Circles-in-the-sky searches and observable cosmic topology in the inflationary limit

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While the topology of the Universe is at present not specified by any known fundamental theory, it may in principle be determined through observations. In particular, a nontrivial topology will generate pairs of matching circles of temperature fluctuations in maps of the cosmic microwave background, the so-called circles-in-the-sky. A general search for such pairs of circles would be extremely costly and would therefore need to be confined to restricted parameter ranges. To draw quantitative conclusions from the negative results of such partial searches for the existence of circles we need a concrete theoretical framework. Here we provide such a framework by obtaining constraints on the angular parameters of these circles as a function of cosmological density parameters and the observer's position. As an example of the application of our results, we consider the recent search restricted to pairs of nearly back-to-back circles with negative results. We show that assuming the Universe to be very nearly flat, with its total matter-energy density satisfying the bounds $0 < |\Omega_0 - 1| \lesssim 10^{-5}$, compatible with the predictions of typical inflationary models, this search, if confirmed, could in principle be sufficient to exclude a detectable nontrivial cosmic topology for most observers. We further relate explicitly the fraction of observers for which this result holds to the cosmological density parameters.

I. INTRODUCTION

One of the central open questions regarding our understanding of the Universe concerns its shape (topology), and, in particular, whether it is finite or infinite (see, e.g., the reviews [1]). A major difficulty in this regard is that general relativity, being a metric theory, does not specify the topology of the Universe.[1] Despite our present-day inability to predict the topology of the Universe from a fundamental theory, it may in principle be determined through observations. This requires high resolution observations, which have increasingly become available in recent years. Most notably, the ongoing accumulation of data by the Wilkinson Microwave Anisotropy Probe (WMAP) [2] and other cosmic microwave background (CMB) surveys have, on the one hand, provided strong support for the inflationary scenario [3], and the very near flatness of the Universe, and on the other hand made it feasible to perform systematic searches for possible evidence of a nontrivial topology of the Universe [4–7] (see also the related Refs. [8]).

In the context of general relativity, the observable Universe seems to be well described by a 4-manifold $\mathcal{M} = \mathbb{R} \times M$ with locally homogeneous and isotropic spatial sections M and endowed with a Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)],$$
(1)

where $a(t)$ is the scale factor and $S_k(\chi)$ takes the forms χ, $\sin\chi$, or $\sinh\chi$ depending upon whether the geometry of the spatial sections is Euclidian, spherical or hyperbolic with curvature parameters $k = 0, \pm1$. These geometries in turn are determined by finding out whether the total energy-matter density of the Universe, Ω_0, is equal to, greater than or smaller than 1. Often the homogeneous and isotropic spatial sections M are assumed to be the simply connected 3-manifolds: Euclidian $\mathbb{R}^3$, spherical $\mathbb{S}^3$ or hyperbolic $\mathbb{H}^3$. However, given the metrical (local) nature of general relativity, they can also be multiply connected 3-manifolds (which we assume to be compact and orientable) $M = \tilde{M}/\Gamma$, where the covering space $\tilde{M}$ is respectively $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$ depending on k, and Γ is a discrete and fixed point-free group of isometries of $\tilde{M}$ called the holonomy group. The local geometry of the spatial sections M thus constrains, but does not dictate, its topology.

The immediate observational consequence of such multiple connectedness is that an observer could potentially detect multiple images of radiating sources. In particular, in a universe with a detectable nontrivial topology the last scattering surface (LSS) intersects its topological images in the so-called circles-in-the-sky [9], i.e., pairs of matching circles of equal radii, centered at different points of the LSS with the same distribution of temperature fluctuations along both circles. In this way, to observationally probe a nontrivial topology on the largest available scales, one

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[1]In this work, in line with the usage in the literature, by topology of the Universe we mean the topology of its three dimensional spatial sections M.

needs to scrutinize the CMB sky maps in order to extract such correlated circles in order to determine the topology of the Universe. Thus, a detectable nontrivial cosmic topology is an observable attribute, which can be probed through the circles-in-the-sky for all locally homogeneous and isotropic universes with no assumptions on the cosmological density parameters.

The conditions for the detectability of cosmic topology were studied in Refs. [10] for classes of hyperbolic and spherical manifolds, as a function of cosmological density parameters. These studies were extended to the case of *generic* manifolds, and general qualitative results were obtained for the *inflationary limit* in Ref. [11] (see also Ref. [12]). Furthermore, the inverse question of whether the detection of a nontrivial cosmic topology can be used to set constraints on cosmological density parameters has been studied for *specific* topologies [13,14].

Our aims here are twofold. First, to extend the previous general results by the authors [11], in order to give, for *generic* detectable nontrivial topologies, a concrete relationship between the angular parameters associated with the circles-in-the-sky and the cosmological density parameters in the *inflationary limit*. Second, by using these relations, and taking into account the observer positions, we provide a setup to interpret the negative results of recent and future searches for circles-in-the-sky in the WMAP data. In this way, we can concretely specify, for example, the extent to which the recent searches for circles-in-the-sky may be used to exclude detectable nontrivial cosmic topologies for most observers.

The structure of the paper is as follows. In Sec. II we give a brief account of the prerequisites necessary for the following sections, including a brief discussion of the so-called inflationary limit, which we use to obtain bounds on the detectability of cosmic topology. In Sec. III we combine these bounds with our previous results to obtain bounds for the parameters of detectable holonomies in the inflationary limit. We quantify these bounds by giving a concrete measure of the observers for whom the detectability holds. In Sec. IV we recast our bounds on detectable holonomies in terms of bounds on the angular parameter that measures the deviation from antipodicity in pairs of matching circles generated by detectable holonomies. These bounds are then compared in Sec. V to the parameters adopted in the recent searches for circles-in-the-sky to determine whether and under which conditions it is possible to rule out a detectable nontrivial cosmic topology given the inflationary limit. Finally, in Sec. VI, we conclude with a brief discussion of the significance of our results to draw quantitative conclusions from such negative results of searches for circles.

II. PRELIMINARIES

A natural way to study the detectability of a topology is through the lengths of its closed geodesics. For a holonomy

$\gamma \in \Gamma$ and a point $x \in M$, the length of the closed geodesic generated by γ is given by its distance function $d(x, \gamma x)$ in the covering space, i.e., the distance between x and its image γx. This readily allows the definition of the local injectivity radius $r_{inj}(x)$ as half the length of the smallest closed geodesic passing through the point x.

A necessary condition for detectability of cosmic topology is then given by

$$r_{inj}(x) < \chi_{obs}, \qquad (2)$$

where χ_{obs} is the redshift distance evaluated at the maximum redshift ($z = z_{obs}$) of the survey used. We assume throughout a ΛCDM model. In globally homogeneous manifolds, $r_{inj}(x)$ is by definition position independent, and in this case it suffices to use the global injectivity radius r_{inj} (defined in general as $r_{inj} = \inf_{x \in M} r_{inj}(x)$, which is the radius of the smallest sphere inscribable in M) to determine sufficient conditions for detectability. There are, however, significant classes of 3-manifolds which are not globally homogeneous, including the totality of hyperbolic manifolds and the majority of the multiply connected spherical manifolds. One must therefore allow for the fact that the detectability of cosmic topology may be dependent on the observer's position.

Generally, a full systematic search for multiple images or pattern repetitions of topological origin is a daunting task, as the very diverse set of potential holonomies generate very different patterns often dependent on the observer's position. Given, however, that the fraction of the Universe which is effectively accessible to observations is limited, we shall in this work concentrate only on detectable holonomies, that is those for which at least two images of some radiating sources may be observable. In this connection, we have shown in Ref. [11] that, assuming

  (i) The spatial sections of the Universe are not exactly flat;
  (ii) The Universe has undergone a phase of inflationary expansion, such that $|\Omega_0 - 1| \ll 1$, which ensures that $\chi_{obs} \ll 1$[2];
  (iii) The cosmic topology, or more properly some element of its holonomy group, is detectable, i.e., Eq. (2) holds for some point $x \in M$;

then any detectable holonomy is (nearly) indistinguishable from a Clifford translation (or CT, defined as a holonomy γ under which the distance between each point x and its image γx is constant for all points in the manifold for most observers. As a consequence, a generic compact nonflat manifold is ''locally'' well approximated by either a cylindrical ($\mathbb{R}^2 \times \mathbb{S}^1$) or toroidal ($\mathbb{R} \times \mathbb{T}^2$) manifold, irrespective of its global topology.

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[2]Here and in the following we express distances in units of the curvature radius $a_0 = |k|H_0^{-1}|\Omega_0 - 1|^{-1/2}$.

In typical inflationary scenarios, achieving a sufficient number of e-foldings required to generically solve the flatness and horizon problems imposes bounds on the total density parameter which is estimated as [3,15]

$$|\Omega_0 - 1| \lesssim 10^{-5} \ll 1. \tag{3}$$

In what follows, we shall refer to this bound as the inflationary limit.

For the bound given by Eq. (3) one can numerically calculate χ_{obs}. By using this value along with Eq. (2) we obtain an estimate of the detectability condition in the inflationary limit

$$r_{inj}(x) \lesssim \chi_{obs} \lesssim 0.01, \tag{4}$$

where we have taken $\Omega_{m0} = 0.28$ and $z_{obs} = 1089$ [2].

III. BOUNDS ON THE HOLONOMY PARAMETERS

As was discussed in the previous section, an important observational signature of any nontrivial detectable topology is the existence of the circles-in-the-sky in the CMB maps, in which matching pairs of circles are identified by a holonomy $\gamma \in \Gamma$. Reciprocally, each detectable holonomy corresponds to a pair of matching circles. Thus the detection of the matching circles in the CMB maps would effectively amount to the detection of the holonomy group (or some subgroup thereof), and in turn the topology of the Universe. In general, for a given radius χ_{obs} of the LSS, the relative position of any pair of matching circles depends on the specific holonomy which connects the circles, the position of the observer and the value of χ_{obs}. In the case of Clifford translations, however, the pairs of circles generated are always antipodal (or back-to-back) for all observers and radii. A search that could be restricted to such back-to-back pairs would of course be far less computationally intensive than a full search covering the entire parameter space.[3] More generally, if one were able to somehow restrict the permissible holonomy parameters from theoretical and observational considerations, one could in turn restrict the parameter space that needs to be spanned, thereby greatly simplifying the search. This is what we aim to do here for most observers by assuming the inflationary limit.

To obtain the holonomy parameters, we recall that the main outcome of our earlier results [11] is that, for most observers (in the sense made precise below), the detectable holonomies of nearly flat manifolds will deviate only by a small amount from being Clifford translations (i.e., are CT-like), in the sense that within the observable Universe the lengths $d(x, \gamma x)$ of the closed geodesics generated by any

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[3]A general search requires the computation of a correlation function spanning a six-dimensional parameter space, whose parameters give the location of the first and the second circles centers [(Θ_1, Φ_1) and (Θ_2, Φ_2), say], their angular radius ν and the relative phase of the two circles ϕ.

detectable holonomy γ will be such that

$$\frac{\Delta d}{d_0} \leq \frac{2\chi_{obs}}{|z_2||z_1|}, \tag{5}$$

and for hyperbolic manifolds

$$\frac{\Delta d}{d_0} \leq 2\chi_{obs}. \tag{6}$$

In these expressions $\Delta d \equiv d_{max} - d_{min}$, where d_{max} and d_{min} are, respectively, the maximum and minimum lengths of the geodesics generated by this holonomy inside the detectable sphere of radius χ_{obs}, d_0 is the distance function of the holonomy γ evaluated at the center of the sphere, and z_1 and z_2 are a pair of complex numbers that parametrize the 3-sphere (thus $|z_1|^2 + |z_2|^2 = 1$), which are used to express the holonomy γ in its canonical form. Of course, for true Clifford translations, $\Delta d/d_0 = 0$ (for more details see Ref. [11]).

For all observers able to detect a holonomy γ in hyperbolic cusplike manifolds, and for most such observers in a spherical manifold (i.e., those bounded away from the equators, so that $|z_1|, |z_2| \gg \chi_{obs}$), the bounds (5) and (6) imply that $\Delta d/d_0$ is very small, which means that the holonomy is CT-like. It is easy to show that the fraction of the volume of the manifold where $|z_1| < Z$ or $|z_2| < Z$, for any $Z \leq 1$, is proportional to Z^2. This can be seen by writing $Z = \sin\xi_{max}$, $|z_1| = \cos\xi$, and $|z_2| = \sin\xi$, where $0 \leq \xi \leq \pi/2$ and $0 \leq \xi_{max} \leq \pi/4$. The holonomy group Γ of manifold $\mathbb{S}^3/\Gamma$ tiles the 3-sphere $\mathbb{S}^3$ into $\mathcal{O}(\Gamma)$ identical copies of the fundamental domain, where $\mathcal{O}(\Gamma)$ is the order of Γ. Therefore, the volume of the region where either $\xi \leq \xi_{max}$ or, conversely, $\xi \geq \pi/2 - \xi_{max}$, corresponds to the sum of the volumes of two tori in the 3-sphere $|z_1|^2 + |z_2|^2 = 1$ centered at $|z_1| = 0$ and $|z_2| = 0$ (with radii ξ_{max} and $\pi/2 - \xi_{max}$ respectively), divided by $\mathcal{O}(\Gamma)$. In toroidal, or Hopf, coordinates, this volume is given by

$$V_{\xi_{max}} = \frac{1}{\mathcal{O}(\Gamma)} \int_0^{2\pi} \int_0^{2\pi} \left(\int_0^{\xi_{max}} + \int_{\pi/2 - \xi_{max}}^{\pi/2} \right)$$
$$\times \cos\xi \sin\xi d\phi_1 d\phi_2 d\xi$$
$$= \frac{4\pi^2 Z^2}{\mathcal{O}(\Gamma)}. \tag{7}$$

Now, the volume of the manifold $\mathbb{S}^3/\Gamma$ is given by the volume of the 3-sphere divided by $\mathcal{O}(\Gamma)$, i.e., $V_{\mathbb{S}^3/\Gamma} = V_{\mathbb{S}^3}/\mathcal{O}(\Gamma) = V_{\pi/2} = 2\pi^2/\mathcal{O}(\Gamma)$. Thus the fraction $\mathcal{R}$ of $\mathbb{S}^3/\Gamma$ where $|z_1| \leq Z$ or $|z_2| \leq Z$ is given by

$$\mathcal{R} = V_{\xi_{max}}/V_{\mathbb{S}^3/\Gamma} = 2Z^2. \tag{8}$$

It is then clear from (8) that the volume of the region where $z_1 \sim \chi_{obs}$ or $z_2 \sim \chi_{obs}$, as a fraction of the volume of the manifold, is of the order of $2\chi_{obs}^2$, which is very small if the inequality $\chi_{obs} \ll 1$ holds. The smallness of $\mathcal{R}$ then im-

plies that the probability of finding an observer in such regions is small (in the same sense as that defined by the so-called injectivity profiles [16]), which means that (5) imposes stringent bounds on $\Delta d/d_0$ in all but a small region of the 3-sphere whose volume is of the order of χ_{obs}^2. In what follows, therefore, we shall confine ourselves to the case of *typical observers*, i.e., those that reside away from these small probability regions.

Now, in the limit (5) and (6), the geometry of the (non-flat) spatial sections of the observable universe will be well approximated by the geometry of flat space $\mathbb{R}^3$. Therefore, any detectable holonomy will likewise be locally well approximated by some isometry of $\mathbb{R}^3$. It is known, however, that any orientation-preserving flat isometry (and, in particular, any flat holonomy) can be expressed as a screw motion [17], which consists of a rotation around a suitable axis followed by a translation along the same axis. In what follows, we shall write a generic nonflat detectable holonomy γ in the limit (5) and (6) as a generic screw motion, and use our previous bounds on nearly flat detectable holonomies to constrain the parameters characterizing such holonomy.

Let us consider, without loss of generality, a typical observer at a point P (see Fig. 1) with coordinates chosen such that $P = (\rho, 0, 0)$, where ρ is the distance from the axis of rotation. Let the image of the point P under the action of a holonomy γ be $P' = \gamma P$ with coordinates $(\rho \cos\alpha, \rho \sin\alpha, L)$, where α is the phase angle and L is the translation length corresponding to the screw motion isometry. The length of the closed geodesic connecting P and P' is given by

$$d_0 = |P' - P| = |\rho(1 - \cos\alpha), \rho \sin\alpha, L|$$

$$= \sqrt{2\rho^2(1 - \cos\alpha) + L^2}. \qquad (9)$$
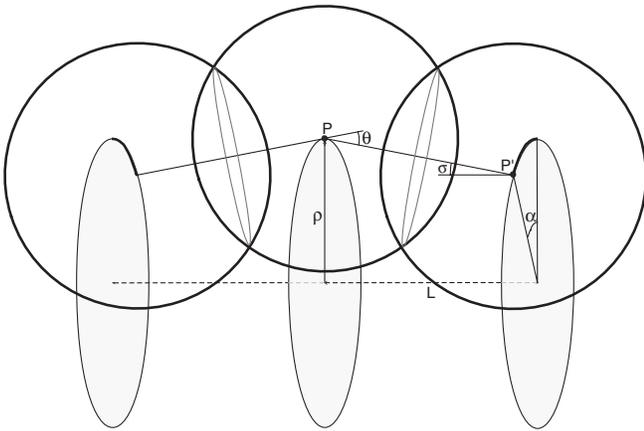
FIG. 1. Depiction of a screw motion isometry γ, whose action takes the point P to P'. This amounts to a translation by L along the axis of isometry (dotted line), and a rotation by α around the same axis. The LSS sphere (thick solid line) centered at P intersects its images (also in thick solid lines) centered at γP and $\gamma^{-1}P$ along two matched circles-in-the-sky.

To proceed, we need to calculate not only the length of the geodesic passing through the observer's position given by Eq. (9), but also the lengths of the longest and shortest closed geodesics within the observable Universe, which are, respectively, given by

$$d_{max} = \sqrt{2(\rho + \chi_{obs})^2(1 - \cos\alpha) + L^2}, \qquad (10)$$

$$d_{min} = \begin{cases} \sqrt{2(\rho - \chi_{obs})^2(1 - \cos\alpha) + L^2} & \text{if } \rho \geq \chi_{obs}, \\ L & \text{if } \rho \leq \chi_{obs}. \end{cases} \qquad (11)$$

Since $L \leq [2(\rho - \chi_{obs})^2(1 - \cos\alpha) + L^2]^{1/2}$ holds identically, it follows that $d_{min} \leq [2(\rho - \chi_{obs})^2 \times (1 - \cos\alpha) + L^2]^{1/2}$. Thus, combining Eqs. (9)–(11) we obtain

$$\frac{\Delta d}{d_0} \geq \sqrt{1 + \frac{1}{d_0^2}(4\rho\chi_{obs} + 2\chi_{obs}^2)(1 - \cos\alpha)}$$

$$- \sqrt{1 + \frac{1}{d_0^2}(-4\rho\chi_{obs} + 2\chi_{obs}^2)(1 - \cos\alpha)}. \qquad (12)$$

The terms on the right-hand side are, respectively, greater and smaller than 1. Also, according to (5) and (6), the difference between the two terms must be $\ll 1$. Therefore, both terms must be ~ 1, which allows them to be expanded to obtain

$$\frac{\Delta d}{d_0} \geq \frac{4\rho\chi_{obs}}{d_0^2}(1 - \cos\alpha) + \mathcal{O}(\alpha^4). \qquad (13)$$

Note that since $4\rho\chi_{obs}/d_0^2 \gg 1$, then in order to keep the first term <1 we must have $(1 - \cos\alpha) \ll 1$, which implies that α is very small.

Now let the angle between the axis of the screw motion and the segment $(P' - P)$ be σ (see Fig. 1), such that $\cos\sigma = L/d_0$. It can then be shown that

$$\tan\sigma = \frac{\rho}{L}\sqrt{2(1 - \cos\alpha)}. \qquad (14)$$

Using (14) to express α in terms of σ, it is possible to write Eq. (13) as

$$\frac{\Delta d}{d_0} \geq \frac{2\chi_{obs}}{\rho}\sin^2\sigma + \mathcal{O}(\alpha^4). \qquad (15)$$

Combining the above results with the bounds (5) and (6) we finally obtain

$$\frac{\rho}{|z_2||z_1|} \geq \sin^2\sigma, \quad \text{for spherical manifolds}, \qquad (16)$$

$$\rho \geq \sin^2\sigma, \quad \text{for hyperbolic manifolds}. \qquad (17)$$

IV. BOUNDS ON THE PARAMETERS OF THE CIRCLES-IN-THE-SKY

In the previous section we used our previous bounds (c.f. [11]) on $\Delta d/d_0$ to constrain the screw motion parameters (ρ, α, L). To relate these to the corresponding parameters (θ, ν, ϕ) for the circles-in-the-sky (see Fig. 2), one needs the relations between the circles-in-the-sky and the screw motion parameters, which are given by [18]

$$\cos\theta = 1 - \sin^2\sigma(1 - \cos\alpha), \qquad (18)$$

and

$$\cos\nu = \frac{L}{2\chi_{\mathrm{obs}}\cos\sigma}. \qquad (19)$$

Now, in the inflationary limit α is small, which according to (18) implies that θ is small. We can then write, up to the second order in θ,

$$\theta = \frac{1}{\sqrt{2}}\sin\sigma\sqrt{1 - \cos\alpha}.$$

Using (14) and employing the bounds on $\sin\sigma$ from (16) and (17), together with the expression (19) for $\cos\nu$, we obtain

$$\theta \leq \frac{\sqrt{2}\cos\nu}{|z_2||z_1|}\chi_{\mathrm{obs}} \quad \text{for spherical manifolds,} \qquad (20)$$

$$\theta \leq \sqrt{2}\cos\nu\,\chi_{\mathrm{obs}} \quad \text{for hyperbolic manifolds.} \qquad (21)$$

Clearly, these inequalities provide upper bounds on the values of the angle θ that characterizes the deviation from
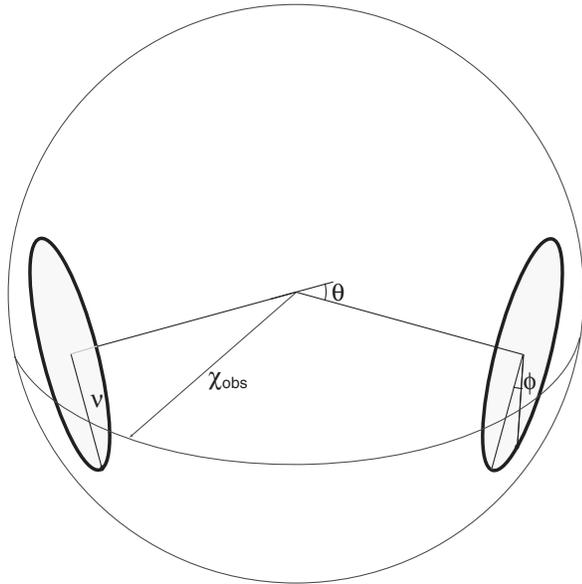
FIG. 2. This figure depicts the circles at the intersection of the LSS with its images. The parameter θ is a measure of the deviation of the circles from being antipodal, and ϕ measures the phase difference between the circles of radius ν.

antipodicity of pairs of circles-in-the-sky as a function of the circles' radii ν, the distance to the LSS χ_{obs}, and the observer's position (for the spherical case). The dependence of these bounds on θ on the density parameters can be made explicit by recalling that for the small values of $|\Omega_0 - 1|$ given by the bound (3), the contour curve $\chi_{\mathrm{obs}}(\Omega_{m0}, \Omega_{\Lambda 0}) = r_{\mathrm{inj}}$ can be well approximated by the secant line joining its intersections with the $\Omega_{m0} = 0$ and $\Omega_{\Lambda 0} = 0$ axes [19]. Using this approximation we obtain

$$\chi_{\mathrm{obs}} \simeq 2\sqrt{\frac{|\Omega_0 - 1|}{\Omega_{m0}}}, \qquad (22)$$

which can be substituted in inequalities (20) and (21) to give the bounds

$$\theta \lesssim \frac{2\sqrt{2}\cos\nu}{|z_2||z_1|}\sqrt{\frac{|\Omega_0 - 1|}{\Omega_{m0}}}, \qquad (23)$$

and

$$\theta \lesssim 2\sqrt{2}\cos\nu\sqrt{\frac{|\Omega_0 - 1|}{\Omega_{m0}}}, \qquad (24)$$

for, respectively, spherical and hyperbolic manifolds.

Thus given observational bounds on the density parameters they allow constraints to be set on the parameter θ for any radii ν of matching circles to be considered in order to perform a comprehensive search for matching circles in the observable Universe.

Finally, as is clear from our discussion in Sec. III, there are some observers in spherical manifolds—namely, those close to the equators ($z_1 \simeq 0$ or $z_2 \simeq 0$)—for whom the bounds on θ derived here are not applicable. However, as can be seen from Eq. (8), the set of such observers is very small in the inflationary limit.

V. IMPLICATIONS OF OUR RESULTS AND SEARCHES FOR CIRCLES-IN-THE-SKY

To illustrate the application of the bounds (20) and (21), we consider the recent search for circles-in-the-sky using the WMAP data [5], in order to determine the sets of topologies that can be ruled out. Given the prohibitive numerical cost of a full search, this search was confined to antipodal or nearly-antipodal circles with deviation from antipodicity $\theta \lesssim 10°$ and with radii $\nu \gtrsim 18°$. This search found no statistically significant pairs of matching circles.

If confirmed, an important question regarding this search would be to determine concretely whether it is sufficient to rule out detectable nontrivial, nonflat, cosmic topologies, assuming the inflationary limit (3). To answer this question we have, employing the inequalities (20) and (21), plotted in Fig. 3 an upper bound, θ_{max}, for the maximum deviation from antipodicity of a pair of matching circles as a function of the fraction of the observers in spherical and hyperbolic manifolds. As can be seen from this figure, and concretely
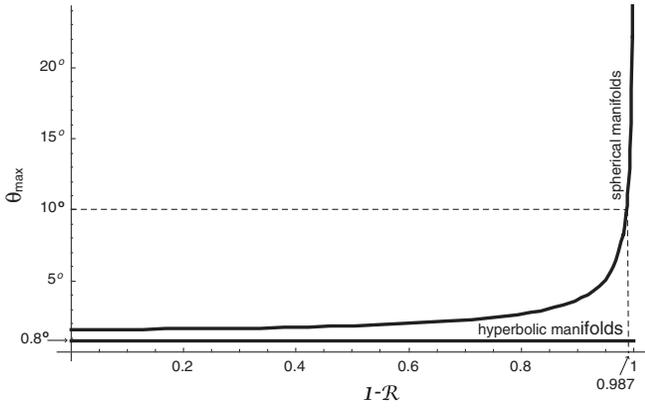
FIG. 3. Upper bound θ_{max} for the maximum deviation from antipodicity as a function of the fraction of potential observers for which $\theta \leq \theta_{max}$ in the inflationary limit. This figure shows, for example, that for 98.7% of all observers in spherical manifolds, $\theta \leq 10°$. It also shows that for all observers in hyperbolic manifolds, $\theta \leq 0.8°$. In this figure, we have taken $\Omega_{m0} = 0.28$ and circles with radii $\nu \geq 18°$, but for circles with arbitrarily small radii, the value of θ_{max} would increase by a factor of at most 1.05.

FIG. 4. Upper bound θ_{max} for the maximum deviation from antipodicity as a function of the total density parameter Ω_0, calculated for the sets of 90%, 99% and 99.9% of observers (respectively, $\mathcal{R} = 0.1$, 0.01 and 0.001), and for the totality of observers in hyperbolic manifolds. The shaded region corresponds to the inflationary limit as defined in the text. Again, we have taken $\Omega_{m0} = 0.28$ and circles with radii $\nu \geq 18°$.

estimated from inequalities (20) and (21), in the inflationary limit (3), the angle θ_{max} is less than 0.8° for any observer in a hyperbolic manifold.[4] On the other hand, for 98.7% of the observers in spherical manifolds $\theta_{max} \leq 10°$. In both cases we have confined ourselves to circles with radii $\nu \geq 18°$ (following [5]); but had we considered circles of arbitrarily small radii, the upper bounds on θ would increase only by a factor of 1.05. This implies that, if one accepts the negative result of the search for circles of Ref. [5], then the assumption of the inflationary limit considered here is sufficient to rule out detectable nontrivial cosmic topologies for the overwhelming majority of the potential observers. Thus to detect a nontrivial topology an observer must either live in a restricted region of some spherical manifolds or the Universe must have a total density parameter which is not too close to the critical density. This latter statement can be made precise by plotting θ_{max} as a function of the density parameter ($|\Omega_0 - 1|$) for different sets of observers (Fig. 4). So, for instance, the search undertaken in Ref. [5] is sufficient to rule out a detectable cosmic topology for all observers in hyperbolic manifolds, and for at least 99.9% of the observers in spherical manifolds, if $|\Omega_0 - 1| \leq 10^{-6}$. On the other hand, if $|\Omega_0 - 1| \leq 10^{-4}$, then such a detectable topology is still definitively excluded for all observers in hyperbolic manifolds, but only for 90% of observers in spherical manifolds.

Future searches for pairs of correlated circles may of course study different regions of the circles-in-the-sky

parameter space. As Fig. 4 illustrates, for all observers in hyperbolic manifolds, even searches restricted to small values of θ (if confirmed) would be sufficient to exclude a detectable nontrivial cosmic topology for relatively large values of total density, i.e., for $1 - \Omega_0 \sim 10^{-4}$. On the other hand, the fraction of observers in spherical manifolds that may observe pairs of circles with $\theta \geq \theta_{max}$ varies significantly with the value of θ_{max}. Thus, for example, a search for pairs with $\theta \leq 5°$ would be able to exclude a detectable nontrivial cosmic topology for 99.9% of observers only if $\Omega_0 - 1 \leq 1.2 \times 10^{-7}$, a value almost 1 order of magnitude smaller than that for $\theta \leq 10°$. Conversely, it would take a search for $\theta \leq 35°$ to exclude a detectable nontrivial topology for the same 99.9% of observers for the range of values of Ω_0 given by the inflationary limit (3).

Of course, for sufficiently low values of $|\Omega_0 - 1|$, a detectable nontrivial (nonflat) topology can be excluded for an arbitrarily small fraction of potential observers and arbitrarily small values of θ_{max}. However, one can not completely exclude all potentially detectable spherical holonomies for all observers, since one can always find one such holonomy that is well approximated in some neighborhood around the equator $z_1 = 0$ by a screw motion of arbitrarily large twist, which will in turn generate, for some values of χ_{obs}, circle pairs with arbitrary deviation from antipodicity.

A nondetection in a limited search for pairs of correlated circles would likewise not rule out the possibility of a detectable flat nontrivial topology. Since flat holonomies lack a characteristic scale factor, there is some considerable leeway in choosing the holonomy parameters, and it is easy to obtain holonomies that produce circle pairs of arbitrary θ for typical observers. Indeed, if one assumes

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[4]We note, however, that for most such observers the topology is undetectable to begin with (see Ref. [16] for details).

the inflationary limit, then the detection of a pair of correlated circles with high θ would strongly suggest that the Universe has flat spatial sections, since pairs of correlated circles with high values of θ would be present for no observers in a hyperbolic Universe, and for only a small fraction of observers in a spherical Universe.

Although it is not possible to exclude the possibility of detection of the cosmic topology by all observers, the combination of such negative results of any search (present or forthcoming), together with the results derived in this paper, would allow precise bounds to be put on the fraction of the potential observers for whom a nontrivial cosmic topology would be ruled out, for any given value of the density parameters.

VI. FINAL REMARKS

An important model-independent observational signature of a detectable nontrivial cosmic topology is the occurrence of pairs of matching circles of temperature fluctuations in maps of the cosmic microwave background radiation. Here, by employing some recent results concerning the local nature of generic nonflat detectable nontrivial topology in the inflationary limit, we have obtained concrete bounds on the angular parameter characterizing the deviation from antipodicity of circles-in-the-sky as a function of the cosmological density parameters and the position of the observer, for any radius ν of the circles.

As an example of the application of our results, we have considered the most recent search restricted to nearly back-to-back circles, which has found no statistically significant pairs of matching circles. Using our bounds we have found

that, assuming the total density parameter satisfies $0 < |\Omega_0 - 1| \lesssim 10^{-5}$, this search, if confirmed, could in principle be sufficient to exclude a detectable nontrivial cosmic topology for most observers.

Our results also provide a framework to draw quantitative conclusions from the negative results of such partial searches, past and future, for circles-in-the-sky as a generic signature of nontrivial cosmic topology. More specifically, they allow us to quantify the fraction of the potential observers for which the absence of pairs of matching circles-in-the-sky in CMB maps rules out a nontrivial (nonflat) topology for the spatial sections of the Universe, as a function of the cosmological density parameters. We emphasize that these results apply generically to all potential nonflat manifolds with nontrivial topology rather than specific classes. In particular, if the negative results of the recent searches are confirmed, then in the inflationary limit a detectable nontrivial (nonflat) topology is excluded for all observers apart from a very small subset, if the Universe has positively curved spatial sections, and for all observers if the spatial sections turn out to be negatively curved.

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