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Friedmann branes with variable tension

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We introduce braneworlds with nonconstant tension, strengthening the analogy with fluid membranes, which exhibit a temperature dependence according to the empirical law established by Eötvös. This new degree of freedom allows for evolving gravitational and cosmological constants, the latter being a natural candidate for dark energy. We establish the covariant dynamics on a brane with variable tension in full generality, by considering asymmetrically embedded branes and allowing for nonstandard model fields in 5-dimensional space-time. Then we apply the formalism for a perfect fluid on a Friedmann brane, which is embedded in a 5-dimensional charged Vaidya-anti-de Sitter space-time.

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I. INTRODUCTION

The hierarchy problem and the failure to quantize gravity in the way other interactions were quantized are but two of the symptoms indicating the need to break out from the established framework of field theories and general relativity towards a more fundamental theory. Current attempts, like string/M theory, require additional spatial dimensions in addition to the $3 + 1$ dimensional space-time. These additional dimensions are compact; however, one of them can be extended (but warped) in the scenario introduced by Randall and Sundrum [1]. Generalizations of this early model, allowing for a curved $3 + 1$ space-time with matter (the brane), embedded in curved 5-dimensional (5D) background [2] have been developed (for a review see [3]). By lifting the symmetry of the embedding and for generic sources in 5D, the dynamics was worked out in detail in [4]. This work generalized previous discussions with asymmetric embedding [5].

For such codimension one braneworlds the gravitational variables (namely, the induced metric and the extrinsic curvature) should satisfy the Israel junction conditions [6]. Standard model matter fields can be introduced on the brane by the Lanczos equation [7], which establishes a connection between the jump of the extrinsic curvature and the brane energy-momentum tensor, similar to how various components of the electromagnetic field exhibit a jump across surfaces with distributional charge or current densities. This mechanism works only for codimension one; therefore, the generalization of these braneworlds with arbitrary Riemannian curvature to higher codimensions seems far from straightforward.

In the context of codimension one braneworlds, black holes were found (in the static [8] and rotating [9] cases, or in cosmological context [10]). Gravitational collapse [11–16] together with various stellar models [17–19] were also studied. The possibility that braneworld effects can replace dark matter in galactic dynamics [20] and the dynamics of clusters of galaxies [21] were also considered. The deflection of light was computed to second order accuracy [22], and a confrontation with Solar System tests has been done [23].

Equally interesting applications of braneworlds arise in cosmology. Modifications to the evolution of the early universe were discussed in [24]. From the thermal radiation of an initial very hot brane, even a black hole can condensate in the extra dimension [25]. This black hole, in turn, modifies the Weyl curvature and backreacts onto the intrinsic curvature (and consequently the gravitation) of the brane. Structure formation has been considered to some extent in [26]. However, the equations do not close on the brane; therefore (despite progress made [27]), a full perturbation formalism on the brane is not yet available. Such a formalism would be necessary in order to discuss density perturbations in relation to the cosmic microwave background (CMB) and structure formation in full generality. Nucleosynthesis constraints [28] and confrontation with distant type Ia supernova data [29] have been employed in order to establish the range of various braneworld model parameters.

A key ingredients in braneworld theories is a positive brane tension. Its value should be enormously high, such that, when fine-tuned to the 5D (negative) cosmological constant, they add up to a small positive cosmological constant in 4 dimensions (4D). There are various lower limits established for the brane tension from measurements on the gravitational constant [30], from the requirement that braneworld effects are already weak at nucleosynthesis [31], and from astrophysical considerations [17] (for a review of these topics see [32]).

The brane tension is the analogue of the tension of a fluid membrane, which, however, is not a constant. In 1886 Eötvös established an empirical law for the temperature

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dependence of fluid membrane tension $\lambda_{\rm fluid}$, known today as Eötvös' law [33]. According to this,

$$\lambda_{\rm fluid} = K(T_c - T), \tag{1}$$

with K being a constant and T_c a critical temperature representing the highest temperature for which the membrane exists.

During cosmological evolution, the temperature of the brane (given by the CMB) changes drastically from very high values to 2.7 K nowadays. The question naturally arises, why should the brane tension stay constant during this spectacular evolution?

In this paper we lift the assumption of constancy of the brane tension. We derive the codimension one braneworld dynamics with variable brane tension. In Sec. II we decompose the 5D Einstein equations with respect to the brane and we perform those transformations which lead to the effective Einstein equation. We give there the complete dynamics in the most generic case of an asymmetric embedding and arbitrary 5D sources. This is given by the effective Einstein, Codazzi, and twice-contracted Gauss equations on the brane. The most interesting applications of the developed formalism would be for cosmological and black hole branes.

We consider cosmological branes in Sec. III, where we specialize our results for a Friedmann brane with perfect fluid. We derive the generalized Raychaudhuri and Friedmann equations, and also give the energy-balance equation, the twice-contracted Bianchi identity, and the Lanczos equation for this case.

It has been proved that the 5D vacuum space-time should be either Schwarzschild-anti de Sitter [34,35] or its horizon metric [36,37]. In Sec. IV, however, we consider the most general 5D space-time with radiation and electromagnetic field which admits Friedmann branes in any point, the charged Vaidya-anti-de Sitter (VAdS5) space-time. Such models with 5D radiation were considered before in [25,32,38–44]. We analyze the geometry, the sources, the embedding, and the dynamics, represented by the Friedmann, Raychaudhuri, and energy-balance equations. Finally, we discuss the implications of the model and we summarize our findings in Sec. V.

The formalism developed in this paper generalizes the results of Ref. [4] for variable brane tension. We also find that the formalism developed in [4] applies only when pieces of interior charged VAdS5 space-time regions are glued together along the brane, whereas the results of the present paper stand for the more generic case when either interior or exterior regions are present on both sides of the brane.

Throughout the paper a tilde distinguishes the quantities defined on the 5-dimensional space-time, the only exception under this notation being the normal n to the leaves of the foliation. Its norm is $n^c n_c = 1$. Latin indices represent abstract indices running from 0 to 5. Vector fields in Lie derivatives are represented by boldface characters. An overbar denotes the average taken over the left (L) and right (R) parts of the brane (with the exception of $\bar{L}_{ab}$, which is rather constructed from averaged quantities); Δ denotes the difference taken between the right and the left values of a quantity, and the superscript TF denotes trace-free (the trace being formed with the brane metric).

II. BRANE-COVARIANT GRAVITATIONAL DYNAMICS

Following [4] we summarize the most generic form of the equations characterizing the evolution of gravitation in a codimension one braneworld, allowing for asymmetric embedding and nonstandard model sources in 5D. For this we write the 5D metric $\tilde{g}_{ab}$ in terms of the brane normal $n^a = (\partial/\partial y)^a$ and the induced metric g_{ab} as

$$\tilde{g}_{ab} = n_a n_b + g_{ab}. \tag{2}$$

The 5D geometry evolves according to the 5D Einstein equation

$$\tilde{G}_{ab} = \tilde{\kappa}^2[-\tilde{\Lambda}\tilde{g}_{ab} + \tilde{T}_{ab} + \tau_{ab}\delta(y - y_b)], \tag{3}$$

where $\tilde{\kappa}^2$ and $\tilde{\kappa}^2\tilde{\Lambda}$ are the 5D gravitational coupling constant and cosmological constant, $\tilde{T}_{ab}$ is the regular part of the 5D energy-momentum tensor representing the contribution of possible nonstandard model fields in 5D (like moduli fields or radiation from off-brane sources), and τ_{ab} is the distributional part localized on the brane (at $y = y_b$), obeying $\tau_{ik}n^i = 0$. Usually the brane energy-momentum tensor τ_{ab} is further decomposed as

$$\tau_{ab} = -\lambda g_{ab} + T_{ab}, \tag{4}$$

where λ is the brane tension and T_{ab} represents ordinary matter on the brane.

A. The effective Einstein equation

The 4D metric g_{ab} evolves on the brane according to the effective Einstein equation

$$G_{ab} = -\Lambda g_{ab} + \kappa^2 T_{ab} + \tilde{\kappa}^4 S_{ab} - \bar{\mathcal{E}}_{ab} + \bar{L}_{ab}^{\rm TF} + \bar{\mathcal{P}}_{ab}. \tag{5}$$

As shown in [4], this equation comes from a combination of the scalar and trace-free parts of the tensorial projections of the 5D Einstein equation (3); from the definition of the electric part $\mathcal{E}_{ac} = \tilde{C}_{abcd}n^b n^d$ of the 5D Weyl curvature; and, finally, from averaging over the two sides of the brane, which requires one to apply the junction conditions.

The 4D gravitational coupling "constant" κ^2 is given by

$$6\kappa^2 = \tilde{\kappa}^4\lambda. \tag{6}$$

For attracting gravity the brane tension should be positive, $\lambda > 0$.

The source term S_{ab} denotes a quadratic expression in the brane energy-momentum tensor T_{ab}:

$$S_{ab} = \frac{1}{4}\left[-T_{ac}T_b^c + \frac{T}{3}T_{ab} - \frac{g_{ab}}{2}\left(-T_{cd}T^{cd} + \frac{T^2}{3}\right)\right]. \quad (7)$$

[The first four source terms of Eq. (5) were first derived for a symmetric embedding in [2].]

The possible asymmetric embedding is characterized by the tensor

$$\overline{L}_{ab} = \overline{K}_{ab}\overline{K} - \overline{K}_{ac}\overline{K}_b^c - \frac{g_{ab}}{2}(\overline{K}^2 - \overline{K}_{cd}\overline{K}^{cd}) \quad (8)$$

(for a symmetric embedding $\overline{K}_{ab} = 0$, thus $\overline{L}_{ab} = 0$) with trace

$$\overline{L} = \overline{K}_{ab}\overline{K}^{ab} - \overline{K}^2, \quad (9)$$

and the trace-free part

$$\overline{L}_{ab}^{\mathrm{TF}} = \overline{K}_{ab}\overline{K} - \overline{K}_{ac}\overline{K}_b^c + \frac{\overline{L}}{4}g_{ab}. \quad (10)$$

Finally, $\overline{\mathcal{P}}_{ab}$ is given by the pullback to the brane of the energy-momentum tensor characterizing possible nonstandard model fields (scalar, dilaton, moduli, radiation of quantum origin) living in 5D:

$$\overline{\mathcal{P}}_{ab} = \frac{2\tilde{\kappa}^2}{3}\overline{(g_a^c g_b^d \tilde{T}_{cd})^{\mathrm{TF}}}, \quad (11)$$

which is traceless by definition.

Another projection $n^c n^d \tilde{T}_{cd}$ of these 5D sources appears in the brane cosmological constant Λ, which, in general, can vary both due to nonstandard model fields and due to the asymmetric embedding through $\overline{L}$,

$$\Lambda = \Lambda_0 - \frac{\overline{L}}{4} - \frac{\tilde{\kappa}^2}{2}\overline{(n^c n^d \tilde{T}_{cd})}, \quad (12)$$

with Λ_0 given as

$$2\Lambda_0 = \kappa^2\lambda + \tilde{\kappa}^2\overline{\tilde{\Lambda}}. \quad (13)$$

For varying λ, even Λ_0 fails to be a constant.

The effective Einstein equation does not represent a closed system. Indeed, among its sources we find embedding variables $\overline{L}_{ab}^{\mathrm{TF}}$ and $\overline{L}$, the nonlocal term $\mathcal{E}_{ac}$, as well as the $n^c n^d \tilde{T}_{cd}$ and $g_a^c g_b^d \tilde{T}_{cd}$ projections of 5D sources (their evolution being intertwined with the evolution of the 5D metric again requires nonlocal knowledge of gravitational dynamics).

B. Difference equations

We can also form the difference over the two sides of the brane for both the scalar and the trace-free part of the tensorial projections of the 5D Einstein equation (3), obtaining

$$-\lambda\overline{K} + T_{ab}\overline{K}^{ab} = \Delta(n^a n^b \tilde{T}_{ab}) - \Delta\tilde{\Lambda},$$

$$\Delta\mathcal{E}_{ab} = \frac{2\tilde{\kappa}^2}{3}\Delta(g_a^c g_b^d \tilde{T}_{cd})^{\mathrm{TF}} \quad (14a)$$

$$- \tilde{\kappa}^2\left[\overline{K}T_{ab} + \frac{T}{3}\overline{K}_{ab} + \frac{2}{3}\lambda\overline{K}_{ab} - 2\overline{K}_{(a}^c T_{b)c}\right]^{\mathrm{TF}}. \quad (14b)$$

For given brane and 5D sources, the first of these equations represents a constraint on the embedding. The second equation gives $\Delta\mathcal{E}_{ab}$ in terms of embedding, brane, and 5D sources.

C. The Codazzi equation

The vectorial projection of the 5D Einstein equation (3) gives a constraint on the brane variables (g_{ab}, K_{ab}), known as the Codazzi equation. When averaging and subtracting the corresponding equations taken on the two sides of the brane, we obtain

$$\nabla_c\overline{K}_a^c - \nabla_a\overline{K} = \tilde{\kappa}^2\overline{(g_a^c n^d \tilde{T}_{cd})}, \quad (15a)$$

$$\nabla_c\tau_a^c = -\Delta(g_a^c n^d \tilde{T}_{cd}). \quad (15b)$$

The first of these, the averaged Codazzi equation, is a constraint on the embedding and 5D sources (for symmetric embedding, only for the latter), while the second, the difference Codazzi equation, gives the energy balance on the brane. Written in more detail, we obtain

$$\nabla_c T_a^c = \nabla_a\lambda - \Delta(g_a^c n^d \tilde{T}_{cd}). \quad (16)$$

This is the first equation, which is modified with respect to the constant λ case by our assumption of a nonconstant brane tension. It tells us that the energy balance on the brane can be changed in two ways: (i) due to specific nonstandard model fields in 5D, like radiation (this has been explored in [25–40]), and (ii) due to the varying brane tension.

D. The twice-contracted Gauss equation

As shown in [4], the scalar projection of the 5D Einstein equation (3) is, by construction, the trace of the effective Einstein equation (5). Then the trace of the tensorial projection gives the remaining independent scalar equation, equivalent to the twice-contracted Gauss equation or [after eliminating R with the help of the trace of Eq. (5)] to

$$E = \tilde{\kappa}^2\left(n^a n^b \tilde{T}_{ab} - \frac{\tilde{T}}{3} + \frac{2\tilde{\Lambda}}{3}\right). \quad (17)$$

We have denoted by E the trace of

$$E_{ab} = K_{ac}K_b^c - \mathcal{L}_{\mathbf{n}}K_{ab} + \nabla_b\alpha_a - \alpha_b\alpha_a, \quad (18)$$

carrying information about the off-brane evolution of K_{ab}.

Here $\alpha^b = n^c\tilde{\nabla}_c n^b = g^b_d\alpha^d$ is the curvature of the congruence n^a.

E. The Lanczos equation

The Lanczos equation was already employed in the averaged equations above, but is also useful for monitoring the off-brane gravitational sector.

The junction conditions across the brane, written in a covariant way by Israel [6], include (a) the continuity of the induced metric $g_{ab} = g^R_{ab} = g^L_{ab}$, and (b) the Lanczos equation [7], which establishes the connection between the jump in the extrinsic curvature and the energy-momentum tensor on the brane:

$$\Delta K_{ab} = -\tilde{\kappa}^2\left(\tau_{ab} - \frac{\tau}{3}g_{ab}\right), \tag{19}$$

or equivalently

$$-\tilde{\kappa}^2\tau_{ab} = \Delta K_{ab} - g_{ab}\Delta K. \tag{20}$$

The extrinsic curvature on the two sides is given by

$$2K^{R,L}_{ab} = 2\overline{K}_{ab} \pm \Delta K_{ab}. \tag{21}$$

Here the second term is determined by the brane energy-momentum tensor through the Lanczos equation, while the first should be a solution of the first Codazzi equation (15a).

F. Off-brane gravitational evolution

The brane gravitational variables are the induced metric g_{ab} and the extrinsic curvature K_{ab}. They can be evolved along the off-brane normal n^a in the following way. Equation (21) gives the off-brane evolution of g_{ab}, cf. the definition of the extrinsic curvature,

$$\mathcal{L}_{\mathbf{n}} g_{ab} = 2K_{ab}, \tag{22}$$

with $\mathcal{L}_{\mathbf{n}}$ denoting the brane-projected Lie derivative along the brane normal.

The off-brane evolution of K_{ab} can be found by rewriting Eq. (18) in the form

$$\mathcal{L}_{\mathbf{n}} K_{ab} = -\left(E^{\mathrm{TF}}_{ab} + \frac{E}{4}g_{ab}\right) + K_{ac}K^c_b + \nabla_b\alpha_a - \alpha_b\alpha_a, \tag{23}$$

where E can be expressed in terms of the 5D sources, cf. Eq. (17). The trace-free part of E_{ab} is found from the definition of $\mathcal{E}_{ab}$ and the trace-free part of the tensorial projection of the 5D Einstein equation [4] as

$$E^{\mathrm{TF}}_{ab} = \mathcal{E}_{ab} - \frac{\tilde{\kappa}^2}{3}[g^c_a g^d_b \tilde{T}_{cd} + \tau_{ab}\delta(y)]^{\mathrm{TF}}. \tag{24}$$

Here $\mathcal{E}_{ab}$ can be further decomposed as $\mathcal{E}^{R,L}_{ab} = \overline{\mathcal{E}}_{ab} \pm \Delta\mathcal{E}_{ab}/2$, with $\Delta\mathcal{E}_{ab}$ given by the difference equation (14b) and $\overline{\mathcal{E}}_{ab}$ by the trace-free part of the effective Einstein equation (5).

G. The brane Bianchi identity

The covariant divergence of the effective Einstein equation gives the twice-contracted Bianchi identity in 4 dimensions, from which the longitudinal part of $(\overline{\mathcal{E}}_{ab} - \overline{L}^{\mathrm{TF}}_{ab} - \overline{\mathcal{P}}_{ab})$ can be expressed as

$$\begin{aligned}\nabla^a(\overline{\mathcal{E}}_{ab} - \overline{L}^{\mathrm{TF}}_{ab} - \overline{\mathcal{P}}_{ab}) &= \frac{\nabla_b\overline{L}}{4} + \frac{\tilde{\kappa}^2}{2}\nabla_b\overline{(n^c n^d\tilde{T}_{cd})} - \kappa^2\Delta(g^c_b n^d\tilde{T}_{cd}) \\ &\quad + \frac{\tilde{\kappa}^4}{4}\left(T^a_b - \frac{T}{3}g^a_b\right)\Delta(g^c_a n^d\tilde{T}_{cd}) \\ &\quad + \frac{\tilde{\kappa}^4}{4}\left[2T^{ac}\nabla_{[b}T_{a]c} + \frac{1}{3}(T_{ab}\nabla^a T - T\nabla_b T)\right] \\ &\quad - \frac{\tilde{\kappa}^4}{12}(T^a_b - Tg^a_b)\nabla_a\lambda.\end{aligned} \tag{25}$$

In deriving this identity we have used the relations $\nabla^a\kappa^2 = (\kappa^2/\lambda)\nabla^a\lambda$ and $\nabla^a\Lambda_0 = \kappa^2\nabla^a\lambda$, deductible from the definitions of κ^2 and Λ_0, Eqs. (6) and (13), respectively, together with the energy-balance equation (16). We note that there are manifest contributions due to the varying brane tension in the covariant divergence of the effective Einstein equation, such that (a) the term $\nabla^a(\kappa^2)T_{ab}$ and (b) the term $-g_{ab}\nabla^a\Lambda = \ldots + \kappa^2\nabla_b\lambda$; however, other $\nabla^a\lambda$ contributions arise when we replace the covariant divergence of T_{ab} through its expression (16). As a result, contribution (b) is canceled, and the coefficient of contribution (a) is changed. The whole contribution to the varying brane tension is encompassed in the last term of Eq. (25). The cosmological implications of the twice-contracted Bianchi identity will be exploited in the next section.

We note that terms due to the variable tension appear only in Eqs. (16) and (25).

III. PERFECT FLUID ON FRIEDMANN BRANE

In this section we consider branes with cosmological symmetry (Friedmann branes), containing perfect fluid; however, we leave unspecified both the 5D geometry and 5D sources, for possible future applications.

The metric on a Friedmann brane can be characterized covariantly as

$$g_{ab} = -u_a u_b + a^2(\tau)h_{ab}, \tag{26}$$

where $a(\tau)$ is the scale factor, τ is cosmological time, and h_{ab} is a 3-metric with *constant* curvature (with curvature index $k = 1, 0, -1$) of the maximally symmetric spacial slices of constant τ. The timelike congruence $u^a = (\partial/\partial\tau)^a$ obeys $u^a u_a = -1$ and $h_{ab}u^a = 0$. From the first condition $u_b\nabla_a u^b = 0$ also follows. In a basis adapted to u^a the vector $u^b\nabla_b u^a$ can be easily shown to vanish. As the vanishing of any tensor is a basis-independent statement, it is generally true that $u^b\nabla_b u^a = 0$.

We define the time derivative with respect to τ (and denote it by a dot) as the Lie derivative in the u^a direction, projected into the hypersurface perpendicular to u^a (of constant τ). The condition $\dot{h}_{ab} = 0$ then gives

$$u^c \nabla_c h_{ab} = -\frac{1}{a^2}(\nabla_a u_b + \nabla_b u_a), \tag{27}$$

with the trace

$$\nabla_a u^a = 3\frac{\dot{a}}{a}. \tag{28}$$

The brane tension should vary only in a way that obeys cosmological symmetries; thus it can depend only on the time τ, such that $\lambda = \lambda(\tau)$. The brane perfect fluid is characterized by the energy-momentum tensor

$$T_{ab} = \rho(\tau) u_a u_b + p(\tau) a^2 h_{ab}, \tag{29}$$

with u^a its 4-velocity. Spatial isotropy and homogeneity imply $h_{ab}\nabla^b a = h_{ab}\nabla^b \rho = h_{ab}\nabla^b p = h_{ab}\nabla^b \lambda = 0$; thus for any of the functions $f = (a, \rho, p, \lambda)$, we have

$$\nabla_a f = g_a^b \nabla_b f = -u_a \dot{f}. \tag{30}$$

A similar relation applies for any other function of τ.

The quadratic term (7) for the perfect fluid takes the form

$$\tilde{\kappa}^4 S_{ab} = \kappa^2 \frac{\rho}{\lambda}\left[\frac{\rho}{2} u_a u_b + \left(\frac{\rho}{2} + p\right) a^2 h_{ab}\right]. \tag{31}$$

For the rest of the trace-free source terms, we introduce the effective energy density U as

$$-\bar{\mathcal{E}}_{ab} + \bar{L}_{ab}^{\mathrm{TF}} + \bar{\mathcal{P}}_{ab} = \kappa^2 U\left(u_a u_b + \frac{a^2}{3} h_{ab}\right). \tag{32}$$

U is composed of nonlocal (Weyl), embedding, and 5D matter contributions.

The Einstein tensor of the metric (26) is

$$G_{ab} = 3\frac{\dot{a}^2 + k}{a^2} u_a u_b - [2a\ddot{a} + \dot{a}^2 + k] h_{ab}. \tag{33}$$

Then the nontrivial projections of the effective Einstein equation (5) combine to the generalized Friedmann and generalized Raychaudhuri equations:

$$3\frac{\dot{a}^2 + k}{a^2} = \Lambda + \kappa^2\left[\rho\left(1 + \frac{\rho}{2\lambda}\right) + U\right], \tag{34}$$

$$6\frac{\ddot{a}}{a} = 2\Lambda - \kappa^2\left[\rho\left(1 + \frac{2\rho}{\lambda}\right) + 3p\left(1 + \frac{\rho}{\lambda}\right) + 2U\right]. \tag{35}$$

The energy-balance equation (16) decouples into temporal and spatial projections:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = -\dot{\lambda} + \Delta(u^c n^d \tilde{T}_{cd}), \tag{36}$$

$$\Delta(h_a^c n^d \tilde{T}_{cd}) = 0. \tag{37}$$

Note that the normal vectors on the two sides of the brane are $n_R = n$ and $n_L = -n$; therefore, the second term on the right-hand side of Eq. (36) can be nonvanishing even in the symmetric case. When both terms on the right-hand side of Eq. (36) vanish, the physical fluid obeys a continuity equation. When the 5D sources obey $\Delta(u^c n^d \tilde{T}_{cd}) = 0$, while the brane tension is varying, Eq. (36) becomes a continuity equation for the fluid with energy density $\rho + \lambda$ and pressure $p - \lambda$.

The twice-contracted Bianchi identity (25) can be specified for the chosen cosmological setup by employing Eqs. (6), (26), (28), (29), (32), and (37) and the fact that the functions U, $\bar{L}$, and $\overline{(n^c n^d \tilde{\Pi}_{cd})}$ depend only on τ in order to fulfill the cosmological symmetries. We find that the space projection identically vanishes, while the temporal projection gives

$$\kappa^2\left(\dot{U} + 4U\frac{\dot{a}}{a} + U\frac{\dot{\lambda}}{\lambda}\right) = \left[\frac{\bar{L}}{4} + \frac{\tilde{\kappa}^2}{2}\overline{(n^c n^d \tilde{T}_{cd})}\right]^{\cdot} - \kappa^2\left(1 + \frac{\rho}{\lambda}\right)\Delta(u^c n^d \tilde{T}_{cd}). \tag{38}$$

Again, some manifest contributions from the twice-contracted Bianchi identity (25), like the one emerging from the last term $-(\tilde{\kappa}^4/4)p\dot{\lambda}u_b$, have canceled with contributions from the term quadratic in T_{ab}, when we employed the energy-balance equation (36). As a consequence, the twice-contracted Bianchi identity (38) has only one explicit $\dot{\lambda}$-term, originating in the derivative of $(\kappa^2 U)$. The equation can be written in a somewhat simpler form by introducing the new function $U_0(\tau)$ with an a^{-4} dependence factorized out,

$$U = U_0\left(\frac{a_0}{a}\right)^4, \tag{39}$$

where a_0 is an integration constant, and by reintroducing the function Λ, cf. Eqs. (12) and (13):

$$\kappa^2\left(\frac{a_0}{a}\right)^4\left(\dot{U}_0 + U_0\frac{\dot{\lambda}}{\lambda}\right) + \dot{\Lambda} - \kappa^2\dot{\lambda} = -\kappa^2\left(1 + \frac{\rho}{\lambda}\right)\Delta(u^c n^d \tilde{T}_{cd}). \tag{40}$$

We have verified that the twice-contracted Bianchi identity (40) can also be deduced in a direct way by taking the time derivative of the generalized Friedmann equation, then employing the Raychaudhuri equation (35) and the energy-balance equation (36). According to this remark, it is obvious that the energy-balance equation (36) is *not* a consequence of the Friedmann and Raychaudhuri equations, as in the standard cosmological model, unless Eq. (40) is identically satisfied. This latter condition holds, provided the functions U_0 and Λ are compatible with the 5D Einstein equation and the embedding of the brane is properly chosen. An example of what this means is provided in the next section.

Finally, we write the Lanczos equation (20) for a perfect fluid on the cosmological brane as

$$\Delta K_{ab} = -\frac{\tilde{\kappa}^2}{3}[(2\rho + 3p - \lambda)u_a u_b + (\rho + \lambda)a^2 h_{ab}], \tag{41}$$

a relation useful in relating brane variables to the extrinsic curvature, and consequently to the study of off-brane gravitational evolution.

In summary, cosmological evolution on the brane is given by a generalized Friedmann equation (34), a generalized Raychaudhuri equation (35), and an energy-balance equation (36). After employing Eq. (39), these equations contain two unspecified functions Λ and U_0, depending on nonlocal contributions, the asymmetry of the embedding, and nonstandard model 5D fields.

IV. CHARGED 5D VAIDYA-ANTI-DE SITTER SPACE-TIME

In this section, in addition to the assumption of a variable tension Friedmann brane with cosmological fluid, employed in the previous section, we also specify the 5D space-time and the 5D sources, together with the possible asymmetry in the embedding. These enable us to fix the functions Λ and U (or U_0) in the generalized Friedmann and Raychaudhuri equations. Fixing them in such a way that the twice-contracted Bianchi relation is obeyed, the energy-balance equation becomes a consequence.

We choose the most generic 5D space-time with electromagnetic field and unpolarized radiation (treated in the geometrical optics approximation), admitting the cosmological symmetries of the brane in each point. The discussion generalizes the one of [4] for variable brane tension.

A. The geometry

The 5D space-time is the charged Vaidya solution (VAdS5). In Eddington-Finkelstein-type coordinates

$$d\tilde{s}^2 = -f(v, r; k)dv^2 + 2\epsilon dv dr + r^2[d\chi^2 + \mathcal{H}^2(\chi; k)(d\theta^2 + \sin^2\theta d\phi^2)], \tag{42}$$

where $\epsilon = 1$ applies for an outgoing null coordinate v (with ingoing $v =$ constant lines), while $\epsilon = -1$ for ingoing v (outgoing $v =$ constant lines). The metric functions are

$$\mathcal{H}(\chi; k) = \begin{cases} \sin\chi & k = 1 \\ \chi & k = 0 \\ \sinh\chi & k = -1, \end{cases} \tag{43}$$

(k being the curvature index of the constant curvature 3-metric h_{ab}), and

$$f(v, r; k) = k - \frac{1}{r^2}\left[2m(v) + \frac{\tilde{\kappa}^2\tilde{\Lambda}}{6}r^4 - \frac{q^2(v)}{r^2}\right]. \tag{44}$$

The functions $m(v)$ and $q(v)$ are freely specifiable. Depending on the value of the parameters, the metric can have one or two horizons or no horizon at all (for a discussion of the simplest such metric with $m =$ constant and $q = 0$, see [37]). Because of the allowed asymmetry, two regions of possibly different VAdS5 space-times can be glued together across the brane, such that the global 5D space-time can contain a charged black hole on none, one, or either side of the brane. We classify the possible left (L) and right (R) regions with an index η_I (with $I = \overline{L, R}$) taking the value 1 if the region contains $r = 0$, and 0 otherwise.

The VAdS5 metric is written covariantly as

$$\tilde{g}_{ab} = -u_a u_b + n_a n_b + r^2 h_{ab}. \tag{45}$$

Instead of the (v, r) coordinate chart, sometimes it is more convenient to use the coordinates (τ, y) adapted to the 4-velocity u^a of the fluid and the (right-pointing) brane normal $n^a = (-1)^\sigma(\partial/\partial y)^a$. The sign $(-1)^\sigma$ was introduced in order to allow the coordinate y either to increase or decrease in the direction of the brane normal, which we choose to be right-pointing. For an *outgoing* coordinate y this is assured by

$$\sigma = \begin{cases} \eta_R & \text{right region} \\ \eta_L + 1 & \text{left region.} \end{cases} \tag{46}$$

We also note that u is timelike (thus it can be interpreted as the 4-velocity) only above the horizons. The dual coordinate bases are related as

$$dv = \dot{v}d\tau + v'dy, \qquad dr = \dot{r}d\tau + r'dy, \tag{47}$$

where the dot and the prime represent derivatives with respect to τ and y, respectively. The vector bases then transform with the transposed inverse matrix; thus we have

$$\begin{aligned} u^{\cdot} &\equiv \frac{\partial}{\partial\tau} = \dot{v}\frac{\partial}{\partial v} + \dot{r}\frac{\partial}{\partial r}, \\ (-1)^\sigma n^{\cdot} &\equiv \frac{\partial}{\partial y} = v'\frac{\partial}{\partial v} + r'\frac{\partial}{\partial r}. \end{aligned} \tag{48}$$

Outside horizons, the unit negative norm of u^a implies

$$f\dot{v} = \epsilon\dot{r} + S_1(\dot{r}^2 + f)^{1/2}, \tag{49}$$

with $S_1^2 = 1$. In contrast to [4], we do not choose the positive sign in front of the square root, but leave it unspecified for later convenience. A simple computation shows

$$\dot{v}^{-1} = -\epsilon\dot{r} + S_1(\dot{r}^2 + f)^{1/2}; \tag{50}$$

therefore, outside the horizon ($f > 0$) the above two equations imply that the sign of $\dot{v}$ is given by S_1, irrespective of the value of ϵ.

The 1-form $u_{\cdot}$ then becomes

$$u_{\cdot} \equiv g(u^{\cdot}, .) = -S_1(\dot{r}^2 + f)^{1/2}dv + \epsilon\dot{v}dr. \tag{51}$$

The condition $g(n^{\cdot}, u^{\cdot}) = 0$ gives

$$r' = \frac{\epsilon f \dot{v} - \dot{r}}{\dot{v}} v' = \epsilon S_1 (\dot{r}^2 + f)^{1/2} \frac{v'}{\dot{v}}, \tag{52}$$

while from the normalization of n^a, by employing Eqs. (49), (50), and (52) we obtain

$$v' = S_2 \dot{v}, \tag{53}$$

with $S_2^2 = 1$. Then the normal form becomes

$$n_{\cdot} \equiv g(n^{\cdot}, .) = (-1)^{\sigma} \epsilon S_2 (-\dot{r} dv + \dot{v} dr). \tag{54}$$

Starting from Eqs. (47) and employing Eq. (49), it is now straightforward to show $u_{\cdot} = -d\tau$ and $n_{\cdot} = (-1)^{\sigma} dy$, these relations being independent of the choices of the signs S_1 and S_2.

The electric part with respect to the brane normal n^a of the Weyl tensor of the space-time (42) can be found by a straightforward calculation:

$$\begin{aligned}\mathcal{E}_{ab} &= \frac{1}{4r^2}\left(r^2 \frac{\partial^2 f}{\partial r^2} - 2r \frac{\partial f}{\partial r} + 2f - 2k\right)\left(u_a u_b + \frac{r^2}{3} h_{ab}\right) \\ &= \frac{3[5q^2(v) - 4m(v) r^2]}{2r^6}\left(u_a u_b + \frac{r^2}{3} h_{ab}\right).\end{aligned} \tag{55}$$

From here the Weyl source term in the effective Einstein equation emerges as

$$\begin{aligned} -\overline{\mathcal{E}}_{ab} &= \kappa^2 U^{Weyl}\left(u_a u_b + \frac{r^2}{3} h_{ab}\right), \\ \kappa^2 U^{Weyl} &= \frac{6\overline{m}}{r^4} - \frac{15}{2r^6}\left(\overline{q}^2 + \frac{(\Delta q)^2}{4}\right).\end{aligned} \tag{56}$$

[We have used that for any quantity h the average of its square is $\overline{h^2} = \overline{h}^2 + (\Delta h)^2/4$.]

B. The 5D sources

The source of the charged VAdS5 metric (42) is the 5D cosmological constant $\tilde{\kappa}^2 \tilde{\Lambda}$, together with a superposition of a radiation stream, which generates $m(v)$ and an electromagnetic field, responsible for $q(v)$.

The electromagnetic field is characterized by the energy-momentum tensor:

$$\tilde{T}_{ab}^{\rm EM} = \frac{3q^2(v)}{\tilde{\kappa}^2 r^6}(u_a u_b - n_a n_b + r^2 h_{ab}), \tag{57}$$

and is generated by a null 5-potential $A_a = l_a q(v)/r^2$.

The radiation stream (null dust in the geometrical optics approximation) is described by the energy-momentum tensor:

$$\tilde{T}_{ab}^{\rm ND} = \frac{3\beta(v, r)}{\tilde{\kappa}^2 r^3} l_a l_b. \tag{58}$$

Such radiation is ingoing (towards $r = 0$) for $\epsilon = 1$ and outgoing for $\epsilon = -1$. This means that the radiation flux can either approach or leave the brane and it should not necessarily be the same on the two sides. From the definitions of ϵ_I and η_I it can be checked that the global sign $\epsilon_I (-1)^{\eta_I}$ is negative for any radiation leaving the brane and positive for any radiation arriving at the brane.

In Eq. (58) l is a null 1-form:

$$l = dv = \dot{v}[(-1)^{\sigma} S_2 n - u]; \tag{59}$$

thus in terms of the fluid 4-velocity and brane normal, the energy-momentum tensor becomes

$$\tilde{T}_{ab}^{\rm ND} = \frac{3\beta \dot{v}^2}{\tilde{\kappa}^2 r^3}[n_a n_b + 2(-1)^{\sigma+1} S_2 u_{(a} n_{b)} + u_a u_b]. \tag{60}$$

The function $\beta(v, r)$ is related to the energy density of radiation (with dimension of linear density of mass), and the 5D Einstein equations imply

$$\epsilon \beta = \frac{dm}{dv} - \frac{q}{r^2}\frac{dq}{dv}. \tag{61}$$

On the brane [where $r = a(\tau)$; see the following subsection] the projection of the total energy-momentum tensor $\tilde{T}_{ab} = \tilde{T}_{ab}^{\rm ND} + \tilde{T}_{ab}^{\rm EM}$ needed in the energy-balance equation (36) becomes

$$u^c n^d \tilde{T}_{cd} = \frac{3\epsilon (-1)^{\sigma} \beta \dot{v}^2}{\tilde{\kappa}^2 a^3}, \tag{62}$$

such that

$$\begin{aligned}\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) &= -\dot{\lambda} + \frac{3}{\tilde{\kappa}^2 a^3}\Delta[\epsilon(-1)^{\sigma}\beta \dot{v}^2] \\ &= -\dot{\lambda} + \frac{3}{\tilde{\kappa}^2 a^3}\sum_{I=L,R}\epsilon_I (-1)^{\eta_I}\beta_I \dot{v}_I^2.\end{aligned} \tag{63}$$

The global sign $\epsilon_I(-1)^{\eta_I}$ is negative if radiation leaves the brane and positive if radiation is absorbed on the brane.

Finally, the source term $\overline{\mathcal{P}}_{ab}$ arising from $\tilde{T}_{cd}$ becomes

$$\begin{aligned}\overline{\mathcal{P}}_{ab} &= \kappa^2 U^{\text{ch-rad}}\left(u_a u_b + \frac{a^2}{3} h_{ab}\right), \\ \kappa^2 U^{\text{ch-rad}} &= \frac{3}{2a^3}\overline{\beta \dot{v}^2} + \frac{3}{a^6}\left(\overline{q}^2 + \frac{(\Delta q)^2}{4}\right).\end{aligned} \tag{64}$$

C. The embedding

The brane is located at $y =$ const; thus in the coordinates (τ, y) its movement is encoded only in the change of the coordinate τ. Therefore the embedding relations are $v = v(\tau)$, given by Eq. (49), and $r = a(\tau)$. The latter simply allows one to replace r and $\dot{r}$ with a and $\dot{a}$ in all expressions of the previous subsection, whenever they are specified on the brane. The normal n and tangent u to the brane are given by Eqs. (48), and the induced metric by Eq. (26). The extrinsic curvature can then be calculated to give

$$K_{ab} = (-1)^{\sigma+1}\epsilon S_1 S_2\left[\frac{2\ddot{a} + \frac{\partial f}{\partial a} - \epsilon\dot{v}^2\frac{\partial f}{\partial v}}{2(\dot{a}^2 + f)^{1/2}}u_a u_b - (\dot{a}^2 + f)^{1/2} a h_{ab}\right]. \tag{65}$$

By introducing the notations

$$2A_I = 2\ddot{a} + \frac{\partial f_I}{\partial a} - \epsilon_I \dot{v}_I^2 \frac{\partial f_I}{\partial v}, \tag{66}$$

$$B_I = (-1)^{\eta_I}(\dot{a}^2 + f_I)^{1/2}, \tag{67}$$

with $I = R$ (right region) or L (left region), the jump and average of the extrinsic curvature can be written as

$$\epsilon S_1 S_2 \Delta K_{ab} = -2\overline{\left(\frac{A}{B}\right)}u_a u_b + 2\bar{B} a h_{ab}, \tag{68}$$

$$2\epsilon S_1 S_2 \overline{K}_{ab} = -\Delta\left(\frac{A}{B}\right)u_a u_b + \Delta B a h_{ab}. \tag{69}$$

From the expression (69) of $\overline{K}_{ab}$ we obtain the embedding contribution to Λ:

$$\overline{L} = -\frac{3\Delta B}{2a}\left[\Delta\left(\frac{A}{B}\right) + \frac{\Delta B}{a}\right], \tag{70}$$

and the embedding trace-free source term in the effective Einstein equation:

$$\overline{L}^{\rm TF}_{ab} = \kappa^2 U^{\rm emb}\left(u_a u_b + \frac{a^2}{3}h_{ab}\right),\qquad \kappa^2 U^{\rm emb} = \frac{3\Delta B}{8a}\left[\frac{\Delta B}{a} - \Delta\left(\frac{A}{B}\right)\right]. \tag{71}$$

Note that the sign ambiguity $\epsilon S_1 S_2$ dropped out from the above expressions of $\overline{L}^{\rm TF}_{ab}$ and $\overline{L}$, as both are quadratic in $\overline{K}_{ab}$.

D. The generalized Friedmann and Raychaudhuri equations

Comparing Eq. (68) with the Lanczos equation on the Friedmann brane, Eq. (41), we identify the averaged quantities $\overline{(A/B)}$ and $\bar{B}$ in terms of the brane tension and fluid variables as

$$\epsilon S_1 S_2\overline{\left(\frac{A}{B}\right)} = \frac{\tilde{\kappa}^2}{6}(2\rho + 3p - \lambda), \tag{72}$$

$$\epsilon S_1 S_2 \bar{B} = -\frac{\tilde{\kappa}^2}{6}(\rho + \lambda)a. \tag{73}$$

Next, by taking the square of Eq. (67) and averaging, and then employing Eqs. (6) and (73), we have

$$\frac{\kappa^2}{6\lambda}(\rho + \lambda)^2 a^2 + \frac{(\Delta B)^2}{4} = \dot{a}^2 + \bar{f}. \tag{74}$$

Finally, by taking into account Eqs. (13) and (44), we obtain the Friedmann equation:

$$\frac{\dot{a}^2 + k}{a^2} = \frac{\Lambda_0}{3} + \frac{\kappa^2\rho}{3}\left(1 + \frac{\rho}{2\lambda}\right) + \frac{2\overline{m}}{a^4} - \frac{\overline{q}^2}{a^6} + \frac{(\Delta B)^2}{4a^2} - \frac{(\Delta q)^2}{4a^6}. \tag{75}$$

Here ΔB can be calculated from the jump of the square of Eq. (67), by employing the expressions of f and $\overline{B}$, Eqs. (44) and (73), and $\Delta(h^2) = 2\overline{h}\Delta h$ applied both for B and q:

$$\epsilon S_1 S_2 \Delta B = \frac{12a^2\Delta m - 12\overline{q}\Delta q + \tilde{\kappa}^2 a^6 \Delta\tilde{\Lambda}}{2\tilde{\kappa}^2 a^5(\rho + \lambda)}. \tag{76}$$

However, we note that the combined sign $\epsilon S_1 S_2$ does not appear in the Friedmann equation, which contains $(\Delta B)^2$.

From the definitions of the functions A, f, and β, Eqs. (44), (61), and (66), we find the average and jump of A as

$$\overline{A} = \ddot{a} + \frac{2\overline{m}}{a^3} - \frac{\tilde{\kappa}^2\overline{\tilde{\Lambda}}}{6}a - \frac{2}{a^5}\left(\overline{q}^2 + \frac{(\Delta q)^2}{4}\right) + \frac{\overline{\beta\dot{v}^2}}{a^2}, \tag{77}$$

$$\Delta A = \frac{2\Delta m}{a^3} - \frac{\tilde{\kappa}^2\Delta\tilde{\Lambda}}{6}a - \frac{4\overline{q}\Delta q}{a^5} + \frac{\Delta(\beta\dot{v}^2)}{a^2}. \tag{78}$$

The previously deduced expressions for ΔA and ΔB obey

$$3\Delta A + \tilde{\kappa}^2 a C = \tilde{\kappa}^2(\rho + \lambda)\epsilon S_1 S_2 \Delta B, \tag{79}$$

with

$$C = \Delta\tilde{\Lambda} + \frac{6\overline{q}\Delta q}{\tilde{\kappa}^2 a^6} - \frac{3\Delta(\beta\dot{v}^2)}{\tilde{\kappa}^2 a^3}. \tag{80}$$

The same relation can also be found from the definitions (66) and (67). Next, from the definition of $\overline{(A/B)}$ we obtain a second expression for $\overline{A}$:

$$\overline{A} = \frac{1}{\overline{B}}\left[\frac{\Delta A\Delta B}{4} + \overline{\left(\frac{A}{B}\right)}\left(\overline{B}^2 - \frac{(\Delta B)^2}{4}\right)\right]. \tag{81}$$

When writing up this relation in detail, by employing Eqs. (72), (73), (76), and (78), the global sign $\epsilon S_1 S_2$ drops out. Comparing the two expressions for $\overline{A}$, by taking into account the relations (6) and (13), the Raychaudhuri equation emerges:

$$\frac{\ddot{a}}{a} = \frac{\Lambda_0}{3} - \frac{\kappa^2}{6}\left[\rho\left(1 + \frac{2\rho}{\lambda}\right) + 3p\left(1 + \frac{\rho}{\lambda}\right)\right] - \frac{2\overline{m}}{a^4} + \frac{2}{a^6}\left(\overline{q}^2 + \frac{(\Delta q)^2}{4}\right) - \frac{\overline{\beta\dot{v}^2}}{a^3} - \frac{3(12a^2\Delta m - 12\overline{q}\Delta q + \tilde{\kappa}^2 a^6\Delta\tilde{\Lambda})}{4\tilde{\kappa}^4 a^9(\rho + \lambda)^2}\Delta(\beta\dot{v}^2) + \frac{3\Delta_2}{2\tilde{\kappa}^4 a^{12}(\rho + \lambda)^3}, \tag{82}$$

with

$$\Delta_2 = 18a^4(p-\lambda)(\Delta m)^2 + 12a^2(\rho - 3p + 4\lambda)\overline{q}\Delta q\Delta m + \tilde{\kappa}^2 a^8(2\rho + 3p - \lambda)\Delta\tilde{\Lambda}\Delta m - 6(2\rho - 3p + 5\lambda) \times \overline{q}^2(\Delta q)^2 - \tilde{\kappa}^2 a^6(\rho + 3p - 2\lambda)\overline{q}\Delta q\Delta\tilde{\Lambda} + \frac{\tilde{\kappa}^4}{24}a^{12}(4\rho + 3p + \lambda)(\Delta\tilde{\Lambda})^2. \tag{83}$$

The Friedmann and Raychaudhuri equations can be derived in an independent way by adding together all contributions to U and Λ. To see this, we derive, from the definition of the jump and by employing Eq. (81),

$$\Delta\left(\frac{A}{B}\right) = \frac{\Delta A}{\overline{B}} - \overline{\left(\frac{A}{B}\right)}\frac{\Delta B}{\overline{B}}. \tag{84}$$

Then, starting from Eq. (71) we compute the detailed expression of $U^{\rm emb}$, as follows

$$\kappa^2 U^{\rm emb} = \frac{9(12a^2\Delta m - 12\overline{q}\Delta q + \tilde{\kappa}^2 a^6\Delta\tilde{\Lambda})}{8\tilde{\kappa}^4 a^9(\rho+\lambda)^2}\Delta(\beta\dot{v}^2) + \frac{9\delta_{2,U}}{8\tilde{\kappa}^4 a^{12}(\rho+\lambda)^3}, \tag{85}$$

with

$$\delta_{2,U} = 12a^4(\rho - 3p + 4\lambda)(\Delta m)^2 - 24a^2(2\rho - 3p + 5\lambda) \times \overline{q}\Delta q\Delta m - 2\tilde{\kappa}^2 a^8(\rho + 3p - 2\lambda)\Delta m\Delta\tilde{\Lambda} + 36(\rho - p + 2\lambda)\overline{q}^2(\Delta q)^2 + 6\tilde{\kappa}^2 a^6(p-\lambda) \times \overline{q}\Delta q\Delta\tilde{\Lambda} - \frac{\tilde{\kappa}^4}{4}a^{12}(\rho + p)(\Delta\tilde{\Lambda})^2. \tag{86}$$

Now we can add up the various contributions to U, Eqs. (56), (64), and (85), finding

$$\kappa^2 U = \frac{6\overline{m}}{a^4} - \frac{9}{2a^6}\left(\overline{q}^2 + \frac{(\Delta q)^2}{4}\right) + \frac{3\overline{\beta\dot{v}^2}}{2a^3} + \frac{9(12a^2\Delta m - 12\overline{q}\Delta q + \tilde{\kappa}^2 a^6\Delta\tilde{\Lambda})}{8\tilde{\kappa}^4 a^9(\rho+\lambda)^2}\Delta(\beta\dot{v}^2) + \frac{9\delta_{2,U}}{8\tilde{\kappa}^4 a^{12}(\rho+\lambda)^3}. \tag{87}$$

The function Λ can be computed starting from its definition, Eq. (12). We find

$$\Lambda = \Lambda_0 + \frac{3}{2a^6}\left(\overline{q}^2 + \frac{(\Delta q)^2}{4}\right) - \frac{3\overline{\beta\dot{v}^2}}{2a^3} - \frac{9(12a^2\Delta m - 12\overline{q}\Delta q + \tilde{\kappa}^2 a^6\Delta\tilde{\Lambda})}{8\tilde{\kappa}^4 a^9(\rho+\lambda)^2}\Delta(\beta\dot{v}^2) + \frac{9\delta_{2,\Lambda}}{8\tilde{\kappa}^4 a^{12}(\rho+\lambda)^3}, \tag{88}$$

with

$$\delta_{2,\Lambda} = 12a^4(\rho + 3p - 2\lambda)(\Delta m)^2 - 72a^2(p-\lambda) \times \overline{q}\Delta q\Delta m + 6\tilde{\kappa}^2 a^8(\rho + p)\Delta m\Delta\tilde{\Lambda} - 12(\rho - 3p + 4\lambda)\overline{q}^2(\Delta q)^2 - 2\tilde{\kappa}^2 a^6(2\rho + 3p - \lambda) \times \overline{q}\Delta q\Delta\tilde{\Lambda} + \frac{\tilde{\kappa}^4}{12}a^{12}(5\rho + 3p + 2\lambda)(\Delta\tilde{\Lambda})^2. \tag{89}$$

Inserting U and Λ in Eqs. (34) and (35) we recover the explicit form of the Friedmann and Raychaudhuri equations derived earlier in this section, Eqs. (75) and (82). In the process we use

$$\delta_{2,\Lambda} + \delta_{2,U} = \frac{2\tilde{\kappa}^4}{3}a^{10}(\rho+\lambda)^3(\Delta B)^2, \tag{90}$$

$$\delta_{2,\Lambda} - \delta_{2,U} = 4\Delta_2. \tag{91}$$

Thus we have the complete set of dynamical equations for the Friedmann brane with variable tension, embedded in charged VAdS5 space-time.

E. The consistency of the model

In Sec. III we argued that the twice-contracted Bianchi equation emerges as a consequence of the Friedmann, Raychaudhuri, and energy-balance equations, provided they hold independently. This was a consequence of the effective energy density U depending, in an unspecified way, on the embedding, 5D sources, and nonlocal sources of the gravitational field (Weyl fluid). In this section, however, we have explicitly constructed the embedding, and assumed that the 5D matter consists of a charged radiation field propagating in interior or exterior pieces of charged VAdS5 space-time. As such, the system of Friedmann, Raychaudhuri, and energy-balance equations become interrelated, as in general relativity.

In order to illustrate this, we restrict ourselves to a particularly simple case, with symmetric embedding, no electric charge, and flat spatial sections $k = 0$. We will prove, for this case, that the Raychaudhuri equation emerges from the Friedmann and the energy-balance equations in the same way as in general relativity. In the process we will be able to remove the sign ambiguity represented by S_1 and S_2 and make further comments on the validity of the model discussed in this section.

The energy-balance equation (63) simplifies to

$$\dot{\rho} + \dot{\lambda} + 3\frac{\dot{a}}{a}(\rho + p) = \epsilon(-1)^\eta\frac{6\beta\dot{v}^2}{\tilde{\kappa}^2 a^3}. \tag{92}$$

As noted before, the global sign $\epsilon(-1)^\eta$ is negative when the brane radiates away energy (for example, by thermic radiation) and positive if radiation is absorbed by the brane (such a radiation can be emitted by a 5D black hole if the 5D region is an interior patch of VAdS5, or may come from the 5D infinity, for exterior regions).

The brane bounds a region of VAdS5 characterized by the mass function $m(v)$ [or equivalently $m(t)$, when seen from the brane]. When the brane radiates away energy into the 5D space-time, this mass function should increase in τ, irrespective of the movement of the brane. As its velocity is subluminal, the brane will never pass its own emitted radiation, even if it moves in the same direction. When the brane absorbs radiation, such that the total energy of the bounded 5D space-time region decreases, the mass $m(\tau)$ will become smaller. In consequence $\mathrm{sgn}(\dot{m}) = -\epsilon(-1)^{\eta}$.

Let us now derive an expression for $\dot{m}$. We start from Eq. (61), which can be rewritten as $\dot{m} = \epsilon\beta\dot{v}$. By remembering that $\mathrm{sgn}(\dot{v}) = S_1$ and that $\beta > 0$ for radiation obeying the energy conditions,[1] we have that $\mathrm{sgn}(\dot{m}) = \epsilon S_1$. Thus we conclude that $S_1 = (-1)^{\eta+1}$.

Under the simplifying assumptions of this subsection, the Friedmann equation is

$$\frac{\dot{a}^2}{a^2} = \frac{\kappa^2}{6\lambda}(\rho + \lambda)^2 + \frac{2m}{a^4} + \frac{\tilde{\kappa}^2\tilde{\Lambda}}{6}. \tag{93}$$

Its time derivative, employing the energy-balance equation (92), gives

$$\frac{\dot{a}}{a}\frac{\ddot{a}}{a} = \frac{\dot{a}}{a}\left\{\frac{\Lambda_0}{3} - \frac{\kappa^2}{6}\left[\rho\left(1 + \frac{2\rho}{\lambda}\right) + 3p\left(1 + \frac{\rho}{\lambda}\right)\right] - \frac{2m}{a^4}\right\} + \frac{\dot{m}}{a^4} + \epsilon(-1)^{\eta}\frac{\tilde{\kappa}^2}{6}(\rho + \lambda)\frac{\beta\dot{v}^2}{a^3}. \tag{94}$$

By employing Eqs. (50) and (93) we obtain

$$\dot{m} = \frac{\epsilon\beta\dot{v}^2}{\dot{v}} = \beta\dot{v}^2 a\left[-\frac{\dot{a}}{a} + \epsilon(-1)^{\eta+1}\frac{\tilde{\kappa}^2}{6}(\rho + \lambda)\right]. \tag{95}$$

The last two terms of Eq. (94), after inserting Eq. (95), reduce to $-(\beta\dot{v}^2/a^3)(\dot{a}/a)$. After simplifying with $\dot{a}/a$ we recover the Raychaudhuri equation:

$$\frac{\ddot{a}}{a} = \frac{\Lambda_0}{3} - \frac{\kappa^2}{6}\left[\rho\left(1 + \frac{2\rho}{\lambda}\right) + 3p\left(1 + \frac{\rho}{\lambda}\right)\right] - \frac{2m}{a^4} - \frac{\beta\dot{v}^2}{a^3}. \tag{96}$$

Having no other constraint on the rest of the signs, we can choose $S_2 = \epsilon$, so that Eq. (53) agrees with the corresponding equation in Ref. [4].

V. DISCUSSION AND CONCLUDING REMARKS

In this paper we have introduced the possibility of a variable brane tension in the context of codimension one braneworlds in which both the brane and the 5D space-time are curved. This possibility has not been explored before, although there is no *a priori* reason why the brane tension should be independent of temperature, while the tension of the fluid membranes definitely is. The evolution of the brane tension triggers a variable gravitational constant [through Eq. (6)] and a variable cosmological constant [through Eq. (13)].

The sources for the curvature in the model are (a) non-standard model fields and a negative cosmological constant in 5D, and (b) distributional standard model sources on the brane, together with the brane tension. We have considered this setup without any particular symmetry assumption or further specification of the sources. Gravitational dynamics regarded from a brane point of view is expressed by an effective Einstein equation, the Codazzi equation, and the twice-contracted Gauss equation. Beyond the distributional standard model sources already mentioned and the effective 4D cosmological constant, other sources appear in the effective Einstein equation, which are as follows: a quadratic source term (arising by replacing the quadratic expressions in the extrinsic curvature with matter terms by use of the junction conditions); a Weyl fluid term (originating in the 5D Weyl curvature); an asymmetry source term (from the possible asymmetry of the embedding); and a 5D matter term (from the pullback of the 5D sources). Various other equations were derived, expressing either constraints on the embedding and 5D matter sources or the off-brane evolution of the Weyl fluid. The Lanczos equation and Bianchi identity were also given. A careful analysis has identified those equations which exhibit a $\dot{\lambda}$ term: these are the twice-contracted Bianchi identity and the Codazzi relation. The manifest form of the other equations is not changed, although their solutions will depend on the specific way the brane tension varies.

Then we have specified the formalism for a cosmological context, discussing Friedmann branes containing perfect fluid. The Codazzi equation generates an energy-balance equation, expressing the possible energy interchange between the brane and the 5D sources, and also depends on the time derivative of the brane tension. (In the absence of such an interchange and for constant brane tension, the fluid obeys the usual continuity equation.) When the collection of Weyl, asymmetry, and 5D matter sources obey the cosmological symmetries, they are characterized by a time-dependent potential. The time evolution of this potential is given by the twice-contracted Bianchi identity.

Next we have considered the most generic 5D space-time with cosmological constant, electromagnetic field, and radiation (the VAdS5 space-time), which has the property that a Friedmann brane can be embedded into it at any point. The detailed discussion of the 5D geometry led to the expression of the Weyl fluid source, not given before. The above-mentioned sources obeying the 5D Einstein equation gave the pullback 5D matter source term and the respective contribution to the 4D cosmological constant. The analysis of the embedding led to the asymmetry source term and the asymmetry contribution to the 4D cosmological constant. These are new results. The explicit knowledge of various contributions to the potential U will

[1] For a discussion of exotic radiation, corresponding to $\beta < 0$ in 4D, see Ref. [45].

allow one to easily apply the formalism for special cases, when either the mass, the cosmological constant or the charge is symmetric, or the masses/charges are missing from the 5D space-time. Gravitational dynamics on the brane was derived in the form of the generalized Friedmann, generalized Raychaudhuri, and energy-balance equations, given explicitly. From among these, only the energy-balance equation has a manifest λ dependence. We have verified these results by an independent derivation of the generalized Friedmann and Raychaudhuri equations.

Finally we have shown in the special case without charge, symmetric embedding, and spatially flat sections, how the Raychaudhuri equation emerges from the derivative of the Friedmann equation, the energy-balance equation, and the expression of $\dot{m}$. This derivation closely resembles the general relativistic result. In the process we have fixed an existing sign ambiguity: $S_1 = (-1)^{\eta+1}$. In the derivation we have assumed the same S_1 on both sides of the brane; therefore, our results hold for a brane which is the common boundary of either two interior ($S_1 = 1$) or two exterior ($S_1 = -1$) regions, whereas the results of [4] hold only for the matching of two interior regions, as in that paper the positive sign was chosen in Eq. (49). (All applications to specific models of the formalism derived in [4] were for a brane bounding interior regions [32,42,43].) The cases characterized by $S_1^L S_1^R = -1$ are not incorporated in the present analysis, their discussion in full detail being beyond the scope of the present paper.

With these the formalism is set for analyzing particular braneworld models with variable tension. In order to do this in a cosmological context, we need the scale-factor dependence of the brane tension. As one of the generic features of the models with temperature-dependent tension is that the 4D gravitational constant (defined in terms of the brane tension) evolves in time, there may be certain similarities with other models with a variable gravitational constant (for some recent proposals see [46,47]).

In a parallel work [48] we have investigated the flat Eötvös brane embedded symmetrically in an uncharged VAdS5 space-time. Such a brane exhibits the Eötvös law for the temperature dependence of the brane. By additionally imposing that the cosmological perfect fluid obeys the continuity equation, the dynamics was considerably simplified. The continuity equation can be obeyed by fine-tuning the energy interchange between the brane and 5D radiation with the evolution of the brane tension. In a reasonable parameter range the emerging cosmology is open, reproducing a decelerated expansion followed by an accelerated phase. In the process the mass of the VAdS5 regions decreases until the 5D radiation is completely fueled out, such that the 5D space-time becomes anti de Sitter. Both the brane tension and the 4D cosmological constant evolve from infinitesimal values at the formation of the brane in a very hot early universe towards late-time constant values. The 4D cosmological constant evolves from huge negative values, contributing to gravitational attraction in the early universe, to a small positive value. Cosmological expansion then continues in a de Sitter phase.

The next simplest model with variable tension would arise for a static 5D space-time. Without energy interchange between the brane and 5D space-time, the energy-balance equation becomes a continuity equation for the perfect fluid with energy density $\rho + \lambda$ and pressure $p - \lambda$. A detailed investigation of this model is in progress.

Various other models with temperature-dependent brane tension can be constructed in the framework of the formalism developed in the present paper, by considering specific radiation fields in VAdS5 or adopting a temperature dependence, which is different from the Eötvös law established for fluid membranes.

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