the total mass, r the orbital separation (in harmonic coordinates), and $v = \sqrt{Gm/r}$ is the orbital velocity. The rate at which the black-hole mass changes as a result of tidal heating can next be obtained from the rigid-rotation relation $\dot{m}_1 c^2 = \bar{\omega}\dot{J}_1$. With Eq. (7.37) we get

$$G\dot{m}_1 = -\frac{8\epsilon}{5c^{12}}\chi(1+3\chi^2)\frac{m_1^5 m_2^2}{m^7}\bigg(\frac{Gm}{r}\bigg)^{15/2} \times\bigg[1-\bigg(\frac{14+57\chi^2}{4+12\chi^2}\frac{m_1^2}{m^2} + \frac{34+132\chi^2}{4+12\chi^2}\frac{m_1 m_2}{m^2} + \frac{18+69\chi^2}{4+12\chi^2}\frac{m_2^2}{m^2}\bigg)(v/c)^2 + O(c^{-4})\bigg], \tag{9.7}$$

where the parameter $\epsilon = \pm 1$ was previously defined by the relation $s^a = \epsilon l^a$. Thus, the black hole *loses mass* when the orbital motion proceeds in the same direction as the spinning motion ($\epsilon = 1$), and it *gains mass* when the orbital motion proceeds in the opposite direction ($\epsilon = -1$). In each case the orbital motion is slower than the spinning motion, and the black hole always loses angular momentum.

Equation (9.7) can be compared with the result obtained by Poisson [30] for a rotating black hole of small mass m_1 moving in the field of a nonrotating black hole of large mass m_2. In geometrized units, and in harmonic coordinates, Poisson's result is

$$\dot{m}_1 = -\frac{8\epsilon}{5}\chi(1+3\chi^2)\bigg(\frac{m_1}{m_2}\bigg)^5\bigg(\frac{m_2}{r}\bigg)^{15/2} \times \frac{(1-m_2/r)(1-\frac{15\chi^2}{4+12\chi^2}m_2/r)}{(1+m_2/r)^{15/2}(1-2m_2/r)^2}.$$

When $m_2/r = (v/c)^2$ is small the relativistic factor becomes

$$1-\frac{18+69\chi^2}{4+12\chi^2}(v/c)^2 + O(c^{-4}),$$

and this expression agrees with Eq. (9.7) when $m_1 \ll m_2$. The same conclusion holds when the small black hole moves in the field of a large rotating black hole. In this situation, the error term in the previous expression is of order c^{-3} instead of order c^{-4}.

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