Rotating black holes on Kaluza-Klein bubbles

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Using the solitonic solution-generating techniques, we generate a new exact solution which describes a pair of rotating black holes on a Kaluza-Klein bubble as a vacuum solution in the five-dimensional Kaluza-Klein theory. We also investigate the properties of this solution. Two black holes with topology S^3 are rotating along the same direction even though the directions of intrinsic spin of the black holes are different. The bubble plays a role in holding two black holes. In the static case, it coincides with the solution found by Elvang and Horowitz.

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I. INTRODUCTION

Solitonic solution-generating methods are powerful tools to generate exact solutions of Einstein equations. They are mainly classified into two types. One is called Bäcklund transformation [1,2], which is basically the technique to generate a new solution of the Ernst equation. The other is the inverse scattering technique, which Belinski and Zakharov [3] developed as another type of solution-generating technique. Both methods have produced vacuum solutions from a certain known vacuum solution and succeeded in generation of some four-dimensional exact solutions. The relation between these methods was discussed in Ref. [4] for four dimensions.

Recently, these techniques have been used to generate five-dimensional black hole solutions. A new stationary and axisymmetric black ring solution with rotating two-sphere was found by two of the authors [5] by applying the former solitonic solution-generating techniques [6] to five dimensions. They also reproduced a black ring solution with S^1-rotation [7] by this method [8] and constructed a black di-ring solution [9]. As to asymptotically flat higher-dimensional black hole/ring solutions, some of solutions have been generated by using the inverse scattering method. As an infinite number of static solutions of the five-dimensional vacuum Einstein equations with axial symmetry, the five-dimensional Schwarzschild solution and the static black ring solution were reproduced [10], which gave the first example of the generation of a higher-dimensional asymptotically flat black hole solution by the inverse scattering method. The Myers-Perry solution with single and double angular momenta were regenerated from the Minkowski seed [11,12] and an unphysical one [13], respectively. The black ring solutions with S^2-rotation [11] and S^1-rotation [14] were also generated by one of the authors. Furthermore, Pomerasky and Sen'kov seem to succeed in generation of a new black ring solution with two angular momentum components [15] by the latter method. Elvang and Figueras also generated a black Saturn solution which describes a spherical black hole surrounded by a black ring [16].

However, from a more realistic view point, we need not impose the asymptotic Minkowski spacetime toward the extra dimensions. In fact, higher-dimensional black holes admit a variety of asymptotic structures. Kaluza-Klein (KK) black hole solutions have the spatial infinity with compact extra dimensions [17,18]. Black hole solutions on the Eguchi-Hanson space have the spatial infinity of topologically various lens spaces [19]. The latter black hole spacetimes have asymptotically and locally Minkowski structure. In spacetimes with such asymptotic structures, black holes themselves have different structures from the one with the asymptotically Minkowski structure. For instance, the Kaluza-Klein black holes [17,18] and the black holes on the Eguchi-Hanson space [19] admit the horizon of lens spaces in addition to S^3. We expect that the solitonic methods also help us generate new black hole solutions which have asymptotic structures different from the Minkowski spacetime. Remarkably, as a vacuum solution in five-dimensional Kaluza-Klein theory, there is a static two black hole solution, which does not have even a conical singularity [20] since a Kaluza-Klein bubble of nothing, which was first found by Witten [21], plays a role in holding two black holes. In Ref. [22], Harmark and Obers constructed all the sequences of black holes and Kaluza-Klein bubbles, generalizing the results in Ref. [20]. In Ref. [23], Harmark *et al.* discussed how Kaluza-Klein bubbles and sequences of bubbles and black holes fit into the general story of Kaluza-Klein black holes. See also the review [24] on the bubbles.

So far most people have not considered the effect of a rotation of black holes. For example, since in general, a rotating black hole has less entropy than a nonrotating black hole for the same mass, it is nontrivial whether a black string spontaneously generates a KK bubble which split the black string with $S^1 \times S^2$ topology into two black holes with an S^3 horizons connected bubble, while in the static solutions in Ref. [20] such a transition is not expected to occur. Hence it is important to construct such solutions which describe rotating black holes on a KK bubble and to investigate the features of such solutions, e.g., the effect of the frame dragging. In this article, we generate a new exact solution which describes a pair of rotating black holes on a Kaluza-Klein bubble by using the two different kinds of solution-generating techniques whose relation was discussed in [25]. In the static case, our solution coincides with the solution found by Elvang and Horowitz [20].

This article is organized as follows: In Sec. II, we give a new solution generated by the solitonic methods. We introduce only the construction by the inverse scattering method in this section, while the other construction is briefly mentioned in the appendix. In Sec. III, we investigate the properties of the solution. In Sec. IV, we give the summary and discussion of this article.

II. SOLUTIONS

Following the techniques in Refs. [11,14,25], we construct a new Kaluza-Klein black hole solution. We consider the five-dimensional stationary and axisymmetric vacuum spacetimes which admit three commuting Killing vectors $\partial/\partial t$, $\partial/\partial \phi$, and $\partial/\partial \psi$, where $\partial/\partial t$ is a Killing vector field associated with time translation, and $\partial/\partial \phi$ and $\partial/\partial \psi$ denote spacelike Killing vector fields with closed orbits. In such a spacetime, the metric can be written in the canonical form as

$$ds^2 = g_{ij}dx^i dx^j + f(d\rho^2 + dz^2), \tag{1}$$

where the metric components g_{ij} and the metric coefficient f are functions which depend on ρ and z only. The metric g_{ij} satisfies the supplementary condition $\det g_{ij} = -\rho^2$. We begin with the following seed:

$$\begin{aligned} ds^2 = &-\frac{R_{\eta_2\sigma} + z - \eta_2\sigma}{R_{\eta_1\sigma} + z - \eta_1\sigma}dt^2 \\ &+ \frac{(R_{\eta_1\sigma} + z - \eta_1\sigma)\rho^2}{R_{\lambda\sigma} + z - \lambda\sigma}d\phi^2 \\ &+ \frac{R_{\lambda\sigma} + z - \lambda\sigma}{R_{\eta_2\sigma} + z - \eta_2\sigma}d\psi^2 + f(d\rho^2 + dz^2), \end{aligned} \tag{2}$$

where R_d is defined as $R_d := \sqrt{\rho^2 + (z - d)^2}$. The parameters η_1, η_2, and λ satisfy the inequality $\eta_1 < \eta_2 < -1 < \lambda < 1$ and $\sigma > 0$. Instead of solving the L-A pair for the seed metric (2), it is sufficient to consider the following metric form:

$$ds^2 = -dt^2 + g_2 d\phi^2 + g_3 d\psi^2 + f(d\rho^2 + dz^2), \tag{3}$$

where g_2 and g_3 are given by

$$\begin{aligned} g_2 &= \frac{(R_{\eta_1\sigma} + z - \eta_1\sigma)^2\rho^2}{(R_{\eta_2\sigma} + z - \eta_2\sigma)(R_{\lambda\sigma} + z - \lambda\sigma)}, \\ g_3 &= \frac{(R_{\lambda\sigma} + z - \lambda\sigma)(R_{\eta_2} + z - \eta_2\sigma)}{(R_{\eta_1\sigma} + z - \eta_1\sigma)^2}. \end{aligned} \tag{4}$$

Let us consider the conformal transformation of the two-dimensional metric $g_{AB}(A, B = t, \phi)$ and the rescaling of the $\psi\psi$-component in which the determinant $\det g$ is invariant

$$g_0 = \mathrm{diag}(-1, g_2, g_3) \to g_0' = \mathrm{diag}(-\Omega, \Omega g_2, \Omega^{-2}g_3), \tag{5}$$

where Ω is the tt-component of the seed (2), i.e.,

$$\Omega = \frac{R_{\eta_2\sigma} + z - \eta_2\sigma}{R_{\eta_1\sigma} + z - \eta_1\sigma}. \tag{6}$$

Then, under this transformation, the three-dimensional metric coincides with the metric (2). On the other hand, as discussed in [25], under this transformation the physical metric of two-solitonic solution is transformed as

$$g = \begin{pmatrix} g_{AB} & 0 \\ 0 & g_3 \end{pmatrix} \to g' = \begin{pmatrix} \Omega g_{AB} & 0 \\ 0 & \Omega^{-2}g_3 \end{pmatrix}. \tag{7}$$

This is why we may perform the transformation (5) for the two-solitonic solution generated from the seed (3) in order to obtain the two-solitonic solution from the seed (2). The generating matrix ψ_0 for this seed metric (3) is computed as follows:

$$\psi_0[\lambda] = \mathrm{diag}(-1, \psi_2[\lambda], \psi_3[\lambda])$$

with

$$\begin{aligned} \psi_2[\lambda] &= \frac{(R_{\eta_1\sigma} + z - \eta_1\sigma + \lambda)^2(\rho^2 - 2z\lambda - \lambda^2)}{(R_{\eta_2\sigma} + z - \eta_2\sigma + \lambda)(R_{\lambda\sigma} + z - \lambda\sigma + \lambda)}, \\ \psi_3[\lambda] &= \frac{(R_{\lambda\sigma} + z - \lambda\sigma + \lambda)(R_{\eta_2\sigma} + z - \eta_2\sigma + \lambda)}{(R_{\eta_1\sigma} + z - \eta_1\sigma + \lambda)^2}. \end{aligned}$$

Then, the two-solitonic solution is obtained as

$$\begin{aligned} g_{tt}^{(\mathrm{phys})} &= -\frac{\Omega G_{tt}}{\mu_1\mu_2\Sigma}, \\ g_{t\phi}^{(\mathrm{phys})} &= -g_2\frac{\Omega(\rho^2 + \mu_1\mu_2)G_{t\phi}}{\mu_1\mu_2\Sigma}, \\ g_{\phi\phi}^{(\mathrm{phys})} &= -g_2\frac{\Omega G_{\phi\phi}}{\mu_1\mu_2\Sigma}, \\ g_{\psi\psi}^{(\mathrm{phys})} &= \Omega^{-2}g_3, \\ g_{\phi\psi}^{(\mathrm{phys})} &= g_{t\psi}^{(\mathrm{phys})} = 0, \end{aligned}$$

where the functions G_{tt}, $G_{t\phi}$, $G_{\phi\phi}$, and Σ are given by

$$G_{tt} = -m_{01}^{(1)2} m_{01}^{(2)2} \psi_2[\mu_1]^2 \psi_2[\mu_2]^2 (\mu_1 - \mu_2)^2 \rho^4 + m_{01}^{(1)2} m_{02}^{(2)2} g_2 \mu_2^2 (\rho^2 + \mu_1\mu_2)^2 \psi_2[\mu_1]^2 + m_{01}^{(2)2} m_{02}^{(1)2} g_2 \mu_1^2 (\rho^2 + \mu_1\mu_2)^2 \psi_2[\mu_2]^2 - m_{02}^{(1)2} m_{02}^{(2)2} g_2^2 \mu_1^2 \mu_2^2 (\mu_1 - \mu_2)^2 - 2 m_{01}^{(1)} m_{01}^{(2)} m_{02}^{(1)} m_{02}^{(2)} g_2 \psi_2[\mu_1] \psi_2[\mu_2] (\rho^2 + \mu_1^2)(\rho^2 + \mu_2^2) \mu_1 \mu_2, \tag{8}$$

$$G_{\phi\phi} = m_{01}^{(1)2} m_{01}^{(2)2} \mu_1^2 \mu_2^2 (\mu_1 - \mu_2)^2 \psi_2[\mu_1]^2 \psi_2[\mu_2]^2 + m_{02}^{(1)2} m_{02}^{(2)2} g_2^2 (\mu_1 - \mu_2)^2 \rho^4 - m_{01}^{(1)2} m_{02}^{(2)2} g_2 \mu_1^2 \psi_2[\mu_1]^2 (\rho^2 + \mu_1\mu_2)^2 - m_{01}^{(2)2} m_{02}^{(1)2} g_2 \mu_2^2 \psi_2[\mu_2]^2 (\rho^2 + \mu_1\mu_2)^2 + 2 m_{01}^{(1)} m_{01}^{(2)} m_{02}^{(1)} m_{02}^{(2)} g_2 \mu_1 \mu_2 \psi_2[\mu_2] \psi_2[\mu_1] (\rho^2 + \mu_1^2)(\rho^2 + \mu_2^2), \tag{9}$$

$$G_{t\phi} = m_{01}^{(1)} m_{01}^{(2)2} m_{02}^{(1)} \mu_2 (\mu_1 - \mu_2) \psi_2[\mu_2]^2 \psi_2[\mu_1] (\rho^2 + \mu_1^2) + m_{01}^{(1)} m_{02}^{(1)} m_{02}^{(2)2} g_2 \mu_2 (\mu_2 - \mu_1) \psi_2[\mu_1] (\rho^2 + \mu_1^2) + m_{01}^{(1)2} m_{01}^{(2)} m_{02}^{(2)} \mu_1 (\mu_2 - \mu_1) \psi_2[\mu_1]^2 \psi_2[\mu_2] (\rho^2 + \mu_2^2) + m_{01}^{(2)} m_{02}^{(1)2} m_{02}^{(2)} \mu_1 g_2 \psi_2[\mu_2] (\rho^2 + \mu_2^2)(\mu_1 - \mu_2), \tag{10}$$

$$\Sigma = m_{01}^{(1)2} m_{01}^{(2)2} \psi_2[\mu_1]^2 \psi_2[\mu_2]^2 (\mu_1 - \mu_2)^2 \rho^2 + m_{02}^{(1)2} m_{02}^{(2)2} g_2^2 (\mu_1 - \mu_2)^2 \rho^2 + m_{01}^{(1)2} m_{02}^{(2)2} g_2 \psi_2[\mu_1]^2 (\rho^2 + \mu_1\mu_2)^2 + m_{02}^{(1)2} m_{01}^{(2)2} g_2 \psi_2[\mu_2]^2 (\rho^2 + \mu_1\mu_2)^2 - 2 m_{01}^{(1)} m_{01}^{(2)} m_{02}^{(1)} m_{02}^{(2)} g_2 \psi_2[\mu_1] \psi_2[\mu_2] (\rho^2 + \mu_1^2)(\rho^2 + \mu_2^2). \tag{11}$$

Here, μ_1 and μ_2 are given by

$$\mu_1(\rho, z) = \sqrt{\rho^2 + (z + \sigma)^2} - (z + \sigma), \qquad \mu_2(\rho, z) = \sqrt{\rho^2 + (z - \sigma)^2} - (z - \sigma). \tag{12}$$

We should note that this three-dimensional metric $g_{ij}^{(\mathrm{phy})}$ satisfies the supplementary condition $\det g_{ij} = -\rho^2$. Next, let us consider the coordinate transformation of the physical metric such that

$$t \to t' = t - C_1 \phi, \qquad \phi \to \phi' = \phi, \tag{13}$$

where C_1 is a constant. Under this transformation, the physical metric becomes

$$\begin{aligned} g_{tt}^{(\mathrm{phys})} &\to g_{tt} = g_{tt}^{(\mathrm{phy})}, \\ g_{t\phi}^{(\mathrm{phys})} &\to g_{t\phi} = g_{t\phi}^{(\mathrm{phy})} + C_1 g_{tt}^{(\mathrm{phys})}, \\ g_{\phi\phi}^{(\mathrm{phy})} &\to g_{\phi\phi} = g_{\phi\phi}^{(\mathrm{phy})} + 2C_1 g_{t\phi}^{(\mathrm{phys})} + C_1^2 g_{tt}^{(\mathrm{phy})}. \end{aligned} \tag{14}$$

Here, we should note that the transformed metric also satisfies the supplementary condition $\det g = -\rho^2$. Though the metric seems to contain the four new parameters $m_{01}^{(1)}$, $m_{01}^{(2)}$, $m_{02}^{(1)}$, and $m_{02}^{(2)}$, it can be written only in terms of the ratios

$$\alpha := -\frac{m_{02}^{(2)}}{2\sigma m_{01}^{(2)}}, \qquad \beta := -\frac{2\sigma m_{01}^{(1)}}{m_{02}^{(1)}}. \tag{15}$$

Using the parameters α and β, we can write all components of the metric. The metric function $f(\rho, z)$ takes the following form:

$$f = \frac{C_2 Y_{\sigma,-\sigma} Y_{\sigma,\eta_1\sigma}}{4 Y_{-\sigma,\eta_1\sigma}} \sqrt{\frac{Y_{-\sigma,\eta_2\sigma} Y_{-\sigma,\lambda\sigma} Y_{\eta_1\sigma,\eta_2\sigma} Y_{\lambda\sigma,\eta_1\sigma} Y_{\lambda\sigma,\eta_2\sigma}}{Y_{-\sigma,-\sigma} Y_{\eta_1\sigma,\eta_1\sigma} Y_{\eta_2\sigma,\eta_2\sigma} Y_{\lambda\sigma,\lambda\sigma} Y_{\sigma,\eta_2\sigma} Y_{\sigma,\lambda\sigma} Y_{\sigma,\sigma}}} \frac{\Omega Y}{(\rho^2 + \mu_1\mu_2)^4 \mu_1^3 \mu_2 \psi_2[\mu_2]^2}, \tag{16}$$

where C_2 is an arbitrary constant, $Y_{c,d}$ is defined as $Y_{c,d} := R_c R_d + (z - c)(z - d) + \rho^2$, and the function Y is given by

$$Y = \rho^2 [-4\beta \mu_1^2 \mu_2^2 \psi_2[\mu_1] \psi_2[\mu_2] + \alpha g_2 (\mu_1 - \mu_2)^2 (\rho^2 + \mu_1\mu_2)^2]^2 + 4 g_2 \mu_1^2 \mu_2^2 (\rho^2 + \mu_1\mu_2)^4 (\psi_2[\mu_2] - \alpha\beta \psi_2[\mu_1])^2.$$

We comment that the constants α and β exactly coincide with the ones that appear in the Bäcklund transformation in the appendix. To assure that the metric asymptotically approaches $\mathcal{M}^{3,1} \times S^1$, where $\mathcal{M}^{3,1}$ denotes the four-dimensional Minkowski spacetime and S^1 is a Kaluza-Klein circle, the constants C_1 and C_2 are chosen as follows:

$$C_1 = -\frac{2\sigma(\alpha - \beta)}{1 + \alpha\beta}, \qquad C_2 = \frac{1}{(1 + \alpha\beta)^2}, \qquad \alpha + \beta = 0, \tag{17}$$

which assure the regular behavior in the asymptotic region.

The third condition of Eq. (17) is needed to assure the staticity of the asymptotic region. This condition reduces the number of parameters and imposes some controls on rotations of black holes.

To avoid a singular behavior of $g_{\phi\phi}$ on the ϕ axis, we also need to impose the following condition on β:

$$\beta^2 = -\frac{(\lambda+1)(1+\eta_2)}{(1+\eta_1)^2}. \tag{18}$$

In this article, we study the solution (14) and (16) satisfying the conditions (17) and (18). As mentioned later, to assure that the Arnowitt-Deser-Misner (ADM) mass is positive, we assume that the parameters η_1, η_2, and λ satisfy $\beta^2 < 1$, i.e.,

$$(1+\eta_1)^2 > -(1+\lambda)(1+\eta_2). \tag{19}$$

III. PROPERTIES

In this section, we investigate the properties of the solution satisfying the conditions (17) and (18). We study the asymptotic structure, the geometry of two black hole horizons and a bubble. We also analyze physical properties of the solution and the limit cases.

A. Asymptotic structure

In order to investigate the asymptotic structure of the solution, let us introduce the coordinate (r, θ) defined as

$$\rho = r\sin\theta, \qquad z = r\cos\theta, \tag{20}$$

where $0 \leq \theta < 2\pi$ and r is a four-dimensional radial coordinate in the neighborhood of the spatial infinity. For the large $r \to \infty$, each component behaves as

$$g_{tt} \simeq -1 - \frac{\eta_1 - \eta_2 - 2 - \beta^2(\eta_1 - \eta_2 + 2)}{1-\beta^2}\frac{\sigma}{r}, \tag{21}$$

$$g_{\rho\rho} = g_{zz} \simeq 1 - \frac{\eta_1 - 2 - \beta^2(\eta_1 - \lambda + 2) - \lambda}{1-\beta^2}\frac{\sigma}{r}, \tag{22}$$

$$g_{t\phi} \simeq -\frac{2\sigma^2\beta(2\eta_1 - \eta_2 - \lambda - 2 - \beta^2(2\eta_1 - \eta_2 - \lambda + 2))\sin^2\theta}{(1-\beta^2)^2 r}, \tag{23}$$

$$g_{\phi\phi} \simeq r^2\sin^2\theta\left(1 - \frac{\eta_1 - \lambda - 2 - \beta^2(\eta_1 - \lambda + 2)}{1-\beta^2}\frac{\sigma}{r}\right), \tag{24}$$

$$g_{\psi\psi} \simeq 1 + \frac{\sigma(\eta_2 - \lambda)}{r}. \tag{25}$$

Hence, the leading order of the metric takes the form

$$ds^2 \simeq -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + d\psi^2. \tag{26}$$

Therefore, the spacetime asymptotically has the structure of the direct product of the four-dimensional Minkowski spacetime and S^1. The S^1 at infinity is parametrized by ψ and the size $\Delta\psi$ is given in III C.

B. Mass and angular momentum

Next, we compute the total mass and the total angular momentum of the spacetime. It should be noted that since the asymptotic structure is $\mathcal{M}^{3,1} \times S^1$, the ADM mass and angular momentum are given by the surface integral over the spatial infinity with the topology of $S^2 \times S^1$. In order to compute these quantities, we introduce asymptotic Cartesian coordinates (x, y, z, ψ), where $x = \rho\cos\phi$ and $y = \rho\sin\phi$. Then, the ADM mass and angular momenta are given by

$$M_{\rm ADM} = \frac{1}{16\pi}\int_{S^2\times S^1} H^{0\alpha 0j}{}_{,\alpha} dS_j, \tag{27}$$

$$J^{\mu\nu} = \frac{1}{16\pi}\int_{S^2\times S^1}(x^\mu H^{\nu\alpha 0j}{}_{,\alpha} - x^\nu H^{\mu\alpha 0j}{}_{,\alpha} + H^{\mu j 0\nu} - H^{\nu j 0\mu})dS_j, \tag{28}$$

respectively. Here $H^{\mu\alpha\nu\beta}$ is defined by

$$H^{\mu\alpha\nu\beta} := -(\bar{h}^{\mu\nu}\eta^{\alpha\beta} + \bar{h}^{\alpha\beta}\eta^{\mu\nu} - \bar{h}^{\alpha\nu}\eta^{\beta\mu} - \bar{h}^{\beta\mu}\eta^{\alpha\nu}), \tag{29}$$

where

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \tfrac{1}{2}h^\alpha{}_\alpha\eta_{\mu\nu}, \tag{30}$$

and $\eta_{\mu\nu}$ is the five-dimensional flat metric with compactified one-dimensional and $h_{\mu\nu} := g_{\mu\nu} - \eta_{\mu\nu}$. The Latin index j runs x, y, z, and ψ and the Greek indices μ, ν, α, and β label t, x, y, z, and ψ. Then, the ADM mass of the solution is computed as

$$M_{\rm ADM} = \frac{\sigma(4 - 2\eta_1 + \eta_2 + \lambda + \beta^2(4 + 2\eta_1 - \eta_2 - \lambda))}{4(1-\beta^2)}\Delta\psi. \tag{31}$$

The nonzero component of the angular momentum becomes

$$J = J^{xy} = -\frac{\beta\sigma^2(2 - 2\eta_1 + \eta_2 + \lambda + \beta^2(2 + 2\eta_1 - \eta_2 - \lambda))}{(1-\beta^2)^2}\Delta\psi. \tag{32}$$

It should be noted that the ADM mass is non-negative when $\beta^2 < 1$.

We can also compute the tension $\mathcal{T}$ from the asymptotic forms of the metric following the formula given by [26],

$$\mathcal{T} = \frac{\sigma}{4}\left(\frac{-\eta_1 - \eta_2 + 2\lambda + 2 + \beta^2(\eta_1 + \eta_2 - 2\lambda + 2)}{(1-\beta^2)}\right). \tag{33}$$

C. Black holes and bubble

Here, for the solution, we consider the rod structure developed by Harmark [27] and Emparan and Reall [28]. The rod structure at $\rho = 0$ is illustrated in Fig. 1. (i) The finite timelike rod $[\eta_1\sigma, \eta_2\sigma]$ and $[\lambda\sigma, \sigma]$ denote the locations of black hole horizons. These timelike rods have directions $v_1 = (1, \Omega_1, 0)$ and $v_2 = (1, \Omega_2, 0)$, where Ω_1 and Ω_2 mean angular velocities of the horizons. These are given by

$$\Omega_1 = \frac{-\beta(1-\beta^2)}{[1 - \eta_1 + \beta^2(1+\eta_1)]^2\sigma}, \tag{34}$$

for $\eta_1\sigma < z < \eta_2\sigma$ and

$$\Omega_2 = \frac{-\beta(1-\beta^2)((1-\eta_2)(1-\lambda) + (1-\eta_1)^2)}{4[(1-\eta_1)^2 + \beta^2(1-\eta_2)(1-\lambda)]\sigma}, \tag{35}$$

for $\lambda\sigma < z < \sigma$. Here, it should be noted that Ω_1 and Ω_2 have the same signature. Therefore, two black holes are rotating along the same direction. (ii) The finite spacelike rod $[\eta_2\sigma, \lambda\sigma]$ which corresponds to a Kaluza-Klein bubble has the direction $v = (0, 0, 1)$. In order to avoid a conical singularity for $z \in [\eta_2\sigma, \lambda\sigma]$ and $\rho = 0$, ψ has the periodicity of

$$\begin{aligned}\frac{\Delta\psi}{2\pi} &= \lim_{\rho\to\infty}\sqrt{\frac{\rho^2 g_{\rho\rho}}{g_{\psi\psi}}} = \frac{2\sigma}{1-\beta^2}\left(\frac{\eta_1 - 1}{\eta_1 + 1}\right)\sqrt{\frac{(\lambda-\eta_1)(\lambda-\eta_2)(\eta_2+1)}{\eta_2 - 1}}\left(1 - \frac{(\eta_1+1)^2(\eta_2-1)}{(\eta_1-1)^2(\eta_2+1)}\beta^2\right)\\ &= 2\sigma\frac{(\eta_1+1)((\eta_1-1)^2 + (\lambda+1)(\eta_2-1))}{(\eta_1-1)((\eta_1+1)^2 + (\lambda+1)(\eta_2+1))}\sqrt{\frac{(\lambda-\eta_1)(\lambda-\eta_2)(\eta_2+1)}{\eta_2-1}}.\end{aligned} \tag{36}$$

(iii) The semi-infinite spacelike rods $[-\infty, \eta_1\sigma]$ and $[\sigma, \infty]$ have the direction $v = (0, 1, 0)$. In order to avoid conical singularity, ϕ has the periodicity of

$$\Delta\phi = 2\pi. \tag{37}$$

Here, we write the induced metrics of the event horizons and the bubble. For $\eta_1\sigma < z < \eta_2\sigma$, the induced metric on the surface with constant t becomes

$$g_{\phi\phi} = \frac{4\sigma^2(z^2-\sigma^2)(z-\eta_1\sigma)(\lambda\sigma - z)(1-\eta_1+\beta^2(1+\eta_1))^4}{(1-\beta^2)^2(\sigma^2[(\eta_1-1)^2(z+\sigma) + \beta^2(1+\eta_1)^2(\sigma-z)]^2 + 4\beta^2 k)}, \tag{38}$$

$$g_{\psi\psi} = \frac{z - \eta_2\sigma}{z - \lambda\sigma}, \tag{39}$$

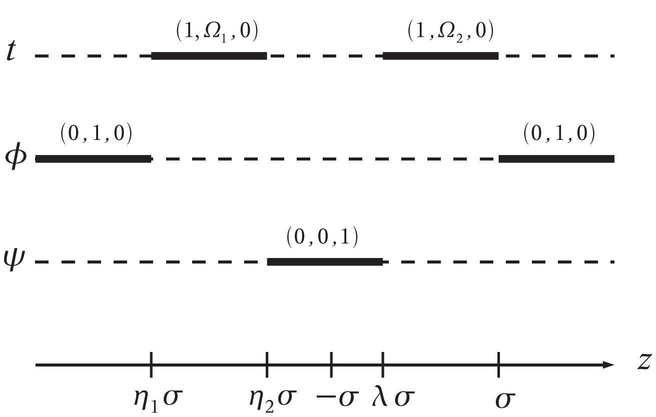

FIG. 1. Rod structure of rotating black holes on a Kaluza-Klein bubble. The finite timelike rods $[\eta_1\sigma, \eta_2\sigma]$ and $[\lambda\sigma, \sigma]$ correspond to rotating black holes with angular velocities Ω_1 and Ω_2, respectively. The finite spacelike rod $[\eta_2\sigma, \lambda\sigma]$ denotes a Kaluza-Klein bubble, where Kaluza-Klein circles shrink to zero.

$$g_{zz} = \frac{(\eta_2 - \eta_1)(\lambda - \eta_1)(\sigma^2[(\eta_1 - 1)^2(z + \sigma) + \beta^2(1 + \eta_1)^2(\sigma - z)]^2 + 4\beta^2 k)}{(1 - \beta^2)^2(\eta_1^2 - 1)^2(z^2 - \sigma^2)(z - \eta_1\sigma)(\eta_2\sigma - z)}, \tag{40}$$

where the function $k(\rho, z)$ is defined as

$$k(\rho, z) := (z - \eta_1\sigma)^2(\eta_2\sigma - z)(\lambda\sigma - z). \tag{41}$$

Since the ψ circles shrink to zero at $z = \eta_2\sigma$ and ϕ circles shrink to zero at $z = \eta_1\sigma$, the spatial cross section of this black hole horizon is topologically S^3. The area of the event horizon is

$$A_1 = 16\pi^2\sigma^3 \frac{(1 + \eta_1)^2(\eta_2 - \eta_1)^{3/2}(\lambda - \eta_1)}{(1 - \eta_1)^2} \times \sqrt{\frac{(\lambda - \eta_2)(\eta_2 + 1)}{(\eta_2 - 1)}} \frac{((1 - \eta_1)^2 - (1 + \lambda)(1 - \eta_2))(1 - \eta_1^2 - (1 + \lambda)(1 + \eta_2))^2}{((1 + \eta_1)^2 + (1 + \lambda)(1 + \eta_2))^3}. \tag{42}$$

For $\lambda\sigma < z < \sigma$, the induced metric takes the following form:

$$g_{\phi\phi} = \frac{16\sigma^2(\sigma - z)(z + \sigma)(z - \eta_1\sigma)(z - \eta_2\sigma)[(-1 + \eta_1)^2 + \beta^2(-1 + \eta_2)(-1 + \lambda)]^2}{(1 - \beta^2)^2[4\sigma^2(-1 + \eta_1)^4(z - \eta_2\sigma)(z - \lambda\sigma) + \beta^2 h]}, \tag{43}$$

$$g_{\psi\psi} = \frac{z - \lambda\sigma}{z - \eta_2\sigma}, \tag{44}$$

$$g_{zz} = \frac{4\sigma^2(\eta_1 - 1)^4(z - \eta_2\sigma)(z - \lambda\sigma) + \beta^2 h}{(1 - \beta^2)^2(\eta_1 - 1)^2(\eta_2 - 1)(\lambda - 1)(z - \eta_1\sigma)(z - \lambda\sigma)(\sigma^2 - z^2)}, \tag{45}$$

where the function h is given by

$$h(\rho, z) := (z - \eta_1\sigma)^2[(\sigma - z)(-1 + \eta_1)^2 - (\eta_2 - 1)(\lambda - 1)(z + \sigma)]^2. \tag{46}$$

Since the ψ circles shrink to zero at $z = \lambda\sigma$ and ϕ circles shrink to zero at $z = \sigma$, the spatial cross section of this black hole horizon is also topologically S^3. The area of this event horizon is

$$A_2 = 32\pi^2\sigma^3 \frac{(1 + \eta_1)^3\sqrt{(\lambda - 1)(\lambda - \eta_1)(\lambda - \eta_2)(1 + \eta_2)}}{(\eta_1 - 1)^2(1 - \eta_2)} \times \frac{((1 - \eta_1)^2 - (1 + \lambda)(1 - \eta_2))((1 - \lambda^2)(1 - \eta_2^2) - (1 - \eta_1^2)^2)}{((1 + \eta_1)^2 + (1 + \lambda)(1 + \eta_2))^3}. \tag{47}$$

For $\eta_2\sigma < z < \lambda\sigma$, the induced metric on the bubble can be written in the form

$$g_{\phi\phi} = \frac{4(1 - c\beta^2)^2(z - \eta_1\sigma)(z - \eta_2\sigma)(\sigma^2 - z^2)(z - \lambda\sigma)}{4\beta^2 d^2(z - \lambda\sigma)(z - \eta_1\sigma)^2 + (z - \eta_2\sigma)(z + \sigma + \beta^2 c(\sigma - z))^2} + \frac{16\beta^2 p}{(1 - \beta^2 c)^2(\sigma^2 - z^2)(z - \eta_1\sigma)}, \tag{48}$$

$$g_{zz} = -\frac{\sigma^2(\lambda - \eta_1)(\lambda - \eta_2)[(-1 + \eta_1)^2(1 + \eta_2) - \beta^2(1 + \eta_1)^2(-1 + \eta_2)]^2}{(1 - \beta^2)^2(-1 + \eta_1^2)^2(-1 + \eta_2^2)(z - \eta_2\sigma)(z - \lambda\sigma)}, \tag{49}$$

where the function p is given by

$$p(\rho, z) := (z - \eta_2\sigma)\left[(z + \sigma + \beta^2 c(\sigma - z))^2 + \frac{4\beta^2 d^2(z - \lambda\sigma)(z - \eta_1\sigma)^2}{(z - \eta_2\sigma)}\right] \times \left[\frac{\sigma}{1 - \beta^2} - \frac{(z + \sigma)}{(z - \eta_2\sigma)(z + \sigma + \beta^2 c(\sigma - z))^2 + 4\beta^2 d^2(z - \lambda\sigma)(z - \eta_1\sigma)^2} q\right]^2, \tag{50}$$

with

$$q(\rho, z) := \frac{2\sigma d(z-\lambda\sigma)(z-\eta_1\sigma)^2(1+\beta^2 c)}{\sigma + z} - (z-\eta_2\sigma)\left(1+\beta^2 c\frac{\sigma - z}{z+\sigma}\right)\left(-\frac{\sigma^2(1+\eta_1)^2}{1+\eta_2} + d\frac{(z-\lambda\sigma)(z-\eta_1\sigma)^2}{(z-\eta_2\sigma)}\right), \tag{51}$$

$$c = \frac{(1+\eta_1)^2(-1+\eta_2)}{(-1+\eta_1)^2(1+\eta_2)}, \qquad d = \frac{-1+\eta_2}{(-1+\eta_1)^2}. \tag{52}$$

The ψ circle vanishes for $z \in [\eta_2\sigma, \lambda\sigma]$, $\rho = 0$, which means that there exists a Kaluza-Klein bubble in this region. Since the ϕ circle does not vanish at $z = \eta_2\sigma$ and $z = \lambda\sigma$, this bubble on the time slice is topologically a cylinder $S^1 \times R$. Therefore, there exists a Kaluza-Klein bubble between two rotating black holes with the topology of S^3. The proper distance between the two black holes is

$$s = \pi\sigma\frac{(\eta_1+1)}{(\eta_1-1)}\sqrt{\frac{(\eta_2+1)(\lambda-\eta_1)(\lambda-\eta_2)}{(\eta_2-1)}}\frac{((1-\eta_1)^2-(1+\lambda)(1-\eta_2))}{((1+\eta_1)^2+(1+\lambda)(1+\eta_2))}. \tag{53}$$

The Kaluza-Klein bubble is significant to keep the balance of two black holes and achieve the solution without any strut structures and singularities. This property resembles that of the solution given by Elvang and Horowitz [20].

D. Intrinsic spin of black holes

To consider the intrinsic spin of the black holes and discuss the frame-dragging effects of them, we calculate Komar angular momenta of the black holes [16,29]. The Komar angular momenta of the left and right black holes are obtained as

$$J_{\mathrm{Komar},1} = \frac{\sigma^2\beta}{(1-\beta^2)^2}\frac{(\eta_2-\eta_1)^2((1+\lambda)(1+\eta_2)-(1-\eta_1^2))^2}{(1+\eta_1)^2(1+\eta_2)((1+\lambda)(1-\eta_2)-(1-\eta_1)^2)}\Delta\psi, \tag{54}$$

$$J_{\mathrm{Komar},2} = -\frac{\sigma^2\beta}{(1-\beta^2)^2}\frac{((1-\lambda)(1+\eta_2)-(1+\eta_1)^2)((1-\lambda^2)(1-\eta_2^2)-(1-\eta_1^2)^2)}{(1+\eta_1)^2(1+\eta_2)((1+\lambda)(1-\eta_2)-(1-\eta_1)^2)}\Delta\psi, \tag{55}$$

respectively. These satisfy that $J_{\mathrm{Komar},1} + J_{\mathrm{Komar},2} = J$. We also observe that the absolute value of the angular momentum of the left black hole is always smaller than that of the right black hole. Note that although two angular velocities of the horizons Ω_1 and Ω_2 have the same signature, the two Komar angular momenta have the inverse signature. As shown in Fig. 2, as η_2 approaches -1, the ratio of the two angular momenta $|J_{\mathrm{Komar},1}/J_{\mathrm{Komar},2}|$ asymptotically approaches 1. Then the total angular momentum J of two black holes can become sufficiently small. However it should be noted that in this limit the periodicity $\Delta\psi$ and the proper distance s between black holes go to zero. Therefore the case of $\eta_2 = -1$ is a singular limit of the solution.

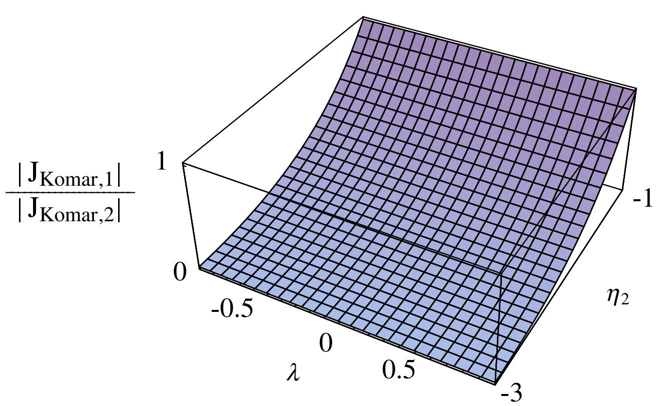

FIG. 2 (color online). Plot of the ratio $|J_{\mathrm{Komar},1}/J_{\mathrm{Komar},2}|$. In this plot, we fix the other parameter as $\eta_1 = -3.0$. This qualitative behavior of the ratio for η_2 does not depend on the choice of the parameter η_1.

Here we can give a brief statement of the rotation of black holes of the solution. These two black holes intrinsically rotate along the opposite directions to each other. However the weakly spinning black hole is turned to rotate along the same direction of the strong one because of the gravitational frame dragging of it. We discuss this effect in the next subsection. For the singular static limit $\eta_2 \to -1$, the frame-dragging effects by two black holes balance with each other, and then the spacetime becomes static.

E. Frame dragging

To confirm the frame-dragging effect, we investigate the ergo regions of black holes. It is expected that the ergo region of the left black hole becomes smaller as the separation s increases or the right black hole rotates slower.

Figure 3 shows the ergo regions for the different values -2.0, -4.0, -6.0 of η_2, where we fix the parameters as $\sigma = 1.0$, $\eta_1 = \eta_2 - 1.0$, and $\lambda = -0.5$. We find that the

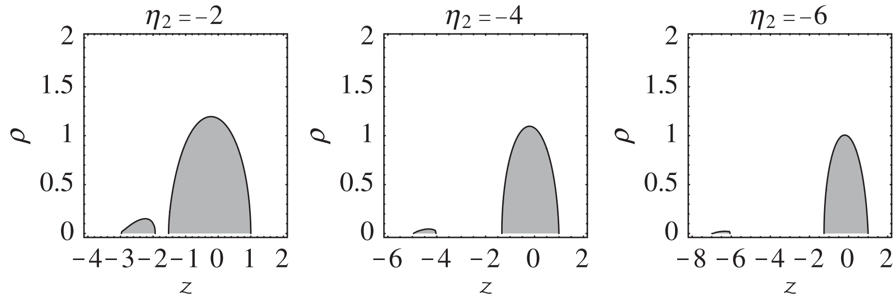

FIG. 3. Plots of ergo regions for $\eta_2 = -2.0, -4.0, -6.0$ on the (ρ, z) plane, where we fix the other parameters as $\sigma = 1.0$, $\eta_1 = \eta_2 - 1.0$, and $\lambda = -0.5$. The regions $\eta_1 < z < \eta_2$, $\rho = 0$, $-0.5 < z < 1.0$, $\rho = 0$ denote the black hole horizon, and the region $\eta_2 < z < \lambda$, $\rho = 0$ denotes the bubble.

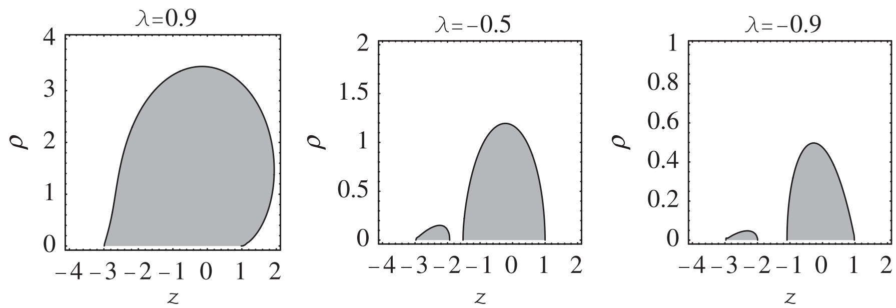

FIG. 4. Plots of ergo regions for $\lambda = 0.9, -0.5, -0.9$ on the (ρ, z) plane, where we fix the other parameters as $\sigma = 1.0$, $\eta_1 = -3.0$, and $\eta_2 = -2.0$. The regions $-3.0 < z < -2.0$, $\rho = 0$, $\lambda < z < 1.0$, $\rho = 0$ denote two black hole horizons, and the region $-2.0 < z < \lambda$, $\rho = 0$ denotes the bubble.

ergo region of the left black hole gradually shrinks as the black holes are away from the each other. Figure 4 shows the ergo regions for the different values $-0.9, -0.5, 0.9$ of λ, where we fix the parameters as $\sigma = 1.0$, $\eta_1 = -3.0$, and $\eta_2 = -2.0$. We find that for $\lambda = 0.9$, an ergo surface encloses the two black holes and the bubble. For $\lambda = -0.5$, two ergo surfaces appear and enclose only each black hole. For $\lambda = -0.9$, the ergo surface surrounding the left black hole becomes small. These facts are consistent with the above expectation.

The left panel of Fig. 4 shows that the KK bubble also rotates because the bubble is enclosed by an ergo region. To confirm the rotation of the bubble, we consider the motion for a zero angular momentum observer (ZAMO) on the bubble. The angular velocity along $\partial/\partial\phi$ of the ZAMO measured with respect to time for a distant observer is given by $-\frac{g_{t\phi}}{g_{\phi\phi}}$. The ZAMO on the bubble has an angular velocity by the effect of the frame dragging. As shown in Fig. 5, the angular velocity of a ZAMO on the bubble is an increasing function of z, where the parameters are fixed as $\sigma = 1.0$, $\lambda = 0.5$, $\eta_1 = -3.0$, and $\eta_2 = -2.0$ and we take a minus sign of β. The angular velocity of the ZAMO on the bubble takes a maximum on the right black hole and a minimum on the left black hole. In the choice of other parameters, this qualitative behavior of the angular velocity does not change so much. This suggests that the left black hole is rotating in the same direction as the right one by the effect of gravitational frame dragging.

We comment on the signs of the Komar masses of two black holes $M_{\mathrm{Komar},i} (i = 1, 2)$, which are given by the following integrals over the horizons:

$$M_{\mathrm{Komar},i} = \frac{3}{32\pi} \int_{bh_i} *d\xi, \tag{56}$$

where $\xi = \partial/\partial t$ and $*$ is a Hodge dual operator. Note that

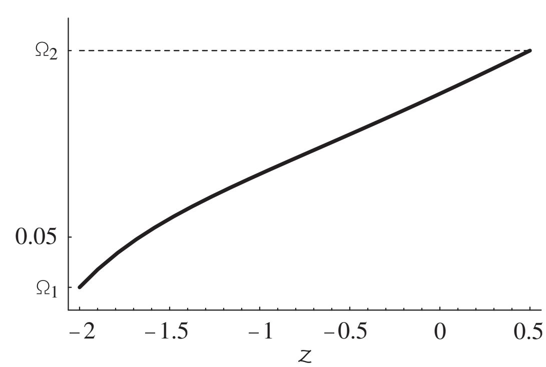

FIG. 5. Angular velocity of the ZAMO on the bubble in the case of $\sigma = 1.0$, $\lambda = 0.5$, $\eta_1 = -3.0$, $\eta_2 = -2.0$, and $\beta < 0$. Near $z = -2$, the graph is bent down by the dragging of the left black hole.

they are related to the ADM mass and tension by

$$M_{\rm ADM} = M_{\rm Komar,1} + M_{\rm Komar,2} + \tfrac{1}{2}\mathcal{T}\Delta\psi, \tag{57}$$

(See Kastor *et al*'s discussions [30] about the relation between ADM mass and tension in black holes–bubbles systems). The Komar masses of the two black holes satisfy the Smarr-type formula

$$M_{{\rm Komar},i} = \tfrac{3}{8}l_i\Delta\psi + \tfrac{3}{2}J_{{\rm Komar},i}\Omega_i, \tag{58}$$

where $l_1 = \sigma(1-\lambda)$ and $l_2 = \sigma(\eta_2-\eta_1)$. Now we consider the case of $\beta > 0$, where $J_{\rm Komar,1} > 0$, $J_{\rm Komar,2} < 0$, $\Omega_1 < 0$, and $\Omega_2 < 0$. As β^2 approaches 1, the absolute value of $J_{\rm Komar,1}\Omega_1(<0)$ becomes sufficiently large. Then the left black hole has negative Komar mass. This situation is similar to the rotational frame dragging in the black Saturn solution [16,31]. In the five-dimensional Einstein-Maxwell-Chern-Simons system, a single black hole can be counterrotating; its horizon rotates in the opposite sense to the angular momentum [32]. This type of counterrotating black hole can also posses a negative Komar mass at the horizon, though the total mass is positive [33].

F. Static case

Let us consider the static case, which can be obtained by the choice of the parameter $\beta = 0$. From Eq. (18) this is achieved by $\eta_2 = -1$ or $\lambda = -1$. As in Sec. III D, the case of $\eta_2 = -1$ corresponds to the singular static limit. Therefore we concentrate our attention on the case of $\lambda = -1$. Let us define the parameters a, b, and c as

$$a = \frac{2-\lambda-\eta_2}{2}\sigma, \qquad b = \frac{\lambda-\eta_2}{2}\sigma, \qquad c = \frac{\lambda+\eta_2-2\eta_1}{2}\sigma. \tag{59}$$

It should be noted that $\lambda = -1$ is equal to the condition $\sigma = (a-b)/2$. Furthermore, let us shift an origin of the z coordinate such that $z \to \tilde z := z - (\eta_2+\lambda)\sigma/2$. Then, we obtain the metric

$$\begin{aligned} ds^2 = &-\frac{(R_b-(\tilde z-b))(R_{-c}-(\tilde z+c))}{(R_a-(\tilde z-a))(R_{-b}-(\tilde z+b))}dt^2 \\ &+ (R_a-(\tilde z-a))(R_{-c}-(\tilde z+c))d\phi^2 \\ &+ \frac{R_{-b}-(\tilde z+b)}{R_b-(\tilde z-b)}d\psi^2 + \frac{Y_{a,-c}Y_{b,-b}}{4R_aR_bR_{-b}R_{-c}} \\ &\times\sqrt{\frac{Y_{a,b}Y_{-b,-c}}{Y_{a,-b}Y_{b,-c}}}\frac{R_a-(\tilde z-a)}{R_{-c}-(\tilde z+c)}(d\rho^2+d\tilde z^2), \end{aligned} \tag{60}$$

where the coordinate z in the definition of R_d is replaced with $\tilde z$. This coincides with the solution obtained by Elvang and Horowitz [20], which describes nonrotating black holes on the Kaluza-Klein bubble.

G. Small black holes

Now we consider the limit of the solutions to small black holes. We introduce new coordinates θ and $\hat\psi$ defined by

$$z = \eta_2\sigma - \epsilon\sin^2\theta(0<\epsilon\ll 1), \tag{61}$$

$$\begin{aligned} \frac{\psi}{\hat\psi} = 2\sigma&\frac{(\eta_1+1)((\eta_1-1)^2+(\lambda+1)(\eta_2-1))}{(\eta_1-1)((\eta_1+1)^2+(\lambda+1)(\eta_2+1))} \\ &\times\sqrt{\frac{(\lambda-\eta_1)(\lambda-\eta_2)(\eta_2+1)}{\eta_2-1}}, \end{aligned} \tag{62}$$

where $0\le\theta\le\pi/2$, $0\le\hat\psi\le 2\pi$. We also put $\eta_1 = \eta_2-\epsilon$. Then, the induced metric on the black hole $\eta_1\sigma < z < \eta_2\sigma$ takes the form of

$$\begin{aligned} ds^2|_{\rm bh1} = 4\sigma^2&\frac{(\eta_2+1)(\eta_2+\lambda)(\lambda-\eta_2)}{(\eta_2-1)(2+\eta_2+\lambda)^2} \\ &\times\epsilon(d\theta^2+\cos^2\theta d\phi^2+\sin^2\theta\hat\psi^2). \end{aligned} \tag{63}$$

This shows that it is a small black hole with topology S^3. On the other hand, for the other black hole $\lambda\sigma < z < \sigma$, let us use the coordinate $z = \lambda\sigma + \epsilon\sigma\sin^2\theta(0<\epsilon\ll 1)$. Then,

$$\begin{aligned} ds^2|_{\rm bh2} = &\frac{8\sigma^2(1-\eta_1)(1+\eta_1)^4}{(3+2\eta_1+\eta_1^2+2\eta_2)^2}(\cot^2\theta d\phi^2+\sin^2\theta d\theta^2) + 4\sigma^2\epsilon \\ &\times\frac{(1+\eta_1)^2(-1+\eta_2)(-1-2\eta_1+\eta_1^2+2\eta_2)^2}{(1-\eta_1)(1-\eta_2)(3+2\eta_1+\eta_1^2+2\eta_2)^2}\sin^2\theta d\psi^2. \end{aligned} \tag{64}$$

This resembles the horizon geometry of the five-dimensional extremally Myers-Perry black hole solutions with a single angular momentum.

H. Big black holes

Next we consider the big black holes limit. For this end, we take $\eta_2-\eta_1$, $1-\lambda \ll \lambda-\eta_2$. In this limit, β vanishes. The induced metric on the horizon $\eta_1\sigma < z \ll \eta_2\sigma$ approximately takes the form of

$$ds^2|_{\rm bh1} \simeq d\psi^2 + 4(\eta_1\sigma - z)zd\phi^2 + \frac{\sigma^2\eta_1^2}{z(z-\eta_1\sigma)}dz^2. \tag{65}$$

Introducing a new coordinate θ defined as $z = \eta_1\sigma(1-\cos\theta)/2$, we can rewrite the metric as

$$ds^2|_{\mathrm{bh}1} \simeq d\psi^2 + \sigma^2\eta_1^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (66)$$

For $\lambda\sigma \ll z < \sigma$, the metric on the horizon behaves as

$$ds^2|_{\mathrm{bh}2} \simeq \frac{1-\lambda}{1-\eta_2}d\psi^2 + 4\frac{-\eta_1}{1-\lambda} \times \left[(\sigma^2 - z^2)d\phi^2 + \sigma^2\frac{dz^2}{\sigma^2 - z^2}\right]. \quad (67)$$

In terms of a coordinate $z = \sigma\cos\theta$, the metric is written as

$$ds^2|_{\mathrm{bh}2} \simeq \frac{1-\lambda}{1-\eta_2}d\psi^2 + 4\frac{-\eta_1}{1-\lambda}[d\theta^2 + \sin^2\theta d\phi^2]. \quad (68)$$

From Eqs. (66) and (68), we find that away from the bubble, the induced metrics on the horizons are the product of a circle and a round two-sphere with large radius. When $z \simeq \eta_2\sigma$ for the left black hole and $z \simeq \lambda\sigma$ for the right black hole, the metrics do not take the form of Eqs. (66) and (68). In this region, the KK circles ψ depend on z and shrink to zero at $z = \eta_2\sigma$ and $z = \lambda\sigma$, respectively. This is how the solutions admit configuration of arbitrarily large black holes with S^3 topology, as is pointed out in the static case [20]. Hence, it is an interesting issue to consider that the solutions admit the deformation from a black string $S^1 \times S^2$ topology into two large black holes with S^3 topology. In the next section, we will discuss the entropy of two black holes on the bubble and that of a black string, and will compare them. The ratio of the distance between two black holes to the size of KK circles is given by

$$\frac{s}{\Delta\psi} = \frac{1}{4}. \quad (69)$$

Therefore, the two black holes cannot approach each other without shrinking the size of the KK circles at infinity. Thus we find that this limit is singular.

I. Entropy

As mentioned in the above, our solutions describe two black holes in equilibrium by the existence of the bubble between them. We compare the total area $A_{2\mathrm{BH}} = A_1 + A_2$ of two black holes with the area of a rotating black string with the same mass (31), the same angular momentum (32), and the same circle (36) at infinity. For simplicity, we consider the five-dimensional rotating black string with a translationally invariant compactified extra dimension. The metric of the rotating black string can be written in the form

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\tilde\theta}{\Sigma}\right)d\tilde t^2 - \frac{2a\sin^2\tilde\theta(\tilde r^2 + a^2 - \Sigma)}{\Sigma}d\tilde t d\tilde\phi + \frac{(\tilde r^2 + a^2)^2 - \Delta a^2\sin^2\tilde\theta}{\Sigma}\sin^2\tilde\theta d\tilde\phi^2 + \frac{\Sigma}{\Delta}d\tilde\phi^2 + \Sigma d\tilde\theta^2 + d\tilde\psi^2, \quad (70)$$

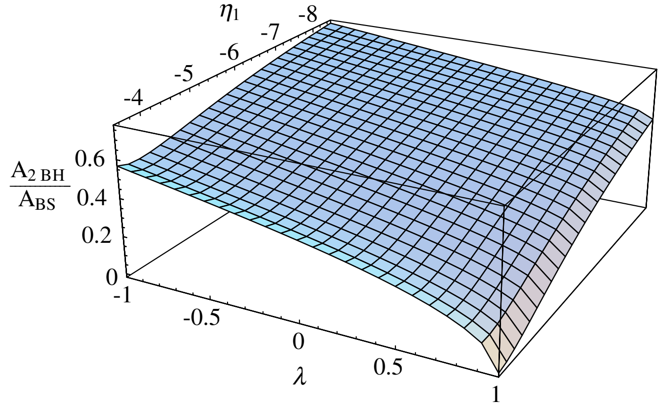

FIG. 6 (color online). Plot of the ratio $A_{2\mathrm{BH}}/A_{\mathrm{BS}}$ on the (η_1, λ) plane. We fix the other parameters as $\eta_2 = -3.5$ and $\sigma = 1.0$.

where $\Sigma = \tilde r^2 + a^2\cos^2\tilde\theta$, $\Delta = \tilde r^2 + a^2 - 2mr$. $\tilde\psi$ has the periodicity of ψ, i.e., $\Delta\tilde\psi = \Delta\psi$. The mass parameter and the angular momentum parameter of the black string are given by $m = M_{\mathrm{ADM}}/\Delta\psi$ and $a = J/M_{\mathrm{ADM}}$, respectively. Then, the area of the rotating black string is given by

$$A_{\mathrm{BS}} = 4\pi[(m + \sqrt{m^2 - a^2})^2 + a^2]\Delta\tilde\psi. \quad (71)$$

Figure 6 shows the plots of the ratio $A_{2\mathrm{BH}}/A_{\mathrm{BS}}$ on the (η_1, λ) plane, where we fix the other parameters as $\eta_2 = -3.5$ and $\sigma = 1.0$. The ratio is smaller than 1 within the range of $\eta_1 < \eta_2 < \lambda$ and $\beta^2 < 1$; we also find that this qualitative result does not depend on the choice of the parameter η_2. Hence like the static case [20], we cannot expect a rotating black string to spontaneously generate a Kaluza-Klein bubble and split into two rotating black holes with topology S^3.

IV. SUMMARY AND DISCUSSION

Using the solitonic solution-generating methods, we generated a new exact solution which describes a pair of rotating black holes on a Kaluza-Klein bubble as a vacuum solution in the five-dimensional Kaluza-Klein theory. We also investigated the properties of this solution, particularly, its asymptotic structure, the geometry of the black hole horizons and the Kaluza-Klein bubble, and the limit of the static case. The asymptotic structure is the S^1 bundle over the four-dimensional Minkowski spacetime. Two black holes have the topological structure of S^3 and the bubble is topologically $S^1 \times R$. The solution describes the physical situation such that two black holes have the angular velocity of the same direction and the bubble plays a role in holding two black holes. In the static case, it coincides with the solution found by Elvang and Horowitz. We also have studied the physical properties of the solutions. It has been shown that the two black holes have inverse intrinsic spins of each other even though the angular velocities are the same sign. This feature is attributed to the gravitational frame dragging of the faster black hole.

The counterrotating black hole can have negative Komar mass. We have also compared the entropy of two black holes on a bubble with a rotating black string with the same mass and the same angular momentum. Like the static solution [20] and the boosted black hole solutions on a bubble [29], we cannot expect that a black string spontaneously generates a Kaluza-Klein bubble and it splits the horizon with the topology $S^1 \times S^2$ into two black holes with the topology S^3.

In this article, we concentrated on the black hole solution with a single angular momentum component. The investigation of the solution with two angular momentum components is enormously challenging. In general, the inverse scattering method can generate a solution with two angular momentum components. However, as discussed in Refs. [11,14], such a solution generated from our seed would have singular behavior on an axis due to the issues of the normalization. In order to obtain a solution with two angular momentum components, we need to change our seed into another seed which does not satisfy the condition $\det g_{ij} = -\rho^2$. We will give such a solution in our future article.

ACKNOWLEDGMENTS

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APPENDIX: SOLUTION BY BÄCKLUND TRANSFORMATION

In this appendix we briefly present the solution obtained by the Bäcklund transformation which was developed to apply the five-dimensional case [5].

The metric of the solitonic solution can be written in the following form:

$$ds^2 = e^{-T}[-e^S(dt - \omega d\phi)^2 + e^{-S}\rho^2(d\phi)^2 + e^{2\gamma - S}(d\rho^2 + dz^2)] + e^{2T}(d\psi)^2. \tag{A1}$$

The function T is derived from the seed metric (2) as

$$T = \tilde{U}_{\lambda\sigma} - \tilde{U}_{\eta_2\sigma}, \tag{A2}$$

where the function $\tilde{U}_d$ is defined as $\tilde{U}_d := \frac{1}{2}\ln[R_d + (z - d)]$. The other metric functions for the five-dimensional metric (A1) are obtained by using the formulas shown by [6],

$$e^S = e^{S^{(0)}}\frac{A}{B}, \tag{A3}$$

$$\omega = 2\sigma e^{-S^{(0)}}\frac{C}{A} - C_1, \tag{A4}$$

$$e^{2\gamma} = C_2(x^2 - 1)^{-1}Ae^{2\gamma'}, \tag{A5}$$

where C_1 and C_2 are constants and A, B, and C are given by

$$A := (x^2 - 1)(1 + ab)^2 - (1 - y^2)(b - a)^2, \tag{A6}$$

$$B := [(x + 1) + (x - 1)ab]^2 + [(1 + y)a + (1 - y)b]^2, \tag{A7}$$

$$C := (x^2 - 1)(1 + ab)[(1 - y)b - (1 + y)a] + (1 - y^2)(b - a)[x + 1 - (x - 1)ab], \tag{A8}$$

and x and y are the prolate-spheroidal coordinates: $\rho = \sigma\sqrt{(x^2 - 1)(1 - y^2)}$, $z = \sigma xy$. Here the function $S^{(0)}$ is a seed function which can be derived from the seed metric (2) as

$$S^{(0)} = \tilde{U}_{\lambda\sigma} - 2\tilde{U}_{\eta_1\sigma} + \tilde{U}_{\eta_2\sigma}. \tag{A9}$$

The functions a and b, which are auxiliary potentials to obtain the new Ernst potential for the seed by the transformation, are given by

$$a = \alpha \cdot \frac{e^{2U_\sigma} + e^{2\tilde{U}_{\lambda\sigma}}}{e^{\tilde{U}_{\lambda\sigma}}} \cdot \frac{e^{2U_\sigma} + e^{2\tilde{U}_{\eta_2\sigma}}}{e^{\tilde{U}_{\eta_2\sigma}}} \cdot \left(\frac{e^{\tilde{U}_{\eta_1\sigma}}}{e^{2U_\sigma} + e^{2\tilde{U}_{\eta_1\sigma}}}\right)^2, \tag{A10}$$

$$b = \beta \cdot \frac{e^{\tilde{U}_{\lambda\sigma}}}{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\lambda\sigma}}} \cdot \frac{e^{\tilde{U}_{\eta_2\sigma}}}{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\eta_2\sigma}}} \cdot \left(\frac{e^{2U_{-\sigma}} + e^{2\tilde{U}_{\eta_1\sigma}}}{e^{\tilde{U}_{\eta_1\sigma}}}\right)^2, \tag{A11}$$

where the function U_d is defined as $U_d := \frac{1}{2}\ln[R_d - (z - d)]$. In addition the function γ' is obtained as

$$\gamma' = \gamma'_{\sigma,\sigma} + \gamma'_{-\sigma,-\sigma} + \gamma'_{\lambda\sigma,\lambda\sigma} + \gamma'_{\eta_1\sigma,\eta_1\sigma} + \gamma'_{\eta_2\sigma,\eta_2\sigma} - 2\gamma'_{\sigma,-\sigma} + \gamma'_{\sigma,\lambda\sigma} - 2\gamma'_{\sigma,\eta_1\sigma} + \gamma'_{\sigma,\eta_2\sigma} - \gamma'_{-\sigma,\lambda\sigma} + 2\gamma'_{-\sigma,\eta_1\sigma} - \gamma'_{-\sigma,\eta_2\sigma} - \gamma'_{\lambda\sigma,\eta_1\sigma} - \gamma'_{\lambda\sigma,\eta_2\sigma} - \gamma'_{\eta_1\sigma,\eta_2\sigma}, \tag{A12}$$

where

$$\gamma'_{cd} = \tfrac{1}{2}\tilde{U}_c + \tfrac{1}{2}\tilde{U}_d - \tfrac{1}{4}\ln[R_cR_d + (z - c)(z - d) + \rho^2]. \tag{A13}$$

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