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Reheating induced by competing decay modes

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We address the problem of studying the decay of the inflaton field ϕ to another scalar field χ through parametric resonance in the case of a coupling that involves several decay modes. This amounts to the presence of extra harmonic terms in the perturbation of the χ field dynamics. For the case of two frequencies we compute the geometry of the resonance regions, which is significantly altered due to the presence of noncuspidal resonance regions associated to higher harmonics and to the emergence of instability ''pockets.'' We discuss the effect of this change in the efficiency of the energy transfer process for the simplest case of a coupling given by a combination of the two interaction terms of homogeneous degree usually considered in the literature. We find that the presence of higher harmonics has limited cosmological implications.

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I. INTRODUCTION

The success of the inflationary paradigm depends to a great extent on the corresponding success of the reheating stage that takes place after inflation [1,2] through which all elementary particles that exist in the universe were created. During inflation the universe expands exponentially, its matter content is diluted, and the temperature decreases as the inverse of the exponential of the number of e-folds $\exp N_e$. Unless some different mechanism is considered for the inflaton decay (as in warm inflation [3,4]) there has to be some process to raise the temperature to the levels required for the nucleosynthesis of the light elements to take place according to the standard thermal history of the big bang universe [5].

In most post-inflationary scenarios reheating occurs due to particle production by an oscillating scalar field ϕ. In the simplest models this field is the inflaton field that drives inflation. After inflation the scalar field ϕ oscillates near a minimum of its potential and this triggers a sequence of processes that produces elementary particles and eventually restores the temperature [6–12].

Since the beginning of the 90's a considerable effort has been devoted to model the reheating process [6,13,14] (a general account can be found in [2]). In most models, the first stage of this complicated sequence involves the excitation by parametric resonance of a second scalar field, here denoted χ, giving rise to an exponential increase in the number of χ boson particles [15,16] (for a comprehensive review see also [17]).

Despite the many contributions regarding the preheating/reheating mechanism itself [18–27] and its observational implications [28–31], some general questions remain to be completely answered. In particular, how does the resonant energy transfer depend on the coupling between ϕ and χ and on the inflaton asymptotic dynamics? The latter issue has been studied in depth for a large variety of polynomial potentials [16,32–34]. The former issue is less well studied and has been the subject of some recent work [35–37].

In the simplest of the preheating scenarios the inflaton couples to the χ field through interaction terms of the form $h\phi\chi^2$ or $g^2\phi^2\chi^2$ that correspond to two different decay modes of the scalar field ϕ into another boson [1,2]. These two coupling terms give rise to the same qualitative effects and are considered as alternative models. Indeed, in both cases the equation for the scalar field χ can be reduced to a Mathieu equation, and thus the parametric resonance follows similar patterns (even though the numerical values of the model's outcome may be slightly different). There are two regimes, a broad resonance regime, in which the amplitude of the periodic perturbation of the χ-field frequency is of the same order as or larger than the frequency of the ϕ scalar field, and a narrow resonance regime where the amplitude of the perturbation is small. The broad resonance region of parameter space includes, for sufficiently large values of the perturbation, the tachyonic resonance regime which has been shown in [35] to be extremely effective in transferring most of the energy of the inflation to the χ field.

The main feature that emerges from these studies of preheating is that the broad and tachyonic resonance regimes give the predominant contribution to the χ field energy density. However, there is always the possibility

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of a contribution to the total particle production in the narrow resonance regime when the coupling parameters characterizing the interaction between ϕ and χ are small and/or in the decay of residual inflaton oscillations. In the particular case of parametric resonance modeled by a Mathieu equation, this contribution is indeed small. The growth of the modes of χ is exponential in the resonance bands or tongues in parameter space, and the first resonance band is the only band wide enough to give rise to significant χ excitations. However, this need not be so when there are more frequencies of excitation of the χ field. Here we show that in this case resonance is governed by a general Hill equation and that other resonances beyond the first may contribute to the amplification of the χ modes. Therefore, there is the possibility that these higher frequency excitations contribute significantly to the overall creation of χ particles. This analysis is the subject of the present work, in the case when the χ field is parametrically forced by two frequencies in a 1:2 ratio. For this case, we show that the contribution for the χ field energy density of the higher harmonic is a small fraction of that of the fundamental frequency.

The outline of the paper is as follows. In Sec. II we review the parametric frequency preheating mechanism and the method to compute the particle production rate in the general framework of Hill's equation. In Sec. III we apply this method to compare the reheating efficiency of two different couplings of the inflaton field to the χ field. In Sec. IV we sum up the conclusions of this analysis.

II. PARTICLE PRODUCTION BY PARAMETRIC RESONANCE

We start by reviewing some properties of the parametric resonance mechanism for a general periodic perturbation of the frequency of the oscillator, which corresponds to Hill's Eq. (3).

Reheating models in inflationary universes start by considering that, at the end of inflation, the inflaton ϕ is in a coherent oscillatory state described by a space-independent expectation value, governed by the equation of motion for the inflaton

$$\ddot{\phi} + 3H\dot{\phi} + a^{-2}\nabla^2\phi + V'(\phi) = 0, \tag{1}$$

where H is the Hubble parameter of a Friedmann-Robertson-Walker metric with scale factor $a(t)$, and $V'(\phi)$ is the derivative of the inflaton's potential $V(\phi)$ with respect to ϕ. It is assumed that $V(\phi)$ has a vanishing minimum for the oscillations to take place, and also that the inflaton couples to another scalar field χ which is then periodically perturbed by the inflaton. The equation of motion of χ is given by

$$\ddot{\chi} + 3H\dot{\chi} + a^{-2}\nabla^2\chi + U'(\chi) + \frac{\partial}{\partial\chi}V_{\rm int}(\phi, \chi) = 0, \tag{2}$$

where $V_{\rm int} = V_{\rm int}(\phi, \chi)$ is the interaction potential between χ and ϕ, and where $U(\chi)$ is a potential that gives mass to χ. At the onset of the process of energy transfer from ϕ, χ is assumed to be at the vanishing minimum of this potential. The oscillations of ϕ around the minimum of its potential are faster than the expansion rate of the universe, so it is meaningful in a relatively short time scale to work within the simplifying assumption that the actual space-time can be approximated with a Minkowski metric. For reasonable choices of a single interaction potential $V_{\rm int}$, namely, cubic interactions of the form $g\phi\chi^2$, or quartic interactions $h\phi^2\chi^2$, the resulting χ Eq. (2) is that of an oscillator with a harmonically perturbed frequency. This yields a Mathieu type equation and provides a well-known mechanism for the parametric resonance of χ and for the exponential amplification of its particle number [17,25].

For more general coupling terms, in particular, nonhomogeneous couplings involving different powers of ϕ, the perturbation equation for χ is parametrically forced by a periodic function (for the quasiperiodic case see for instance [38]) and the equations of motion for the χ modes are of the form (more details in Sec. III)

$$\ddot{\chi}_k + (\omega_k^2 + \epsilon F(t))\chi_k = 0, \tag{3}$$

where $F(t)$ is a periodic function with period 2π, and we may set $F(t) = F(-t)$, $\int_0^{2\pi} F(t)dt = 0$, and ϵ a small positive parameter related with the amplitude of the inflaton oscillations.

For the general Hill Eq. (3), Floquet's theorem states that the solutions are of the form

$$\chi_k(t) = e^{\mu_k t}\chi_k^{(0)}(t), \tag{4}$$

where $\chi_k^{(0)}(t)$ is a periodic function with the same period as $F(t)$ and μ_k is one of the two characteristic exponents, which are both real or complex conjugate.

Clearly, the lines $\mathrm{Re}(\mu_k) = 0$ divide the (ω_k, ϵ) parameter plane in unstable regions and stable ones, defining instability bands which become narrow close to the $\epsilon = 0$ axis, producing what is often called a structure of tongues [39]. For a perturbation of period 2π these tongues end on the $\epsilon = 0$ axis at the points ω_k that satisfy the parametric resonance condition $\omega_k = n/2$ [see Fig. 1(a)]. These modes are amplified by an arbitrarily small parametric forcing of period 2π.

To determine the characteristic exponent of the solutions of Eq. (3) we follow Hill's method of solution. Given that the function F is periodic we write it as

$$F(t) = \frac{1}{2}\sum_{k=-s}^{s} c_k e^{ikt}, \tag{5}$$

where $c_0 = 0$ (due to the parity of F) and $s \in \mathbb{N}\bigcup\{\infty\}$. This (finite or infinite) Fourier series expansion of $F(t)$, (5), together with Floquet's theorem suggests looking for a solution of (3) of the form

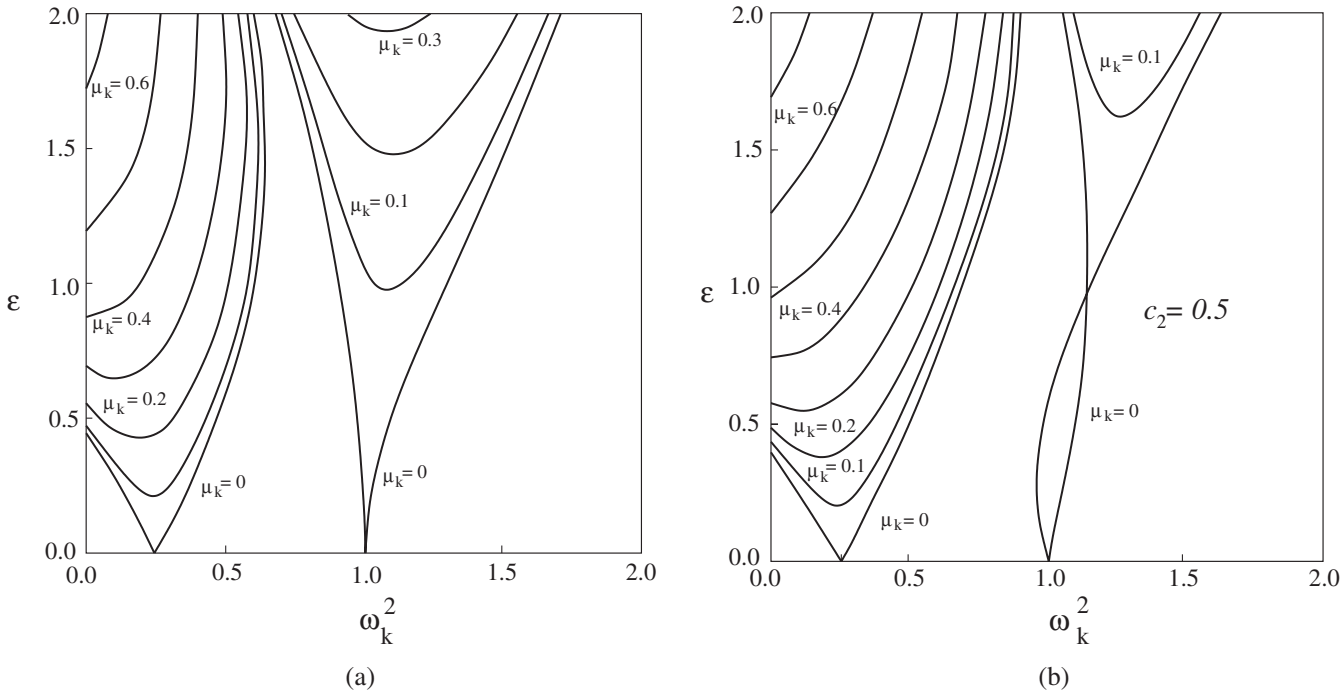

FIG. 1. (a) Instability diagram for Mathieu equation (26); (b) Instability diagram for Hill equation (32). Notice the emergence of an instability pocket in the second band in Fig. 1(b) and the modifications in the level curves of the characteristic exponent μ. These plots were obtained using Hill's method of solution.

$$\chi_k(t) = e^{\mu_k t} \sum_{n=-\infty}^{+\infty} b_n e^{\text{int}}. \tag{6}$$

Inserting this expression and Eq. (5) in (3) and equating the coefficients of $e^{(\mu_k + in)t}$, we derive a homogeneous system with infinitely many linear equations

$$(\mu_k + in)^2 b_n + \frac{\epsilon}{2} \sum_{m=-\infty}^{+\infty} c_m b_{n-m} = 0, \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots \tag{7}$$

Eliminating the coefficients b_n in (7), a nontrivial solution exists if the characteristic exponent satisfies an infinite determinantal equation (called Hill's determinantal equation),

$$\Delta(\epsilon, \mu_k) = |B_{rs}| = 0, \tag{8}$$

where the elements B_{rs} of $\Delta(\epsilon, \mu_k)$ are given by

$$B_{rs} = \begin{cases} 1 & \text{if } r = s \\ \frac{\epsilon c_{r-s}/2}{(\mu + ir)^2 + \omega_k} & \text{if } r \neq s \end{cases}. \tag{9}$$

Here an infinite determinant $D = |B_{mn}|$, $(m, n = -\infty, \ldots, +\infty)$ is defined as the limit of $D_m = \det(B_{ij}) \times (i, j = -m, \ldots, m)$ as $m \to +\infty$, if it exists.

Equation (8) can be reduced to the simpler form of a transcendental equation in μ_k [40]

$$\sin^2(i\pi\mu_k) = \Delta(\epsilon, 0)\sin^2(\pi\omega_k). \tag{10}$$

This determines a characteristic exponent μ_k, which in turn determines the b_n coefficients of (7), and hence, a formal solution (6) of Hill's equation. The real part of μ_k determines the growth factor of the solutions (6).

In the narrow resonance regime the phenomenon of parametrically resonant excitations, where $\epsilon \ll 1$, it is possible to derive an approximate expression in closed form for μ_k as a function of ϵ. First, notice that $\Delta(0, 0) = 1$. Second, we see from (9) that the values of B_{rs} depend linearly on ϵ and that $\Delta(\epsilon, 0) = 1 + O(\epsilon^2)$. This means $\Delta(\epsilon, 0) \simeq 1 + \alpha\epsilon^2$, where α depends on $\mathbf{c} = (c_{-s}, c_{-s+1}, \ldots, c_{-1}, c_1, \ldots, c_{s-1}, c_s)$ and on ω_k.

Inverting (10) using $\arcsin z = \ln(iz + \sqrt{1 - z^2})$ and expanding in powers of ϵ, yields for the growth factor

$$\text{Re}\,(\mu_k) \simeq \epsilon\alpha(\mathbf{c}, \omega_k)|\sin(\pi\omega_k)|. \tag{11}$$

This expression will be used later to determine numerically the resonant tongues for a particular function $F(t)$.

For periodic perturbations with finite Fourier expansions (5), $s \in \mathbb{N}$, it can be shown that the asymptotic form of the width L_n of the nth interval of instability is given by [40]

$$L_n = \frac{8s^2}{[(p-1)!]}\left(\frac{|c_s\epsilon|}{8s^2 4^p}\right)^p + O(\epsilon^{p+1}), \qquad n = sp \tag{12}$$

$$L_n = a_n \frac{|\epsilon|^p}{4^p} + O(\epsilon^{p+1}), \qquad s(p-1) < n < sp, \tag{13}$$

where p is the integer defined in these equations and where

a_n is a constant independent of ϵ. One of the remarkable consequences of these formulae is that the widths of all the resonant bands up to the sth band depend linearly on the perturbation amplitude, while the resonances of order higher than s are associated with thinner instability regions.

This has direct impact in the computation of the energy production by parametric resonance. For small ϵ the main contributions to the energy density of the χ field

$$\rho_\chi(t) = \frac{1}{2}\int d^3\mathbf{k}[\omega_k^2|\chi_k|^2 + |\dot{\chi}_k|^2] \tag{14}$$

$$\simeq \int d^3\mathbf{k}\,\omega_k^2|\chi_k(t)|^2 \tag{15}$$

come from the bands with widths depending linearly on the perturbation amplitude ϵ, since the other have relatively negligible widths due their nonlinear cuspidal form.

This yields

$$\rho_\chi(t) \simeq 4\pi \sum_{m=1}^{s} k^2\omega_k^2 L_m|\chi_k(0)|^2 e^{\mu_k t}\Big|_{\omega_k\simeq m/2}, \tag{16}$$

where the value of $\chi_k(0)$ is evaluated at the center of the resonances bands.

If all the modes start out with an amplitude $|\chi_k(0)|$ then, by the virial theorem, or assuming that there is one χ particle on each mode,

$$|\chi_k(0)|^2 = \frac{1}{\omega_k}, \tag{17}$$

and so

$$\rho_\chi(t) \simeq 4\pi \sum_{i=1}^{s} k^2\omega_k L_i e^{\mu_k t}\Big|_{\omega_k\simeq i/2}. \tag{18}$$

The growth of the χ field modes persists in an expanding universe, if the time scale for the resonance is much shorter than the expanding time scale [14]. If we change the field to

$$\psi_k = a\chi_k \tag{19}$$

and introduce conformal time η, defined by $d\eta = ma^{-1}dt$, then Eq. (2) for ψ_k becomes

$$\psi_k'' + \left(\frac{k^2}{m^2} + \epsilon F(\eta)a^2 - \frac{a''}{m^2 a}\right)\psi_k = 0, \tag{20}$$

where the prime denotes the derivative with respect to η.

Assuming that $m \gg H$, we can treat the expansion of the universe adiabatically, and thus, at any given time, its effect is a shift in the oscillatory frequency

$$k^2 \to k^2 - \frac{a''}{a}, \tag{21}$$

and also an adiabatic increase of the amplitude of the driving force. In the expanding universe the parametric resonance analysis is only applicable if several conditions are satisfied: (i) $\epsilon/m^2 \ll 1$ (perturbative regime), (ii) the expansion of the universe can be neglected, that is $H \ll m$, (iii) we also need the time scale on which the instable solution grows to be smaller than H, so that expansion is unimportant, $H/m \ll \epsilon$, and finally, (iv) that the frequency does not redshift out of the resonance band in a time interval shorter than the amplification period $m/(2\mu_k)$. This latter condition, for the i band, can be written in the form

$$\left|\frac{m}{2\mu_k}\frac{d}{d\eta}\omega_k\right| < L_i, \tag{22}$$

where μ_k and ω_k are evaluated at the center of the resonance ith band.

The regime where these conditions can be easily satisfied is the period after inflation where naturally one has $H/m \ll 1$ and where the equation of motion, in conformal time, for $\bar{\phi} = a\phi$, is given by

$$\bar{\phi}'' + \bar{\phi}(1 - \delta) = 0, \tag{23}$$

with

$$\delta = \frac{1}{m^2}\frac{a''}{a} \ll 1. \tag{24}$$

III. INSTABILITY POCKETS IN HIGHER ORDER RESONANCES

We now apply the results of the previous section to the χ dynamics in preheating, by considering two cases: (i) a cubic interaction term, $g\phi\chi^2$, between the ϕ and χ fields, and (ii) the more general case of a cubic plus a quartic interaction, $g\phi\chi^2 + h\phi^2\chi^2$. The former case is one of the models that yield a Mathieu equation for the χ modes, and is here only briefly considered for the purpose of illustrating the usual analysis and for comparison with our extended model (ii), in which the χ field equation is of the form (3) with $F(t)$ given by (5) and $s = 2$, the simplest possible extension of the usual single frequency Mathieu model. According to (13) and (18) one would expect that, for comparable cubic and quartic terms, the efficiency of the energy transfer process should approximately double in the more general case.

Assume that the background field ϕ oscillates with frequency large compared with the Hubble expansion rate, which is always satisfied asymptotically [14,41]. If we neglect the expansion in the ϕ dynamics then the solution of (1) can be given by, with a specific set of initial conditions,

$$\phi(t) = A\cos(mt). \tag{25}$$

A. Single frequency interaction: the Mathieu equation case

Consider the interaction potential $g\phi\chi^2$. This yields for the k-mode equation, using $\tau = mt$,

$$\chi_k'' + \left[\frac{k^2}{m^2} + \frac{gA}{m^2}\cos(\tau)\right]\chi_k = 0. \tag{26}$$

In this case one has $s = 1$, and

$$\omega_k^2 = \frac{k^2}{m^2}, \tag{27}$$

$$c_1 = 1, \tag{28}$$

$$\epsilon = \frac{gA}{m^2}, \tag{29}$$

and thus the width of the first instability band, when it is linear on ϵ as given by (12), is

$$L_1 = \frac{gA}{4m^2}. \tag{30}$$

Using (18), the energy density in this case is given by

$$\rho_\chi^0(\tau) = \frac{\pi}{8} gA e^{\mu_1^0 \tau/m}, \tag{31}$$

where μ_1^0 is the characteristic exponent of the first band, evaluated at the center of the band at height ϵ.

B. Multifrequency interactions: the Hill equation case

For the interaction potential $g\phi\chi^2 + h\phi^2\chi^2$ the equation for the kth χ mode is, with $\tau = mt$,

$$\chi_k'' + \left[\frac{k^2}{m^2} + \frac{hA^2}{2m^2} + \frac{gA}{m^2}\cos(\tau) + \frac{hA^2}{2m^2}\cos(2\tau)\right]\chi_k = 0. \tag{32}$$

In this case one has (3) and (5) with $s = 2$, and

$$\omega_k^2 = \frac{k^2}{m^2} + \frac{hA^2}{2m^2}, \tag{33}$$

$$c_1 = 1, \tag{34}$$

$$c_2 = \frac{hA}{2g}, \tag{35}$$

$$\epsilon = \frac{gA}{m^2}. \tag{36}$$

Equations (13) for the widths of the first and second instability bands are given by (12) and (13) in the regions where they are linear in ϵ, and yield

$$L_1 = \frac{gA}{4m^2}, \tag{37}$$

$$L_2 = \frac{hA^2}{8m^2}. \tag{38}$$

The energy density in this case is given by, using (18),

$$\rho_\chi^1(\tau) = \frac{\pi}{8} gA e^{\mu_1^0 \tau/m} + \frac{\pi}{2} hA^2 e^{\mu_2^0 \tau/m}, \tag{39}$$

where μ_1^0 and μ_2^0 are the characteristic exponents of the first and second bands, evaluated at the center of the bands at height ϵ.

C. Reheating efficiency of the two couplings

To proceed further in the comparison of the growth of the χ field energy density generated by the two different couplings considered in this section, we must compute the relevant characteristic exponents in both cases.

In order to determine the value of the real part of the characteristic value we use Eq. (11), where the matrix $[B_{rs}]$ was truncated at a size of 11×11 after numerical accuracy tests. Then we have for the value of α as a function of ω_k and of c_2

$$\alpha(c_2, \omega_k) = -\frac{1}{2}\left(\sum_{n=0}^{4} \frac{1}{(\omega_k^2 - n^2)(\omega_k^2 - (n+1)^2)} + \sum_{n=0}^{3} \frac{c_2^2}{(\omega_k^2 - n^2)(\omega_k^2 - (n+2)^2)}\right) - \frac{c_2^2}{4(w_k^2 - 1)^2} \tag{40}$$

and for the value for the growth factor

$$\mathrm{Re}\,(\mu_k) = \epsilon\alpha \sin(\pi\omega_k). \tag{41}$$

Notice that if we set $c_2 = 0$ in the latter expressions we recover the Mathieu case.

In Fig. 1 we show the stability diagrams for the two cases given above. Notice the appearance of an ''instability pocket'' in the second resonance band of Eq. (32) in Fig. 1 (b). This phenomenon, which is much less well known than the appearance of additional noncuspidal instability tongues associated with higher harmonics of the parametric forcing term, was studied in depth in [42,43]. It plays a major role in explaining why the contributions of higher order resonances may be neglected and why the χ field excitations are essentially single mode.

As shown in Fig. 1, there are two major differences when comparing the bifurcation diagram of the two-frequencies Hill case with the Mathieu case. On the one hand, the width of the first instability band is slightly larger and its level curves of constant μ are slighted tilted when compared to the Mathieu case [Fig. 1(a)]. On the other hand, the shape of the second instability band is distorted giving rise to the emergence of a pocket, and its level curves of constant μ are lifted up. Also, the values of μ crossed by straight lines of fixed ϵ in the first instability band are significantly larger

than those crossed in the second band, with the exception of the small region close to $\epsilon = 0$.

In order to compare systematically the efficiency of the two couplings in transferring energy from the inflaton to the χ field, we have computed the time that it takes to reach an e-fold increase of the total number of particles as a function of the coupling strength ϵ, for $c_2 = 0.5$. Instead of Eqs. (31) and (39), which are valid only in the limit of small ϵ, we use the general Eqs. (18) with L_i taken as the numerical value of the band widths for each model.

In Fig. 2(a), we see that the e-fold time is essentially determined by the contribution of the first instability band, except in the region of very small values of ϵ. In this region, the multifrequency coupling becomes more efficient than the Mathieu model because of the contribution of an additional linear instability band, but the effect has no cosmological implications since the energy transfer achieved by both couplings is negligible for these parameter values.

In Fig. 2(b) the overall behavior of the e-fold time as a function of ϵ is shown for the two models. The kinks that can be seen in the two curves correspond to the values of ϵ where the first band hits the $\omega_k = 0$ axis. Because of the formation of the pocket in the second instability band, the multifrequency model becomes actually less efficient than the single frequency excitation for moderate and large values of the coupling.

This effect is also of limited cosmological relevance, since the e-fold times of the two models are of the same order of magnitude for similar values of the coupling strength. However, it is somehow unexpected that an additional linear (as opposed to cuspidal) instability band may translate into a less efficient resonance mechanism for most parameter values. This is of course a consequence of the pocket formation phenomenon, and it shows that the conclusions based on the analytic expressions for the asymptotic behavior of the instability bands for small values of ϵ cannot be extrapolated.

D. Reheating efficiency in the expanding universe

We now take into consideration the expansion of the universe and discuss how this affects the contribution of the resonant bands found for the case of the two different couplings.

The condition which should be satisfied for the resonance mechanism to work is given by Eq. (22). As discussed in Sec. II (see [14]), it translates the requirement that the frequency of the χ particles should not be redshifted out of the resonance band in a time interval shorter than the amplification period $m/(2\mu_k)$.

Using Eqs. (21) and (22) can be recast as

$$\left|\frac{1}{m^2}\left(\frac{a''}{a}\right)'\right| < |\omega_k|\left(\frac{2\mu_k}{m}\right)L_i \tag{42}$$

for the ith instability band.

According to the analysis of this section, significant particle production occurs, for either model, only for values of ϵ such that the asymptotic approximations (13) no longer hold, and that the instability pockets of Fig. 1(b) are instead fully formed. Therefore, for a given ϵ, the two-frequency model will exhibit a combination of lower values of μ_k and smaller width L_2 than the single frequency model. Both effects contribute to making condition (42) harder to meet. For the same coupling strength and increasing the expansion rate, the drift across the second instability band will become swifter than the amplification time for the two-frequency model first.

Therefore, when the expansion of the universe is taken into consideration, the conclusion that the reheating

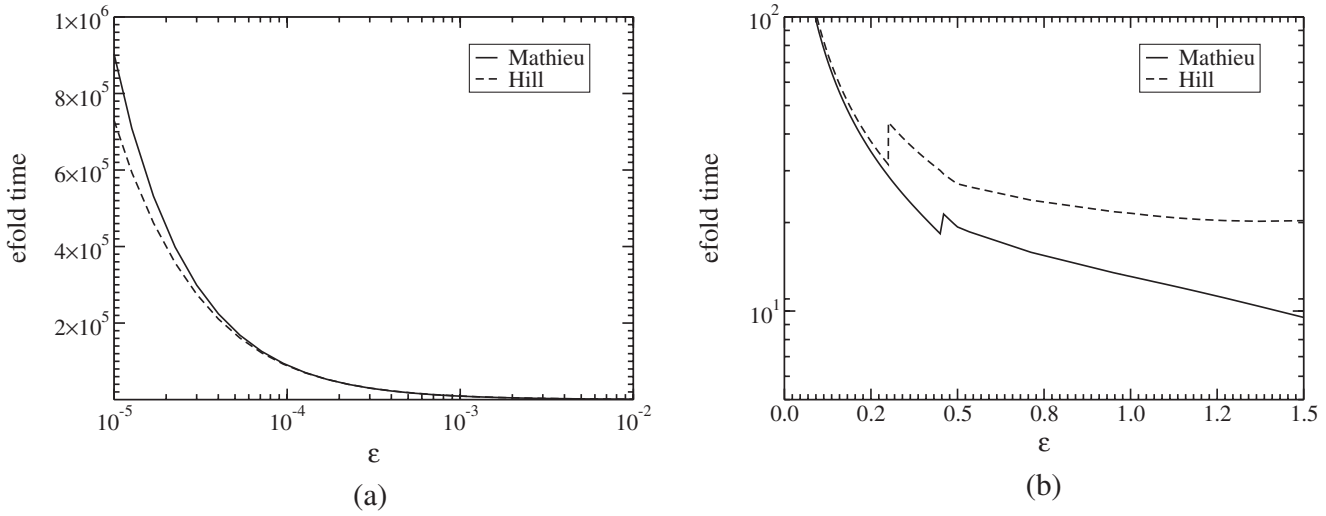

FIG. 2. Time of one e-fold increase in the total number of particles of the χ field as a function of the coupling strength ϵ, for $c_2 = 0.5$.

mechanism is weaker in the multifrequency case than in the Mathieu case is reinforced. Since, however, only the contribution of the first instability band is accounted for in the efficiency estimates found in the literature, these estimates remain valid for the multifrequency model.

Now Eqs. (31) and (39) are valid as long as any given mode remains in the resonance band. Because of the expansion, the time interval $\Delta\tau$ during which a mode remains in the band is

$$\Delta\tau \simeq \frac{L}{Hm^2}. \tag{43}$$

As long as the total time is small compared with H^{-1}/m, the total energy produced during the time interval $N\Delta\tau$ is approximately given by $N\rho_\chi$. In this scenario, reheating is efficient if the ratio

$$\frac{N\rho_\chi}{\rho_\phi} \tag{44}$$

becomes of order one after a time smaller than the Hubble time. Otherwise a significant fraction of the original energy density is redshifted away.

For the first band (as we have seen, the only one that might contribute), the latter quantity (44) reaches the value one for

$$N = \frac{\rho_\phi}{\rho_\chi} \simeq \frac{m^2A^2}{\frac{\pi}{8}gAe^{\mu\Delta\tau/m}}. \tag{45}$$

Therefore the condition for sufficient reheating $N\Delta\tau < H^{-1}/m$ becomes

$$\mu\frac{\Delta\tau}{m}e^{\mu\Delta\tau/m} > \frac{24}{H^3\alpha}\pi m^2 \tag{46}$$

or

$$\mu\frac{\Delta\tau}{m} > W\left(\frac{24}{H^3\alpha}\pi m^2\right), \tag{47}$$

where W is Lambert function [44]. Equations (43) and (47) establish a relation between the free parameters for sufficient reheating to take place that holds for both models considered in this section.

IV. CONCLUSIONS

We have shown how the consideration of extra frequencies in the χ equation of motion can be found in a simple model with $V(\phi) = m^2\phi^2/2$ for the inflaton potential and the interaction potential $V(\phi, \chi) = g\phi\chi^2 + h\phi^2\chi^2$. In the narrow resonance regime the phenomenon of parametrically resonant excitations of the scalar field χ by the inflaton's oscillations is then governed by a Hill equation, where the forcing term in the χ equation has two distinct and commensurable frequencies.

As a result of the presence of various harmonics in the equation of motion of the χ field, the geometrical features of the resonant bands in the bifurcation parameter space are modified. Two main changes take place with respect to the single frequency (Mathieu) case. On the one hand, for small amplitudes of excitation, there are two resonant tongues with linear dependences on the amplitude, rather than just one as in the Mathieu case. On the other hand, closely related to the previous effect, there is a distortion of this additional band which gives rise to the formation of instability pockets. Because of the simultaneous presence (and interference) of two excitation frequencies, the lines in parameter space that correspond to the periodic solutions ($\mu = 0$) and define the boundary of the instability band cross each other for values of the forcing amplitude of order one.

For the two cases under consideration we have evaluated and compared the particle production rates. We have considered first that the inflaton field ϕ oscillates around the minimum of its potential much faster than the expansion rate of the universe, $m_\phi \gg H$, so that the expansion of the universe may be neglected. We have shown that, in general, the presence of an additional excitation frequency hinders, rather than favors, the efficiency of parametric resonance as an energy transfer mechanism, and that this is a consequence of the instability pockets in the bifurcation diagram of the general Hill's equation. The enhancement of particle production due to the presence of a second linear, as opposed to cuspidal, instability band is shown to occur only for extremely small coupling strengths, for which both models yield negligible rates of χ particle creation. We then argue that the effects of an expanding universe further justify the approximation of neglecting the contribution of the second noncuspidal instability band.

In conclusion, our detailed analysis of two different coupling terms supports and justifies the usual approach in the literature, where the efficiency of reheating by parametric resonance is evaluated by considering the simplest form of the parametrically forced equation, and only the dominant contribution of the first instability band.

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