PHYSICAL REVIEW D **78,** 076010 (2008)

Studies on $\pi^+\pi^-$ phase motion in the $\Psi' \to J/\Psi\pi^+\pi^-$ process

Z. X. Zhang,[1,2] J. J. Sanz-Cillero,[3] X. Y. Shen,[2] N. Wu,[2] L. Y. Xiao,[4] and H. Q. Zheng[1]

[1]*Department of Physics, Peking University, Beijing 100871, People's Republic of China*

[2]*Institute of High Energy Physics, Chinese Academy of Science, Beijing 100049, People's Republic of China*

[3]*IFAE, Universitat Autonoma de Barcelona, 08193 Bellaterra, Barcelona, Spain*

[4]*Olivet Institute of Technology, Olivet University, San Francisco, California 94103, USA*

(Received 20 May 2008; revised manuscript received 25 August 2008; published 31 October 2008)

We propose a measurement of the elastic $\pi\pi$ scattering phase-shift difference $\delta_0^0 - \delta_0^2$ through the $\Psi' \to J/\Psi\pi^+\pi^-$ process in the future high statistics BES-III experiment. The decay amplitude is constructed with seven Lorentz invariant form factors and is compared with their theoretical estimation. Based on a Monte Carlo study, it is found that the phase-shift difference can be indeed recovered and, hence, it is expected to be measured in the energy region between 350 and 550 MeV at future BES-III.

DOI: 10.1103/PhysRevD.78.076010 PACS numbers: 11.80.Et, 13.20.Gd, 13.75.Lb

I. INTRODUCTION

In recent years, a number of high precision, high statistics experimental machines, varying from fixed target experiments to B factories, has opened a new era of precision hadronic experiments. Based on that, both experimental and theoretical studies on low energy $\pi\pi$ and πK system from production processes have also received revived interest. The importance of these studies follows from the fact that when the final state theorem applies, one can extract low energy elastic scattering phase shifts in the related channels through a partial-wave analysis. The information on the $\pi\pi$ and πK phase shifts then provides a crucial ingredient for understanding the dynamics of the Goldstone bosons and the spontaneous breaking of chiral symmetry.

The experimental and theoretical activities in the last few years have been mainly focused on semileptonic and hadronic D decays (see, for example, Refs. [1–4]). It is known that in D semileptonic decays the p wave dominates, and the more interesting s-wave component is small. In this paper we reinvestigate the $\pi\pi$ final state interactions in the $\Psi' \to J/\Psi\pi^+\pi^-$ process. Here the s wave dominates and the next contribution comes from the tiny d wave. The existence of the latter is, however, crucial for exploring the s-wave phase motion through interference effects. The decay product under concern is a three body final state. The J/Ψ particle is, however, irrelevant to any final state interactions here. Because of color transparency, the effect from rescatterings between the J/Ψ and one of the pions is expected to be negligible. Another important fact is that, in the kinematic region under concern, between the initial Ψ' and the final $J/\Psi\pi\pi$ there is no other on-shell intermediate hadronic state available (or it is doubly the Okubo-Zweig-Iizuka suppressed and, hence, negligible). Thus, the final state theorem is applicable to the $\pi\pi$ system in the $\Psi' \to J/\Psi\pi^+\pi^-$ process.

This decay has been the subject of a number of previous publications [5–7]. In Ref. [8], the author proposed a method to extract the $\pi\pi$ phase shift from $\Psi' \to J/\Psi\pi^+\pi^-$—similar to the Pais-Treiman method [9] for obtaining the $\pi\pi$ phase shifts from K_{l4} decays—but only three partial-wave amplitudes were considered in order to reduce the difficulty of the analysis. A similar method [10] was also proposed for the $\Upsilon(3S) \to \Upsilon(1S)\pi\pi$ process but only the lowest order in the pion momentum expansion was considered. In this work we are able to provide a more general parametrization to the decay amplitude and compare it with previous works. Our parametrization will be discussed in Sec. II. Furthermore we will also provide a Monte Carlo study in Sec. III to test the stability and reliability of our parametrization in order to extract the phase-shift difference.

II. GENERAL STRUCTURE OF THE $\Psi' \to J/\Psi\pi^+\pi^-$ DECAY AMPLITUDE

A. The Lorentz invariant form factors

There are 3 independent momenta p_{π^+}, p_{π^-}, $p_{J/\Psi} = p_3$, which can be reexpressed in 3 variables $q = p_{\pi^+} + p_{\pi^-}$, $p = p_{\pi^+} - p_{\pi^-}$, and p_3. The three independent momenta can form 2 independent Lorentz invariant products, chosen as q^2 and $p \cdot p_3$ here. Then,

$$q^2 = s, \tag{1}$$

$$p^2 = -s\rho^2 = 4m_\pi^2 - s, \tag{2}$$

$$p_3^2 = M_\Psi^2, \tag{3}$$

$$q \cdot p = 0, \tag{4}$$

$$q \cdot p_3 = \frac{1}{2}(M_{\Psi'}^2 - M_\Psi^2 - s), \tag{5}$$

with the kinematical factor of the di-pion system $\rho = \sqrt{1 - \frac{4m_\pi^2}{s}}$, and the J/Ψ energy and three-momentum in the lab frame (the Ψ' rest frame), respectively $E_3 = (M_{\Psi'}^2 + M_\Psi^2 - s)/(2M_{\Psi'})$ and $|\vec{p}_3| = \sqrt{E_3^2 - M_\Psi^2}$,

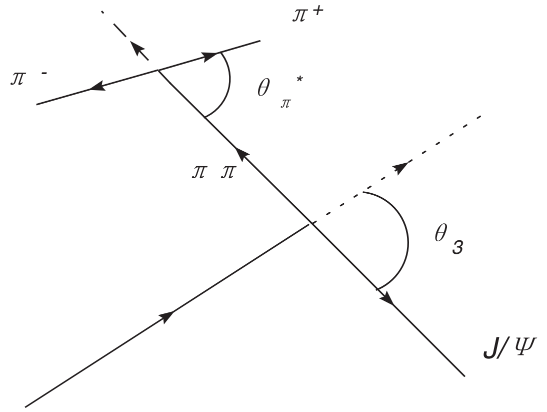

FIG. 1. θ_3 is the angle between the beam and J/Ψ in the Ψ' rest frame. θ^*_π is the angle between the J/Ψ direction and the π^+ in the di-pion rest frame, ϕ is the azimuthal angle between the beam J/Ψ plane and $\pi^+\pi^-$ plane in the Ψ' rest frame (not drawn in the figure).

being functions of q^2. Moreover, $p \cdot p_3$ can be expressed in the Ψ' rest frame in terms of variables of the di-pion rest frame,

$$p \cdot p_3 = 2\gamma|\vec{p}^*_{\pi\pi}|\cos\theta^*_\pi(\beta E_3 + |\vec{p}_3|), \tag{6}$$

where $|\vec{p}^*_{\pi\pi}| = \rho\sqrt{s}/2$ is the pion three momentum in the di-pion rest frame and θ^*_π is the angle between the direction of π^+ and direction opposite to the final J/Ψ in the di-pion rest frame (see Fig. 1 for illustration). $\beta = \sqrt{1 - \frac{1}{\gamma^2}}$ is the boost factor from the di-pion rest frame to the lab frame with $\gamma = (M^2_{\Psi'} - M^2_\Psi + s)/(2M_{\Psi'}\sqrt{s})$.

Denoting the polarization vector of Ψ' and J/Ψ by ϵ' and ϵ respectively, we can form five invariants bilinear in $\epsilon'\epsilon^*$:

$$(\epsilon \cdot \epsilon'), \qquad (\epsilon \cdot q)(\epsilon' \cdot q), \qquad (\epsilon \cdot p)(\epsilon' \cdot p),$$
$$(\epsilon \cdot q)(\epsilon' \cdot p), \qquad (\epsilon \cdot p)(\epsilon' \cdot q).$$

Hence, the amplitude have five independent structure:

$$T = (\epsilon \cdot \epsilon')F_0 + (\epsilon \cdot q)(\epsilon' \cdot q)F_1 + (\epsilon \cdot p)(\epsilon' \cdot p)F_2 + (\epsilon \cdot q)(\epsilon' \cdot p)F_3 + (\epsilon \cdot p)(\epsilon' \cdot q)F_4. \tag{7}$$

The F_i are functions of s and $p \cdot p_3$, allowing us to do a partial-wave decomposition in terms of the angle θ^*_π.

B. Partial-wave decomposition

From Ref. [11], we obtain the basis of tensors

$$\tilde{t}^{(0)} = 1, \tag{8}$$

$$\tilde{t}^{(1)} = p^\mu, \tag{9}$$

$$\tilde{t}^{(2)} = p^\mu p^\nu - \frac{1}{3}p^2\tilde{g}^{\mu\nu}, \quad \text{with} \quad \tilde{g}^{\mu\nu} = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}, \tag{10}$$

where every tensor $\tilde{t}^{(L)}$ transforms irreducibly as a tensor of spin L. In the present problem we have the four vectors, q^μ, p_3^μ, ϵ'^μ and ϵ^μ, which are independent on θ^*_π. We can build the independent Lorentz scalars:

$$q^2, q \cdot \epsilon, q \cdot \epsilon' \quad \text{and} \quad \epsilon \cdot \epsilon'.$$

The available Lorentz vectors would be

$$q^\mu, \qquad p_3^\mu, \qquad \epsilon'^\mu, \qquad \epsilon^\mu.$$

With these ingredients we build first the $L = 0$ quantities. This can only be obtained through $\tilde{t}^{(0)}$ and taking into account that the polarization vectors ϵ' and ϵ must be always contracted at the end of the day,

$$S = \tilde{t}^{(0)} \cdot \{I_1^{(0)}(s)(\epsilon' \cdot \epsilon) + I_2^{(0)}(s)(\epsilon' \cdot q)(\epsilon \cdot q)\}, \tag{11}$$

which can be expressed through $S = \epsilon'^\mu\epsilon^\nu S_{\mu\nu}$, with $S_{\mu\nu} = I_1(s)g_{\mu\nu} + I_2(s)q_\mu q_\nu/s$.

The d wave is more complicated, since we have to use $\tilde{t}^{\mu\nu}_{(2)}$ and the number of contractions gets larger. The available four-vectors, q^μ, p_3^μ, ϵ'^μ, ϵ^μ, are then contracted with $\tilde{t}^{\mu\nu}_{(2)}$ in all the different possible ways:

$$\begin{aligned} D = \tilde{t}^{\mu\nu}_{(2)} \cdot \{&I_3^{(0)}(s)(\epsilon' \cdot \epsilon)q^\mu q^\nu + I_4^{(0)}(s)(\epsilon' \cdot \epsilon)p_3^\mu p_3^\nu + I_5^{(0)}(s)(\epsilon' \cdot \epsilon)(p_3^\mu q^\nu + q^\mu p_3^\nu) + I_6^{(0)}(s)(\epsilon' \cdot q)(\epsilon \cdot q)q^\mu q^\nu \\ &+ I_7^{(0)}(s)(\epsilon' \cdot q)(\epsilon \cdot q)p_3^\mu p_3^\nu + I_8^{(0)}(s)(\epsilon' \cdot q)(\epsilon \cdot q)(p_3^\mu q^\nu + q^\mu p_3^\nu) + I_9^{(0)}(s)(\epsilon \cdot q)(\epsilon'^\mu q^\nu + q^\mu\epsilon'^\nu) \\ &+ I_{10}^{(0)}(s)(\epsilon \cdot q)(\epsilon'^\mu p_3^\nu + p_3^\mu\epsilon'^\nu) + I_{11}^{(0)}(s)((\epsilon' \cdot q)(\epsilon^\mu q^\nu + q^\mu\epsilon^\nu)) + I_{12}^{(0)}(s)(\epsilon' \cdot q)(\epsilon^\mu p_3^\nu + p_3^\mu\epsilon^\nu) \\ &+ I_{13}^{(0)}(s)(\epsilon'^\mu\epsilon^\nu + \epsilon^\mu\epsilon'^\nu)\}. \end{aligned} \tag{12}$$

It is not difficult to realize that there are not actually so many independent Lorentz structure. Thus, it can be rewritten in a more compact way through $D = \epsilon'^\mu\epsilon^\nu D'_{\mu\nu}$, with

$$D'^{\mu\nu} = I'_3(s)\frac{1}{s^2}g^{\mu\nu}\left[(p_3\cdot p)^2 - \frac{1}{3}p^2(\tilde{p}_3)^2\right] + I'_4(s)\frac{1}{s^3}q^\mu q^\nu\left[(p_3\cdot p)^2 - \frac{1}{3}p^2(\tilde{p}_3^2)\right] + I'_5(s)\frac{1}{s^2}\left[q^\mu p^\nu(p_3\cdot p) - \frac{1}{3}p^2 q^\mu \tilde{p}_3^\nu\right] + I'_6(s)\frac{1}{s^2}\left[p^\mu q^\nu(p_3\cdot p) - \frac{1}{3}p^2\tilde{p}_3^\mu q^\nu\right] + I'_7(s)\frac{1}{s}\left[p^\mu p^\nu - \frac{1}{3}p^2\tilde{g}^{\mu\nu}\right]. \tag{13}$$

In order to avoid that any form factor becomes artificially large or small, we extract Lorentz structures that are numerically order one. The amplitudes are then given by

$$S^{\mu\nu} = I_1(s)g^{\mu\nu} + I_2(s)\frac{1}{s}q^\mu q^\nu, \tag{14}$$

$$D^{\mu\nu} = I_3(s)g^{\mu\nu}\left[\cos^2\theta^*_\pi - \frac{1}{3}\right] + I_4(s)\frac{1}{s}q^\mu q^\nu\left[\cos^2\theta^*_\pi - \frac{1}{3}\right] + I_5(s)\frac{1}{s^{3/2}M_\Psi}\left[q^\mu p^\nu(p_3\cdot p) - \frac{1}{3}p^2 q^\mu\tilde{p}_3^\nu\right] + I_6(s)\frac{1}{s^{3/2}M_\Psi}\left[p^\mu q^\nu(p_3\cdot p) - \frac{1}{3}p^2\tilde{p}_3^\mu q^\nu\right] + I_7(s)\frac{1}{s}\left[p^\mu p^\nu - \frac{1}{3}p^2\tilde{g}^{\mu\nu}\right]. \tag{15}$$

Until this point the derivation is completely general. Now, we are going to make the main assumption: we will consider that no further rescattering occurs between the J/Ψ and the $\pi\pi$ system. Hence, the phase shift of the amplitude is due to the $\pi\pi$ final state interaction. This allows us to use the Watson theorem for the elastic scattering region (from the practical point of view, up to the $K\bar{K}$ threshold). The decay amplitude can be then decomposed into partial waves ($S, D\ldots$) with their phase shifts equal to those in $\pi\pi$ scattering (respectively, δ_0, $\delta_2\ldots$):

$$T = \epsilon'^\mu\epsilon^\nu[S_{\mu\nu}e^{i\delta_0} + D_{\mu\nu}e^{i\delta_2}]. \tag{16}$$

The D wave is supposed to be suppressed with respect to the $J = 0$ component and higher partial waves are neglected.

C. Theoretical estimation to the leading contributions

Starting from the effective Lagrangian in Ref. [12], based on chiral and heavy quark symmetries, for soft pion momenta the amplitude shows the form,

$$\mathcal{A}(\Psi' \to \Psi\pi^+\pi^-) = -\frac{4}{F_0^2}\left\{\left[\frac{g}{2}(q^2 - 2m_\pi^2) + g_1 E_{\pi^+}E_{\pi^-}\right]\epsilon'\cdot\epsilon + g_2[p^\mu_{\pi^+}p^\nu_{\pi^-} + p^\mu_{\pi^-}p^\nu_{\pi^+}]\epsilon'_\mu\epsilon_\nu\right\}, \tag{17}$$

where $p_{\pi^\pm} = (E_{\pi^\pm}, \vec{p}_{\pi^\pm})$ in the Ψ' rest frame. We can then calculate the leading contributions to our form factors $I_i(s)$:

$$I_1(s) = -\frac{4}{F_0^2}\left[\frac{g}{2}(s - 2m_\pi^2) + \frac{sg_1\gamma^2}{4}\left(1 - \frac{\rho(s)^2}{3}\beta^2\right) + g_2\frac{s\rho(s)^2}{6}\right], \tag{18}$$

$$I_2(s) = -\frac{2s}{F_0^2}g_2\left(1 - \frac{\rho(s)^2}{3}\right), \tag{19}$$

$$I_3(s) = \frac{1}{F_0^2}g_1\rho(s)^2 s\beta^2\gamma^2, \tag{20}$$

$$I_7(s) = \frac{2s}{F_0^2}g_2, \tag{21}$$

with the remaining ones vanishing at leading order. This calculation suggests that the form factors $I_4(s)$, $I_5(s)$, $I_6(s)$ are small quantities. This theoretical prediction can be checked in future experiments as, in principle, the data can also determine the $I_i(s)$.

D. Expressions for angular distribution

For three body decays, one has the differential decay rate [13],

$$d\Gamma = \frac{1}{(2\pi)^5}\frac{1}{16M^2}|\mathcal{M}|^2|\vec{p}^*_{\pi\pi}||\vec{p}_3|dsd\Omega^*_{\pi\pi}d\Omega_{J/\Psi}, \tag{22}$$

where $(\vec{p}^*_{\pi\pi}, \Omega^*_{\pi\pi})$ is the π^+ momentum and angle in the di-pion rest frame, and $\Omega_{J/\Psi}$ is the J/Ψ angle in the Ψ' rest frame. It can be expressed in the form

$$\frac{d\Gamma}{dsd\cos\theta^*_\pi d\cos\theta_3 d\phi^*_\pi} \propto \sum_{klm}G_{klm}(s)\cos^k\phi^*_\pi\cos^l\theta_3\cos^m\theta^*_\pi + \cos\phi^*_\pi\sin\theta_3\cos\theta_3\sin\theta^*_\pi\cos\theta^*_\pi(\tilde{G}_0(s) + \tilde{G}_2(s)\cos^2\theta^*_\pi), \tag{23}$$

where k, $l = 0, 2$, and $m = 0, 2, 4$. The $G_{klm}(s)$ are known functions of the $I_i(s)$ and $\cos(\delta_0 - \delta_2)$.[1]

If one can determine $G_{klm}(s)$ experimentally, then it is possible to extract both the $I_i(s)$ and the phase-shift difference $\delta_0 - \delta_2$.

Alternatively, this information can be found in the partial distributions,

$$\frac{d^2\Gamma}{dsd\cos\theta_\pi^*} = A_0(s) + A_2(s)\cos^2\theta_\pi^* + A_4(s)\cos^4\theta_\pi^*, \quad (24)$$

$$\frac{d^2\Gamma}{dsd\cos\theta_3} = B_0(s) + B_2(s)\cos^2\theta_3, \quad (25)$$

$$\frac{d^2\Gamma}{dsd\cos\phi} = C_0(s) + C_2(s)\cos^2\phi, \quad (26)$$

and another weighted distribution,

$$\begin{aligned} W[s,\cos\theta_\pi^*] &= \int_{-\pi}^{+\pi} d\phi_\pi^* \int_{-1}^{1} d\cos\theta_3 \frac{d\Gamma}{dsd\cos\theta_\pi^* d\cos\theta_3 d\phi_\pi^*}\cos\phi_\pi^*\cos\theta_3 \\ &\propto \int_{-\pi}^{+\pi} d\phi_\pi^* \int_{-1}^{1} d\cos\theta_3 \cos^2\phi_\pi^* \sin\theta_3\cos^2\theta_3 \sin\theta_\pi^*\cos\theta_\pi^*(\tilde{G}_0(s) + \tilde{G}_2(s)\cos^2\theta_\pi^*) \\ &= \sin\theta_\pi^*\cos\theta_\pi^*(\tilde{G}_0(s) + \tilde{G}_2(s)\cos^2\theta_\pi^*)\int_{-\pi}^{+\pi} d\phi_\pi^* \int_{-1}^{1} d\cos\theta_3 \cos^2\phi_\pi^*\sin\theta_3\cos^2\theta_3 \\ &= \frac{\pi^2}{4}\sin\theta_\pi^*\cos\theta_\pi^*(\tilde{G}_0(s) + \tilde{G}_2(s)\cos^2\theta_\pi^*). \end{aligned} \quad (27)$$

$A_i(s)$, $B_i(s)$, $C_i(s)$ and $\tilde{G}_i$ are functions of $I_i(s)$ and $\cos(\delta_0(s) - \delta_2(s))$. If those are fitted precisely, it is not be difficult to recover the values of the I_i and $\delta_0 - \delta_2$ at a given energy.

The $A_i(s)$, $B_i(s)$, $C_i(s)$ can also be written as combinations of G_{klm}:

$$A_0 = \frac{2\pi}{3}(6G_{000} + 2G_{020} + 3G_{200} + G_{220}), \quad (28)$$

$$A_2 = \frac{2\pi}{3}(6G_{002} + 2G_{022} + 3G_{202} + G_{222}), \quad (29)$$

$$A_4 = \frac{2\pi}{3}(6G_{004} + 2G_{024} + 3G_{204} + G_{224}), \quad (30)$$

$$B_0 = \frac{2\pi}{15}(30G_{000} + 10G_{002} + 6G_{004} + 15G_{200} + 5G_{202} + 3G_{204}), \quad (31)$$

$$B_2 = \frac{2\pi}{15}(30G_{020} + 10G_{022} + 6G_{024} + 15G_{220} + 5G_{222} + 3G_{224}), \quad (32)$$

$$C_0 = \frac{4}{45}(45G_{000} + 15G_{002} + 9G_{004} + 15G_{020} + 5G_{022} + 3G_{024}), \quad (33)$$

$$C_2 = \frac{4}{45}(45G_{200} + 15G_{202} + 9G_{204} + 15G_{220} + 5G_{222} + 3G_{224}). \quad (34)$$

III. EFFICIENCY CORRECTIONS

The partial and weighted distributions in Eqs. (24)–(27) were based on a theoretical integration of the partial decay rate over different angular variables. However, the experimental situation is slightly different from this. In general, the detector is not able to cover the whole solid angle and, moreover, the detection efficiency is not the same in all directions but it is a rather complicate function $w(\Omega)$.[2] The partial decay rate detected in the experimental analysis is not that in Eq. (23) but the efficiency corrected one,

$$\frac{d\Gamma}{dsd\cos\theta_\pi^* d\cos\theta_3 d\phi_\pi^*}\bigg|_{\rm det} = w(\Omega) \times \frac{d\Gamma}{dsd\cos\theta_\pi^* d\cos\theta_3 d\phi_\pi^*}. \quad (35)$$

The calculation of the corrected functions corresponding to the distributions in Eqs. (24)–(27) is more tedious but it does not introduce any important complication. In order to ease the understanding of the procedure, we present a detailed calculation for $d^2\Gamma/dsd\cos\theta_\pi^*$. We integrate the detected partial rate in Eq. (35) over θ_3 and ϕ_π^* and we integrate separately every monomial

[1]The Mathematica notebook and fortran programs can be obtained from zxzhang@mail.ihep.ac.cn, xiaoly@pku.edu.cn and cillero@ifae.es. Detailed formula can be found in arXiv:0805.2780v1.

[2]Although *a priori* we will assume $w(\Omega) = w(\theta_\pi^*, \phi_\pi^*, \theta_3)$, notice that for asymmetric detectors the efficiency could also depend on the azimuth angle ϕ_3

$\cos^k\phi_\pi^*\cos^l\theta_3\cos^m\theta_\pi^*$:

$$\frac{d^2\Gamma}{dsd\cos\theta_\pi^*}\bigg|_{\rm det} = \sum_{k,l,m} G_{klm}(s)\int d\cos\theta_3 d\phi_\pi^* w(\Omega)\cos^k\phi_\pi^*\cos^l\theta_3\cos^m\theta_\pi^* + \tilde{G}_0(s)\int d\cos\theta_3 d\phi_\pi^* w(\Omega)\cos\phi_\pi^*\sin\theta_3\cos\theta_3\sin\theta_\pi^*\cos\theta_\pi^* + \tilde{G}_2(s)\int d\cos\theta_3 d\phi_\pi^* w(\Omega)\cos\phi_\pi^*\sin\theta_3\cos\theta_3\sin\theta_\pi^*\cos^3\theta_\pi^*. \quad (36)$$

Since the integral is on θ_3 and ϕ_π^*, it is possible to reexpress it in the form

$$\frac{d^2\Gamma}{dsd\cos\theta_\pi^*}\bigg|_{\rm det} = \sum_{m=0,2,4} A_m cos^m\theta_\pi^* + \sin\theta_\pi^*\cos\theta_\pi^*[\tilde{A}_0 + \tilde{A}_2\cos^2\theta_\pi^*], \quad (37)$$

where we have defined a new set of coefficients A_i, $\tilde{A}_j$ given by

$$A_m = \sum_{k,l=0,2} G_{klm}(s)\int d\cos\theta_3 d\phi_\pi^* w(\Omega)\cos^k\phi_\pi^*\cos^l\theta_3, \quad (38)$$

$$\tilde{A}_0 = \tilde{G}_0(s)\int d\cos\theta_3 d\phi_\pi^* w(\Omega)\cos\phi_\pi^*\sin\theta_3\cos\theta_3, \quad (39)$$

$$\tilde{A}_2 = \tilde{G}_2(s)\int d\cos\theta_3 d\phi_\pi^* w(\Omega)\cos\phi_\pi^*\sin\theta_3\cos\theta_3. \quad (40)$$

In the case of perfect efficiency, $w(\Omega) = 1$, one finds $\tilde{A}_0 = \tilde{A}_2 = 0$ and the different A_i become those provided in Eqs. (28)–(34). The dependence on s is implicitly assumed. Furthermore, if the efficiency depends on θ_π^* then the coefficients A_i, $\tilde{A}_j$ are also functions of this angle. In this case, when analyzing the experimental data one should compute these integrals for every θ_π^*. The simplest procedure to evaluate these integrals is through the Monte Carlo method, where we have for instance

$$\int d\cos\theta_3 d\phi_\pi^* w(\Omega)\cos^k\phi_\pi^*\cos^l\theta_3 \simeq \frac{1}{N_{\rm tot}}\sum_{a=1}^{N_{\rm MC}}\cos^k\phi_{\pi,a}^*\cos^l\theta_{3,a}. \quad (41)$$

In the case of a θ_π^*-dependent efficiency, this integral also depends on this angle and it must be repeated for every point in the fit analysis.

Through a similar procedure, one also recovers the detected distributions corresponding to those in Eqs. (25)–(27):

$$\frac{d^2\Gamma}{dsd\cos\theta_3}\bigg|_{\rm det} = \sum_{l=0,2} B_l\cos^l\theta_3 + \tilde{B}\sin\theta_3\cos\theta_3, \quad (42)$$

$$\frac{d^2\Gamma}{dsd\cos\phi_\pi^*}\bigg|_{\rm det} = \sum_{k=0,2} C_k\cos^k\phi_\pi^* + \tilde{C}\cos\phi_\pi^*, \quad (43)$$

$$W[s,\theta_\pi^*]|_{\rm det} = \sum_{m=0,2,4} W_k\cos^m\theta_\pi^* + \sin\theta_\pi^*\cos\theta_\pi^*[\tilde{W}_0 + \tilde{W}_2\cos^2\theta_\pi^*], \quad (44)$$

with the coefficients

$$B_l = \sum_{k=0,2}\sum_{m=0,2,4} G_{klm}(s) \times \int d\cos\theta_\pi^* d\phi_\pi^* w(\Omega)\cos^k\phi_\pi^*\cos^m\theta_\pi^*, \quad (45)$$

$$\tilde{B} = \int d\cos\theta_\pi^* d\phi_\pi^* w(\Omega)\cos\phi_\pi^*\sin\theta_\pi^* \times \cos\theta_\pi^*[\tilde{G}_0(s) + \tilde{G}_2(s)\cos^2\theta_\pi^*], \quad (46)$$

$$C_k = \sum_{l=0,2}\sum_{m=0,2,4} G_{klm}(s) \times \int d\cos\theta_\pi^* d\cos\theta_3 w(\Omega)\cos^l\theta_3\cos^m\theta_\pi^*, \quad (47)$$

$$\tilde{C} = \int d\cos\theta_\pi^* d\cos\theta_3 w(\Omega) \times \sin\theta_3\cos\theta_3\sin\theta_\pi^*\cos\theta_\pi^*[\tilde{G}_0(s) + \tilde{G}_2(s)\cos^2\theta_\pi^*], \quad (48)$$

$$W_m = \sum_{k,l=0,2} G_{klm}(s) \times \int d\cos\theta_3 d\phi_\pi^* w(\Omega)\cos^{k+1}\phi_\pi^*\cos^{l+1}\theta_3, \quad (49)$$

$$\tilde{W}_0 = \tilde{G}_0(s) \int d\cos\theta_3 d\phi^*_\pi w(\Omega) \cos^2\phi^*_\pi \sin\theta_3 \cos^2\theta_3, \tag{50}$$

$$\tilde{W}_2 = \tilde{G}_2(s) \int d\cos\theta_3 d\phi^*_\pi w(\Omega) \cos\phi^*_\pi \sin\theta_3 \cos^2\theta_3. \tag{51}$$

As it happened before, for a general efficiency $w(\Omega)$, these coefficients are not simply functions of the energy but they also have a residual dependence on the corresponding angle.

IV. MONTE CARLO STUDY

Clear signals of σ and κ have been found in Beijing spectrometer (BES) data [6,14,15]. These studies have tried to measure the $\pi\pi/\pi K$ S-wave phase shift but, due to the limited statistics, no meaningful results were obtained. In the $J/\Psi \to \omega\pi\pi$ channel, there are resonances in the $\omega\pi$ spectrum, which affect the S-wave phase in the $\pi\pi$ spectrum. Its contribution to the $\pi\pi$ S-wave phase shift is hard to be estimated theoretically, making the measurement of the $\pi\pi$ S-wave phase shift in the $J/\Psi \to \omega\pi\pi$ channel problematic. However, all these troubles do not exist in the $\Psi' \to \pi\pi J/\Psi$ channel, as the energy of the $\pi J/\Psi$ system is too low and no resonances exist in the $\pi J/\Psi$ spectrum. The channel $\Psi' \to \pi\pi J/\Psi$, where $J/\Psi \to \mu^+\mu^-$, has been studied by BES-II [6], and a global partial-wave analysis has been performed. After introducing a wide 0^{++} background which strongly interfered destructively with σ particle, the $\pi\pi$ spectrum could be well fitted. The pole position measured in this channel is consistent with that measured in the $J/\Psi \to \omega\pi\pi$ channel. Though a global partial-wave analysis fit obtained reasonable results on the σ particle, the $\pi\pi$ S-wave phase shift could not be well determined. The reason is that the statistics in BES-II data is too low to perform a reasonable fit on the phase shift, which is studied in this paper. Thus, we will use Monte Carlo techniques to generate data with different D-wave components and different statistics. Then the method proposed in this paper will be used to fit the data.

It is expected that BES-III will collect a huge number of Ψ' events. The statistics of BES-III data will be about 200 times that of BES-II data. For example, the statistics of $\Psi' \to \pi\pi J/\Psi$ in a 10 MeV bin at 500 MeV in the $\pi\pi$ spectrum is about 1000 for BES-II. If BES-III statistics is 200 times larger, we will have around 200 000 statistics in a 10 MeV bin. Hence, about 200 000 Monte Carlo events are generated in our simulation. The method proposed in this paper is used to fit the data to check whether the phase shift can be reasonably recovered or not. The S-wave phase motion is a known input of the Monte Carlo, so we can test the above method by comparing the fitted results with the original value in the simulation.

In the Monte Carlo simulation, we need first to fix the amount of D-wave component, or the percentage of D-wave component in the total Monte Carlo data sample. According to literature [5], the ratios of D-wave component to S-wave component in the $m_{\pi\pi}$ range from 340 MeV to 600 MeV are in the range from 4.7% to 31.9%, and the ratio decreases as $m_{\pi\pi}$ increases. In order to simplify the problem, we generate Monte Carlo data in the energy range between 500 and 510 MeV with different D-wave components. Five independent Monte Carlo data samples are generated with D-component 2%, 4%, 8%, 20%, and 45%, respectively. In our Monte Carlo, we first generate the $\Psi' \to \pi\pi J/\Psi$ events and then we perform the BES-III detector simulation. After this, the number of events for each Monte Carlo is $N_{\rm MC}$, which is listed for each case in Table I. For each data sample, the method proposed in this paper is used to fit the data. A scan on the phase shift is performed. For each data sample, the parameters $I_i(s)$ and $(\delta_0 - \delta_2)$ are fitted. For that, in a first step, we fix $(\delta_0 - \delta_2)$ to a value and fit the parameters $I_i(s)$, where the best fit yields a χ^2 value. Repeating this procedure several times, one obtains a curve that shows the variation of the χ^2 with $(\delta_0 - \delta_2)$. This procedure is called a phase scan. The results for each Monte Carlo data sample are shown in Fig. 2. Observing these figures, one can see that there is a minimum in the smooth curve of the scan. The value of the phase at the minimum is the output phase of each fit. In general, except for the 45% case, the input phases are 1.17 (which corresponds to 67°). In the 45% case the input phase is taken as 1.03 radians. The fit results are listed in Table I. It can be seen that the fit results are quite close to the Monte Carlo inputs and the phase-shift determinations obtained by the method in this paper are then reasonable.

The amount of statistics in the data samples is crucial for the study of the phase difference. In the analysis above, the statistics of the Monte Carlo simulations were, respec-

TABLE I. Scan results on $\delta_0^0 - \delta_0^2$ for different D-wave components. Errors are only statistical. The input phase difference of Monte Carlo data are 1.03 for the 45% case and 1.17 for the rest. $N_{\rm MC}$ is the number of MC events used in the fit.

Ratio of D wave	2%	4%	8%	20%	45%
$\delta_0^0 - \delta_0^2$	1.28 ± 0.08 1.52 ± 0.04	1.10 ± 0.16	1.24 ± 0.08	1.18 ± 0.04	1.04 ± 0.04
$N_{\rm MC}$	6×10^5	2.5×10^5	2×10^5	2×10^5	1.5×10^4

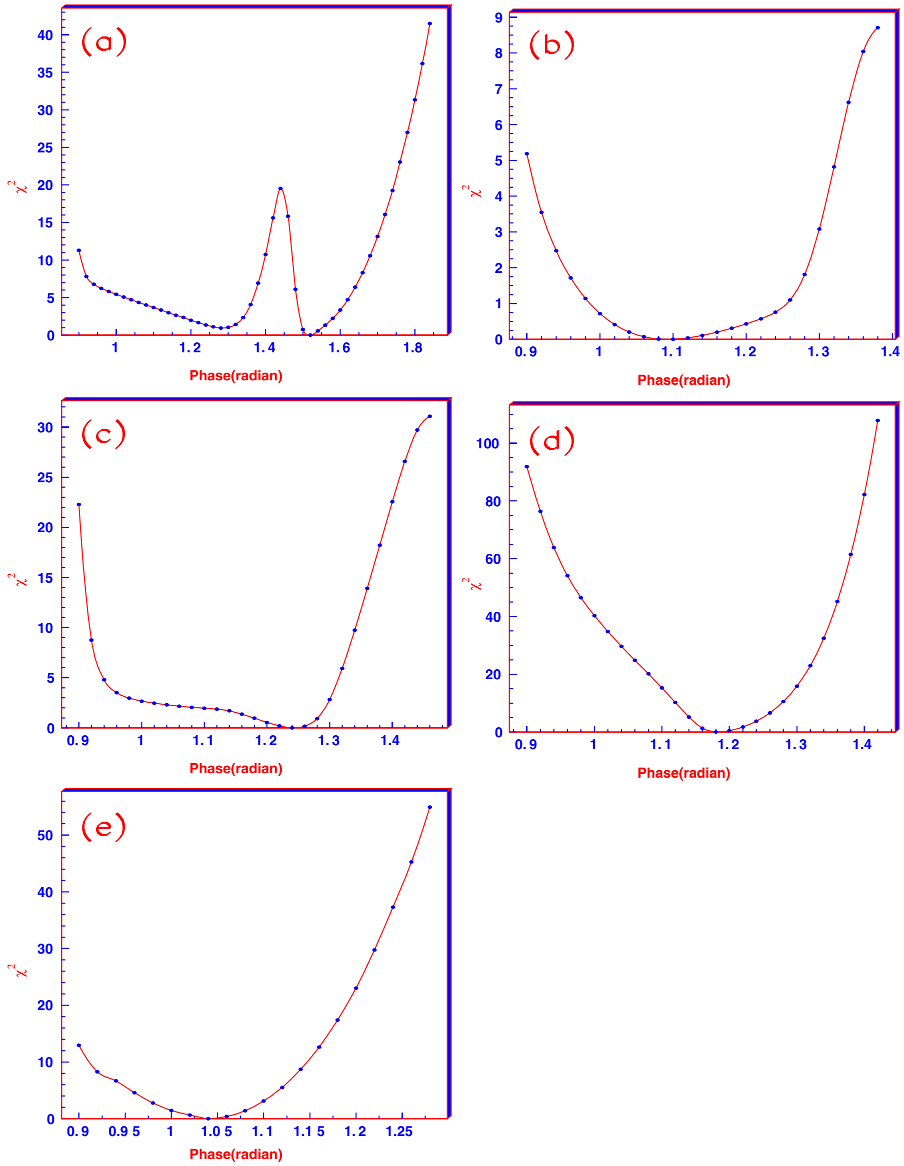

FIG. 2 (color online). Scan results for the I, $J = 0$, 0 phase. (a), (b), (c), (d), and (e) correspond, respectively, to the D-components 2%, 4%, 8%, 20%, and 45%. The value at minimum of the curve is the output phase shown in Table I. The number of events for each cases can be found in Table I.

tively, 600 000, 250 000, 200 000, 200 000, and 15 000 for the D-wave component 2%, 4%, 8%, 20%, and 45%. When the ratio of D-wave component is small, we need much higher statistics. Otherwise the likelihood function becomes not sensitive to the change of S-wave phase. In the BES-II data, there are only around 1000 events in a 10 MeV bin when $m_{\pi\pi} = 500$ MeV and the fit is not sensitive at all to the phase-shift difference. In the Monte Carlo study, we found similar results, that is, when the D-wave component was 4% and the amount of statistics of the Monte Carlo data was below 10 000, the likelihood function remained essentially unchanged when we changed the S-wave phase to another value. Therefore, the reason why we could not obtain a reasonable result for the S-wave phase shift is that the statistics of the BES-II data was indeed too low. It is expected that BES-III will collect 200 times more ψ' data in the near future. And the BES-III detector has much higher selection efficiency than that of the BES-II detector. So, we will have more than 400 000 events in a 10 MeV bin when $m_{\pi\pi}$ is at 500 MeV. Our Monte Carlo study shows that, if the ratio of D-wave component is above 2%, a reasonable S-wave phase shift can be obtained based on BES-III data. In the lower $m_{\pi\pi}$ mass region, such as in the 350 MeV region, the amount of statistics is about a 3.3% of that in the 500 MeV range. In order to achieve enough statistics in the 350 MeV region, more running on Ψ' data will be needed. If, for any reason, we were not able to obtain enough statistics in the

350 MeV region, it would be still possible to select a wider bin for this energy range in order to increase the statistics. For instance, the width of the bin could be increased from 10 to 30 MeV.

V. CONCLUSIONS

A method is proposed in this paper to measure the $\pi\pi$ phase shift. It is tested by means of a Monte Carlo simulation, finding that if the D-wave component is above 2% and the statistics in one 10 MeV bin is above roughly 200 000, a reasonable determination of the $\pi\pi$ phase shift can be performed. It is expected that BES-III will collect enough data and, based on them, we expect to be able to measure the $\pi\pi$ phase-shift difference in the mass region from 350 to 550 MeV.

ACKNOWLEDGMENTS

This work is supported in part by National Nature Science Foundations of China under Contracts No. 10575002, No. 10491306, and No. 10721063, by the EU-RTN Programme, Contract No. MRTN-CT-2006-035482, Flavianet, and by the Spanish Consolider-Ingenio 2010 Program CPAN (CSD2007-00042).

APPENDIX: KINEMATICS

ϵ' and ϵ denote the polarizations of Ψ' and J/Ψ respectively. Here we choose

$$\epsilon_0^\mu = \frac{1}{M}(|\vec{p}_3|, 0, 0, -E_3), \qquad \epsilon_\pm^\mu = \frac{\sqrt{2}}{2}(0, 1, \pm i, 0),$$

and

$$\begin{aligned}
\epsilon_1^{\prime\mu} &= (0, -\cos\theta_3\cos\phi, -\cos\theta_3\sin\phi, -\sin\theta_3),\\
\epsilon_2^{\prime\mu} &= (0, -\sin\phi, \cos\phi, 0),\\
p_3\cdot q &= \sqrt{s}(E_3 + |\vec{p}_3|\beta),\\
p_3\cdot\epsilon_1' &= -|\vec{p}_3|\sin\theta_3,\\
p\cdot\epsilon_1' &= 2|\vec{p}^*_{\pi\pi}|(\cos\theta_3\cos\phi\sin\theta^*_\pi + \gamma\cos\theta^*_\pi\sin\theta_3),\\
q\cdot\epsilon_1' &= \sqrt{s}\beta\gamma\sin\theta_3,\\
p_3\cdot\epsilon_2' &= 0,\\
p\cdot\epsilon_2' &= 2|\vec{p}^*_{\pi\pi}|\sin\theta^*_\pi\sin\phi,\\
q\cdot\epsilon_2' &= 0,\\
p\cdot\epsilon_+ &= -\sqrt{2}|\vec{p}^*_{\pi\pi}|\sin\theta^*_\pi,\\
p\cdot\epsilon_- &= -\sqrt{2}|\vec{p}^*_{\pi\pi}|\sin\theta^*_\pi,\\
p\cdot\epsilon_0 &= 2|\vec{p}^*_{\pi\pi}|\cos\theta^*_\pi\gamma(E_3 + |\vec{p}_3|\beta)\frac{1}{M_\Psi},\\
q\cdot\epsilon_+ &= 0,\\
q\cdot\epsilon_- &= 0,\\
q\cdot\epsilon_0 &= s\gamma(|\vec{p}_3| + E_3\beta)\frac{1}{M_\Psi}\epsilon_1'\cdot\epsilon_+\\
&= \cos\theta_3(\cos\phi + i\sin\phi)/\sqrt{2},\\
\epsilon_1'\cdot\epsilon_- &= \cos\theta_3(\cos\phi - i\sin\phi)/\sqrt{2},\\
\epsilon_1'\cdot\epsilon_0 &= -E_3\sin\theta_3/M_\Psi,\\
\epsilon_2'\cdot\epsilon_+ &= (-i\cos\phi + \sin\phi)/\sqrt{2},\\
\epsilon_2'\cdot\epsilon_- &= (i\cos\phi + \sin\phi)/\sqrt{2},\\
\epsilon_2'\cdot\epsilon_0 &= 0.
\end{aligned}$$

[1] S. Malvezzi (FOCUS Collaboration), AIP Conf. Proc. **1030**, 64 (2008); M. R. Pennington and the FOCUS Collaboration, Phys. Lett. B **653**, 1 (2007); M. R. Pennington, *International Workshop on Tau-Charm Physics* (Charm, Beijing, 2006); Int. J. Mod. Phys. A **21**, 5503 (2006).
[2] B. Meadows, arXiv:0712.1605.
[3] L. Edera and M. R. Pennington, Phys. Lett. B **623**, 55 (2005).
[4] I. Caprini, Phys. Lett. B **638**, 468 (2006).
[5] J. Z. Bai *et al.* (BES Collaboration), Phys. Rev. D **62**, 032002 (2000).
[6] M. Ablikim *et al.* (BES Collaboration), Phys. Lett. B **645**, 19 (2007).
[7] L. S. Brown and R. N. Cahn, Phys. Rev. Lett. **35**, 1 (1975).
[8] R. N. Cahn, Phys. Rev. D **12**, 3559 (1975).
[9] A. Pais and S. B. Treiman, Phys. Rev. **168**, 1858 (1968).
[10] S. Chakravarty and P. Ko, Phys. Rev. D **48**, 1205 (1993); S. Chakravarty, S. M. Kim, and P. Ko, Phys. Rev. D **50**, 389 (1994).
[11] S. U. Chung, Phys. Rev. D **48**, 1225 (1993).
[12] T. Mannel and R. Urech, Z. Phys. C **73**, 541 (1997).
[13] W. M. Yao *et al.*, J. Phys. G **33**, 1 (2006).
[14] M. Ablikim *et al.* (BES collaboration), Phys. Lett. B **598**, 149 (2004).
[15] M. Ablikim *et al.* (BES collaboration), Phys. Lett. B **633**, 681 (2006).