Study of the preheating phase of chaotic inflation

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Particle production and its effects on the inflaton field are investigated during the preheating phase of chaotic inflation using a model consisting of a massive scalar inflaton field coupled to N massless quantum scalar fields. The effects of spacetime curvature and interactions between the quantum fields are ignored. A large N expansion is used to obtain a coupled set of equations including a backreaction equation for the classical inflaton field. Previous studies of preheating using these equations have been done. Here the first numerical solutions to the full set of equations are obtained for various values of the coupling constant and the initial amplitude of the inflaton field. States are chosen so that initially the backreaction effects on the inflaton field are small and the mode equations for the quantum fields take the form of Mathieu equations. Potential problems relating to the parametric amplification of certain modes of the quantum fields are identified and resolved. A detailed study of the damping of the inflaton field is undertaken. Some predictions of previous studies are verified and some new results are obtained.

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I. INTRODUCTION

In all inflationary models there must be a period after inflation in which a substantial amount of particle production occurs in order to repopulate the universe with matter and radiation. At the end of this process it is crucial that the universe reheats to a temperature that is not too large and not too small [1]. If the temperature is too large, there are problems with the creation of monopoles and domain walls that have not been observed, and if the temperature is too low there can be problems with baryogenesis [2]. Thus, to assess the viability of a particular model of inflation, it is necessary to obtain an accurate picture of the amount of particle production that occurs and the rate that it occurs at. This phase of particle production after inflation is usually called reheating [3,4].

Both the mechanism for inflation and the way in which reheating occurs are different for different inflationary models. In both new inflation [5] and chaotic inflation [6], inflation is due to a classical scalar field called the inflaton field. When inflation is over, this field oscillates in time as it effectively "rolls back and forth" in its potential [6–8]. The coupling of this classical field to various quantum fields, and in some cases to its own quantum fluctuations, results in the production of particles which are thermalized by their mutual interactions. The backreaction of the quantum fields on the classical inflaton field causes its oscillations to damp [1].

The original particle production calculations for new inflation and chaotic inflation [9] were done using perturbation theory. The damping of the inflaton field was taken into account through the use of a dissipative term that was added to its wave equation. Later, the mode equations for the quantum fields coupled to the inflaton field were solved, and it was found that the time dependent part of the modes can, in some cases, undergo parametric amplification due to the periodic behavior of the inflaton field [10–12]. Since the rate of particle production is extremely rapid in this case and the particles do not have time to thermalize while it is occurring, this initial phase of the particle production process was dubbed "preheating" [10].

The process of preheating is interesting because particle production occurs very rapidly with the result that quantum effects are large and the backreaction on the inflaton field is important. This pushes the semiclassical approximation, which is what is typically used in these calculations, to the limit. In fact, to take such large effects into account it is necessary to go beyond the ordinary loop expansion. The Hartree approximation and the large N expansion are two ways in which such effects have been treated [13,14].

The original calculations of preheating [10,11] involved solving the mode equations for the quantum fields in the presence of a background inflaton field which obeyed a classical scalar field wave equation. Later estimates of the effects of the quantized fields on the inflaton field were made using a variety of techniques [15–20]. In Refs. [21,22] the first fully nonlinear calculations of classical inflaton decay were carried out by means of statistical field theory lattice numerical simulations. The numerical simulations in [21] and the later ones in [23–28] were implemented by means of the lattice code LATTICEASY [29]. These simulations are important for the study of the development of equilibrium after preheating, when scattering effects cannot be neglected [23]. Recently, the same

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type of calculation has been used to investigate the possibility that nonequilibrium dynamics, such as that of parametric resonance during preheating, can produce non-Gaussian density perturbations [30].

The full set of coupled equations relating the inflaton field to the quantum fields were solved numerically for both the Hartree approximation with a single massive scalar field with a quartic self-coupling and the $O(N)$ model using a large N expansion truncated at leading order. The $O(N)$ model consists of N identical massive scalar fields each with a quartic self-coupling and a quadratic coupling to every other field. As a result there are N inflaton fields in this model. The first set of calculations were done in a Minkowski spacetime background [13,31] and then followed by radiation and matter dominated Friedmann-Robertson-Walker backgrounds [32]. As the fields have an energy-momentum tensor associated with them, the expansion of the universe is also affected. This was taken into account in Ref. [33]. A numerical calculation in the context of the SU(2) Higgs model was done in Ref. [34]. A calculation within the context of the $O(N)$ model that takes scattering of the produced particles into account was done in Ref. [35]. Numerical calculations in models with two scalar fields, with backreaction effects taken into account using the Hartree approximation, were done in Ref. [36].

There are some potential problems with using the $O(N)$ model as a model for chaotic inflation. One is that it results in N inflaton fields rather than the single inflaton field that is usually postulated. Another is that the fields have a quartic self-coupling as well as couplings to each other. There is evidence from the WMAP III data [37] that if chaotic inflation occurred, the inflaton field is a massive scalar field with no quartic self-coupling. Finally, in the $O(N)$ model or in a model in which the primary quantum fluctuations come from the inflaton field, the quantum fields are massive with the same mass as the inflaton field. However, in a realistic situation there would also be massless or effectively massless fields coupled to the inflaton field.

A possibly more realistic model for chaotic inflation consists of a single classical inflaton field coupled to various quantum fields, including its own quantum fluctuations. This is the type of model considered by Kofman, Linde, and Starobinsky [10,11,15]. They used solutions of the mode equations of the quantum fields to estimate both the amount of particle production that would occur in various cases and the amount of damping of the inflaton field that occurs due to the backreaction of the quantum fields on it.

In this paper we consider a model similar to the one used by Kofman, Linde, and Starobinsky, henceforth referred to as KLS, in their analysis of preheating in chaotic inflation [15]. The inflaton field is a classical massive scalar field with minimal coupling to the scalar curvature. It has no quartic or higher order self-coupling. But it is coupled to N identical massless quantized scalar fields with arbitrary curvature coupling ξ. As discussed in Sec. II, in the large N limit, the system effectively reduces to one consisting of a classical inflaton field coupled to one quantized massless scalar field with arbitrary curvature coupling. The effects of quantum fluctuations of the inflaton field come in at next to leading order so we do not take them into account.

We investigate some of the details of the preheating process by numerically integrating both the mode equations for the quantized field and the backreaction equation for the inflaton field. These are the first solutions to the full set of coupled equations relating the inflaton field to the quantum field that have been obtained for this model. To simplify the calculations we ignore interactions between the created particles. Calculations that have taken such interactions into account [21,22,26,27,35] indicate that they can be ignored during the first stages of preheating but eventually become important. Another simplification is that, as a first step, we work in a Minkowski spacetime background. It has previously been shown that the expansion of the universe can significantly affect the evolution of the inflaton field during preheating if this field is of Planck scale or larger at the onset of inflation [15]. This in turn can substantially affect the details of the particle creation process. Thus, some of our results will not be relevant for most models of chaotic inflation. However, for the rapid damping phase which occurs in many models, the amplitude of the inflaton field changes rapidly on time scales which are small compared to the Hubble time. Thus, it should be possible to ignore the expansion of the universe during this phase [15].

If, for the model we are considering, one neglects the backreaction effects of the quantized fields on the inflaton field, then the inflaton field undergoes simple harmonic motion with a frequency equal to its mass. As shown in Sec. III, this means that the time dependent part of the modes of the quantized field obey a Mathieu equation. Thus, parametric amplification will occur for modes in certain energy bands. There is an infinite number of these bands, and some occur for arbitrarily large values of the energy. This could, in principle, result in divergences of quantities such as $\langle \psi^2 \rangle$, with ψ the quantum field. For example, if the contribution to the mode integral in $\langle \psi^2 \rangle$ [see Eq. (2.11)] for each band of modes undergoing parametric amplification was the same at a given time, then $\langle \psi^2 \rangle$ would diverge. Another issue that must be addressed is that even if $\langle \psi^2 \rangle$ is finite, one must be certain that all significant contributions to it are accounted for when making numerical computations; in other words, one must be certain that no important bands are missed. These issues are addressed in Sec. III.

When the backreaction of the quantized fields is included in the wave equation for the inflaton field, then, as expected, the inflaton field's amplitude damps as the amplitudes of some of the modes of the quantized field grow. KLS predicted that if all the instability bands are narrow then the damping is relatively slow, while if one or more of the instability bands are wide then there is a period of rapid damping. We find this to be correct, and for a Minkowski spacetime background, we find a more precise criterion which determines whether a phase of rapid damping will occur. In cases where there is rapid damping, we find a second criterion that must be satisfied before it takes place. In a Minkowski spacetime background, this latter criterion explains the differences that are observed in the time that it takes for the period of rapid damping to begin.

In their study of the rapid damping phase, KLS gave a criterion for when the damping should cease and used it to predict how much damping should occur. We find something similar except that the amount of damping that is observed to occur is larger than they predict. We also find that the rapid damping actually occurs in two phases separated by a short time. An explanation for this is provided in Sec. IV D.

If most of the damping occurs gradually then it is observed that the frequency of the oscillations of the inflaton field changes slowly as its amplitude is damped. There is also a significant transfer of energy away from the inflaton field as would be expected. However, if most of the damping occurs rapidly then during the rapid damping phase the frequency of the oscillations of the inflaton field increases significantly. As a result there is less energy permanently transferred away from the inflaton field than might be otherwise expected. This result was seen previously in the classical lattice simulation of Prokopec and Roos [22].

No significant further damping of the amplitude of the inflaton field was observed to occur after the rapid damping phase in all cases in which such a phase occurs. Instead, both the amplitude and frequency of the inflaton field were observed to undergo periodic modulations and a significant amount of energy was continually transferred away from and then back to the inflaton field. However, the actual evolution of the inflaton field after the rapid damping phase is likely to be very different because interactions that we are neglecting are expected to be important during this period [21–23,26,27,35].

In Sec. II the details of our model are given and the coupled equations governing the inflaton field and the quantum fields are derived. In Sec. III a detailed study is made of parametric amplification and the bands that undergo parametric amplification, in order to address the issues discussed above relating to the finiteness of certain quantities and the potential accuracy of numerical computations of these quantities. In Sec. IV some of our numerical solutions to the coupled equations governing the modes of the quantum field and the behavior of the inflaton field are presented and discussed. Our results are summarized in Sec. V. In the Appendix we provide the details of the renormalization and covariant conservation of the energy-momentum tensor for the system under consideration. Throughout this paper we use units such that $\hbar = c = G = 1$. The metric signature is $(-+++)$.

II. THE MODEL

We consider a single inflaton field Φ and N identical scalar fields Ψ_j, that represent the quantum matter fields present during the inflationary phase. We assume that the inflaton field is massive (with mass m) and minimally coupled to the scalar curvature, R, and that the quantum scalar fields are massless with arbitrary coupling ξ to the scalar curvature. We also assume that the interaction between the classical inflaton and the quantum matter fields is given by $g^2\Phi^2\Psi_j^2/2$. We study the dynamics of Φ and Ψ_j in a Minkowski spacetime background and denote the metric by $\eta_{\mu\nu}$.

The action for the system is given by

$$S[\Phi, \Psi_j] = -\frac{1}{2}\int d^4x(\eta^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi + m^2\Phi^2) - \frac{1}{2}\sum_{j=1}^{N}\int d^4x(\eta^{\mu\nu}\nabla_\mu\Psi_j\nabla_\nu\Psi_j + g^2\Phi^2\Psi_j^2). \tag{2.1}$$

One can divide the fields into classical and quantum parts by writing $\Phi = \phi_c + \hat{\phi}$ and $\Psi_j = \hat{\psi}$ with $\langle\Phi\rangle = \phi_c$ and $\langle\Psi_j\rangle = 0$. Then if the inflaton field is rescaled so that $\phi_c \rightarrow \sqrt{N}\phi_c$ it is possible to carry out a large N expansion[1] using the closed time path formalism [39]. The result to leading order is that the system is equivalent to a two field system with the classical inflaton field ϕ_c coupled to a single quantized field $\hat{\psi}$, henceforth referred to simply as ϕ and ψ, respectively. The equations of motion for these fields are

$$(-\Box + m^2 + g^2\langle\psi^2\rangle_B)\phi = 0, \tag{2.2a}$$

$$(-\Box + g^2\phi^2)\psi = 0, \tag{2.2b}$$

where $\langle\psi^2\rangle_B$ is the bare (unrenormalized) expectation value of ψ^2. Restricting to the case of a homogeneous classical inflaton field in Minkowski spacetime, the equation of motion simplifies to

$$\ddot{\phi}(t) + (m^2 + g^2\langle\psi^2\rangle_B)\phi(t) = 0. \tag{2.3}$$

Equation (2.2b) is separable and the quantum field $\psi(x)$ can then be expanded as

$$\psi(\mathbf{x}, t) = \int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}[a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}f_k(t) + a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}f_k^*(t)]. \tag{2.4}$$

The time dependent modes, $f_k(t)$, satisfy the following

[1]The expansion here is similar in nature to one that was carried out for quantum electrodynamics with N fermion fields in Ref. [38].

ordinary differential equation,

$$\ddot{f}_k(t) + [k^2 + g^2\phi^2(t)]f_k(t) = 0, \tag{2.5}$$

and the Wronskian (normalization) condition

$$f_k \frac{\mathrm{d}}{\mathrm{dt}} f_k^* - f_k^* \frac{\mathrm{d}}{\mathrm{dt}} f_k = i. \tag{2.6}$$

The WKB approximation for these modes is useful both for renormalization and for fixing the state of the field. It is obtained by writing

$$f_k(t) = \frac{1}{\sqrt{2W_k(t)}} \exp\left[-i\int_{t_0}^{t} \mathrm{d}t' W_k(t')\right]. \tag{2.7}$$

Substitution of Eq. (2.7) into Eq. (2.5) gives

$$W_k^2 = k^2 + g^2\phi^2 - \frac{1}{2}\left(\frac{\ddot{W}_k}{W_k} - \frac{3}{2}\frac{\dot{W}_k^2}{W_k^2}\right). \tag{2.8}$$

One solves this equation by iteration. Upon each iteration one obtains a WKB approximation which is higher by two orders than the previous one. Note that the order depends both upon the number of time derivatives and on the power of g. The zeroth order approximation is just $W_k^{(0)} = k$, while the second order approximation is

$$W_k^{(2)} = k + \frac{g^2\phi^2}{2k}, \tag{2.9}$$

and the fourth order one is

$$W_k^{(4)} = k + \frac{g^2\phi^2}{2k} - \frac{g^4\phi^4}{8k^3} - \frac{g^2}{4k^3}(\phi\ddot{\phi} + \dot{\phi}^2). \tag{2.10}$$

Equation (2.3) for the inflaton field involves the quantity $\langle\psi^2\rangle_{\mathrm{B}}$. Using (2.4) one finds

$$\langle\psi^2\rangle_{\mathrm{B}} = \frac{1}{2\pi^2}\int_0^{+\infty} \mathrm{d}k k^2 |f_k(t)|^2. \tag{2.11}$$

This quantity is divergent and must be regularized. We use the method of adiabatic regularization [40–43] which, for free scalar fields in Robertson-Walker spacetimes, has been shown to be equivalent to the covariant scheme of point splitting [44,45].

In adiabatic regularization the renormalization counterterms are obtained by using a WKB approximation for the modes of the quantized field [42–47]. One works with a massive field and then takes the zero mass limit at the end of the calculation. Expressions for the renormalized values of both $\langle\psi^2\rangle$ and the expectation value of the energy-momentum tensor $\langle T_{\mu\nu}\rangle$ for a system similar to the one we are using[2] have been obtained in Ref. [47]. Some details of the renormalization procedure are given in the Appendix. The result for our system is

$$\langle\psi^2\rangle_{\mathrm{ren}} = \frac{1}{2\pi^2}\int_0^{\epsilon} \mathrm{d}k k^2\left(|f_k(t)|^2 - \frac{1}{2k}\right) + \frac{1}{2\pi^2}\int_{\epsilon}^{+\infty} \mathrm{d}k k^2\left(|f_k(t)|^2 - \frac{1}{2k} + \frac{g^2\phi^2}{4k^3}\right) + \langle\psi^2\rangle_{\mathrm{an}}, \tag{2.12a}$$

$$\langle\psi^2\rangle_{\mathrm{an}} = -\frac{g^2\phi^2}{8\pi^2}\left[1 - \log\left(\frac{2\epsilon}{M}\right)\right], \tag{2.12b}$$

where ϵ is a lower limit cutoff that is placed in the integrals that are infrared divergent and M is an arbitrary parameter with dimensions of mass which typically appears when the massless limit is taken [46,47]. In principle, the value of M should be fixed by observations. For simplicity, we set it equal to the mass of the inflaton field, m, in the calculations below. Note that the value of $\langle\psi^2\rangle_{\mathrm{ren}}$ is actually independent of the value of the infrared cutoff ϵ.

It is useful for both the analytic and numerical calculations to scale the mass of the inflaton field, m, out of the problem. This can be done by defining new dimensionless variables as follows:

$$\bar{t} = mt, \tag{2.13a}$$

$$\bar{\phi} = \frac{\phi}{m}, \tag{2.13b}$$

$$\bar{k} = \frac{k}{m}, \tag{2.13c}$$

$$\bar{f}_k = \sqrt{m} f_k, \tag{2.13d}$$

$$\bar{M} = \frac{M}{m}, \tag{2.13e}$$

$$\bar{\epsilon} = \frac{\epsilon}{m}. \tag{2.13f}$$

If one substitutes $\langle\psi^2\rangle_{\mathrm{ren}}$ for $\langle\psi^2\rangle_B$ in Eq. (2.3), writes Eqs. (2.3), (2.5), (2.6), and (2.12) in terms of the dimensionless variables, rescales $\langle\psi^2\rangle_{\mathrm{ren}}$ so that $\langle\psi^2\rangle_{\mathrm{ren}} \to m^{-2}\langle\psi^2\rangle_{\mathrm{ren}}$, and then drops the bars, the equation for the inflaton field [Eq. (2.3)] becomes

$$\ddot{\phi}(t) + [1 + g^2\langle\psi^2\rangle_{\mathrm{ren}}]\phi(t) = 0, \tag{2.14}$$

while Eqs. (2.5), (2.6), and (2.12) remain the same. These four equations in terms of the dimensionless variables are the ones that are solved in the following sections.

III. BACKGROUND FIELD APPROXIMATION

In the background field approximation the wave equation for the inflaton field is solved without taking into account the backreaction effects of the quantum fields on the inflaton field. However, the effect of the inflaton field on the mode equations for the quantum fields is taken into account. For our model the wave equation for the inflaton field (2.14) then is a simple harmonic oscillator equation, and the mode equation (2.5) is a Mathieu equation.

[2] In Ref. [47], if the scale factor, $a(t)$, is set equal to 1, the coupling of the inflaton field to the scalar curvature is set to zero, the coupling constant λ is set equal to $2g^2$, the mass of the quantum field is set to zero, and the $\lambda\phi^3/3!$ term in the equation of motion for the ϕ field is dropped; then the two systems are equivalent.

It is well known that for the Mathieu equation there are regions in parameter space for which there are solutions which grow exponentially due to a process called parametric amplification. For the mode equation (2.5) the last term is proportional to $g^2\phi_0^2$, with ϕ_0 the amplitude of the oscillations of the inflaton field. Thus, for a given $g^2\phi_0^2$, it is the parameter k which determines which modes undergo parametric amplification. There are bands of values of k for which this occurs and we shall call them instability bands. The instability band which contains the smallest values of k will be called the first band.

For a given value of $g^2\phi_0^2$ there is an infinite number of instability bands, some of which contain modes with arbitrarily large values of k. In principle, this could lead to divergences in quantities such as $\langle\psi^2\rangle_{\rm ren}$ and the expectation value of the energy-momentum tensor, $\langle T_{\mu\nu}\rangle_{\rm ren}$. The reason is that, as can be seen from Eq. (2.12a), the renormalization counterterms do not undergo any exponential growth, while bands of modes with arbitrarily large values of k do. One might also be concerned that, even if $\langle\psi^2\rangle_{\rm ren}$ and $\langle T_{\mu\nu}\rangle_{\rm ren}$ are finite, in order to compute them one must take into account modes in instability bands with arbitrarily large values of k. This is clearly not possible for purely numerical computations, which must necessarily include only a finite number of modes and for which there must be an ultraviolet cutoff. In this section we use known properties of solutions to the Mathieu equation to show that both $\langle\psi^2\rangle_{\rm ren}$ and $\langle T_{\mu\nu}\rangle_{\rm ren}$ are finite at any finite time. We also address the question of how to make sure that the contributions of all of the instability bands which contribute significantly to $\langle\psi^2\rangle_{\rm ren}$ and $\langle T_{\mu\nu}\rangle_{\rm ren}$ are included in numerical calculations of these quantities.

The Mathieu equation

In the background field approximation, without loss of generality (due to time translation invariance), we can take the solution to Eq. (2.14) to be

$$\phi(t) = \phi_0 \cos(t). \tag{3.1}$$

The mode equation is then

$$\ddot{f}_k(t) + [k^2 + g^2\phi_0^2\cos^2(t)]f_k(t) = 0. \tag{3.2}$$

Using the identity

$$\cos^2(t) = \tfrac{1}{2}[1 + \cos(2t)], \tag{3.3}$$

the mode equation can be put into a standard form for the Mathieu equation[48]

$$\frac{d^2 f_k(t)}{dt^2} + [a - 2q\cos(2t)]f_k(t) = 0, \tag{3.4}$$

with

$$a = k^2 + \tfrac{1}{2}g^2\phi_0^2, \tag{3.5a}$$

$$q = -\tfrac{1}{4}g^2\phi_0^2. \tag{3.5b}$$

Floquet's theorem applies to the Mathieu equation [49]. It implies the existence of two solutions of Eq. (3.4) of the form

$$f_1(t) = e^{\mu t}h_1(t), \tag{3.6}$$

$$f_2(t) = e^{-\mu t}h_2(t), \tag{3.7}$$

where $h_1(t)$ and $h_2(t)$ are periodic functions of period 2π, and μ is, in general, a complex number [49]. If μ has a nonvanishing real part, it is easy to see that one solution will grow exponentially at late times.

Our goal is to find which ranges of values of k lead to exponentially growing modes. This can be accomplished by first fixing the value of q and then finding the values of the parameter a for which the Mathieu equation has solutions with $\mu = 0$. Those values of a which correspond to even solutions are traditionally labeled a_r, and those which correspond to odd solutions are labeled b_r, with r a positive integer. The pattern of instability and stability regions in the (a, q) plane is symmetric about $q = 0$. Thus, for simplicity we shall consider $q > 0$ in what follows, even though $q < 0$ for the equations we are concerned with [see Eq. (3.5b)]. For $q > 0$, it is found that $\mu^2 > 0$ for values of a which are between a_r and b_r for the same value of r [48]. Further, one has the relation $a_0 < b_1 < a_1 < b_2 < \ldots$.

For the purpose of assessing the contribution of unstable modes with large values of k to Eq. (2.12), it is necessary to know both the range of values of k in the unstable bands and the maximum value of μ in a band containing large values of k. The ranges of k in the large r limit are given by the relation [50]

$$a_r - b_r = O\left(\frac{q^r}{r^{r-1}}\right). \tag{3.8}$$

Thus, for fixed q, the width of the unstable bands in the (a, q) plane becomes arbitrarily small for bands with large values of r and therefore becomes arbitrarily small for large values of k. This means that for unstable modes at large values of k to contribute significantly to the integrals in Eq. (2.12), it would be necessary that these modes grow much faster than the unstable modes in bands with smaller values of k, which are wider.

An approximation to the value of μ in unstable bands is [48]

$$\mu_r^2 = \frac{(a_r + b_r - 2a - 4r^2)}{2} \pm \frac{\sqrt{a_r^2 - 2a_r b_r - 8r^2(a_r + b_r - 2a) + b_r^2 + 16r^4}}{2}, \tag{3.9}$$

where the boundaries of the unstable band of index r are given by the pair of eigenvalues b_r and a_r with $a_r > b_r$. Some details of the derivation of this approximation can be found in Ref. [51]. Note that there are two possible solutions for μ_r for fixed values of r and a. Given that for large r the instability bands become narrower, and that $b_r \leq a \leq a_r$, it is easy to show that for large r only the plus sign gives $\mu_r^2 > 0$. It is also easy to show that

$$|2a - a_r - b_r| \ll 4r^2, \tag{3.10}$$

and thus that

$$\mu_r \simeq \frac{[(a_r - a)(a - b_r)]^{1/2}}{2r}. \tag{3.11}$$

For this approximation the maximum value of μ_r is at the midpoint between b_r and a_r. Since, for large r the width of the band, $a_r - b_r$, gets very small [see Eq. (3.8)], it is clear that the maximum value of μ in a band is smaller for bands with very large values of k than for bands with smaller values of k.

The relative contribution of unstable modes to the integral in Eq. (2.12) at a given time t depends in part on how much they have grown compared to unstable modes in lower bands. It is clear from the above analysis that modes in higher bands grow significantly slower than those in lower bands. Also for a given time t, there will be some value, $k = K$, for which unstable bands with larger values of k have not grown much at all. This, coupled with the fact that the widths of the unstable bands get narrower at larger values of k, means that at any given time t the modes in bands with $k > K$ will not contribute significantly to the mode integral in Eq. (2.12). Thus, $\langle\psi^2\rangle_{\rm ren}$ and $\langle T_{\mu\nu}\rangle_{\rm ren}$ are finite.

From the above analysis it should be clear that in numerical computations it is possible to obtain a good approximation to $\langle\psi^2\rangle_{\rm ren}$ and $\langle T_{\mu\nu}\rangle_{\rm ren}$ by putting in a cutoff at some large value of k. Since the location of the bands at small values of k can be determined using standard techniques [48], then one simply needs to use a high enough density of modes so that those bands with smaller values of k are adequately covered.

The maximum value of μ occurs for the first unstable band which has the smallest values of k. Once the modes in this band have become sufficiently large, they make the major contribution to $\langle\psi^2\rangle_{\rm ren}$ and $\langle T_{\mu\nu}\rangle_{\rm ren}$.

IV. BACKREACTION ON THE INFLATON FIELD

A. Initial conditions

A FORTRAN program was written to simultaneously solve the mode equations for the ψ field, compute the quantity $\langle\psi^2\rangle_{\rm ren}$, and solve the equation for the inflaton field ϕ in terms of the dimensionless variables (2.13). For this program to run, initial values must be given for ϕ and its first time derivative, as well as each mode function f_k and its first time derivative. Since the inflaton field is in its rapid oscillation phase, what is important is the initial amplitude of its oscillations. This is most easily obtained by starting with $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$.

The initial values for the mode functions of the ψ field are determined by the state of the field. As the quantum field is coupled to the inflaton field, there is no preferred vacuum state. Instead, one can use an adiabatic vacuum state. Such states have been discussed in detail for cosmological spacetimes [52]. They are based on the WKB approximation for the modes. It is necessary to choose a state that is of adiabatic order two or higher for the renormalized value of $\langle\psi^2\rangle$ to be finite. For the renormalized energy-momentum tensor $\langle T_{\mu\nu}\rangle$ to be finite, it is necessary to have at least a fourth order adiabatic state. For such a state, in the large k limit,

$$|f_k|^2 \to \frac{1}{2k} - \frac{g^2\phi^2}{4k^3} + \frac{3g^4\phi^4}{16k^5} + \frac{g^2(\dot{\phi}^2 + \phi\ddot{\phi})}{8k^5} + O(k^{-6}). \tag{4.1}$$

Although the large k behavior of an adiabatic state of a given order is constrained by requiring that renormalization of the energy-momentum tensor be possible, there are no such constraints on modes with smaller values of k. Thus, one can construct fourth order adiabatic states that would give virtually any value for $\langle\psi^2\rangle_{\rm ren}$ or $\langle T_{\mu\nu}\rangle_{\rm ren}$ at some initial time t. However, if the value of $g^2\langle\psi^2\rangle_{\rm ren}$ is too large at the beginning of preheating, then backreaction effects will be important immediately. In fact, if $g^2\langle\psi^2\rangle_{\rm ren} \gg 1$, then backreaction effects are so strong initially that parametric amplification is unlikely to occur. If $g^2\langle\psi^2\rangle_{\rm ren} \sim 1$, backreaction effects will be important, but it is possible that parametric amplification might still occur. On the other hand, if $g^2\langle\psi^2\rangle_{\rm ren} \ll 1$ at the onset of preheating, then backreaction effects are not important initially and parametric amplification will occur. Since our goal is to study the preheating process in detail, we restrict attention to this latter case.

A natural way to construct a fourth order adiabatic state is by using a fourth order WKB approximation to fix the initial values of the modes, f_k, and their derivatives, $\dot{f}_k$, for all values of k. For the numerical calculations done in this paper, the state used was an adiabatic vacuum state with

$$f_k(t=0) = \sqrt{\frac{Y}{2}}, \tag{4.2a}$$

$$\dot{f}_k(t=0) = -\frac{i}{\sqrt{2Y}}, \tag{4.2b}$$

$$Y = \frac{1}{\Omega_0} + \frac{g^2\phi_0\ddot{\phi}_0}{4\Omega_0^5}, \tag{4.2c}$$

$$\Omega_0 = \sqrt{k^2 + g^2\phi_0^2}. \tag{4.2d}$$

It is easy to show that this is a fourth order adiabatic state by recalling that $\dot{\phi}(0) = 0$. The initial values have been chosen in part to make it possible to analytically compute the initial value of $\langle\psi^2\rangle_{\rm ren}$, and in part so that the Wronskian condition (2.6) is satisfied exactly. For the fourth order adiabatic state that we are using, at the initial time $t = 0$

$$g^2\langle\psi^2(t=0)\rangle_{\rm ren} = \frac{g^2\ddot{\phi}_0}{48\pi^2\phi_0} - \frac{g^4\phi_0^2}{16\pi^2}\left[1 - \log\left(\frac{g^2\phi_0^2}{M^2}\right)\right]. \tag{4.3}$$

Using this initial value for $g^2\langle\psi^2\rangle_{\rm ren}$ one finds that the initial value for $\ddot{\phi}(t)$ is

$$\ddot{\phi}(t=0) = \frac{1}{1+\frac{g^2}{48\pi^2}}\left[-1 + \frac{g^4\phi_0^2}{16\pi^2}\left(1 - \log\left(\frac{g^2\phi_0^2}{M^2}\right)\right)\right]\phi_0. \tag{4.4}$$

Thus, whether backreaction effects are important initially depends on the values of both ϕ_0 and g. For the values of g considered here, $\frac{g^2}{48\pi^2} \ll 1$. From Eq. (4.4) it can be seen that backreaction effects for the states we are considering are likely to be important if

$$g^2(g^2\phi_0^2) \gtrsim 16\pi^2. \tag{4.5}$$

As an example, for realistic models of chaotic inflation, KLS [15] chose

$$m = 10^{-6}M_p, \qquad \phi_0 = \frac{M_p}{5m}. \tag{4.6}$$

The values they used for g range from $\sim 10^{-4}$ to 10^{-1}. For $g = 10^{-4}$, $g^2\phi_0^2 = 400$, while for $g = 10^{-1}$, $g^2\phi_0^2 = 4 \times 10^8$. Applying the criterion (4.5) one finds that our fourth order adiabatic states result in significant backreaction initially for these values of m and ϕ_0 if $g \gtrsim 8 \times 10^{-3}$.

B. Parametric amplification with backreaction

If the initial conditions are such that $g^2\langle\psi^2\rangle_{\rm ren} \ll 1$, then examination of Eq. (2.14) shows that there is no significant backreaction at early times, and the solution of the inflaton field will be approximately equal to Eq. (3.1). When, primarily through parametric amplification of certain modes, $g^2\langle\psi^2\rangle_{\rm ren} \sim 1$, one expects backreaction effects to be important and to make the amplitude of the oscillations of the inflaton field, ϕ, decrease. It has been observed in the numerical computations that $\langle\psi^2\rangle_{\rm ren}$ oscillates about an average value which changes in time. The effective mass of the inflaton field can be defined in terms of this average value:

$$m_{\rm eff}^2 = 1 + g^2\overline{\langle\psi^2\rangle}_{\rm ren}. \tag{4.7}$$

It is clear that an increase in $m_{\rm eff}^2$ results in an increase of the frequency of oscillations of ϕ. This should happen slowly at first, so one would expect the basic character of the mode equation (2.5) to be a Mathieu equation, except that the effective value of q would change [15]. A zeroth order WKB analysis of the solutions to Eq. (2.14) shows that the effective frequency of oscillations can be approximated by

$$\omega_{\rm eff} \approx \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} dt\sqrt{1 + g^2\langle\psi^2\rangle_{\rm ren}}, \tag{4.8}$$

with $t_1 < t < t_2$ for some time interval $t_2 - t_1$ equal to or larger than an oscillation period of the inflaton field. A derivation similar to that at the beginning of Sec. III then shows that

$$a_{\rm eff} \approx \frac{k^2}{\omega_{\rm eff}^2} + 2q_{\rm eff}, \tag{4.9a}$$

$$q_{\rm eff} \approx \frac{g^2A^2}{4\omega_{\rm eff}^2}, \tag{4.9b}$$

where A is the amplitude of the inflaton field and we have defined $q_{\rm eff}$ so that it is positive.[3] As the inflaton field evolves, A will tend to slowly decrease and $\omega_{\rm eff}$ will tend to slowly increase. Thus, the resulting changes in the instability bands should be small enough that there would be no problem in having enough modes to accurately determine the primary contributions to $\langle\psi^2\rangle_{\rm ren}$. Our numerical work described below seems to bear this out.

Once the backreaction is significant, it is possible for the character of the equation to change substantially. If ϕ varies in a periodic manner, then it is still possible for parametric amplification to occur due to Floquet's theorem [49]. It would be difficult in this more complicated situation to analyze the behavior of the solutions to the extent that solutions to the Mathieu equation have been analyzed. However, it seems quite likely that, as for the Mathieu equation, the most important instability bands will be at relatively small values of k, and that if an instability band is extremely narrow, then it probably will not have a large effect on $\langle\psi^2\rangle_{\rm ren}$ for a very long time. It is likely that interactions which are neglected in our model will be important on such long time scales [21–23,26,27,35].

C. Some numerical results

Since the mode equation (2.5) depends on $g^2\phi^2$ and the equation of motion of the inflaton field (2.14) depends on $g^2\langle\psi^2\rangle_{\rm ren}$, one would expect that the effects of the values of g and $g^2\phi_0^2$ decouple, at least to some extent. As can be seen in Figs. 1 and 2 this is the case, with the quantity $g^2\phi_0^2$ determining the type of evolution the inflaton field undergoes and g having its strongest effect on the time scale of that evolution.

In Fig. 1 plots of solutions to the backreaction equation (2.14) for the case in which $g = 10^{-3}$ are shown, with

[3]The actual derivation results in a negative value for $q_{\rm eff}$ but the symmetry between positive and negative values of q for the Mathieu equation allows us to simply consider positive values.

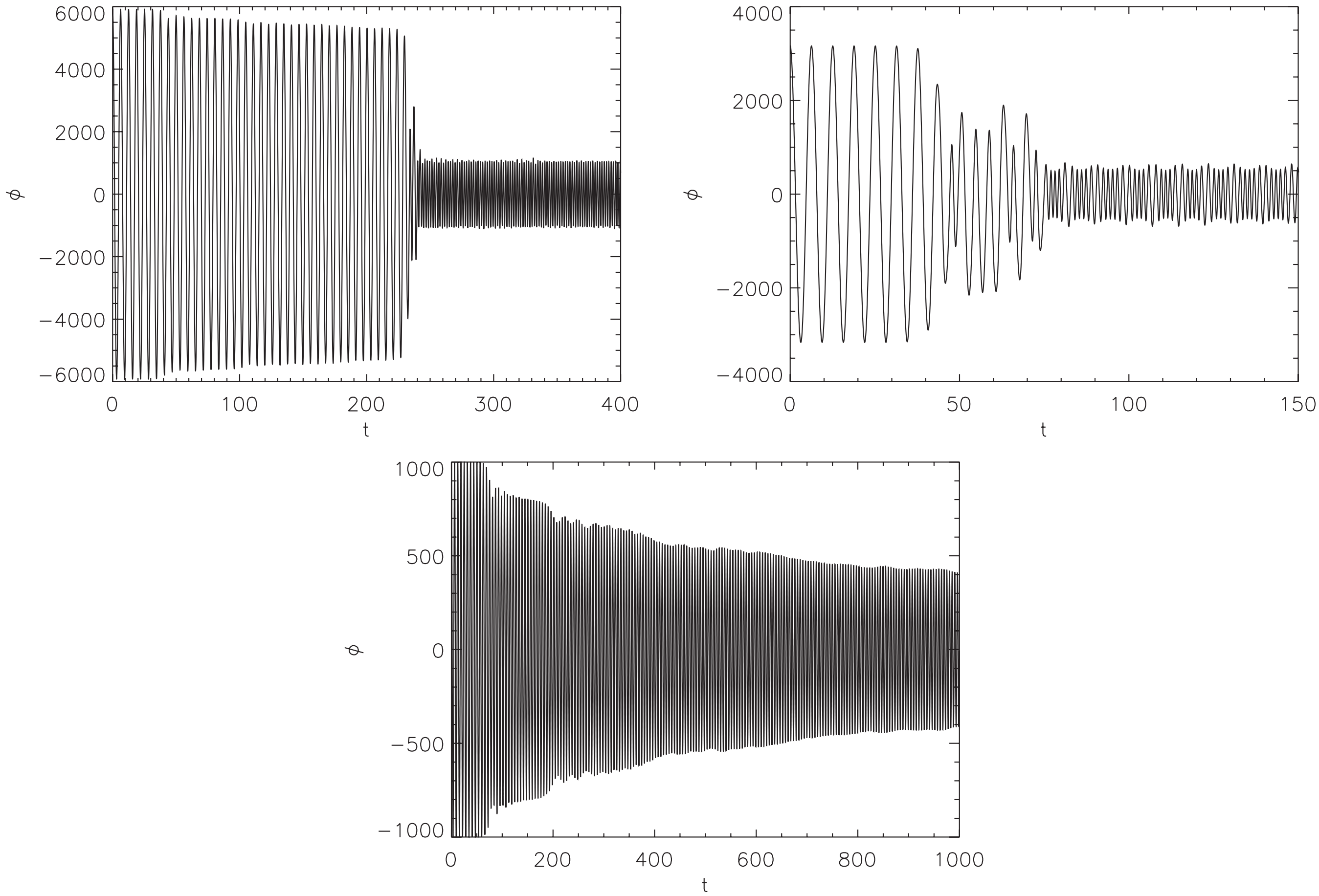

FIG. 1. The evolution of the inflaton field for $g = 10^{-3}$. From left to right the top plots are for $g^2\phi_0^2 = 35$ and 10. The bottom plot is for $g^2\phi_0^2 = 1$.

$g^2\phi_0^2$ ranging from 35 to 1. For the first two plots most of the damping occurs over a very short period. After that, the field does not appear to damp significantly. Instead, it continues to oscillate but with a much higher frequency than before. The envelope of its oscillations also oscillates but with a much smaller frequency. This behavior has been observed for all cases investigated with $g^2\phi_0^2 \gtrsim 2$. Conversely, for the plot on the bottom ($g^2\phi_0^2 = 1$), it is clear that the damping is much slower and occurs over a much longer period of time. This behavior is a verification

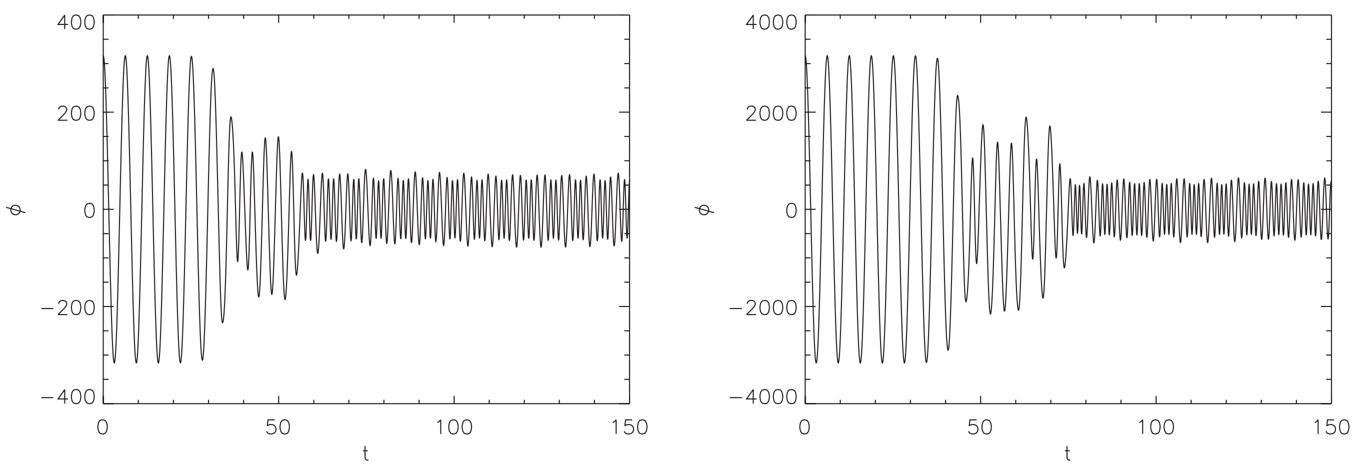

FIG. 2. The evolution of the inflaton field for $g^2\phi_0^2 = 10$. The plot on the left is for $g = 10^{-2}$ and the one on the right is for 10^{-3}.

of the prediction by KLS [15] that a period of rapid damping would occur for $g^2\phi_0^2 \gg 1$ and that only relatively slow damping occurs for $g^2\phi_0^2 \ll 1$. A detailed analysis of the evolution of the inflaton field is given in Sec. IV D.

In Fig. 2 plots of solutions to the backreaction equation for the case in which $g^2\phi_0^2 = 10$ and $g = 10^{-2}$ and 10^{-3} are shown. The amount of damping does not appear to depend in any strong way upon the value of the coupling constant g. The amount of time it takes for significant damping to begin is longer for the smaller value of g. This is easily explained by the fact that initially $g^2\langle\psi^2\rangle_{\rm ren} \ll 1$, so that only negligible damping of the inflaton field occurs at early times. During this period the mode equations, and thus the growth of $\langle\psi^2\rangle_{\rm ren}$, is approximately independent of the value of g. Significant backreaction begins to occur when $g^2\langle\psi^2\rangle_{\rm ren} \sim 1$, and this will obviously take longer for smaller values of g, since $\langle\psi^2\rangle_{\rm ren}$ will have to grow larger before it can occur.

The natural explanation for what is happening in all of these cases is that particle production occurs and the backreaction of the produced particles on the inflaton field causes its amplitude to be damped. However, if this picture is correct, then one would expect that a significant amount of energy would be transferred from the inflaton field to the modes of the quantum field. To check this, one can compute the time evolution of the energy density of the inflaton field. Since the gravitational background is Minkowski spacetime, the energy density is conserved as is shown explicitly in the Appendix. As is also shown in the Appendix, the energy density can be broken into two different contributions. The first one is the energy density the inflaton field would have if there was no interaction ($g = 0$). It is given by

$$\rho_\phi = \tfrac{1}{2}(\dot{\phi}^2 + m^2\phi^2). \tag{4.10}$$

The other contribution is the energy density of the quantum field shown in Eq. (A6c). It contains terms that would be there if there was no interaction along with terms that explicitly depend on the interaction. This split is useful because almost all of the energy is initially in the inflaton field, and thus, one can clearly see how much has been transferred to the quantum field ψ and to the interaction between the two fields as time goes on.

In Fig. 3 the left-hand plot shows the evolution of the energy density of the inflaton field for the case $g = 10^{-3}$ and $g^2\phi_0^2 = 1$. Comparison with the bottom plot in Fig. 1 shows that as the inflaton field is damped, a large portion of its energy is permanently transferred away from it.

For all cases investigated in which rapid damping of the inflaton field occurs, it was found that much less energy is permanently transferred away from the inflaton field than one might expect, given the amount by which its amplitude has been damped. This effect was seen previously in the classical lattice simulation of Prokopec and Roos [22]. An example is the case $g = 10^{-3}$, $g^2\phi_0^2 = 10$ for which the evolution of the energy density of the inflaton field is shown in the plot on the right panel of Fig. 3. Comparison with the top right plot of Fig. 2 shows that after the rapid damping has finished, the inflaton field permanently loses some energy because the maximum of its energy density is lower than before, but more than half of the original energy is transferred back and forth between the inflaton field, the quantum field, and the interaction between the two. However, it is important to note that in cases in which rapid damping occurs, our results cannot be trusted after the rapid damping phase has finished because, as previously mentioned, there is evidence that the interactions we neglect become important [21–23,26,27,35].

Another effect that was observed to occur in all cases in which the inflaton field undergoes rapid damping is an extreme sensitivity to initial conditions. If the initial conditions are changed by a small amount, then initially the

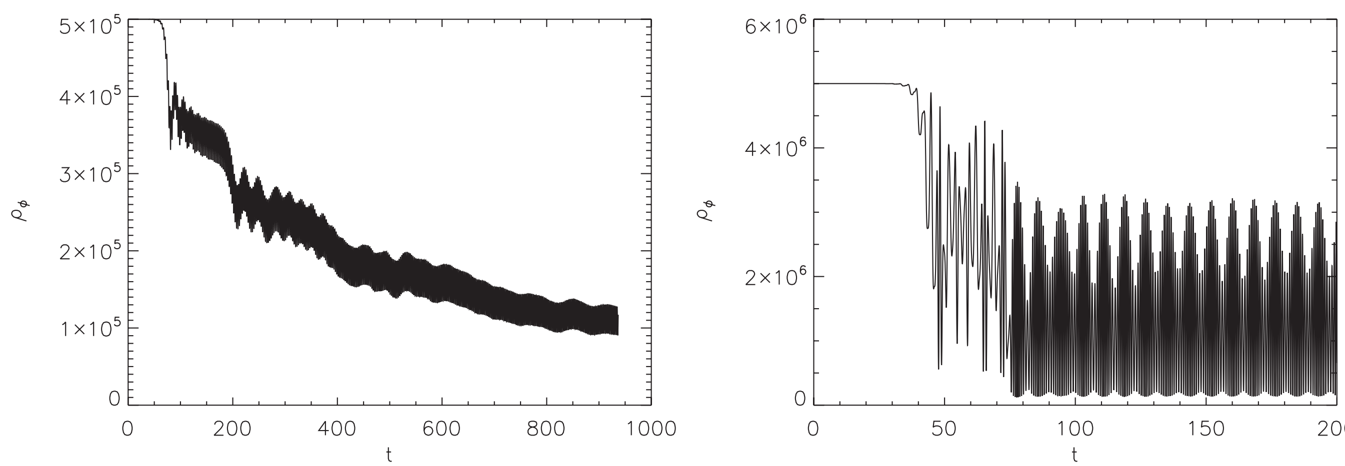

FIG. 3. The evolution of the energy density of the inflaton field, ρ_ϕ. Both plots are for $g = 10^{-3}$. The one on the left is for $g^2\phi_0^2 = 1$ and the one on the right is for $g^2\phi_0^2 = 10$.

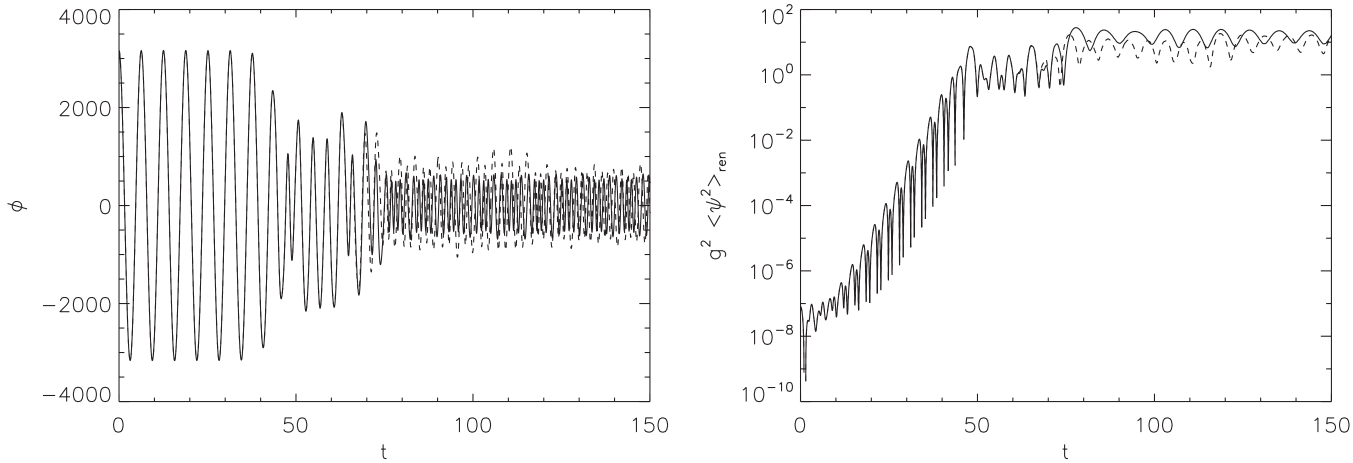

FIG. 4. This figure shows differences in ϕ and $g^2\langle\psi^2\rangle_{\rm ren}$ which occur for $g = 10^{-3}$ and $g^2\phi_0^2 = 10$ and $g^2\phi_0^2 = 10(1 + 10^{-5})$. In each plot the dashed curve corresponds to $g^2\phi_0^2 = 10(1 + 10^{-5})$.

evolution of the inflaton field and the quantity $\langle\psi^2\rangle_{\rm ren}$ are nearly the same as before. However, at some point the evolution of $\langle\psi^2\rangle_{\rm ren}$ begins to change significantly and that is followed by a significant change in the evolution of ϕ. An example is shown in Fig. 4. This phenomenon is discussed in more detail below.

D. Detailed analysis

As discussed in Sec. IV B, as $\langle\psi^2\rangle_{\rm ren}$ increases due to parametric amplification the amplitude of the field damps and its oscillation frequency increases. Thus, it is useful to define an effective value for $|q|$, which we call $q_{\rm eff}$, and whose definition is given in Eq. (4.9b). If $q_{\rm eff} \gg 4k^2 > 0$ for some or all of the modes in an instability band, the oscillation frequency for the modes is much larger than that for the inflaton field. KLS point out that this allows one to use the Mathieu equation to provide a reasonable description of what happens since $q_{\rm eff}$ varies slowly compared to the oscillation time for the modes. If $q_{\rm eff}$ is small enough, then it seems likely that a description in terms of the Mathieu equation breaks down from a quantitative, and perhaps also from a qualitative, point of view.

Our explanations of the details of the damping process are based on the basic scenario outlined by KLS in which, as the damping proceeds, $q_{\rm eff}$ decreases and there are various instability bands that cross over a given set of modes. The damping stops when $q_{\rm eff}$ gets small enough. The locations of the five lowest instability bands in the (k^2, q) plane are shown in Fig. 5.

There are four primary factors that go into our explanations:

(1) The lowest instability band, i.e., that with the smallest values of k, has the largest value of $\mu_{\rm max}$ and is the widest. Thus the modes near its center undergo the largest amount of parametric amplification.

(2) As $q_{\rm eff}$ decreases the tendency is for instability bands to move to the right along the k axis and for new ones to appear when $k^2 = \omega_{\rm eff}^2(a_{\rm eff} - 2q_{\rm eff})$ becomes positive for part of the band.

(3) Modes near the center of an instability band will continue to undergo significant parametric amplification (compared with the rest of the band) until the backreaction on the inflaton field causes $q_{\rm eff}$ to decrease enough that these modes are either near the edge of the instability band (where μ is small) or no longer in it, due to the shift of the band along the k axis.

(4) Once the lowest instability band shifts significantly so that the original modes near the center stop growing rapidly, new modes will begin to undergo a significant amount of parametric amplification. However, it takes a while for the contributions of

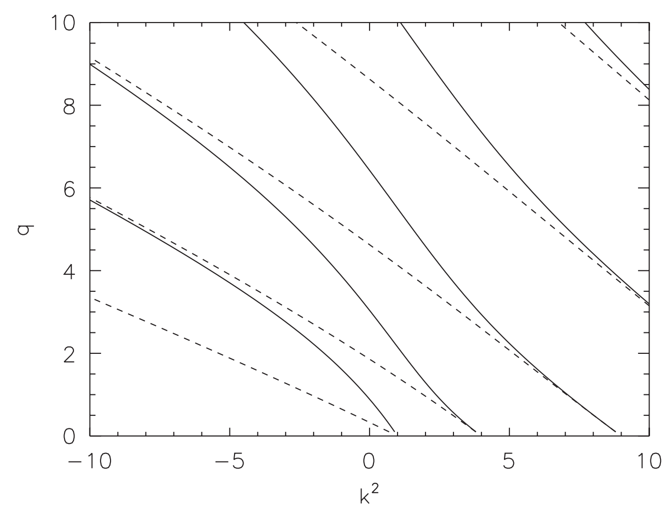

FIG. 5. This figure shows the boundaries of the five lowest instability bands for the Mathieu equation using (3.5a) and (3.5b). The lower bound of a given band is the dashed curve and the upper bound is the solid curve immediately above it. Note that there are no modes of the quantum field with $k^2 < 0$.

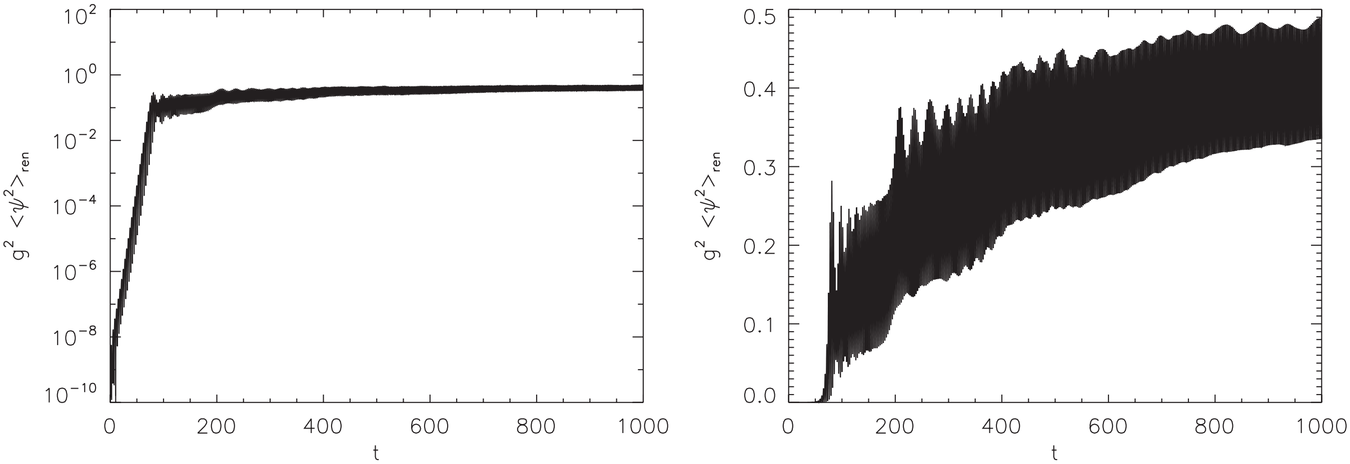

FIG. 6. These plots show $g^2\langle\psi^2\rangle_{\rm ren}$ for $g = 10^{-3}$ and $g^2\phi_0^2 = 1$. The plots are identical except that the one on the left is plotted on a logarithmic scale and the one on the right is plotted on a linear scale.

these new modes to the value of $\langle\psi^2\rangle_{\rm ren}$ to become significant compared with that of the modes which were previously near the center of the first instability band, and have now stopped undergoing significant parametric amplification.

In what follows we first discuss the case in which no significant amount of rapid damping occurs. For the cases in which rapid damping does occur, the pre-rapid damping, rapid damping, and post-rapid damping phases are discussed separately.

1. $g^2\phi_0^2 \lesssim 1$

KLS predict and we observe that there is no significant amount of rapid damping for $g^2\phi_0^2 \lesssim 1$. However, the details are different because they consider an expanding universe. In the Minkowski spacetime case, as can be seen by comparing the lower plot of Fig. 1 and the plots of Fig. 6, there is actually a small amount of rapid damping that occurs for $g^2\phi_0^2 = 1$ once $g^2\langle\psi^2\rangle_{\rm ren} \sim 10^{-2}$. What appears to be happening is that modes near the center of the first instability band cause $\langle\psi^2\rangle_{\rm ren}$ to grow exponentially until the band has shifted to the right along the k axis enough so that these modes no longer grow so rapidly. Because of the exponential growth of $g^2\langle\psi^2\rangle_{\rm ren}$, the damping of the inflaton field is very rapid once it begins to occur in a significant way. However, as $|q| = 0.25$ in this case, the first instability band is relatively narrow and thus not much damping of the inflaton field needs to occur for this band to shift by the required amount. Once it has shifted, new modes begin to undergo significant parametric amplification, and as they start to contribute to $\langle\psi^2\rangle_{\rm ren}$, the amplitude of the inflaton field continues to decrease and

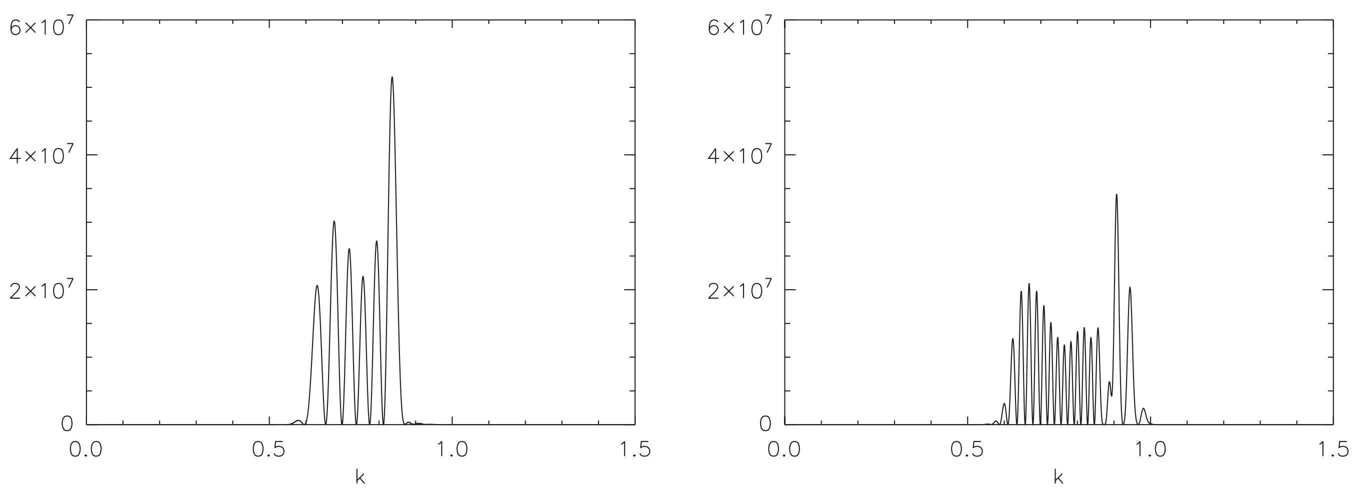

FIG. 7. This figure shows the integrand in Eq. (2.12a) for the case $g = 10^{-3}$, $g^2\phi_0^2 = 1$, for the times $t = 150$ (left panel) and $t = 250$ (right panel). Note that at the later time the band of modes that contributes significantly to the integral includes modes with larger values of k than the band at the earlier time.

the position of the instability band continues to move slowly to the right along the k axis. One can see evidence of this shift by comparing plots of the integrand for $\langle\psi^2\rangle_{\rm ren}$ at two different times, as is done in Fig. 7.

As the inflaton field continues to damp and $q_{\rm eff}$ continues to decrease, the value of μ at the center of the band should also be decreasing. Thus, the time scale for the new modes undergoing parametric amplification to contribute significantly to the damping increases and the rate of damping decreases.

2. $g^2\phi_0^2 \gtrsim 2$

For $g^2\phi_0^2 \gtrsim 2$ rapid damping has occurred in every case that has been numerically investigated. However, the times at which the rapid damping occurs, and the details relating to both how the rapid damping occurs and how much damping there is, vary widely. To understand the reasons for this, it is useful to break the discussion up into the pre-rapid damping, rapid damping, and post-rapid damping phases.

a. Pre-rapid damping phase.—Since $g^2\langle\psi^2\rangle_{\rm ren} \ll 1$ initially for all cases considered in this paper, there is no significant amount of backreaction on the inflaton field at first. Therefore, parametric amplification of the modes near the center of the first instability band, i.e., the one which encompasses the smallest values of k, occurs and the contribution of these modes quickly dominates all other contributions to $\langle\psi^2\rangle_{\rm ren}$.[4] (As mentioned previously this is in contrast to the case of an expanding universe where the expansion causes a decrease in the amplitude of the inflaton field and thus in $q_{\rm eff}$.) As before, $\langle\psi^2\rangle_{\rm ren}$ grows exponentially until it becomes of order 10^{-2}, at which point significant damping of the inflaton field begins to occur. What happens next is observed to depend upon the location on the k axis of the first instability band.

(1) If $k^2 < 0$ for a significant fraction of the band and $k^2 > 0$ for a significant fraction, then the inflaton field goes directly into the rapid damping phase which is discussed next. A good example of this is the case $g^2\phi_0^2 = 10$ which is shown in Fig. 4. The average value of $g^2\langle\psi^2\rangle$ increases exponentially until it gets large enough to significantly affect the inflaton field and cause the first phase of rapid damping to occur. Thus there is no significant amount of gradual damping that occurs before the rapid damping phase.

(2) If $k^2 > 0$ for most or all of the band, then as $q_{\rm eff}$ decreases, the shift in the band causes the modes near the center to stop increasing rapidly in amplitude. In that case the exponential increase in $\langle\psi^2\rangle_{\rm ren}$ ceases. At this point there are two possibilities that have been observed:

(a) If the decrease in $q_{\rm eff}$ has resulted in $k^2 > 0$ for part or all of an instability band in the (a, q) plane for which previously $k^2 < 0$, then the modes which are now in that band will begin to increase in amplitude exponentially and will do so at a faster rate than modes in what was previously the first instability band.

(b) If the decrease in $q_{\rm eff}$ is not large enough for a new instability band to appear, then slow damping will start to occur due to parametric amplification of those modes that are now near the center of the first instability band but which were not close to it before. As they become large enough to have an effect, $q_{\rm eff}$ will gradually decrease and the slow damping which occurs is of the type discussed for the $g^2\phi_0^2 \lesssim 1$ case. Eventually $q_{\rm eff}$ will become small enough that an instability band that previously had $k^2 < 0$ will now encompass $k^2 = 0$ and parametric amplification will occur, again at a faster rate than that which is occurring in what was previously the first instability band. The modes will grow exponentially until rapid damping begins. This is what happens for $g^2\phi_0^2 = 35$. A careful examination of the first plot in Fig. 1 shows the gradual damping phase followed by the rapid damping phase.

b. Rapid damping phase.—In the rapid damping phase, as mentioned previously, KLS point out that a Minkowski spacetime approximation is valid, and they suggest that the value of $q_{\rm eff}$ should decrease until it reaches about $1/4$, at which point there are only narrow instability bands, so the rapid growth of $g^2\langle\psi^2\rangle_{\rm ren}$ ceases, as does the fast damping of the inflaton field. They predict that the ratio of the amplitude of the inflaton field after the rapid damping to that before the rapid damping is

$$\left(\frac{A}{A_0}\right)_{\rm KLS} = \left(\frac{4}{g^2A_0^2}\right)^{1/4}, \qquad (4.11)$$

with A the amplitude of the inflaton field ϕ after the rapid damping, and A_0 the amplitude just before the rapid damping begins. We consistently find that after the rapid damping has finished $q_{\rm eff} \lesssim 10^{-2}$. This is significantly smaller than the value of $1/4$ that they predict. As a result, the amount of damping which the inflaton field undergoes is much larger than the amount predicted by KLS. After the rapid damping, the value of $q_{\rm eff}$ is observed to vary periodically as the envelope and frequency of the oscillations of the inflaton field change in time.

A careful examination of the plots showing the evolution of ϕ indicates that the rapid damping always seems to occur in two different phases separated by a short time. After the first phase the field is damped by a significant amount, but $q_{\rm eff}$ is still large enough that significant parametric amplification can occur for modes in the lowest

[4]Of course, if initially $k^2 > 0$ for only a small part of the first instability band, then the modes in this band may make a comparable or even smaller contribution to $\langle\psi^2\rangle$ than the modes in the second instability band at early times.

instability band. As it is occurring, but before it gets large enough to have a significant effect on $g^2\langle\psi^2\rangle_{\rm ren}$, the inflaton field does not undergo any more noticeable damping. Once there is enough of an effect, the second phase of rapid damping takes place, and after it is over, $q_{\rm eff}$ becomes small enough so that no more significant damping appears to occur even after a large number of oscillations of the inflaton field.

It is during this period, begun by the first rapid damping of the inflaton field and finished by the second, that the extreme sensitivity to initial conditions that we have found seems to manifest. A useful way to study the sensitivity is to look at the time evolution of the quantities ϕ, $g^2\langle\psi^2\rangle_{\rm ren}$, and f_k for two cases in which the initial data are almost but not quite identical. For the cases discussed here, ϕ_0 is different by a small amount, which then generates small changes in the initial values of the modes f_k and thus in $g^2\langle\psi^2\rangle_{\rm ren}$. If the relative differences in the quantities $\delta\phi/\phi$, $\delta\langle\psi^2\rangle_{\rm ren}/\langle\psi^2\rangle_{\rm ren}$, and $\delta f_k/f_k$ are plotted as in Figs. 8 and 9, then it is seen that an exponential increase in their amplitudes takes place between the time when $g^2\langle\psi^2\rangle_{\rm ren} \sim 2$, which is when the rapid damping begins, and the time when the average of $g^2\langle\psi^2\rangle_{\rm ren}$ reaches its maximum value, which is when the rapid damping ends.

The mechanism for this instability is not completely understood. However, from Figs. 8 and 9 it can be seen that the exponential increase in the average value of $|\delta\phi/\phi|$ is tied to that of the average value of $|\delta\langle\psi^2\rangle/\langle\psi^2\rangle|$. Further, there is evidence that the average value of $|\delta f_k/f_k|$ undergoes a similar exponential increase during the same time period even for modes which have never undergone parametric amplification. Finally, it is clear from the plots in Figs. 4, 8, and 9 that the exponential increases occur at times when ϕ, $\langle\psi^2\rangle_{\rm ren}$, and, at least for the mode shown, f_k, are not increasing exponentially.

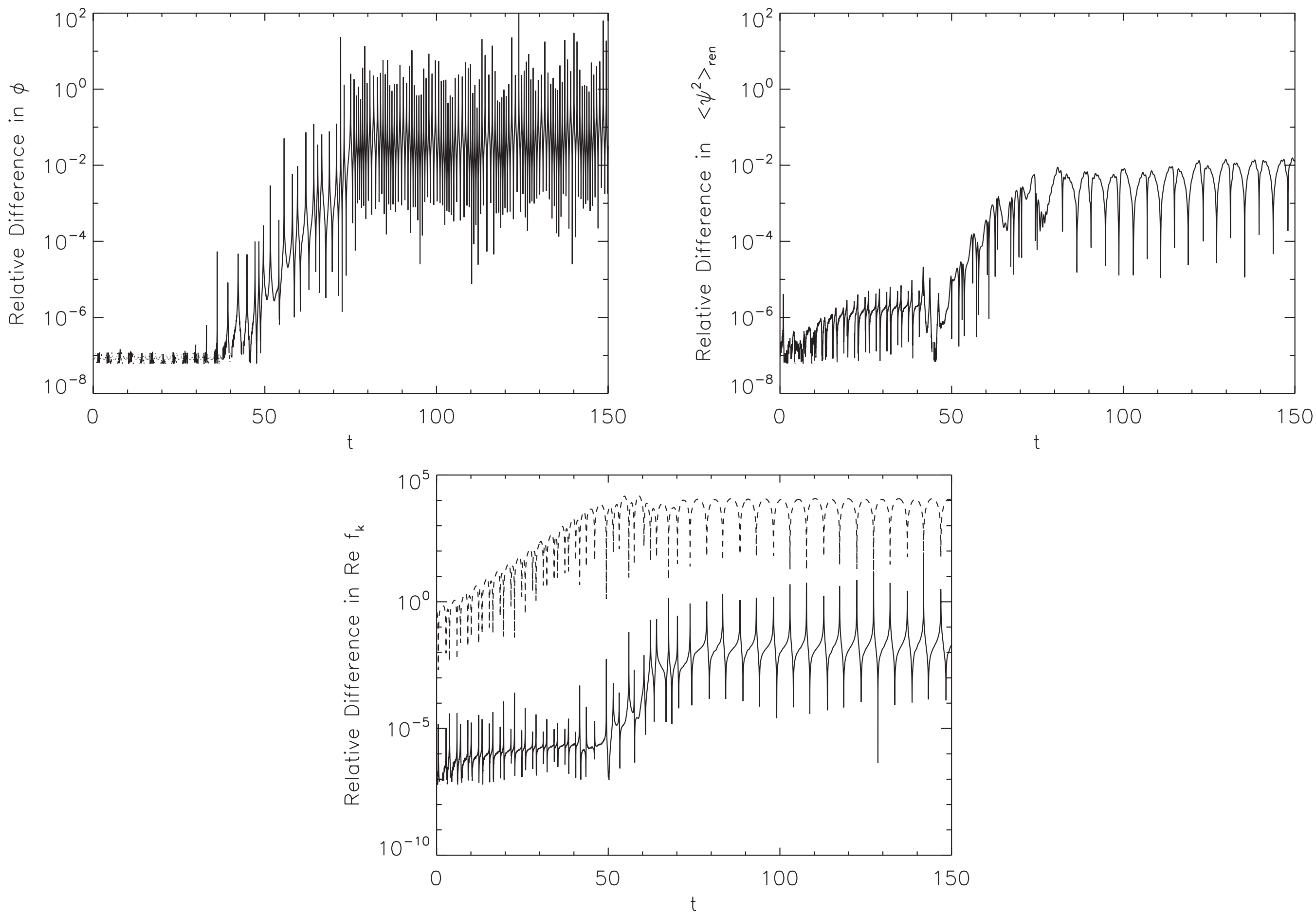

FIG. 8. The time evolution is shown for the case in which $g = 10^{-3}$ and $g^2\phi_0^2 = 10$ for one solution and $10(1 + 10^{-7})$ for the other. The top plots show the evolution of $|\delta\phi/\phi|$ and $|\delta\langle\psi^2\rangle_{\rm ren}/\langle\psi^2\rangle_{\rm ren}|$. The bottom plot shows the evolution of $|\delta{\rm Re}f_k/({\rm Re}f_k)|$ for $k = 0.5$. The dashed curve corresponds to $|{\rm Re}f_k|$ and is not a relative difference. Note that the growth of the relative difference for each quantity shuts off before its average becomes of order unity. The regions of the plots where the average evolves as a straight line with a nonzero slope correspond to times when the average relative differences are growing exponentially.

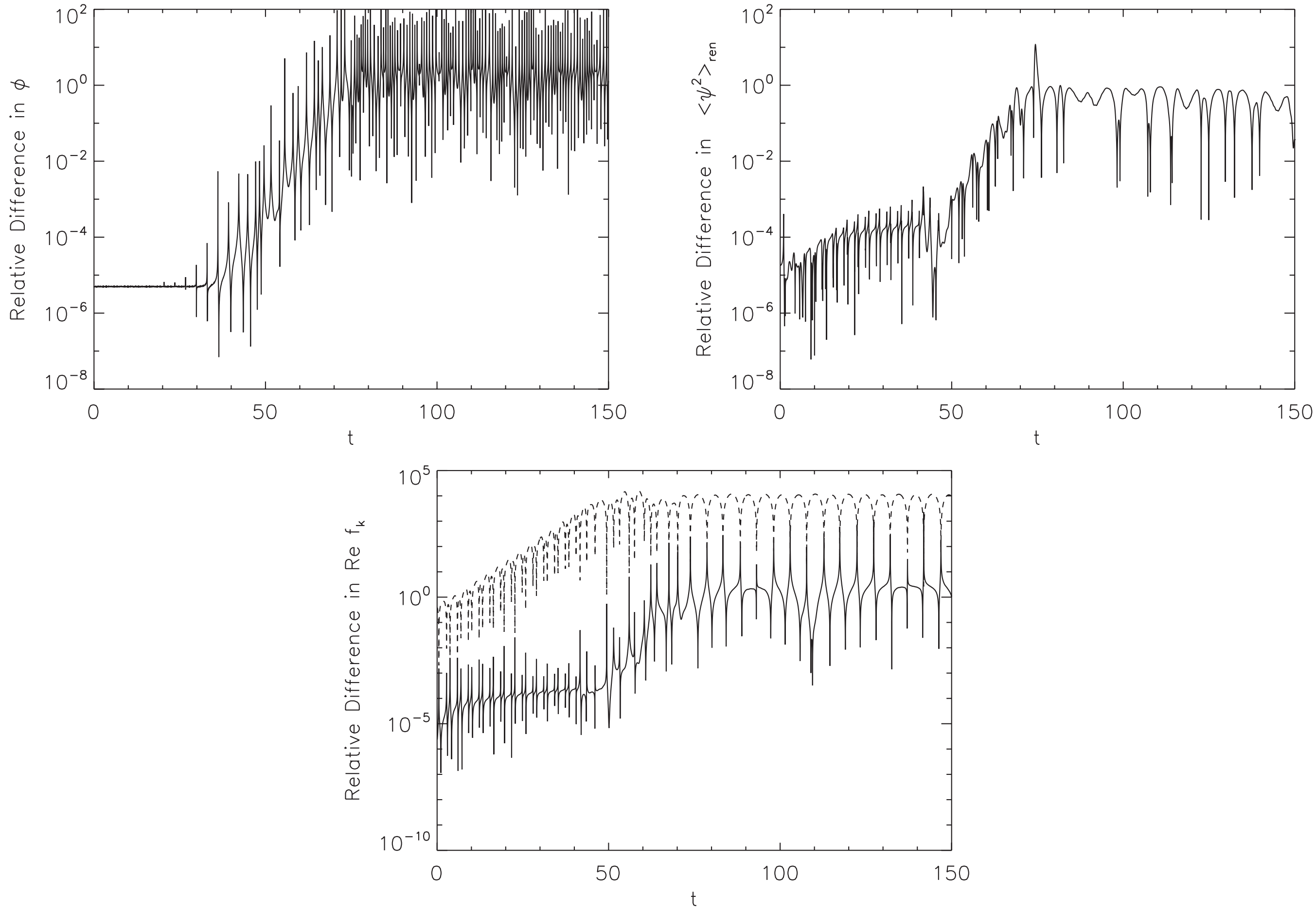

FIG. 9. The time evolution is shown for the case in which $g = 10^{-3}$ and $g^2\phi_0^2 = 10$ for one solution and $10(1 + 10^{-5})$ for the other. The top plots show the evolution of $|\delta\phi/\phi|$, $|\delta\langle\psi^2\rangle_{\rm ren}/\langle\psi^2\rangle_{\rm ren}|$. The bottom plot shows the evolution of $|\delta {\rm Re} f_k/({\rm Re} f_k)|$ for $k = 0.5$. The dashed curve corresponds to $|{\rm Re} f_k|$ and is not a relative difference. The regions of the plots where the average evolves as a straight line with a nonzero slope correspond to times when the average relative differences are growing exponentially.

To understand how this might occur, consider two cases, labeled by the subscripts 1 and 2 such that $\delta\phi = \phi_1 - \phi_2$. Then the equations for the evolution of $\delta\phi$ and δf_k are

$$\delta\ddot{\phi} = -(1 + g^2\langle\psi_1^2\rangle_{\rm ren})\delta\phi - g^2\delta\langle\psi^2\rangle_{\rm ren}\phi_2, \tag{4.12a}$$

$$\delta\ddot{f}_k = -(k^2 + g^2\phi_1^2)\delta f_k - (f_k)_2 g^2(\phi_1^2 - \phi_2^2) \approx -(k^2 + g^2\phi_1^2)\delta f_k + 2(f_k)_2 g^2\phi_1\delta\phi. \tag{4.12b}$$

Thus, it is clear that $\delta\langle\psi^2\rangle$ is a source for $\delta\phi$ and $\delta\phi$ is a source for δf_k. In this way, one can understand how the exponential increases could be correlated.

It is not clear what is driving the exponential increase or what stops it. However, it is clear that it starts when back-reaction effects from the quantum field on the inflaton field become very important, which is just before the rapid damping occurs. It is also clear that it stops once the rapid damping is finished. At this point $q_{\rm eff} \ll 1$.

This appears to be a different type of sensitivity to initial conditions than that found by KLS. Theirs is related to sensitivity of the evolution of the phase of ϕ and the value of the exponential parameter μ to absolute changes in the parameter q which are of order unity, but which correspond to small relative changes in q for large values of q. The sensitivity discussed here has been observed to occur for $0.5 \lesssim q \lesssim 25$. The only larger value of q that was investigated was $q = 10^4$. For $q = 10^4$ a sensitivity to initial conditions was observed which becomes important at earlier times than the ones we found for smaller values of q. Thus it seems likely that this is the sensitivity discussed by KLS. The sensitivity to initial conditions that we have found results in significant changes in the evolution of the inflaton field for changes in q which have been observed in some cases to be less than one part in 10^8, for values of q of order unity.

c. Post-rapid damping phase.—As mentioned in the Introduction, the model considered here does not include quantum fluctuations of the inflaton field and also does not include interactions between the created particles or between those particles and quantum fluctuations of the inflaton field. It would naturally be expected that scattering due to these interactions would become important at some point and that they would lead to the thermalization of the particles. Various studies which have taken such interactions into account [21,22,26,27,35] indicate that scattering effects will become important at intermediate times. In some cases this may occur before the rapid damping phase has ended [22,27]. Once the scattering becomes important the evolution should change significantly from that found using our model.

Nevertheless, it is of some interest to see what happens in our model at late times because of the insight it gives into the effects of particle production in the context of the semiclassical approximation. First, since quantum fluctuations of the inflaton field are ignored, one would expect that the damping of the inflaton field is due to the production of particles of the quantum fields. However, in the semiclassical approximation when the classical field is rapidly varying, different definitions of particle number can give different answers [52]. Even if one has a good definition, the particles may not behave like ordinary particles. This is the case here, as can be seen in the right-hand plot of Fig. 3. Recall that in Minkowski spacetime the total energy density is conserved. Then it is clear from the plot that, after the rapid damping phase, most of the energy is transferred back and forth between the inflaton field and the quantum field. Thus, the created particles have an energy density that is very different from what they would have if the inflaton field was slowly varying.

Also as mentioned previously, damping of the inflaton field is observed to be negligible for a large number of oscillations after the rapid damping phase has finished. However, one would expect that since the quantum fields are massless, particle production should continue to occur [53]. Examination of the behaviors of modes with various frequencies shows that there are some modes which do continue to grow during the post-rapid damping phase. One would expect that these modes would have $k^2 \sim g^2A^2/\omega_{\rm eff}^2 = 4q_{\rm eff}$. The left-hand plot of Fig. 10 shows that such growth does occur. However, it does not appear to be exponential, and this may be the reason that these modes have not grown enough to contribute significantly to $g^2\langle\psi^2\rangle_{\rm ren}$, even after a large number of oscillations of the inflaton field. For other modes, such as that in the plot on the right of Fig. 10, significant growth ceases after the rapid damping of the inflaton field. Finally, as would be expected for modes with large values of k, no significant growth in the amplitude of the oscillations was observed to occur.

V. DISCUSSION AND CONCLUSIONS

We have investigated preheating in chaotic inflation using the large N expansion for a classical inflaton field coupled to N identical massless quantum scalar fields in a Minkowski spacetime background. The inflaton field is minimally coupled to the gravitational field.

When backreaction effects on the inflaton field are ignored, the mode equations for the quantum fields are Mathieu equations. We have shown that, even though parametric amplification occurs in an infinite number of bands of modes with arbitrarily large frequencies, at any finite time quantities such as $\langle\psi^2\rangle_{\rm ren}$ and $\langle T_{\mu\nu}\rangle_{\rm ren}$ are finite due to the fact that the widths of the bands become negligible at very high frequencies and the maximum value of the exponential parameter μ in a band becomes very small at very high frequencies.

We have investigated the backreaction effects of the quantum fields on the inflaton field in detail by numerically solving the full set of equations that couple the inflaton field to the quantum field. These are the first such solutions to these equations for the model we are considering. We have found that, in agreement with the prediction by KLS, there is a period of rapid damping of the inflaton field if $g^2\phi_0^2 \gtrsim 2$ and none otherwise. In both cases the details of the damping can be understood through an analysis similar

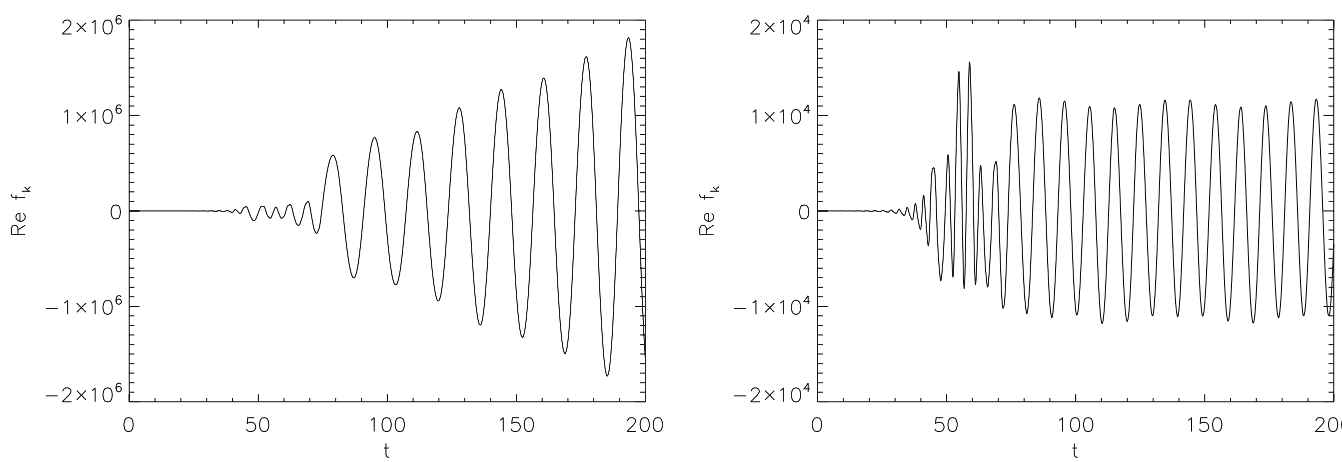

FIG. 10. Shown is the real part of two mode functions f_k for the case $g = 10^{-3}$, $g^2\phi_0^2 = 10$. For the left plot $k = 10^{-4}$ and for the right plot $k = 0.5$.

to that done by KLS; this analysis involves studying the way in which the instability bands change as the inflaton field damps. For $g^2\phi_0^2 \gtrsim 2$ the situation depends upon the locations of the instability bands, and therefore, there are different ways in which the damping proceeds. For $g^2\phi_0^2 \lesssim 1$ there is a brief period in which the amplitude of the inflaton field is rapidly damped but only by a very small amount. This is followed by a long period of slow damping. Thus, we distinguish this from the cases in which the inflaton field is rapidly damped by a significant amount.

The analysis by KLS shows that significant differences in the evolution occur for realistic values of the initial conditions and parameters when the expansion of the universe is taken into account. However, in cases where rapid damping of the inflaton field occurs, it is possible to neglect this expansion during the rapid damping phase. We find that during the rapid damping phase there is more damping of the inflaton field than was predicted originally by KLS. We also find that the rapid damping occurs in two bursts separated by a period of time which is generally longer than the time it takes for each period of rapid damping to occur. Finally, we find that there is an extreme sensitivity to initial conditions which manifests in significant changes in the evolution of the inflaton field and the modes of the quantum fields during the period of rapid damping. This can occur for changes in the initial amplitude of the inflaton field as small as, or even smaller than, one part in 10^8. This sensitivity is not well understood. Since it occurs for values of $g^2\phi_0^2 \gtrsim 2$, it would seem to be different than the one discussed by KLS, which only occurs for large values of $g^2\phi_0^2$.

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APPENDIX: RENORMALIZATION AND CONSERVATION OF THE ENERGY-MOMENTUM TENSOR

In this appendix the renormalization and conservation of the energy-momentum tensor for the system we are considering are discussed. As discussed in Sec. II, at leading order in a large N expansion, the system is equivalent to a massive classical scalar field coupled to a single massless quantum field. That in turn is equivalent to the system discussed in Ref. [47], if the scale factor $a(t)$ is set equal to 1, the coupling of the inflaton field to the scalar curvature is set to zero, the coupling constant λ is set equal to $2g^2$, the mass of the quantum field is set to zero, and the $\lambda\phi^3/3!$ term in the equation of motion for the ϕ field is dropped. Because of this, we will use several results from Ref. [47] in what follows.

We start with the quantum field theory in terms of bare parameters. The bare equations of motion for the homogeneous classical inflaton field and the quantum field modes are given in Eqs. (2.3) and (2.5), and an expression for the quantity $\langle\psi^2\rangle_{\rm B}$ is given in Eq. (2.11). It is assumed throughout that (i) the inflaton field is homogeneous and thus depends only on the time t and that (ii) the quantum field ψ is in a homogeneous and isotropic state so that $\langle\psi^2\rangle$ depends only on time, and $\langle T_{\mu\nu}\rangle$, when expressed in terms of the usual Cartesian components, is diagonal and depends only on time.

The total bare energy-momentum tensor of the system is given by [47]

$$T^B_{\mu\nu} = T^{C,B}_{\mu\nu} + \langle T_{\mu\nu}\rangle^{Q,B}, \tag{A1a}$$

$$T^{C,B}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi - \frac{m_B^2}{2}\eta_{\mu\nu}\phi^2, \tag{A1b}$$

$$\begin{aligned}\langle T_{\mu\nu}\rangle^{Q,B} = {}& (1-2\xi_B)\langle\partial_\mu\psi\partial_\nu\psi\rangle \\ & + \left(2\xi_B - \frac{1}{2}\right)\eta_{\mu\nu}\eta^{\sigma\tau}\langle\partial_\sigma\psi\partial_\tau\psi\rangle \\ & - 2\xi_B\langle\psi\partial_\mu\partial_\nu\psi\rangle + 2\xi_B\eta_{\mu\nu}\langle\psi\Box\psi\rangle \\ & - \frac{g_B^2\phi^2}{2}\eta_{\mu\nu}\langle\psi^2\rangle.\end{aligned} \tag{A1c}$$

We first show that the total energy-momentum tensor is covariantly conserved. Since it is diagonal and depends only on time, the conservation equation in Minkowski spacetime is trivially satisfied for all of the components of $T^B_{\mu\nu}$, except for T^B_{tt}. This component has the explicit form

$$T_{tt}^{C,B} = \frac{1}{2}\dot{\phi}^2 + \frac{m_B^2}{2}\phi^2, \tag{A2a}$$

$$\langle T_{tt}\rangle^{Q,B} = \frac{1}{4\pi^2}\int_0^{+\infty} dk k^2[|\dot{f}_k|^2 + (k^2 + g_B^2\phi^2)|f_k|^2]. \tag{A2b}$$

The conservation condition is

$$\partial_t T_{tt}^B = 0. \tag{A3}$$

It is easy to show, making use of the equations of motion for $\phi(t)$ and $f_k(t)$, that

$$\partial_t T_{tt}^{C,B} = -g_B^2\phi\dot{\phi}\langle\psi^2\rangle_B, \tag{A4a}$$

$$\partial_t\langle T_{tt}\rangle^{Q,B} = g_B^2\phi\dot{\phi}\langle\psi^2\rangle_B, \tag{A4b}$$

with $\langle\psi^2\rangle_B$ given by Eq. (2.11). This implies that the total bare energy-momentum tensor is covariantly conserved.

Renormalization is accomplished through the method of adiabatic regularization [40–43]. The details are given in Ref. [47]. The result for $\langle\psi^2\rangle_B$ is

$$\begin{aligned}\langle\psi^2\rangle_{\rm ren} &= \langle\psi^2\rangle_{\rm B} - \langle\psi^2\rangle_{\rm ad}\\ &= \frac{1}{2\pi^2}\int_0^{\epsilon} dk k^2\left(|f_k(t)|^2 - \frac{1}{2k}\right)\\ &\quad + \frac{1}{2\pi^2}\int_{\epsilon}^{+\infty} dk k^2\left(|f_k(t)|^2 - \frac{1}{2k} + \frac{g_R^2\phi^2}{4k^3}\right)\\ &\quad + \langle\psi^2\rangle_{\rm an},\end{aligned} \tag{A5a}$$

$$\langle\psi^2\rangle_{\rm an} = -\frac{g^2\phi^2}{8\pi^2}\left[1 - \log\left(\frac{2\epsilon}{M}\right)\right]. \tag{A5b}$$

For the energy-momentum tensor

$$\langle T_{tt}\rangle_{\rm ren} = T_{tt}^C + \langle T_{tt}\rangle_{\rm ren}^Q = T_{tt}^C + \langle T_{tt}\rangle_B^Q - \langle T_{tt}\rangle_{\rm ad}^Q, \tag{A6a}$$

with

$$T_{tt}^C = \frac{1}{2}\dot{\phi}^2 + \frac{m^2}{2}\phi^2, \tag{A6b}$$

$$\begin{aligned}\langle T_{tt}\rangle_{\rm ren}^Q &= \frac{1}{4\pi^2}\int_0^{+\infty} dk k^2[|\dot{f}_k|^2 + (k^2 + g^2\phi^2)|f_k|^2]\\ &\quad - \frac{1}{4\pi^2}\int_0^{\epsilon} dk k^2\left(k + \frac{g^2\phi^2}{2k}\right) - \frac{1}{4\pi^2}\\ &\quad \times\int_{\epsilon}^{+\infty} dk k^2\left(k + \frac{g^2\phi^2}{2k} - \frac{g^4\phi^4}{8k^3}\right) + \langle T_{tt}\rangle_{\rm an}^Q,\end{aligned} \tag{A6c}$$

$$\langle T_{tt}\rangle_{\rm an}^Q = -\frac{g^4\phi^4}{32\pi^2}\left[1 - \log\left(\frac{2\varepsilon}{M}\right)\right]. \tag{A6d}$$

We note that in Minkowski spacetime the bare and the adiabatic energy density of the quantum field do not depend on the value of the coupling ξ of the quantum field to the scalar curvature. Thus, the renormalized value of the energy density does not depend on ξ.

Using Eqs. (A5) and (A6), and the equations of motion for ϕ and f_k, it is easy to derive the following identities:

$$\partial_t T_{tt}^C = -g^2\phi\dot{\phi}\langle\psi^2\rangle_{\rm ren}, \tag{A7a}$$

$$\partial_t\langle T_{tt}\rangle_{\rm B}^Q = g^2\phi\dot{\phi}\langle\psi^2\rangle_{\rm B}, \tag{A7b}$$

$$\partial_t\langle T_{tt}\rangle_{\rm ad}^Q = g^2\phi\dot{\phi}\langle\psi^2\rangle_{\rm ad}. \tag{A7c}$$

They can be used to show that

$$\begin{aligned}\partial_t\langle T_{tt}\rangle_{\rm ren} &= \partial_t\langle T_{tt}\rangle^C + \partial_t\langle T_{tt}\rangle_{\rm B}^Q - \partial_t\langle T_{tt}\rangle_{\rm ad}^Q\\ &= g^2\phi\dot{\phi}[-\langle\psi^2\rangle_{\rm ren} + \langle\psi^2\rangle_{\rm B} - \langle\psi^2\rangle_{\rm ad}] = 0.\end{aligned} \tag{A8}$$

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