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Four-loop screened perturbation theory

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We study the thermodynamics of massless ϕ^4-theory using screened perturbation theory. In this method, the perturbative expansion is reorganized by adding and subtracting a thermal mass term in the Lagrangian. We calculate the free energy through four loops expanding in a double power expansion in m/T and g^2, where m is the thermal mass and g is the coupling constant. The expansion is truncated at order g^7 and the loop expansion is shown to have better convergence properties than the weak-coupling expansion. The free energy at order g^6 involves the four-loop triangle sum-integral evaluated by Gynther, Laine, Schröder, Torrero, and Vuorinen using the methods developed by Arnold and Zhai. The evaluation of the free energy at order g^7 requires the evaluation of a nontrivial three-loop sum-integral, which we calculate by the same methods.

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I. INTRODUCTION

In recent years there has been significant progress in the understanding of thermal field theories in equilibrium [1–4]. For example, the thermodynamic functions can be calculated as power series in the coupling constant g at weak coupling and advanced calculational techniques have been developed in order to go beyond the first few corrections. The pressure has been calculated through order g^5 for massless ϕ^4-theory [5,6], massless QED [7–9], and massless non-Abelian gauge theories [10–12]. Very recently, the calculation frontier has been pushed to order g^6 in massless ϕ^4-theory by Gynther, Laine, Schröder, Torrero, and Vuorinen [13]. The calculation in Ref. [13] involves the computation of complicated four-loop vacuum diagrams and was motivated by the corresponding problem in non-Abelian gauge theories: There are three momentum scales—hard momenta of order T, soft momenta of order gT, and supersoft momenta of order g^2T, which give contributions to the free energy. The contribution from the hard scale T to the free energy can be calculated as a power series in g^2 using naïve perturbation theory without resummed propagators. The order g^6 is the first order at which all three momentum scales in QCD contribute to the free energy and so it is important to calculate the full g^6 term. Such a calculation involves the evaluation of four-loop vacuum diagrams in four dimensions.

However, it is well-known that the weak-coupling expansion is very sensitive to the renormalization scale, and it is furthermore convergent only if the coupling constant is tiny. The physical origin of this instability does not seem to be related to the magnetic mass problem in QCD, as it appears in ϕ^4-theory and QED as well. Rather, it seems to be associated with screening effects and quasiparticles.

In recent years there have been large efforts to reorganize the perturbative series such that it has improved convergence properties. Several of these methods are variational in nature, in which the thermodynamic potential Ω depends on one or more variational parameters m_i. The pressure and other thermodynamic quantities are then found by evaluating Ω and its derivatives at the variational point where $\delta\Omega/\delta m_i = 0$.

One of these methods is screened perturbation theory (SPT) which in the context of hot ϕ^4-theory was introduced by Karsch, Patkós, and Petreczky [14] (see also Refs. [15–17]). In this approach, one introduces a single variational parameter m^2 which is added to and subtracted from the original Lagrangian. The added piece is kept as a part of the free Lagrangian and the subtracted piece is treated as an interaction. The parameter m^2 has a simple interpretation of a thermal mass and satisfies a variational equation. SPT has been applied to calculate the pressure to three-loop order [18] and the convergence properties of the successive approximations are dramatically improved as compared to the weak-coupling expansion. The mass parameter is of order g and so it might be reasonable to carry out an additional expansion of the Feynman diagrams in powers of m/T, and truncate at the appropriate order. This was done in Ref. [19] and it was demonstrated that the double expansion in m/T and g converges quickly to the numerically exact result even for large values of the coupling.

The generalization of SPT to gauge theories cannot simply be made by adding and subtracting a local mass term as this would violate gauge invariance. Instead one adds and subtracts to the Lagrangian a hard thermal loop (HTL) improvement term [20]. The free piece of the Lagrangian includes the HTL self-energies, while the remaining terms are treated as perturbations. Hard thermal loop perturbation theory is a manifestly gauge invariant approach that can be used to calculate static as well as

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dynamic quantities in a systematic expansion. HTL perturbation theory has been applied to calculate the pressure to two-loop order [21–25] in an m/T expansion and the convergence properties of the successive approximations are again improved as compared to the weak-coupling expansion.

Another variational method in which the propagator is a variational function was constructed by Luttinger and Ward [26] and by Baym [27] for nonrelativistic fermions in the early 1960s. Later, it was generalized to relativistic quantum field theories by Cornwall, Jackiw, and Tomboulis [28]. The approach is based on the fact that the thermodynamic potential Ω can be written in terms of the two-particle irreducible (2PI) vacuum diagrams. The propagator D satisfies the variational equation $\delta\Omega/\delta D = 0$. The 2PI effective action formalism is also referred to as Φ-derivable approximations.

Since the 2PI effective action formalism involves an effective propagator, a truncated calculation in the loop expansion or $1/N$-expansion involves a selective resummation of diagrams from all orders of perturbation theory. This fact makes renormalization of Φ-derivable approximations highly nontrivial. In recent years, there have been large efforts to prove renormalizability in the loop expansion, $1/N$-expansion, or the Hartree approximation, and, in particular, to prove that the counterterms are medium independent, i.e. independent of temperature and chemical potential [29–32].

The second issue is that of gauge-fixing dependence. While the exact 2PI effective action is gauge independent at the stationary point, this property is often lost in approximations. The problem has been examined by Arrizabalaga and Smit [33] as well as Carrington *et al.* [34]. In Ref. [33], it was shown that the n-loop Φ-derivable approximation, which is defined by the truncation of the action functional after n loops, has a gauge dependence that shows up at order g^{2n}. Furthermore, if the nth order solution to the gap equation is used to evaluate the complete effective action, the gauge dependence first shows up at order g^{4n}. Explicit examples of the gauge dependence of the three-loop Φ-derivable approximation can be found in Ref. [35].

The Φ-derivable approach has been used by Blaizot, Iancu, and Rebhan [36–38] and by Peshier [39] to calculate the thermodynamic quantities at the two-loop level in scalar field theory as well as in gauge theories. The calculations are based on the fact that the solution to the gap equation for the propagator for soft momenta is given by the HTL self-energies. Three-loop calculations have been performed in scalar field theory by Braaten and Petitgirard [40], and in QED in Ref. [35] using an m/T expansion similar to that employed in SPT in Ref. [19]. The convergence of the successive approximations to the pressure is improved significantly compared to the weak-coupling expansion and the sensitivity to the renormalization scale is also reduced. In Ref. [41], the authors carried out a numerically exact three-loop calculation of the pressure in ϕ^4-theory. Similarly, numerically exact two-loop calculations of the pressure in QED including an analysis of the gauge dependence of the results can be found in Ref. [42]. In these calculations no attempts to compare with the m/T expansions of Refs. [35,40] were made.

Finally, we mention other related resummation methods that have been applied in recent years, namely, the 2-particle point irreducible (2PPI) method [43,44] as well as the linear delta-expansion [45–48]. These methods are also variational in spirit. Moreover, it has been shown that they correctly predict a second-order phase transition when applied to ϕ^4-theory. In the case of the linear delta-expansion, the successive approximations of e.g. the pressure are remarkably stable as compared to the weak-coupling expansion.

The article is organized as follows. In Sec. II, we briefly discuss the systematics of screened perturbation theory. In Sec. III, we calculate the pressure to four-loop order in a double expansion in m/T and g^2. In Sec. IV, we discuss different gap equations that are used to determine the mass parameter in screened perturbation theory. We also present our numerical results and compare them with the weak-coupling expansion. In Sec. V, we summarize. In Appendix A and B, we list the sum-integrals and the integrals that we need. In Appendix C, we discuss the m/T expansion of typical sum-integrals that appear in the calculation. In Appendix D , we calculate explicitly a new three-loop sum-integral that contributes to order g^7 in the m/T expansion.

II. SCREENED PERTURBATION THEORY

The Lagrangian density for a massless scalar field with a ϕ^4 interaction is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{g^2}{24}\phi^4 + \Delta\mathcal{L}, \tag{1}$$

where g is the coupling constant and $\Delta\mathcal{L}$ includes counterterms. Renormalizability guarantees that $\Delta\mathcal{L}$ is of the form

$$\Delta\mathcal{L} = \frac{1}{2}\Delta Z\partial_\mu\phi\partial^\mu\phi - \frac{1}{24}\Delta g^2\phi^4. \tag{2}$$

Screened perturbation theory, which was introduced in thermal field theory by Karsch, Patkós, and Petreczky [14], is simply a reorganization of the perturbation series for thermal field theory. It can be made more systematic by using a framework called "optimized perturbation theory" that Chiku and Hatsuda [49] have applied to a spontaneously broken scalar field theory. The Lagrangian density is written as

$$\mathcal{L}_{\rm SPT} = -\mathcal{E}_0 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(m^2 - m_1^2)\phi^2 - \frac{g^2}{24}\phi^4 + \Delta\mathcal{L} + \Delta\mathcal{L}_{\rm SPT}. \tag{3}$$

Here, $\mathcal{E}_0$ is the vacuum energy density term, and we have added and subtracted mass terms. If we set $\mathcal{E}_0 = 0$ and $m_1^2 = m^2$, we recover the original Lagrangian Eq. (1). Screened perturbation theory is defined by taking m^2 to be of order unity and m_1^2 to be of order g^2, expanding systematically in powers of g^2 and setting $m_1^2 = m^2$ at the end of the calculation. This defines a reorganization of the perturbative series in which the expansion is about the free field theory defined by

$$\mathcal{L}_{\text{free}} = -\mathcal{E}_0 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{4}$$

The interacting term is

$$\mathcal{L}_{\text{int}} = \frac{1}{2}m_1^2\phi^2 - \frac{g^2}{24}\phi^4 + \Delta\mathcal{L} + \Delta\mathcal{L}_{\text{SPT}}. \tag{5}$$

Screened perturbation theory generates new ultraviolet divergences, but they can be canceled by the additional counterterm in $\Delta\mathcal{L}_{\text{SPT}}$. If we use dimensional regularization and minimal subtraction, the coefficients of these operators are polynomials in g^2 and $(m^2 - m_1^2)$. The counterterm $\Delta\mathcal{L}$ is

$$\Delta\mathcal{L} = -\frac{\Delta g^2}{24}\phi^4. \tag{6}$$

The additional counterterms required to remove the new divergences are

$$\Delta\mathcal{L}_{\text{SPT}} = -\Delta\mathcal{E}_0 - \frac{1}{2}(\Delta m^2 - \Delta m_1^2)\phi^2. \tag{7}$$

Several terms in the power series expansions of the counterterms are known from previous calculations at zero temperature. The counterterms Δg^2 and Δm^2 are known to order α^5, where $\alpha = g^2/(4\pi)^2$ [50]. We will need the coupling constant counterterm to next-to-leading order in α:

$$\Delta g^2 = \left[\frac{3}{2\epsilon}\alpha + \left(\frac{9}{4\epsilon^2} - \frac{17}{12\epsilon}\right)\alpha^2 + \cdots\right]g^2. \tag{8}$$

We need the mass counterterms Δm^2 and Δm_1^2 to next-to-leading order in α:

$$\Delta m^2 = \left[\frac{1}{2\epsilon}\alpha + \left(\frac{1}{2\epsilon^2} - \frac{5}{24\epsilon}\right)\alpha^2 + \cdots\right]m^2, \tag{9}$$

$$\Delta m_1^2 = \left[\frac{1}{2\epsilon}\alpha + \left(\frac{1}{2\epsilon^2} - \frac{5}{24\epsilon}\right)\alpha^2 + \cdots\right]m_1^2. \tag{10}$$

The counterterm for $\Delta\mathcal{E}_0$ has been calculated to order α^4 [51]. We will need its expansion only to first order in α and second order in m_1^2:

$$(4\pi)^2\Delta\mathcal{E}_0 = \left[\frac{1}{4\epsilon} + \frac{1}{8\epsilon^2}\alpha\right]m^4 - 2\left[\frac{1}{4\epsilon} + \frac{1}{8\epsilon^2}\alpha\right]m_1^2m^2 + \left[\frac{1}{4\epsilon} + \frac{1}{8\epsilon^2}\alpha\right]m_1^4. \tag{11}$$

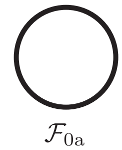

FIG. 1. One-loop vacuum diagram.

III. FREE ENERGY TO FOUR LOOPS

In this section, we calculate the m/T expansions of the pressure to four loops in screened perturbation theory. In performing the truncation, m is treated as a quantity that is $\mathcal{O}(g)$ and we include all terms which contribute to order g^7.

A. One-loop free energy

The free energy at leading order in g^2 is

$$\mathcal{F}_0 = \mathcal{E}_0 + \mathcal{F}_{0a} + \Delta_0\mathcal{E}_0, \tag{12}$$

where $\Delta_0\mathcal{E}_0$ is the term of order g^0 in the vacuum energy counterterm Eq. (11).

The expression for diagram $\mathcal{F}_{0a}$ in Fig. 1 is

$$\mathcal{F}_{0a} = \frac{1}{2}\sum\!\!\!\!\!\!\!\int_P \log[P^2 + m^2], \tag{13}$$

where the symbol $\sum\!\!\!\!\!\!\!\int_P$ is defined in Appendix A.

Treating m as $\mathcal{O}(gT)$ and including all terms which contribute through $\mathcal{O}(g^7)$, we obtain

$$\mathcal{F}_{0a} = \frac{1}{2}I_0' + \frac{1}{2}m^2 I_1 + \frac{1}{2}T\mathcal{I}_0' - \frac{1}{4}m^4 I_2 + \frac{1}{6}m^6 I_3, \tag{14}$$

where the sum-integrals I_0' and I_n are defined in Appendix A and the integral $\mathcal{I}_0'$ is defined in Appendix B. In Appendix C, we illustrate the m/T expansion of simple one-loop sum-integrals such as the one appearing in Eq. (13). We also note that most of the multiloop diagrams are products of simple one-loop sum-integrals.

The term I_2 is logarithmically divergent and the pole in ϵ is canceled by the zeroth-order term $\Delta_0\mathcal{E}_0$ in Eq. (11). The final result for the truncated one-loop free energy is

$$\mathcal{F}_0 = -\frac{\pi^2T^4}{90}\left[1 - 15\hat{m}^2 + 60\hat{m}^3 + 45(L + \gamma_E)\hat{m}^4 - \frac{15}{2}\zeta(3)\hat{m}^6\right], \tag{15}$$

where $\hat{m} = \frac{m}{2\pi T}$ and $L = \log\frac{\mu}{4\pi T}$.

B. Two-loop free energy

The contribution to the free energy at two loops is given by

$$\mathcal{F}_1 = \mathcal{F}_{1a} + \mathcal{F}_{1b} + \Delta_1\mathcal{E}_0 + \frac{\partial\mathcal{F}_{0a}}{\partial m^2}\Delta_1 m^2, \tag{16}$$

where $\Delta_1\mathcal{E}_0$ and $\Delta_1 m^2$ are the vacuum and mass counter-

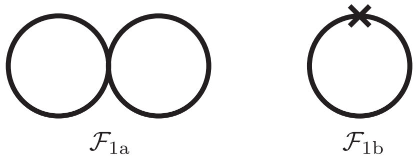

FIG. 2. Two-loop vacuum diagrams. The cross denotes a mass insertion.

terms of order g^2, respectively. The expressions for the diagrams $\mathcal{F}_{1a}$ and $\mathcal{F}_{1b}$ in Fig. 2 are

$$\mathcal{F}_{1a} = \frac{1}{8} g^2 \bigg(\sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2 + m^2} \bigg)^2, \tag{17}$$

$$\mathcal{F}_{1b} = -\frac{1}{2} m_1^2 \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2 + m^2}. \tag{18}$$

Expanding the sum-integrals in Eqs. (17) and (18) to order $\mathcal{O}(g^7)$ yields

$$\mathcal{F}_{1a} = \frac{1}{8} g^2 [I_1^2 + 2T I_1 I_1 - 2m^2 I_1 I_2 + T^2 I_1^2 - 2m^2 I_1 T I_2 + 2m^4 I_1 I_3 + m^4 I_2^2 + 2m^4 T I_1 I_3], \tag{19}$$

$$\mathcal{F}_{1b} = -\frac{1}{2} m_1^2 [I_1 + T I_1 - m^2 I_2 + m^4 I_3], \tag{20}$$

where the integral I_n is defined in Appendix B.

The poles in ϵ in Eqs. (19) and (20) are canceled by the counterterms in Eq. (16). The final result for the two-loop contribution to the free energy is

$$\mathcal{F}_1 = \frac{\pi^2 T^4}{90} \alpha \bigg[\frac{5}{4} - 15\hat{m} - 15(L + \gamma_E - 3)\hat{m}^2 + 90(L + \gamma_E)\hat{m}^3 + 45\bigg((L + \gamma_E)^2 + \frac{1}{12}\zeta(3) \bigg)\hat{m}^4 - \frac{45}{2}\zeta(3)\hat{m}^5 \bigg] - \frac{\pi^2 T^4}{90} 15\hat{m}_1^2 \bigg[1 - 6\hat{m} - 6(L + \gamma_E)\hat{m}^2 + \frac{3}{2}\zeta(3)\hat{m}^4 \bigg]. \tag{21}$$

Note that we here and in the following have pulled out a factor of $\mathcal{F}_{\rm ideal} = -\pi^2 T^4/90$.

C. Three-loop free energy

The contribution to the free energy at three loops is

$$\mathcal{F}_2 = \mathcal{F}_{2a} + \mathcal{F}_{2b} + \mathcal{F}_{2c} + \mathcal{F}_{2d} + \Delta_2 \mathcal{E}_0 + \frac{\partial \mathcal{F}_{0a}}{\partial m^2} \Delta_2 m^2 + \frac{1}{2} \frac{\partial^2 \mathcal{F}_{0a}}{(\partial m^2)^2} (\Delta_1 m^2)^2 + \bigg(\frac{\partial \mathcal{F}_{1a}}{\partial m^2} + \frac{\partial \mathcal{F}_{1b}}{\partial m^2} \bigg) \Delta_1 m^2 + \frac{\mathcal{F}_{1a}}{g^2} \Delta_1 g^2 + \frac{\mathcal{F}_{1b}}{m_1^2} \Delta_1 m_1^2, \tag{22}$$

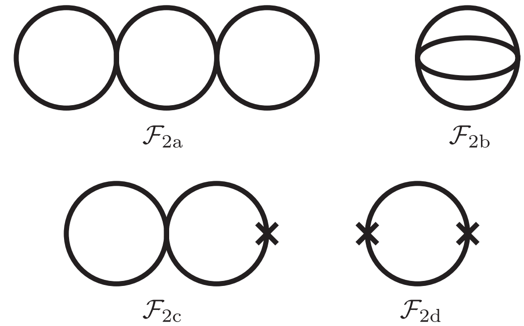

FIG. 3. Three-loop vacuum diagrams.

where we have included all necessary counterterms. The expressions for the diagrams $\mathcal{F}_{2a}$, $\mathcal{F}_{2b}$, $\mathcal{F}_{2c}$, and $\mathcal{F}_{2d}$ in Fig. 3 are

$$\mathcal{F}_{2a} = -\frac{1}{16} g^4 \bigg(\sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2 + m^2} \bigg)^2 \sum\!\!\!\!\!\!\!\int_Q \frac{1}{(Q^2 + m^2)^2}, \tag{23}$$

$$\mathcal{F}_{2b} = -\frac{1}{48} g^4 \sum\!\!\!\!\!\!\!\int_{PQR} \frac{1}{P^2 + m^2} \frac{1}{Q^2 + m^2} \frac{1}{R^2 + m^2} \times \frac{1}{(P + Q + R)^2 + m^2}, \tag{24}$$

$$\mathcal{F}_{2c} = \frac{1}{4} g^2 m_1^2 \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2 + m^2} \sum\!\!\!\!\!\!\!\int_Q \frac{1}{(Q^2 + m^2)^2}, \tag{25}$$

$$\mathcal{F}_{2d} = -\frac{1}{4} m_1^4 \sum\!\!\!\!\!\!\!\int_P \frac{1}{(P^2 + m^2)^2}. \tag{26}$$

Expanding in powers of m^2 to the appropriate order gives[1]

$$\mathcal{F}_{2a} = -\frac{1}{16} g^4 [T I_1^2 I_2 + I_1^2 I_2 + 2T^2 I_1 I_1 I_2 + T^3 I_1^2 I_2 + 2T I_1 I_1 I_2 - 2m^2 T I_1 I_2 I_2 + T^2 I_2 I_1^2 - 2m^2 I_1^2 I_3 - 2m^2 T^2 I_2 I_1 I_2 - 2m^2 I_1 I_2^2 - 4m^2 T I_1 I_1 I_3 - 2m^2 T I_1 I_2^2 + m^4 T I_2 I_2^2 + 2m^4 T I_2 I_1 I_3], \tag{27}$$

$$\mathcal{F}_{2b} = -\frac{1}{48} g^4 \bigg[I_{\rm ball} + T^3 I_{\rm ball} + 4T I_1 I_{\rm sun} + 6T^2 I_2 I_1^2 - 4m^2 I'_{\rm ball} - 8m^2 T I_1 \sum\!\!\!\!\!\!\!\int_{QR} \frac{Q^2 + (2/d)\mathbf{q}^2}{Q^6 R^2 (Q + R)^2} \bigg], \tag{28}$$

[1]Notice that the term $T I_1 I_{\rm sun}$ in $\mathcal{F}_{2b}$ in Eq. (28) vanishes. However, we include this term because it gives rise to a finite term at four loops when renormalizing the coupling constant g.

$$\mathcal{F}_{2c} = \frac{1}{4} g^2 m_1^2 [T I_1 I_2 + I_1 I_2 + T^2 I_1 I_2 + T I_2 I_1 - m^2 T I_2 I_2 - m^2 I_2^2 - 2m^2 I_1 I_3 - 2m^2 T I_1 I_3 + m^4 T I_2 I_3], \quad (29)$$

$$\mathcal{F}_{2d} = -\frac{1}{4} m_1^4 [T I_2 + I_2 - 2m^2 I_3], \quad (30)$$

where $I_{\rm sun}$, $I_{\rm ball}$, and $I'_{\rm ball}$ are defined in Appendix A, and $I_{\rm ball}$ is defined in Appendix B.

The poles in ϵ in Eqs. (27)–(30) are canceled by the counterterms in Eq. (22).

The final result for the three-loop contribution to the free energy is

$$\begin{aligned}
\mathcal{F}_2 = &-\frac{\pi^2 T^4}{90} \frac{5}{8} \frac{1}{\hat{m}} \alpha^2 \bigg[1 - 2\bigg(\frac{59}{15} - \gamma_E - 3L - 4\frac{\zeta'(-1)}{\zeta(-1)} + 2\frac{\zeta'(-3)}{\zeta(-3)} \bigg) \hat{m} \\
&- 12\hat{m}^2 \bigg(5 + 7L + 3\gamma_E - 8\log\hat{m} - 8\log 2 - 4\frac{\zeta'(-1)}{\zeta(-1)} \bigg) \\
&+ \bigg(268(L+\gamma_E) - 48(L+\gamma_E)^2 + \frac{\zeta'(-1)}{\zeta(-1)}(34 + 12\gamma_E) + 12\frac{\zeta''(-1)}{\zeta(-1)} + \gamma_E(17 - 21\gamma_E) + 34 + \frac{9\pi^2}{2} - 48\gamma_1 \\
&- \zeta(3) - 6C'_{\rm ball} \bigg) \hat{m}^3 + (89 + 120(L+\gamma_E) + [18(L+\gamma_E)]^2 + 15\zeta(3))\hat{m}^4 \bigg] \\
&+ \frac{\pi^2 T^4}{90} \frac{15}{2} \frac{\hat{m}_1^2}{\hat{m}} \alpha \bigg[1 + 2(L + \gamma_E - 3)\hat{m} - 18(L+\gamma_E)\hat{m}^2 - (12(L+\gamma_E)^2 + \zeta(3))\hat{m}^3 + \frac{15}{2}\zeta(3)\hat{m}^4 \bigg] \\
&- \frac{\pi^2 T^4}{90} \frac{45}{2} \frac{\hat{m}_1^4}{\hat{m}} [1 + 2(L+\gamma_E)\hat{m} - \zeta(3)\hat{m}^3]. \qquad (31)
\end{aligned}$$

Here $C'_{\rm ball} = 48.7976$ is the numerical constant in $I'_{\rm ball}$ [13].

D. Four-loop free energy

The contributions to the free energy at four loops are

$$\begin{aligned}
\mathcal{F}_3 = &\mathcal{F}_{3a} + \mathcal{F}_{3b} + \mathcal{F}_{3c} + \mathcal{F}_{3d} + \mathcal{F}_{3e} + \mathcal{F}_{3f} + \mathcal{F}_{3g} + \mathcal{F}_{3h} + \mathcal{F}_{3i} + \mathcal{F}_{3j} + \Delta_3\mathcal{E}_0 + \frac{\partial\mathcal{F}_{0a}}{\partial m^2}\Delta_3 m^2 + \frac{1}{6}\frac{\partial^3\mathcal{F}_{0a}}{(\partial m^2)^3}(\Delta_1 m^2)^3 \\
&+ \frac{\partial^2\mathcal{F}_{0a}}{(\partial m^2)^2}(\Delta_1 m^2)(\Delta_2 m^2) + \bigg(\frac{\partial\mathcal{F}_{1a}}{\partial m^2} + \frac{\partial\mathcal{F}_{1b}}{\partial m^2}\bigg)\Delta_2 m^2 + \frac{\mathcal{F}_{1a}}{g^2}\Delta_2 g^2 + \bigg(2\frac{\mathcal{F}_{2a}}{g^2} + 2\frac{\mathcal{F}_{2b}}{g^2} + \frac{\mathcal{F}_{2c}}{g^2}\bigg)\Delta_1 g^2 \\
&+ \frac{1}{2}\bigg(\frac{\partial\mathcal{F}_{1a}^2}{(\partial m^2)^2} + \frac{\partial\mathcal{F}_{1b}^2}{(\partial m^2)^2}\bigg)(\Delta_1 m^2)^2 + \frac{\mathcal{F}_{1b}}{m_1^2}\Delta_2 m_1^2 + \frac{\partial\mathcal{F}_{1b}}{m_1^2\partial m^2}(\Delta_1 m^2)(\Delta_1 m_1^2) + \frac{1}{g^2}\frac{\partial\mathcal{F}_{1a}}{\partial m^2}(\Delta_1 g^2)(\Delta_1 m^2) \\
&+ \bigg(\frac{\mathcal{F}_{2c}}{m_1^2} + 2\frac{\mathcal{F}_{2d}}{m_1^2}\bigg)\Delta_1 m_1^2 + \bigg(\frac{\partial\mathcal{F}_{2a}}{\partial m^2} + \frac{\partial\mathcal{F}_{2b}}{\partial m^2} + \frac{\partial\mathcal{F}_{2c}}{\partial m^2} + \frac{\partial\mathcal{F}_{2d}}{\partial m^2}\bigg)\Delta_1 m^2. \qquad (32)
\end{aligned}$$

Note that some of the terms first contribute at order g^8 or higher. For example, the vacuum counterterm $\Delta_3\mathcal{E}_0$ first contributes at order $m^4\alpha^2 \sim g^8$.

The expressions for the diagrams $\mathcal{F}_{3a}$–$\mathcal{F}_{3j}$, in Fig. 4 are

$$\mathcal{F}_{3a} = \frac{1}{32} g^6 \bigg(\sum\!\!\!\!\!\!\int_P \frac{1}{P^2 + m^2} \bigg)^2 \bigg(\sum\!\!\!\!\!\!\int_Q \frac{1}{(Q^2 + m^2)^2} \bigg)^2, \quad (33)$$

$$\mathcal{F}_{3b} = \frac{1}{48} g^6 \bigg(\sum\!\!\!\!\!\!\int_P \frac{1}{P^2 + m^2} \bigg)^3 \sum\!\!\!\!\!\!\int_Q \frac{1}{(Q^2 + m^2)^3}, \quad (34)$$

$$\mathcal{F}_{3c} = \frac{1}{24} g^6 \sum\!\!\!\!\!\!\int_{PQR} \frac{1}{(P^2 + m^2)^2} \frac{1}{Q^2 + m^2} \frac{1}{R^2 + m^2} \times \frac{1}{(P+Q+R)^2 + m^2} \sum\!\!\!\!\!\!\int_S \frac{1}{S^2 + m^2} \quad (35)$$

$$\mathcal{F}_{3d} = \frac{1}{48} g^6 \sum\!\!\!\!\!\!\int_{PQRS} \frac{1}{Q^2 + m^2} \frac{1}{(P+Q)^2 + m^2} \frac{1}{R^2 + m^2} \times \frac{1}{(P+R)^2 + m^2} \frac{1}{S^2 + m^2} \frac{1}{(P+S)^2 + m^2}, \quad (36)$$

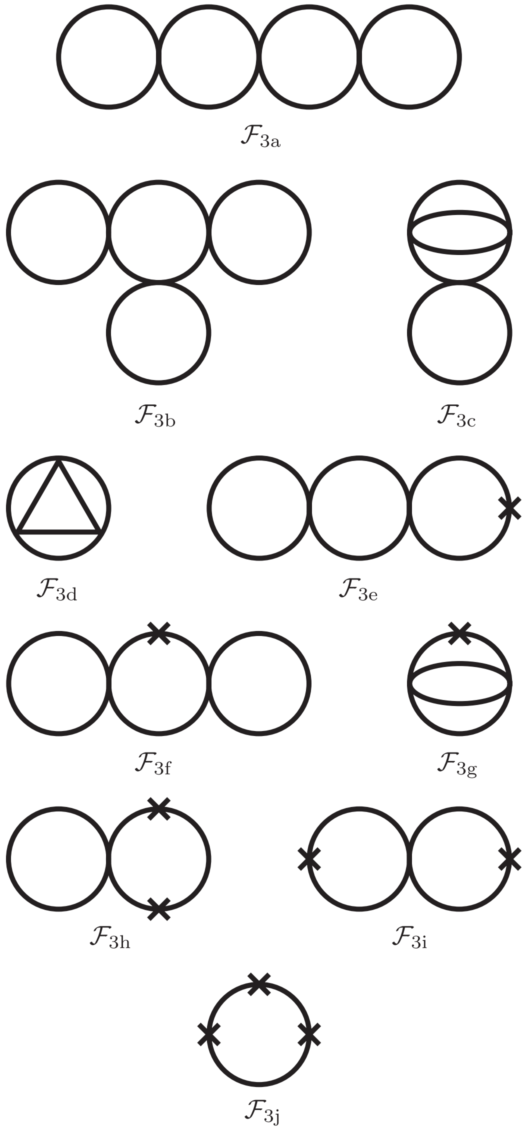

FIG. 4. Four-loop vacuum diagrams.

$$\mathcal{F}_{3e} = -\frac{1}{8} g^4 m_1^2 \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2 + m^2} \left(\sum\!\!\!\!\!\!\!\int_Q \frac{1}{(Q^2 + m^2)^2} \right)^2, \tag{37}$$

$$\mathcal{F}_{3f} = -\frac{1}{8} g^4 m_1^2 \left(\sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2 + m^2} \right)^2 \sum\!\!\!\!\!\!\!\int_Q \frac{1}{(Q^2 + m^2)^3}, \tag{38}$$

$$\mathcal{F}_{3g} = -\frac{1}{12} g^4 m_1^2 \sum\!\!\!\!\!\!\!\int_{PQR} \frac{1}{(P^2 + m^2)^2} \frac{1}{Q^2 + m^2} \frac{1}{R^2 + m^2} \times \frac{1}{(P + Q + R)^2 + m^2} \tag{39}$$

$$\mathcal{F}_{3h} = \frac{1}{4} g^2 m_1^4 \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2 + m^2} \sum\!\!\!\!\!\!\!\int_Q \frac{1}{(Q^2 + m^2)^3}, \tag{40}$$

$$\mathcal{F}_{3i} = \frac{1}{8} g^2 m_1^4 \left(\sum\!\!\!\!\!\!\!\int_P \frac{1}{(P^2 + m^2)^2} \right)^2, \tag{41}$$

$$\mathcal{F}_{3j} = -\frac{1}{6} m_1^6 \sum\!\!\!\!\!\!\!\int_P \frac{1}{(P^2 + m^2)^3}. \tag{42}$$

Expanding the sum-integrals in powers of m^2 to the appropriate order gives

$$\begin{aligned} \mathcal{F}_{3a} = \frac{1}{32} g^6 [& T^2 I_2^2 \mathcal{I}_1^2 + 2T^3 I_1 I_2^2 \mathcal{I}_1 + 2T I_2 \mathcal{I}_1^2 \mathcal{I}_2 + \mathcal{I}_1^2 \mathcal{I}_2^2 \\ & + T^4 I_1^2 I_2^2 + 2T I_1 \mathcal{I}_1 \mathcal{I}_2^2 - 2m^2 T^2 I_2^2 \mathcal{I}_1 \mathcal{I}_2 \\ & - 2m^2 T^3 I_1 I_2^2 \mathcal{I}_2 + 4T^2 I_1 I_2 \mathcal{I}_1 \mathcal{I}_2 - 4m^2 T I_2 \mathcal{I}_1 \mathcal{I}_2^2 \\ & + 2T^3 I_1^2 I_2 \mathcal{I}_2 - 4m^2 T I_2 \mathcal{I}_1^2 \mathcal{I}_3], \end{aligned} \tag{43}$$

$$\begin{aligned} \mathcal{F}_{3b} = \frac{1}{48} g^6 [& T I_3 \mathcal{I}_1^3 + 3T^2 I_1 I_3 \mathcal{I}_1^2 + 3T^3 I_3 I_1^2 \mathcal{I}_1 \\ & - 3m^2 T I_3 \mathcal{I}_1^2 \mathcal{I}_2 + \mathcal{I}_3 \mathcal{I}_1^3 + T^4 I_1^3 I_3 \\ & - 6m^2 T^2 I_1 I_3 \mathcal{I}_1 \mathcal{I}_2 + 3T I_1 \mathcal{I}_1^2 \mathcal{I}_3 - 3m^2 T^3 I_1^2 I_3 \mathcal{I}_2 \\ & + 3m^4 T I_3 \mathcal{I}_1^2 \mathcal{I}_3 + 3m^4 T I_3 \mathcal{I}_1 \mathcal{I}_2^2], \end{aligned} \tag{44}$$

$$\begin{aligned} \mathcal{F}_{3c} = \frac{1}{24} g^6 \Big[& (\mathcal{I}_1 + T I_1 - m^2 \mathcal{I}_2) T^3 I'_{\rm ball} + (\mathcal{I}_1 + T I_1) \mathcal{I}'_{\rm ball} \\ & + 3T^2 I_1 I_2 \mathcal{I}_1 \mathcal{I}_2 - m^2 T I_2 \mathcal{I}_2 \mathcal{I}_{\rm sun} + 3T^3 I_1^2 I_2 \mathcal{I}_2 \\ & + 2\mathcal{I}_1 (T I_1 - m^2 T I_2) \sum\!\!\!\!\!\!\!\int_{QR} \frac{Q^2 + (2/d)\mathbf{q}^2}{Q^6 R^2 (Q + R)^2} \Big], \end{aligned} \tag{45}$$

$$\begin{aligned} \mathcal{F}_{3d} = \frac{1}{48} g^6 \Big[& \sum\!\!\!\!\!\!\!\int_P [\Pi(P)]^3 + T^4 I_{\rm triangle} \\ & + 6T I_1 \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} [\Pi(P)]^2 + 3T^3 \mathcal{I}_2 I_{\rm ball} \Big], \end{aligned} \tag{46}$$

$$\begin{aligned} \mathcal{F}_{3e} = -\frac{1}{8} g^4 m_1^2 [& T^2 I_2^2 \mathcal{I}_1 + T^3 I_1 I_2^2 + 2T I_2 \mathcal{I}_1 \mathcal{I}_2 \\ & - 2m^2 T I_2 \mathcal{I}_2^2 + \mathcal{I}_1 \mathcal{I}_2^2 + 2T^2 I_1 I_2 \mathcal{I}_2 - m^2 T^2 I_2^2 \mathcal{I}_2 \\ & + T I_1 \mathcal{I}_2^2 - 4m^2 T I_2 \mathcal{I}_1 \mathcal{I}_3], \end{aligned} \tag{47}$$

$$\begin{aligned} \mathcal{F}_{3f} = -\frac{1}{8} g^4 m_1^2 [& T I_3 \mathcal{I}_1^2 + 2T^2 I_3 I_1 \mathcal{I}_1 + T^3 I_3 I_1^2 \\ & - 2m^2 T I_3 \mathcal{I}_1 \mathcal{I}_2 + \mathcal{I}_3 \mathcal{I}_1^2 - 2T^2 m^2 I_3 I_1 \mathcal{I}_2 \\ & + 2T I_1 \mathcal{I}_1 \mathcal{I}_3 + 2m^4 T I_3 \mathcal{I}_1 \mathcal{I}_3 + m^4 T I_3 \mathcal{I}_2^2], \end{aligned} \tag{48}$$

$$\begin{aligned} \mathcal{F}_{3g} = -\frac{1}{12} g^4 m_1^2 \Big[& T^3 I'_{\rm ball} + \mathcal{I}'_{\rm ball} + +3T^2 I_1 I_2 \mathcal{I}_2 \\ & + 2(T I_1 - m^2 T I_2) \sum\!\!\!\!\!\!\!\int_{QR} \frac{Q^2 + (2/d)\mathbf{q}^2}{Q^6 R^2 (Q + R)^2} \Big], \end{aligned} \tag{49}$$

$$\mathcal{F}_{3h} = \frac{1}{4} g^2 m_1^4 [T I_3 I_1 + T^2 I_1 I_3 + T I_1 I_3 - m^2 T I_2 I_3 + I_1 I_3 + m^4 T I_3 I_3], \tag{50}$$

$$\mathcal{F}_{3i} = \frac{1}{8} g^2 m_1^4 [T^2 I_2^2 + 2 T I_2 I_2 + I_2^2 - 4 m^2 T I_2 I_3], \tag{51}$$

$$\mathcal{F}_{3j} = -\frac{1}{6} m_1^6 [T I_3 + I_3], \tag{52}$$

where the self-energy $\Pi(P)$ is defined in Eq. (D2) and the integrals $I'_{\rm ball}$ and $I_{\rm triangle}$ are defined in Appendix B. The poles in Eqs. (43)–(52) are canceled by the counterterms in Eq. (32). The final result for the four-loop contribution to the free energy is

$$\begin{aligned}
\mathcal{F}_3 = {} & \frac{\pi^2 T^4}{90} \frac{5}{288} \frac{\alpha^3}{\hat{m}^3} \Bigg[1 + 18\left(11L + 3\gamma_E - 6 - 16\log 2 - 16\log\hat{m} - 8\frac{\zeta'(-1)}{\zeta(-1)}\right)\hat{m}^2 \\
& + \Bigg(1236 + 108 C^a_{\rm triangle} + 36 C'_{\rm ball} + 288\gamma_1 - \frac{9198}{5}\gamma_E + 450\gamma_E^2 - \frac{6456}{5}L + 432\gamma_E L + 648 L^2 + 135\pi^2 \\
& - 54\pi^2 C^b_{\rm triangle} - 216\pi^2\gamma_E + (2100 - 72\gamma_E + 1728L)\frac{\zeta'(-1)}{\zeta(-1)} + 432\left(\frac{\zeta'(-1)}{\zeta(-1)}\right)^2 - 432(\gamma_E + 2L)\frac{\zeta'(-3)}{\zeta(-3)} \\
& + 360\frac{\zeta''(-1)}{\zeta(-1)} + 1728\log 2 + 216\pi^2\log 2 + 432(4 - \pi^2)\log\hat{m} - 4534\zeta(3)\Bigg)\hat{m}^3 \\
& + \frac{9}{2}\Bigg(3742 - 288 C_I - 48 C'_{\rm ball} - 8064\gamma_1 - 6072\gamma_E - 2544\gamma_E^2 - 3904L - 1872\gamma_E L - 2184L^2 + 900\pi^2 \\
& + (1808 + 1824\gamma_E + 2496L)\frac{\zeta'(-1)}{\zeta(-1)} - 288\frac{\zeta''(-1)}{\zeta(-1)} + 2688\gamma_E\log 2 + 4992L\log 2 + 4992(\gamma_E + L)\log\hat{m} \\
& - 2304\gamma_E\log\pi + 2304\log^2(2\pi) - 15\zeta(3)\Bigg)\hat{m}^4\Bigg] \\
& - \frac{\pi^2 T^4}{90}\frac{5}{16}\frac{\alpha^2\hat{m}_1^2}{\hat{m}^3}\Bigg[1 + \left(84L + 36\gamma_E - 96\log\hat{m} - 36 - 96\log 2 - 48\frac{\zeta'(-1)}{\zeta(-1)}\right)\hat{m}^2 \\
& + 2\Bigg(48(L + \gamma_E)^2 - 268(L + \gamma_E) - \gamma_E(17 - 21\gamma_E) + 48\gamma_1 - 34 - \frac{9\pi^2}{2} - \frac{\zeta'(-1)}{\zeta(-1)}(34 + 12\gamma_E) \\
& - 12\frac{\zeta''(-1)}{\zeta(-1)} + \zeta(3) + 6C'_{\rm ball}\Bigg)\hat{m}^3 - 3(89 + 120(L + \gamma_E) + [18(L + \gamma_E)]^2 + 15\zeta(3))\hat{m}^4\Bigg] \\
& + \frac{\pi^2 T^4}{90}\frac{15}{8}\frac{\alpha\hat{m}_1^4}{\hat{m}^3}\left[1 + 18(L + \gamma_E)\hat{m}^2 + [24(L + \gamma_E)^2 + 2\zeta(3)]\hat{m}^3 - \frac{45}{2}\zeta(3)\hat{m}^4\right] - \frac{\pi^2 T^4}{90}\frac{15}{4}\frac{\hat{m}_1^6}{\hat{m}^3}[1 + 2\zeta(3)\hat{m}^3],
\end{aligned} \tag{53}$$

where the constants are

$$C'_{\rm ball} = 48.7976, \tag{54}$$

$$C^a_{\rm triangle} = -25.7055, \tag{55}$$

$$C^b_{\rm triangle} = 28.9250, \tag{56}$$

$$C_I = -38.5309. \tag{57}$$

There are a couple of calculational details that are worthwhile pointing out. The g^6 contribution arising from diagram $\mathcal{F}_{3d}$ when all momenta are hard (h) reads

$$\mathcal{F}_{3d}^{\rm (hhhh)} = \sum\!\!\!\!\!\!\!\int_P [\Pi(P)]^3. \tag{58}$$

This term can be combined with the g^6 term arising from the counterterm $\mathcal{F}_{2b}\Delta_1 g^2 = -g^4 I_{\rm ball}\Delta_1 g^2/48$ and gives

$$\sum\!\!\!\!\!\!\!\int_P \left\{[\Pi(P)]^3 - \frac{3}{(4\pi)^2\epsilon}[\Pi(P)]^2\right\}. \tag{59}$$

This particular combination was first calculated by Gynther *et al.* [13] using the methods of Arnold and Zhai. Similarly, we combine the g^7 term from $\mathcal{F}_{3d}$ with the term $T I_1 I_{\rm sun}$ from $\mathcal{F}_{2b}\Delta_1 g^2$, which gives

$$\sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2}\left\{[\Pi(P)]^2 - \frac{2}{(4\pi)^2\epsilon}[\Pi(P)]\right\}. \tag{60}$$

We calculate this sum-integral in Appendix D. Finally, the term from $\mathcal{F}_{2b}\Delta_1 m^2$ which involves $I_{\rm sun}$ can be combined with the term $-m^2 I_2 I_2 I_{\rm sun}$ arising from $\mathcal{F}_{3c}$ to give

$$\frac{1}{24} g^6 m^2 I_2 \left(\frac{1}{(4\pi)^2}\frac{1}{\epsilon} - I_2\right) I_{\rm sun}. \tag{61}$$

Since $\mathcal{I}_{\rm sun}$ vanishes at order ϵ^0 and the term inside the parenthesis is finite, the particular combination (61) vanishes in the limit $\epsilon \to 0$.

E. Pressure to four loops

The pressure $\mathcal{P}$ is given by $-\mathcal{F}$. The contributions to the pressure of zeroth, first, second order, and third order in g^2 are given by Eqs. (15), (21), (31), and (53), respectively. Adding these contributions and setting $\mathcal{E}_0 = 0$ and $m_1^2 = m^2$, we obtain approximations to the pressure in screened perturbation theory which are accurate to $\mathcal{O}(g^7)$.

The one-loop approximation to the pressure is

$$\mathcal{P}_0 = \mathcal{P}_{\rm ideal}\bigg[1 - 15\hat{m}^2 + 60\hat{m}^3 + 45\hat{m}^4(L+\gamma_E) - \frac{15}{2}\zeta(3)\hat{m}^6\bigg], \tag{62}$$

where $\mathcal{P}_{\rm ideal} = \pi^2 T^4/90$ is the pressure of an ideal gas of massless particles.

The two-loop approximation to the pressure is obtained by adding Eq. (21) with $m_1^2 = m^2$:

$$\mathcal{P}_{0+1} = \mathcal{P}_{\rm ideal}\bigg\{1 - \frac{5}{4}\alpha + 15\hat{m}\alpha + 15\hat{m}^2(L+\gamma_E-3)\alpha - 30\hat{m}^3[1+3(L+\gamma_E)\alpha] - 45\hat{m}^4\bigg[(L+\gamma_E) + \bigg((L+\gamma_E)^2 + \frac{1}{12}\zeta(3)\bigg)\alpha\bigg] + \frac{45}{2}\zeta(3)\hat{m}^5\alpha + 15\zeta(3)\hat{m}^6\bigg\}. \tag{63}$$

The three-loop approximation to the pressure is obtained by adding Eq. (31) with $m_1^2 = m^2$:

$$\begin{aligned}\mathcal{P}_{0+1+2} = \mathcal{P}_{\rm ideal}\bigg\{&1 + \frac{5}{8\hat{m}}\alpha^2 - \frac{5}{4}\alpha + \bigg(-\frac{59}{12} + \frac{15}{4}L + \frac{5}{4}\gamma_E + 5\frac{\zeta'(-1)}{\zeta(-1)} - \frac{5}{2}\frac{\zeta'(-3)}{\zeta(-3)}\bigg)\alpha^2 \\ &+ \frac{15}{2}\hat{m}\bigg[1 - \bigg(5 + 3\gamma_E + 7L - 8\log\hat{m} - 8\log 2 - 4\frac{\zeta'(-1)}{\zeta(-1)}\bigg)\alpha\bigg]\alpha \\ &+ \frac{5}{8}\hat{m}^2\bigg(268(L+\gamma_E) - 48(L+\gamma_E)^2 + \frac{\zeta'(-1)}{\zeta(-1)}(34+12\gamma_E) + 12\frac{\zeta''(-1)}{\zeta(-1)} + \gamma_E(17-21\gamma_E) + 34 \\ &+ \frac{9\pi^2}{2} - 48\gamma_1 - \zeta(3) - 6C'_{\rm ball}\bigg)\alpha^2 - \frac{15}{2}\hat{m}^3\bigg[1 - 6(L+\gamma_E)\alpha - \frac{1}{12}\bigg(89 + 120(L+\gamma_E) + [18(L+\gamma_E)]^2 \\ &+ 15\zeta(3)\bigg)\alpha^2\bigg] + 45\hat{m}^4\bigg((L+\gamma_E)^2 + \frac{1}{12}\zeta(3)\bigg)\alpha - \frac{135}{4}\hat{m}^5\alpha\zeta(3) - \frac{15}{2}\hat{m}^6\zeta(3)\bigg\}.\end{aligned} \tag{64}$$

The four-loop approximation to the pressure is obtained by adding Eq. (53) to Eq. (64), with $m_1^2 = m^2$:

$$\begin{aligned}\frac{\mathcal{P}_{0+1+2+3}}{\mathcal{P}_{\rm ideal}} = 1 &- \frac{5}{288}\frac{\alpha^3}{\hat{m}^3} + \frac{15}{16}\frac{1}{\hat{m}}\bigg[\alpha^2 + \frac{1}{3}\bigg(16\log\hat{m} + 6 - 3\gamma_E - 11L + 8\frac{\zeta'(-1)}{\zeta(-1)} + 16\log 2\bigg)\alpha^3\bigg] \\ &- \frac{5}{4}\bigg[\alpha - \bigg(3L - \frac{59}{15}\gamma_E + 4\frac{\zeta'(-1)}{\zeta(-1)} - 2\frac{\zeta'(-3)}{\zeta(-3)}\bigg)\alpha^2 + \frac{1}{72}\bigg(1236 + 36C'_{\rm ball} + 108C^a_{\rm triangle} + 288\gamma_1 \\ &- \frac{9198}{5}\gamma_E + 450\gamma_E^2 - \frac{6456}{5}L + 432\gamma_E L + 648L^2 + 135\pi^2 - 54\pi^2 C^b_{\rm triangle} - 216\pi^2\gamma_E \\ &+ (2100 - 72\gamma_E + 1728L)\frac{\zeta'(-1)}{\zeta(-1)} + 432\bigg(\frac{\zeta'(-1)}{\zeta(-1)}\bigg)^2 - 432(\gamma_E + 2L)\frac{\zeta'(-3)}{\zeta(-3)} + 360\frac{\zeta''(-1)}{\zeta(-1)} \\ &+ 1728\log 2 + 216\pi^2\log 2 + 432(4-\pi^2)\log\hat{m} - 4534\zeta(3)\bigg)\alpha^3\bigg] \\ &+ \frac{45}{8}\hat{m}\bigg[\alpha - \frac{2}{3}\bigg(13 + 3\gamma_E + 7L - 4\frac{\zeta'(-1)}{\zeta(-1)} - 8\log 2 - 8\log\hat{m}\bigg)\alpha^2 \\ &- \frac{1}{72}\bigg(3742 - 288C_I - 48C'_{\rm ball} - 8064\gamma_1 - 6072\gamma_E - 2544\gamma_E^2 - 3904L - 1872\gamma_E L - 2184L^2 + 900\pi^2 \\ &+ (1808 + 1824\gamma_E + 2496L)\frac{\zeta'(-1)}{\zeta(-1)} - 288\frac{\zeta''(-1)}{\zeta(-1)} + 2688\gamma_E\log 2 + 4992L\log 2 + 4992(\gamma_E + L)\log\hat{m} \\ &- 2304\gamma_E\log\pi + 2304\log^2(2\pi) - 15\zeta(3)\bigg)\alpha^3\bigg] - \frac{15}{4}\hat{m}^3\bigg[1 - 3(L+\gamma_E)\alpha + \frac{1}{12}(89 + 120(L+\gamma_E) \\ &+ [18(L+\gamma_E)]^2 + 15\zeta(3))\alpha^2\bigg] + \frac{135}{16}\zeta(3)\hat{m}^5\alpha.\end{aligned} \tag{65}$$

The final result for the pressure is given by Eq. (65). If we use the weak-coupling expansion for the mass parameter, $\hat{m}^2 = \alpha/6$, our result reduces to the weak-coupling expansion result through order α^3.[2] Inserting $\hat{m}^2$ into Eq. (65), we obtain

$$\mathcal{P} = \mathcal{P}_{\rm ideal}\Big\{1 - \frac{5}{4}\alpha + \frac{5\sqrt{6}}{3}\alpha^{3/2} + \frac{15}{4}\Big[\log\frac{\mu}{4\pi T} + C_4\Big]\alpha^2 - \frac{15\sqrt{6}}{2}\Big[\log\frac{\mu}{4\pi T} - \frac{2}{3}\log\alpha + C_5\Big]\alpha^{5/2} - \frac{45}{4}\Big[\log^2\frac{\mu}{4\pi T} - \frac{1}{3}\Big(\frac{269}{45} - 2\gamma_E - 8\frac{\zeta'(-1)}{\zeta(-1)} + 4\frac{\zeta'(-3)}{\zeta(-3)}\Big)\log\frac{\mu}{4\pi T} + \frac{1}{3}(4-\pi^2)\log\alpha + C_6\Big]\alpha^3\Big\}, \tag{66}$$

where the constants C_4–C_6 are

$$C_4 \equiv -\frac{59}{45} + \frac{1}{3}\gamma_E + \frac{4}{3}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{2}{3}\frac{\zeta'(-3)}{\zeta(-3)}, \tag{67}$$

$$C_5 \equiv \frac{5}{6} + \frac{1}{3}\gamma_E - \frac{2}{3}\log\frac{2}{3} - \frac{2}{3}\frac{\zeta'(-1)}{\zeta(-1)}, \tag{68}$$

$$C_6 \equiv \frac{1}{3}(4-\pi^2)\log\frac{2}{3} + \frac{103}{54} + \frac{1}{18}C'_{\rm ball} - \frac{1}{6}C^a_{\rm triangle} - \frac{\pi^2}{12}C^b_{\rm triangle} + \frac{4}{9}\gamma_1 - \frac{511}{180}\gamma_E + \frac{25}{36}\gamma_E^2 + \frac{5\pi^2}{24} - \frac{\pi^2}{3}\gamma_E + \pi^2\log 2 + \Big(\frac{175}{54} - \frac{1}{9}\gamma_E\Big)\frac{\zeta'(-1)}{\zeta(-1)} + \frac{2}{3}\Big(\frac{\zeta'(-1)}{\zeta(-1)}\Big)^2 + \frac{5}{9}\frac{\zeta''(-1)}{\zeta(-1)} - \frac{2}{3}\gamma_E\frac{\zeta'(-3)}{\zeta(-3)} - \frac{2267}{324}\zeta(3). \tag{69}$$

The numerical values of C_4–C_6 are

$$C_4 = 1.09775, \tag{70}$$

$$C_5 = -0.0273205, \tag{71}$$

$$C_6 = -6.59363. \tag{72}$$

Gynther *et al.* [13] have calculated the pressure for an $O(N)$-symmetric theory at weak coupling through order g^6 using effective field theory methods. Our result agrees with theirs for $N = 1$.

Using the renormalization group equation for the running coupling constant to next-to-leading order,

$$\mu\frac{d\alpha}{d\mu} = 3\alpha^2 - \frac{17}{3}\alpha^3, \tag{73}$$

it is straightforward to verify that the result (66) is independent of the renormalization scale μ through order $g^6\log g$.

[2] It is important to point out that we have only calculated part of the g^7 term in the weak-coupling expansion. See the discussion in Sec. V.

IV. GAP EQUATIONS AND NUMERICAL RESULTS

The mass parameter m in screened perturbation theory is completely arbitrary. In order to complete a calculation using SPT, we need a prescription for the mass parameter m as a function of g and T. One of the complications which arises from the ultraviolet divergences is that the parameters $\mathcal{E}_0$, m^2, m_1^2, and g^2 are all running parameters that depend on the renormalization scale μ.

The prescription of Karsch, Patkós, and Petreczky for $m_*(T)$ is the solution to the one-loop gap equation

$$m_*^2 = \frac{1}{2}\alpha(\mu_*)\Big[J_1(\beta m_*)T^2 - \Big(2\log\frac{\mu_*}{m_*} + 1\Big)m_*^2\Big], \tag{74}$$

where μ^* is the renormalization scale and $J_1(\beta m)$ is the function

$$J_1(\beta m) = 8\beta^2\int_0^\infty \frac{dp\,p^2}{(p^2+m^2)^{1/2}}\frac{1}{e^{\beta(p^2+m^2)^{1/2}} - 1}. \tag{75}$$

Their choice for the scale was $\mu_* = T$. In the weak-coupling limit, the solution to (74) is $m_* = g(\mu_*)T/\sqrt{24}$. The gap equation (74) is the renormalized version of the following equation:

$$m^2 = \frac{1}{2}g^2 \sum\!\!\!\!\!\!\int_P \frac{1}{P^2 + m^2}. \tag{76}$$

There are many possibilities for generalizing (74) to higher orders in g. We will consider three different possibilities in the following.

A. Debye mass

One class of possibilities is to identify m_* with some physical mass in the system. The simplest choice is the Debye mass m_D defined by the location of the pole in the static propagator:

$$p^2 + m^2 + \Sigma(0, p) = 0, \qquad p^2 = -m_D^2. \tag{77}$$

The Debye mass is a well-defined quantity in scalar field theory and Abelian gauge theories at any order in perturbation theory. However, in non-Abelian gauge theories, it is plagued by infrared divergences beyond leading order [52].

B. Tadpole mass

The *tadpole mass* is another generalization of Eq. (74) to higher loops. It can be calculated by taking the partial derivative of the free energy $\mathcal{F}$ with respect to m^2 *before* setting $m_1 = m$:

$$m_t^2 = g^2 \frac{\partial \mathcal{F}}{\partial m^2}\bigg|_{m_1=m}. \tag{78}$$

From this equation, we see that m_t^2 is proportional to the expectation value $\langle\phi^2\rangle$. The tadpole mass is well-defined at all orders in scalar field theory, but the generalization to gauge theories is problematic. The natural replacement of $\langle\phi^2\rangle$ would be $\langle A_\mu A_\mu\rangle$, which is a gauge-variant quantity.

C. Variational mass

There is another class of prescriptions that is variational in spirit. The results of SPT would be independent of m if they were calculated to all orders. This suggests choosing m to minimize the dependence of some physical quantity on m. The *variational mass* is defined by minimizing the free energy:

$$\frac{\partial \mathcal{F}}{\partial m^2} = 0. \tag{79}$$

The variational mass has the benefit that it is well-defined at all orders in perturbation theory and can easily be generalized to gauge theories.

D. Comparison

At one loop, the three different prescriptions give the same gap equation, Eq. (74). Moreover, it turns out that the two-loop tadpole mass coincides with the one-loop tadpole mass [18]. However, at two loops the screening and variational masses are ill-behaved [18]. The screening mass solution ceases to exist beyond $g \sim 2.6$ and the variational gap equation only has solutions in the vicinity of $g = 0$ for some values of L. In the following, we therefore restrict ourselves to the tadpole gap equation.

E. Tadpole gap equation through three loops

At one loop, the renormalized gap equation follows from Eq. (15) upon differentiation with respect to m^2 and can be written as

$$0 = \hat{m}^2 - \frac{1}{6}\alpha\left[1 - 6\hat{m} - 6\hat{m}^2(L+\gamma_E) + \frac{3}{2}\zeta(3)\hat{m}^4\right]. \tag{80}$$

At two loops, the renormalized gap equation follows from differentiating the sum of Eqs. (15) and (21) with respect to m, and setting $m_1 = m$. It can be written in the form

$$\begin{aligned} 0 = {}& \hat{m}^2 + \frac{\alpha^2}{12\hat{m}} - \frac{\alpha}{6}[1 + \alpha(3 - \gamma_E - L)] \\ & + \frac{1}{2}\hat{m}\alpha[1 - 3\alpha(\gamma_E + L)] \\ & - \hat{m}^2\alpha^2\left[(\gamma_E + L)^2 + \frac{\zeta(3)}{12}\right] + \frac{5}{8}\hat{m}^3\alpha^2\zeta(3) \\ & + \frac{1}{4}m^4\alpha\zeta(3). \end{aligned} \tag{81}$$

At three loops, the renormalized gap equation follows from differentiating the sum of Eqs. (15), (21), and (31) and setting $m_1 = m$. This yields

$$\begin{aligned} 0 = {}& \hat{m}^2 + \frac{1}{8}\frac{\alpha^2}{\hat{m}}\left\{1 + \alpha\left[1 - \gamma_E - \frac{7}{3}L + \frac{4}{3}\frac{\zeta'(-1)}{\zeta(-1)} + \frac{8}{3}\log 2 + \frac{8}{3}\log\hat{m}\right]\right\} - \frac{\alpha}{6}\left\{1 - \alpha(L + \gamma_E - 3)\right. \\ & + \alpha^2\left[2(L+\gamma_E)^2 - \frac{17}{12} + 2\gamma_1 - \frac{67}{6}(L+\gamma_E) - \frac{1}{24}\gamma_E(17 - 21\gamma_E) - \frac{3\pi^2}{16} - \frac{17}{12}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{1}{2}\gamma_E\frac{\zeta'(-1)}{\zeta(-1)}\right. \\ & \left.\left. - \frac{1}{2}\frac{\zeta''(-1)}{\zeta(-1)} + \frac{1}{24}\zeta(3) + \frac{1}{4}C'_{\rm ball}\right]\right\} + \frac{3}{8}\hat{m}\alpha\left\{1 - 2\alpha(L+\gamma_E) + \alpha^2\left[9(L+\gamma_E)^2 + \frac{10}{3}(L+\gamma_E) + \frac{89}{36} + \frac{5}{12}\zeta(3)\right]\right\} \\ & - \frac{5}{16}\hat{m}^3\alpha^2\zeta(3). \end{aligned} \tag{82}$$

F. Numerical results

The two-loop SPT-improved approximation to the pressure is obtained by inserting the solution to the one-loop gap equation (80) into the two-loop pressure (63). In Fig. 5(a) we show the various truncations to the two-loop SPT-improved approximation to the $\mathcal{P}/\mathcal{P}_{\rm ideal}$ as a function of $g(2\pi T)$. We notice that the various truncations converge quickly. The order-g^4 to order-g^7 results are almost indistinguishable and essentially equal to the exact numerical two-loop result in Ref. [18]. In the three-loop case, we insert the solution to the two-loop gap equation (81) into the three-loop pressure (64). In Fig. 5(b), we show the various truncations to the three-loop SPT-improved approximation to $\mathcal{P}/\mathcal{P}_{\rm ideal}$ as a function of $g(2\pi T)$. The three-loop result also converges to the exact numerical three-loop result, albeit not as fast as in the two-loop case. At four loops, we insert the solution to the three-loop gap equation (82) into the four-loop pressure (65). In Fig. 5(c), we show the various truncations to the four-loop SPT-improved approximation to $\mathcal{P}/\mathcal{P}_{\rm ideal}$ as a function of $g(2\pi T)$. Although we cannot compare our successive approximations with a numerically exact four-loop result for the pressure, we expect them to converge reasonably fast. Based on the experience with the two- and three-loop approximations, we expect that the g^7 truncation provides a good approximation to the numerically exact result.

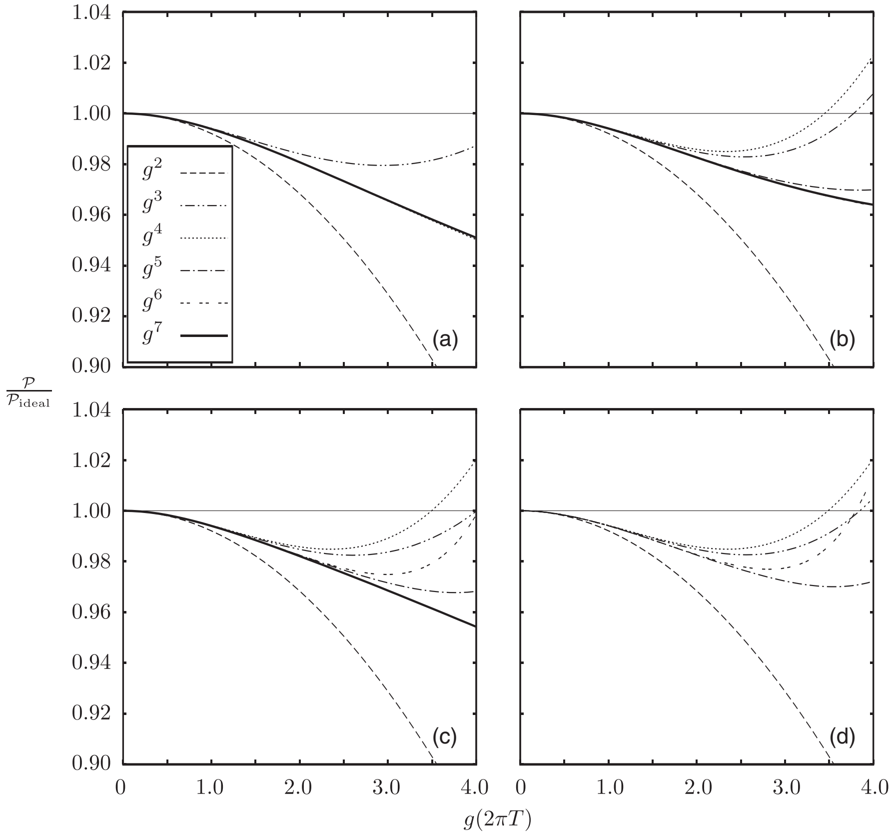

FIG. 5. (a) Two-loop pressure, (b) three-loop pressure, (c) four-loop pressure, (d) weak-coupling expansion of the pressure, all normalized to $\mathcal{P}_{\rm ideal}$.

Clearly, however, only a calculation through g^8 can settle this issue firmly. In Fig. 5(d), we show the weak-coupling expansion of $\mathcal{P}/\mathcal{P}_{\rm ideal}$ to orders g^2, g^3, g^4, g^5, and g^6 as a function of $g(2\pi T)$ for comparison. Note that the results to order g^2 are identical in SPT and in the weak-coupling expansion since there is no m-dependence at this order.

In Fig. 6(a), we show the two-, three-, and four-loop pressure through order g^7 normalized to $\mathcal{P}/\mathcal{P}_{\rm ideal}$ as a function of $g(2\pi T)$. In Fig. 6(b), we show the weak-coupling expansion of $\mathcal{P}/\mathcal{P}_{\rm ideal}$ to orders g^2, g^3, g^4, g^5, and g^6 as a function of $g(2\pi T)$ for comparison. The successive approximations using screened perturbation theory have better convergence properties than the weak-coupling results. The improved stability is partly due to the fact that we are using a thermal mass determined by a gap equation and not by the perturbative value for the Debye mass.

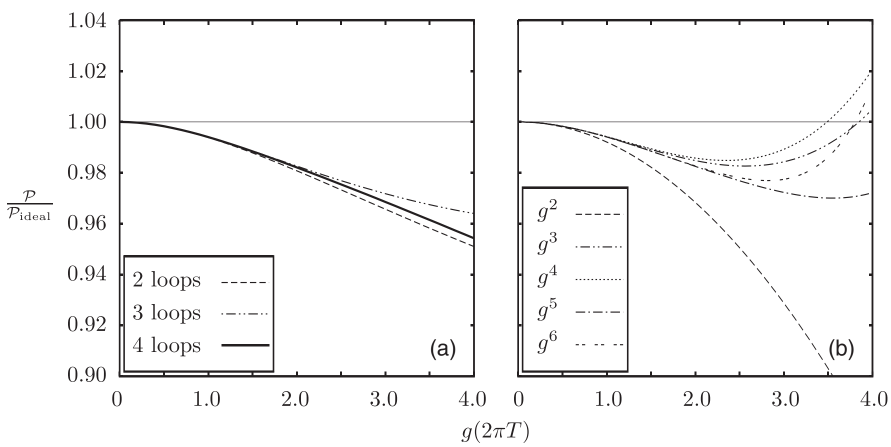

FIG. 6. (a) Pressure normalized to $\mathcal{P}_{\rm ideal}$ through g^7 for various loop orders, (b) weak-coupling pressure at various orders of g.

V. SUMMARY AND OUTLOOK

In this paper, we have calculated the pressure of massless scalar field theory to four loops using screened perturbation theory expanding in a double expansion in powers of g^2 and m/T. Treating m as $\mathcal{O}(gT)$, we truncated our expansion at order g^7. The expansion required the evaluation of a new nontrivial three-loop diagram, where we evaluated the sum-integral using the techniques developed in Ref. [10]. We have seen that the successive approximations are more stable than the weak-coupling expansion. In particular, it is interesting to note that the four-loop curve lies between the two-loop curve and the three-loop curve. The apparent improved convergence seemed to be linked to the fact that SPT basically is an expansion about an ideal gas of massive particles instead of an expansion about an ideal gas of massless particles which is the case for the weak-coupling expansion.

Using the weak-coupling value for the mass parameter m, our result reduces to the weak-coupling result for the pressure through g^6. In particular, we have reproduced the pressure at weak coupling for $N = 1$ obtained by Gynther *et al.* [13]. Using effective-field theory methods, the authors in Ref. [13] have calculated the hard and soft contributions to the pressure through order g^6 separately. It appears that the convergence properties in the hard sector are better than in the soft sector even for moderate values of the coupling.

We have mentioned that our result only includes part of the full g^7 term in the weak-coupling expansion. This is straightforward to see, if one uses the effective-field theory approach developed in [6]. The contributions to the free energy comes from the two momentum scales T and gT. The contribution from the hard scale T can be calculated by evaluating the sum-integrals with bare propagators and so is therefore a series in g^2 starting at order g^0. The contribution to the free energy from the soft scale gT can be calculated using an effective Euclidean three-dimensional field theory whose coefficients depend on g and T. This contribution to the free energy is a series in g starting at g^3. The contributions to the free energy that are odd in powers in g are therefore entirely coming from three-dimensional vacuum diagrams and power-counting tells you immediately that part of the g^7 term is arising from the five-loop vacuum diagrams. Our four-loop calculation therefore agrees with the weak-coupling expansion through order g^6.

In order to evaluate the free energy to order g^7, we must determine all the coefficients in the effective theory to sufficiently high order in g. The only nontrivial calculation that is required is to determine the mass parameter in the effective theory to order g^6. This involves the expression for the diagram calculated in Appendix D i.e. the sum-integral

$$I \equiv \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2}\left\{[\Pi(P)]^2 - \frac{2}{(4\pi)^2\epsilon}\Pi(P)\right\}. \qquad (83)$$

The evaluation of the free energy to order g^7 is in progress [53].

ACKNOWLEDGMENTS

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APPENDIX A: SUM-INTEGRALS

In the imaginary-time formalism for thermal field theory, the four-momentum $P = (P_0, \mathbf{p})$ is Euclidean with $P^2 = P_0^2 + \mathbf{p}^2$. The Euclidean energy p_0 has discrete values: $P_0 = 2n\pi T$ for bosons, where n is an integer. Loop diagrams involve sums over P_0 and integrals over $\mathbf{p}$. With dimensional regularization, the integral is generalized to $d = 3 - 2\epsilon$ spatial dimensions. We define the dimensionally regularized sum-integral by

$$\sum\!\!\!\!\!\!\!\int_P \equiv \left(\frac{e^{\gamma}\mu^2}{4\pi}\right)^{\epsilon} T \sum_{P_0=2n\pi T} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}, \qquad (A1)$$

where $3 - 2\epsilon$ is the dimension of space and μ is an arbitrary momentum scale. The factor $(e^{\gamma}/4\pi)^{\epsilon}$ is introduced so that, after minimal subtraction of the poles in ϵ due to ultraviolet divergences, μ coincides with the renormalization scale of the $\overline{\text{MS}}$ renormalization scheme.

1. One-loop sum-integrals

The massless one-loop sum-integral is given by

$$\begin{aligned} I_n &\equiv \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^{2n}} \\ &= (e^{\gamma_E}\mu^2)^{\epsilon}\frac{\zeta(2n-3+2\epsilon)}{8\pi^2}\frac{\Gamma(n-\frac{3}{2}+\epsilon)}{\Gamma(\frac{1}{2})\Gamma(n)} \\ &\quad \times (2\pi T)^{4-2n-2\epsilon}, \end{aligned} \qquad (A2)$$

where $\zeta(x)$ is Riemann's zeta function. Specifically, we need the sum-integrals:

$$I_0' \equiv \sum\!\!\!\!\!\!\!\int_P \log P^2 = -\frac{\pi^2 T^4}{45}[1 + \mathcal{O}(\epsilon)], \qquad (A3)$$

$$\begin{aligned} I_1 = \frac{T^2}{12}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\Bigg[1 &+ \left(2 + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon \\ &+ \left(4 + \frac{\pi^2}{4} + 4\frac{\zeta'(-1)}{\zeta(-1)} + 2\frac{\zeta''(-1)}{\zeta(-1)}\right)\epsilon^2 + \mathcal{O}(\epsilon^3)\Bigg], \end{aligned} \qquad (A4)$$

$$I_2 = \frac{1}{(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\left[\frac{1}{\epsilon} + 2\gamma_E + \left(\frac{\pi^2}{4} - 4\gamma_1\right)\epsilon + \mathcal{O}(\epsilon^2)\right], \qquad (A5)$$

$$\mathcal{I}_3 = \frac{1}{(4\pi)^4 T^2}[2\zeta(3) + \mathcal{O}(\epsilon)]. \tag{A6}$$

2. Two-loop sum-integrals

We need two two-loop sum-integrals that are listed below:

$$\mathcal{I}_{\rm sun} = \sum\!\!\!\!\!\!\int_{PQ} \frac{1}{P^2Q^2(P+Q)^2} = \mathcal{O}(\epsilon), \tag{A7}$$

$$\sum\!\!\!\!\!\!\int_{PQ} \frac{P^2 + (2/d)p^2}{P^6Q^2(P+Q)^2} = \frac{3}{4(4\pi)^4}\left(\frac{\mu}{4\pi T}\right)^{4\epsilon} \times \left[\frac{1}{\epsilon^2} + \left(\frac{5}{6} + 4\gamma_E\right)\frac{1}{\epsilon} + \frac{89}{36} + \frac{\pi}{2} + \frac{10}{3}\gamma_E + 4\gamma_E^2 - 8\gamma_1 + \mathcal{O}(\epsilon)\right]. \tag{A8}$$

The setting-sun sum-integral was first calculated by Arnold and Zhai in Ref. [10], while Eq. (A8) was calculated in Ref. [40].

3. Three-loop sum-integrals

We need the following three-loop sum-integrals:

$$\mathcal{I}_{\rm ball} = \sum\!\!\!\!\!\!\int_{PQR} \frac{1}{P^2Q^2R^2(P+Q+R)^2} = \frac{T^4}{24(4\pi)^2}\left(\frac{\mu}{4\pi T}\right)^{6\epsilon}\left[\frac{1}{\epsilon} + \frac{91}{15} + 8\frac{\zeta'(-1)}{\zeta(-1)} - 2\frac{\zeta'(-3)}{\zeta(-3)} + \mathcal{O}(\epsilon)\right], \tag{A9}$$

$$\mathcal{I}'_{\rm ball} = \sum\!\!\!\!\!\!\int_{PQR} \frac{1}{P^4Q^2R^2(P+Q+R)^2} = \frac{T^2}{8(4\pi)^4}\left(\frac{\mu}{4\pi T}\right)^{6\epsilon}\left[\frac{1}{\epsilon^2} + \left(\frac{17}{6} + 4\gamma_E + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon} + \frac{1}{2}\gamma_E\left(17 + 15\gamma_E + 12\frac{\zeta'(-1)}{\zeta(-1)}\right) + C'_{\rm ball} + \mathcal{O}(\epsilon)\right], \tag{A10}$$

$$\sum\!\!\!\!\!\!\int_{P} \frac{1}{P^2}\left\{[\Pi(P)]^2 - \frac{2}{(4\pi)^2\epsilon}\Pi(P)\right\} = -\frac{T^2}{4(4\pi)^4}\left(\frac{\mu}{4\pi T}\right)^{6\epsilon}\left\{\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left[\frac{4}{3} + 2\frac{\zeta'(-1)}{\zeta(-1)} + 4\gamma_E\right] + \frac{1}{3}\left[46 - 8\gamma_E - 16\gamma_E^2 - 104\gamma_1 - 24\gamma_E\log(2\pi) + 24\log^2(2\pi) + \frac{45\pi^2}{4} + 24\frac{\zeta'(-1)}{\zeta(-1)} + 2\frac{\zeta''(-1)}{\zeta(-1)} + 16\gamma_E\frac{\zeta'(-1)}{\zeta(-1)}\right] + C_I + \mathcal{O}(\epsilon)\right\}, \tag{A11}$$

where $C'_{\rm ball} = 48.7976$ and $C_I = -38.5309$. The massless basketball sum-integral was first calculated in Ref. [10] and $\mathcal{I}'_{\rm ball}$in Ref. [13]. The expression for the sum-integral Eq. (A11) appears here for the first time and is calculated in Appendix D.

4. Four-loop sum-integrals

We also need a single four-loop sum-integral which was calculated in Ref. [13]:

$$\sum\!\!\!\!\!\!\int_{P}\left\{[\Pi(P)]^3 - \frac{3}{(4\pi)^2\epsilon}[\Pi(P)]^2\right\} = -\frac{T^4}{16(4\pi)^4}\left[\frac{1}{\epsilon^2} + \left(\frac{10}{3} + 4\frac{\zeta'(-1)}{\zeta(-1)} + 4L\right)\frac{1}{\epsilon} + (2L+\gamma_E)^2 + \left(\frac{6}{5} - 2\gamma_E + 4\frac{\zeta'(-3)}{\zeta(-3)}\right)(2L+\gamma_E) + C^a_{\rm triangle}\right] - \frac{T^4}{512(4\pi)^2}\left[\frac{1}{\epsilon} + 8L + 4\gamma_E + C^b_{\rm triangle}\right] + \mathcal{O}(\epsilon), \tag{A12}$$

where $C^a_{\rm triangle} = -25.7055$ and $C^b_{\rm triangle} = 28.9250$.

APPENDIX B: THREE-DIMENSIONAL INTEGRALS

Dimensional regularization can be used to regularize both the ultraviolet divergences and infrared divergences in three-dimensional integrals over momenta. The spatial dimension is generalized to $d = 3 - 2\epsilon$ dimensions. Integrals are evaluated at a value of d for which they converge and then analytically continued to $d = 3$. We use the integration measure

$$\int_p \equiv \left(\frac{e^\gamma\mu^2}{4\pi}\right)^\epsilon \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}. \tag{B1}$$

1. One-loop integrals

The one-loop integral is given by

$$I_n \equiv \int_p \frac{1}{(p^2+m^2)^n} = \frac{1}{8\pi}(e^{\gamma_E}\mu^2)^\epsilon \frac{\Gamma(n - \frac{3}{2} + \epsilon)}{\Gamma(\frac{1}{2})\Gamma(n)} m^{3-2n-2\epsilon}. \tag{B2}$$

Specifically, we need:

$$I'_0 \equiv \int_p \log(p^2+m^2) = -\frac{m^3}{6\pi}\left(\frac{\mu}{2m}\right)^{2\epsilon}\left[1 + \frac{8}{3}\epsilon + \left(\frac{52}{9} + \frac{\pi^2}{4}\right)\epsilon^2 + \mathcal{O}(\epsilon^3)\right], \tag{B3}$$

$$I_1 = -\frac{m}{4\pi}\left(\frac{\mu}{2m}\right)^{2\epsilon}\left[1 + 2\epsilon + \left(4 + \frac{\pi^2}{4}\right)\epsilon^2 + \mathcal{O}(\epsilon^3)\right], \tag{B4}$$

$$I_2 = \frac{1}{8\pi m}\left(\frac{\mu}{2m}\right)^{2\epsilon}\left[1 + \frac{\pi^2}{4}\epsilon^2 + \mathcal{O}(\epsilon^3)\right], \tag{B5}$$

$$I_3 = \frac{1}{32\pi m^3}\left(\frac{\mu}{2m}\right)^{2\epsilon}\left[1 + 2\epsilon + \frac{\pi^2}{4}\epsilon^2 + \mathcal{O}(\epsilon^3)\right]. \tag{B6}$$

2. Three-loop integrals

We need two three-loop integrals:

$$\begin{aligned} I_{\rm ball} &= \int_{pqr}\frac{1}{p^2+m^2}\frac{1}{q^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} \\ &= -\frac{m}{(4\pi)^3}\left(\frac{\mu}{2m}\right)^{6\epsilon}\left[\frac{1}{\epsilon} + 8 - 4\log 2 \right. \\ &\quad \left. + 4\left(13 + \frac{17}{48}\pi^2 - 8\log 2 + \log^2 2\right)\epsilon + \mathcal{O}(\epsilon^2)\right], \end{aligned} \tag{B7}$$

$$\begin{aligned} I'_{\rm ball} &= \int_{pqr}\frac{1}{(p^2+m^2)^2}\frac{1}{q^2+m^2}\frac{1}{r^2+m^2} \\ &\quad \times \frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} \\ &= \frac{1}{8m(4\pi)^3}\left(\frac{\mu}{2m}\right)^{6\epsilon}\left[\frac{1}{\epsilon} + 2 - 4\log 2 \right. \\ &\quad \left. + 4\left(1 + \frac{17}{48}\pi^2 - 2\log 2 + \log^2 2\right)\epsilon + \mathcal{O}(\epsilon^2)\right]. \end{aligned} \tag{B8}$$

The massive basketball was calculated in Ref. [6] to order ϵ^0, and to order ϵ in Ref. [54]. The other three-loop integral is obtained by differentiating the massive basketball with respect to the mass m.

3. Four-loop integrals

We need a single four-loop integral, namely, the triangle integral. This integral was calculated in Ref. [54] and reads

$$\begin{aligned} I_{\rm triangle} &= \int_{pqrs}\frac{1}{q^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2}\frac{1}{r^2+m^2} \\ &\quad \times \frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2}\frac{1}{s^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{s})^2+m^2} \\ &= \frac{\pi^2}{32(4\pi)^4}\left(\frac{\mu}{2m}\right)^{8\epsilon}\left[\frac{1}{\epsilon} + 2 + 4\log 2 - \frac{84}{\pi^2}\zeta(3) \right. \\ &\quad \left. + \mathcal{O}(\epsilon)\right]. \end{aligned} \tag{B9}$$

APPENDIX C: m/T EXPANSIONS

In this appendix, we list the m/T expansions of the sum-integrals we need. The sum-integrals include sums over the Matsubara frequencies $P_0 = 2\pi nT$ and integrals over the three-momentum $\mathbf{p}$. In the sum-integrals, two important mass scales appear. These are the *hard* scale $2\pi T$ and the *soft* scale m. The soft scale m is of order gT and at weak coupling this scale is well-separated from the hard scale, $m \ll 2\pi T$. We can therefore expand the sum-integrals as a Taylor series in powers of m/T.

First consider the simple one-loop sum-integral appearing in the expression for the one-loop free energy in Eq. (13):

$$\begin{aligned} \mathcal{F}_{0\rm a} &= \frac{1}{2}\sum\!\!\!\!\!\!\int_P \log[P^2 + m^2] \\ &= \frac{1}{2}\sum\!\!\!\!\!\!\int_P^{(\rm h)} \log[P^2 + m^2] + \frac{1}{2}\sum\!\!\!\!\!\!\int_P^{(\rm s)} \log[P^2 + m^2], \end{aligned} \tag{C1}$$

where the superscripts (h) and (s) denote the hard and soft contributions, respectively. In the hard region, the momentum P is of order T and so we can expand in powers of m^2/P^2. This yields

$$\begin{aligned} \sum\!\!\!\!\!\!\int_P^{(\rm h)} \log[P^2 + m^2] &= \sum\!\!\!\!\!\!\int_P \log P^2 + m^2 \sum\!\!\!\!\!\!\int_P \frac{1}{P^2} \\ &\quad - \frac{1}{2}m^4 \sum\!\!\!\!\!\!\int_P \frac{1}{P^4} + \cdots. \end{aligned} \tag{C2}$$

The contribution from soft momenta is given by the $p_0 = 0$ mode alone and reads

$$\sum\!\!\!\!\!\!\int_P^{(\rm s)} \log[P^2 + m^2] = T\int_p \log(p^2 + m^2). \tag{C3}$$

The other simple one-loop sum-integrals are expanded in a similar manner.

We next consider the massive basketball diagram in Eq. (24):

$$\mathcal{I}_{\rm ball}(m^2) = \sum\!\!\!\!\!\!\int_{PQR} \frac{1}{(P^2+m^2)(Q^2+m^2)(R^2+m^2)[(P+Q+R)^2+m^2]}. \tag{C4}$$

Equation (C4) involves three sum-integrals and so receives contributions from four momentum regions: (hhh), (hhs), (hss), and (sss). In the first case, where all the loop momenta are hard, we can expand the sum-integral in powers of m^2. This yields

$$\mathcal{I}_{\rm ball}^{\rm (hhh)}(m^2) = \sum\!\!\!\!\!\!\!\int_{PQR} \frac{1}{P^2Q^2R^2(P+Q+R)^2} - 4m^2 \sum\!\!\!\!\!\!\!\int_{PQR} \frac{1}{P^4Q^2R^2(P+Q+R)^2} + \cdots. \tag{C5}$$

When two momenta are hard and one is soft, the contribution reads

$$\begin{aligned}\mathcal{I}_{\rm ball}^{\rm (hhs)}(m^2) &= 4T\int_p \frac{1}{p^2+m^2} \sum\!\!\!\!\!\!\!\int_{QR} \frac{1}{Q^2+m^2}\frac{1}{R^2+m^2}\frac{1}{(\mathbf{p}+Q+R)^2+m^2} \\ &= 4T\int_p \frac{1}{p^2+m^2} \sum\!\!\!\!\!\!\!\int_{QR} \frac{1}{Q^2R^2(Q+R)^2} - 8m^2T\int_p \frac{1}{p^2+m^2}\left[\sum\!\!\!\!\!\!\!\int_{QR} \frac{Q^2+(2/d)\mathbf{q}^2}{Q^6R^2(Q+R)^2}\right] + \cdots. \end{aligned} \tag{C6}$$

When one momentum is hard and two are soft, the contribution is given by

$$\mathcal{I}_{\rm ball}^{\rm (hss)}(m^2) = 6T^2\int_{pq}\frac{1}{p^2+m^2}\frac{1}{q^2+m^2}\sum\!\!\!\!\!\!\!\int_R \frac{1}{R^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q}+R)^2+m^2} = 6T^2\int_{pq}\frac{1}{p^2+m^2}\frac{1}{q^2+m^2}\sum\!\!\!\!\!\!\!\int_R\frac{1}{R^4} + \cdots. \tag{C7}$$

Finally, when all momenta are soft, the contribution is given by the massive basketball diagram $I_{\rm ball}$ in three dimensions:

$$\mathcal{I}_{\rm ball}^{\rm (sss)}(m^2) = T^3\int_{pqr}\frac{1}{p^2+m^2}\frac{1}{q^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2}. \tag{C8}$$

The basketball diagram with a single mass insertion $\mathcal{I}'_{\rm ball}(m^2)$ can be calculated by differentiating the massive basketball diagram with respect to m^2. This yields

$$\begin{aligned}\mathcal{I}'_{\rm ball}(m^2) &= \sum\!\!\!\!\!\!\!\int_{PQR}\frac{1}{(P^2+m^2)^2}\frac{1}{Q^2+m^2}\frac{1}{R^2+m^2}\frac{1}{(P+Q+R)^2+m^2} \\ &= \sum\!\!\!\!\!\!\!\int_{PQR}\frac{1}{P^4Q^2R^2(P+Q+R)^2} + T\int_p\frac{1}{(p^2+m^2)^2}\sum\!\!\!\!\!\!\!\int_{QR}\frac{1}{Q^2R^2(Q+R)^2} \\ &\quad + 2T\int_p\frac{p^2}{(p^2+m^2)^2}\left[\sum\!\!\!\!\!\!\!\int_{QR}\frac{Q^2+(2/d)\mathbf{q}^2}{Q^6R^2(Q+R)^2}\right] + 3T^2\int_{pq}\frac{1}{p^2+m^2}\frac{1}{(q^2+m^2)^2}\sum\!\!\!\!\!\!\!\int_R\frac{1}{R^4} \\ &\quad + T^3\int_{pqr}\frac{1}{(p^2+m^2)^2}\frac{1}{q^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2} + \cdots. \end{aligned} \tag{C9}$$

Note that the second term is formally of order g^5, but it vanishes at order ϵ^0 due to the fact that $\mathcal{I}_{\rm sun} = \mathcal{O}(\epsilon)$.

The massive four-loop triangle sum-integral reads

$$\mathcal{I}_{\rm triangle}(m^2) = \sum\!\!\!\!\!\!\!\int_{PQRS}\frac{1}{Q^2+m^2}\frac{1}{(P+Q)^2+m^2}\frac{1}{R^2+m^2}\frac{1}{(P+R)^2+m^2}\frac{1}{S^2+m^2}\frac{1}{(P+S)^2+m^2}. \tag{C10}$$

When all four momenta are hard, the leading contribution is given by setting $m=0$, i.e.

$$\mathcal{I}_{\rm triangle}^{\rm (hhhh)}(m^2) = \sum\!\!\!\!\!\!\!\int_{PQRS}\frac{1}{Q^2(P+Q)^2R^2(P+R)^2S^2(P+S)^2}. \tag{C11}$$

When one of the momenta is hard and three are soft, we find

$$\mathcal{I}_{\rm triangle}^{\rm (hsss)}(m^2) = 3T^3\int_{pqr}\frac{1}{p^2+m^2}\frac{1}{q^2+m^2}\frac{1}{r^2+m^2}\frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{r})^2+m^2}\sum\!\!\!\!\!\!\!\int_S\frac{1}{S^4} + \cdots.$$

This contribution is of order g^7. When one momentum is soft and three momenta are hard, the contribution is

$$\begin{aligned}\mathcal{I}_{\rm triangle}^{\rm (shhh)}(m^2) &= 6T\int_s\frac{1}{s^2+m^2}\sum\!\!\!\!\!\!\!\int_{PQR}\frac{1}{P^2+m^2}\frac{1}{Q^2+m^2}\frac{1}{(P+Q)^2+m^2}\frac{1}{R^2+m^2}\frac{1}{(P+R)^2+m^2} \\ &= 6T\int_s\frac{1}{s^2+m^2}\sum\!\!\!\!\!\!\!\int_{PQR}\frac{1}{P^2Q^2R^2(P+Q)(P+R)^2} + \cdots. \end{aligned} \tag{C12}$$

This contribution is of order g^7. When all four loop momenta are soft, the contribution is given by the massive three-dimensional triangle diagram $I_{\rm triangle}$:

$$I^{(ssss)}_{\text{triangle}}(m^2) = T^4 \int_{pqrs} \frac{1}{q^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2} \frac{1}{r^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{r})^2+m^2} \frac{1}{s^2+m^2} \frac{1}{(\mathbf{p}+\mathbf{s})^2+m^2}. \tag{C13}$$

This contribution is of order g^6. Finally, we notice that the contribution when two momenta are soft and two momenta are hard, is of higher order in the coupling g.

APPENDIX D: EXPLICIT CALCULATIONS

In this appendix, we illustrate the use of the calculational techniques developed by Arnold and Zhai in Ref. [10] to evaluate complicated multiloop diagrams. The strategy is to rewrite the original sum-integral into two sets of terms. The first type is ultraviolet divergent, but is sufficiently simple to be evaluated analytically using dimensional regularization. The second type is finite both in the ultraviolet and the infrared, but is normally so complicated that it must be evaluated numerically. In order to isolate the divergences in terms that are tractable, typically one or more subtractions are required.

We need to calculate the following three-loop diagram:

$$I \equiv \sum\!\!\!\!\!\!\int_P \frac{1}{P^2}\left\{[\Pi(P)]^2 - \frac{2}{(4\pi)^2\epsilon}\Pi(P)\right\}, \tag{D1}$$

where the self-energy $\Pi(P)$ is defined by

$$\Pi(P) = \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2(P+Q)^2}. \tag{D2}$$

The first term in Eq. (D1) arises from the m/T-expansion of the triangle sum-integral in four dimensions, while the second term arises from the term $TI_1 I_{\text{sun}}$ which is a part of the counterterm $\mathcal{F}_{2b}\Delta_1 g^2/g^2$.

At zero temperature, the self-energy is denoted by $\Pi^0(P)$ and reads

$$\Pi^0(P) = \frac{1}{(4\pi)^2}\left(\frac{e^{\gamma_E}\mu^2}{P^2}\right)^{\epsilon} \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}. \tag{D3}$$

In order to isolate the UV divergences and simplify the calculations, we write the self-energy as

$$\Pi(P) = \frac{1}{(4\pi)^2\epsilon} + \Pi^0_s(P) + \Pi^T(P), \tag{D4}$$

where $\Pi^0_s(P)$ is the finite part of $\Pi^0(P)$, i.e. we have subtracted the divergent piece in Eq. (D3) from $\Pi^0(P)$:

$$\Pi^0_s(P) = \frac{1}{(4\pi)^2}\left\{\left(\frac{e^{\gamma_E}\mu^2}{P^2}\right)^{\epsilon} \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} - \frac{1}{\epsilon}\right\}, \tag{D5}$$

and $\Pi^T(P)$ is the finite-temperature piece of $\Pi(P)$. In three dimensions, $\Pi^T(P)$ reads [10]

$$\Pi^T(P) = \frac{T}{(4\pi)^2}\int \frac{d^3r}{r^2} e^{i\mathbf{p}\cdot\mathbf{r}}\left(\coth\bar{r} - \frac{1}{\bar{r}}\right)e^{-|p_0|r}, \tag{D6}$$

where $\bar{r} = 2\pi Tr$. In the following we need the UV limit of $\Pi^T(P)$. This happens to be given by the UV limit of the full self-energy (D2) and is given by [10]

$$\Pi^T_{\text{UV}}(P) = \frac{2}{P^2}\sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2}. \tag{D7}$$

Using the decomposition (D4), the integral in Eq. (D1) can be written as

$$I = -\frac{1}{(4\pi)^4\epsilon^2}\sum\!\!\!\!\!\!\int_P \frac{1}{P^2} + \sum\!\!\!\!\!\!\int_P \frac{1}{P^2}[\Pi^0_s(P)]^2 + 2\sum\!\!\!\!\!\!\int_P \frac{1}{P^2}\Pi^0_s(P)\Pi^T(P) + \sum\!\!\!\!\!\!\int_P \frac{1}{P^2}[\Pi^T(P)]^2. \tag{D8}$$

We now consider the different contributions to I. The first term in Eq. (D8) is a simple one-loop sum-integral and reads

$$\begin{aligned} I_1 &= -\frac{1}{(4\pi)^4\epsilon^2}\sum\!\!\!\!\!\!\int_P \frac{1}{P^2} \\ &= -\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\frac{T^2}{12(4\pi)^4}\left[\frac{1}{\epsilon^2} + 2\left(1 + \frac{\zeta'(-1)}{\zeta(-1)}\right)\frac{1}{\epsilon} + \frac{\pi^2}{4} + 4 + 4\frac{\zeta'(-1)}{\zeta(-1)} + 2\frac{\zeta''(-1)}{\zeta(-1)} + \mathcal{O}(\epsilon)\right]. \end{aligned} \tag{D9}$$

The second term in Eq. (D8) contains no logarithmic UV divergences and so it is finite in dimensional regularization:

$$\begin{aligned} I_2 &= \sum\!\!\!\!\!\!\int_P \frac{1}{P^2}[\Pi^0_s(P)]^2 \\ &= \frac{T^2}{12(4\pi)^4}\left[4 + \frac{\pi^2}{3} + 8\frac{\zeta'(-1)}{\zeta(-1)}\left(2 + \log\frac{\mu}{4\pi T}\right) + 4\frac{\zeta''(-1)}{\zeta(-1)} + 4\left(2 + \log\frac{\mu}{4\pi T}\right)^2\right] + \mathcal{O}(\epsilon). \end{aligned} \tag{D10}$$

The third term requires a little more thought. Since the UV behavior of $\Pi^T(P)$ is $1/P^2$, the integrand $\Pi^0_s(P)\Pi^T(P)/P^2$ is logarithmically divergent in the ultraviolet. In order to isolate this divergence, we add and subtract $\Pi^T_{\text{UV}}(P)$ from $\Pi^0_s(P)\Pi^T(P)/P^2$. Thus the third sum-integral in Eq. (D8) becomes

$$\begin{aligned} I_3 &= 2\sum\!\!\!\!\!\!\int_P \frac{1}{P^2}\Pi^0_s(P)\Pi^T(P) \\ &= 2\sum\!\!\!\!\!\!\int_P' \frac{1}{P^2}\Pi^0_s(P)[\Pi^T(P) - \Pi^T_{\text{UV}}(P)] + 2\sum\!\!\!\!\!\!\int_P' \frac{1}{P^2}\Pi^0_s(P)\Pi^T_{\text{UV}}(P) \\ &\quad + 2T\int_p \frac{1}{p^2}\Pi^0_s(p_0=0,p)\Pi^T(p_0=0,p), \end{aligned} \tag{D11}$$

where we have isolated the contribution from the $p_0 = 0$ term since the contribution to I_3 from this term is infrared divergent. In order to calculate the first term in Eq. (D11), we need $\Pi^T_{\rm UV}(P)$ in coordinate space. It is given by the small-r behavior of $\Pi^T(P)$ and reads

$$\Pi^T_{\rm UV}(P) = \frac{T}{(4\pi)^2}\int \frac{d^3r}{r^2} e^{i\mathbf{p}\cdot\mathbf{r}} \frac{\bar{r}}{3} e^{-|p_0|r}. \tag{D12}$$

This yields

$$\begin{aligned} I_3^{\rm a} &= 2\sum\!\!\!\!\!\!\int_P{}' \frac{1}{P^2}\Pi_s^0(P)[\Pi^T(P) - \Pi^T_{\rm UV}(P)] \\ &= \frac{2T^2}{(4\pi)^4}\int d^3r \frac{1}{r^2}\left(\coth\bar{r} - \frac{1}{\bar{r}} - \frac{\bar{r}}{3}\right)\sum_{p_0\neq 0} e^{-|p_0|r} \\ &\quad \times \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p_0^2+p^2}\left(2 + \log\frac{\mu^2}{p_0^2+p^2}\right). \end{aligned} \tag{D13}$$

The integral over three-momentum can be done analytically. We write it as

$$\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p_0^2+p^2}\left(2 + 2\log\frac{\mu}{4\pi T} + \log\frac{(4\pi T)^2}{p_0^2+p^2}\right), \tag{D14}$$

where the first two terms in the parentheses are independent of p, making this part of the integral a simple Fourier transform:

$$\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p_0^2+p^2}\left(2 + 2\log\frac{\mu}{4\pi T}\right) = \frac{e^{-|p_0|r}}{4\pi r}\left(2 + 2\log\frac{\mu}{4\pi T}\right). \tag{D15}$$

Averaging over angles, the last term can be rewritten as

$$\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p_0^2+p^2}\log\frac{(4\pi T)^2}{p_0^2+p^2} = \frac{1}{4\pi^2 ir}\int_{-\infty}^{\infty} dp\, p \frac{e^{ipr}}{p_0^2+p^2}\log\frac{(4\pi T)^2}{p_0^2+p^2}. \tag{D16}$$

The integrand has a branch cut starting at $p = i|p_0|$ running to $p = i\infty$, and a pole in $p = i|p_0|$. The contour can be deformed to wrap around the pole and the branch cut, and taking care to include contributions from both, one arrives at the result

$$\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p_0^2+p^2}\log\frac{(4\pi T)^2}{p_0^2+p^2} = \frac{e^{-|p_0|r}}{4\pi r}\left(\log\frac{2\bar{r}}{|\bar{p}_0|} + \gamma_E + e^{2|p_0|r}{\rm Ei}(-2|p_0|r)\right), \tag{D17}$$

where $\bar{p}_0 = p_0/2\pi T = n$ and the exponential-integral function ${\rm Ei}(z)$ is defined as

$${\rm Ei}\,(z) = -\int_{-z}^{\infty}\frac{dte^{-t}}{t}. \tag{D18}$$

Thus Eq. (D13) can be rewritten as

$$\begin{aligned} I_3^a &= \frac{2T^2}{(4\pi)^4}\int d^3r \frac{1}{r^2}\left(\coth\bar{r} - \frac{1}{\bar{r}} - \frac{\bar{r}}{3}\right)\sum_{p_0\neq 0}\frac{e^{-2|p_0|r}}{4\pi r} \\ &\quad\times\left(2 + \gamma_E + 2\log\frac{\mu}{4\pi T} + \log\frac{2\bar{r}}{|\bar{p}_0|} + e^{2|p_0|r}{\rm Ei}(-2|p_0|r)\right). \end{aligned} \tag{D19}$$

The first three terms in the last parentheses are independent of r and p_0 and, for these terms, the integral over r and the sum over Matsubara modes can be evaluated analytically. In particular, we are able to find the coefficient of $\log\mu$. This is fortunate, because it allows us to check the consistency of our final result for the free energy. Let

$$\xi \equiv \frac{2T^2}{(4\pi)^4}\int d^3r\frac{1}{r^2}\left(\coth\bar{r} - \frac{1}{\bar{r}} - \frac{\bar{r}}{3}\right)\sum_{p_0\neq 0}\frac{e^{-2|p_0|r}}{4\pi r}. \tag{D20}$$

Integrating over angles and summing over Matsubara frequencies yields

$$\begin{aligned} \xi &= \frac{2T^2}{(4\pi)^4}\int_0^\infty \frac{d\bar{r}}{\bar{r}}\left(\coth\bar{r} - \frac{1}{\bar{r}} - \frac{\bar{r}}{3}\right)\frac{2}{e^{2\bar{r}}-1} \\ &= \frac{4T^2}{(4\pi)^4}\int_0^\infty \frac{d\bar{r}}{\bar{r}}\left(\frac{2}{e^{2\bar{r}}-1} + 1 - \frac{1}{\bar{r}} - \frac{\bar{r}}{3}\right)\frac{1}{e^{2\bar{r}}-1}. \end{aligned} \tag{D21}$$

The integral above is finite, but the individual terms are divergent for small $\bar{r}$. We therefore regulate them by multiplying by an extra factor $(2\bar{r})^\alpha$ and taking the limit $\alpha \to 0$ in the end. The basic integrals we need are

$$\int_0^\infty \frac{dtt^x}{e^t - 1} = \Gamma(x+1)\zeta(x+1), \tag{D22}$$

$$\int_0^\infty \frac{dtt^x}{(e^t-1)^2} = \Gamma(x+1)[\zeta(x) - \zeta(x+1)]. \tag{D23}$$

This yields

$$\begin{aligned} \xi = \frac{4T^2}{(4\pi)^4}\Big[&2\Gamma(\alpha)[\zeta(\alpha-1) - \zeta(\alpha)] + \Gamma(\alpha)\zeta(\alpha) \\ &- 2\Gamma(\alpha-1)\zeta(\alpha-1) - \frac{1}{6}\Gamma(\alpha+1)\zeta(\alpha+1)\Big]. \end{aligned} \tag{D24}$$

The limit $\alpha \to 0$ is regular, and we obtain

$$\xi = -\frac{2T^2}{3(4\pi)^4}\left(1 + \gamma_E - 3\log(2\pi) + 2\frac{\zeta'(-1)}{\zeta(-1)}\right). \tag{D25}$$

The remaining integral over the coordinate r as well as the Matsubara sum in Eq. (D19) must be done numerically. Equation (D19) can then be written as

$$\begin{aligned} I_3^{\rm a} = -\frac{2T^2}{3(4\pi)^4}\Big[&\left(2 + \gamma_E + 2\log\frac{\mu}{4\pi T}\right) \\ &\times\left(1 + \gamma_E - 3\log(2\pi) + 2\frac{\zeta'(-1)}{\zeta(-1)}\right) + C\Big], \end{aligned} \tag{D26}$$

where the numerical constant C is

$$C = -\frac{3}{4\pi}\int\frac{d^3r}{r^3}\left(\coth\bar{r} - \frac{1}{\bar{r}} - \frac{\bar{r}}{3}\right)\sum_{p_0\neq 0}\left(e^{-2|p_0|r}\log\frac{2\bar{r}}{|\bar{p}_0|} + \text{Ei}(-2|p_0|r)\right) = 0.003\,481\,4. \tag{D27}$$

The subtraction term in Eq. (D11) can be calculated with dimensional regularization and reads

$$\begin{aligned} I_3^{\rm b} &= 2\sum\!\!\!\!\!\!\int_P \frac{1}{P^2}\Pi_s^0(P)\Pi_{\rm UV}^T(P) \\ &= \frac{4}{(4\pi)^2}\sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2}\sum\!\!\!\!\!\!\int_P \frac{1}{P^4}\left\{\left(\frac{e^{\gamma_E}\mu^2}{P^2}\right)^\epsilon\frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} - \frac{1}{\epsilon}\right\} \\ &= -\frac{T^2}{6(4\pi)^4}\left[\frac{1}{\epsilon^2} + \left(2\log\frac{\mu}{4\pi T} + 2\frac{\zeta'(-1)}{\zeta(-1)} + 1\right)\frac{1}{\epsilon} - 2\log^2\frac{\mu}{4\pi T} - 2\log\frac{\mu}{4\pi T}\left(1 + 4\gamma_E - 2\frac{\zeta'(-1)}{\zeta(-1)}\right) \right. \\ &\quad \left. + 2\frac{\zeta'(-1)}{\zeta(-1)} + 2\frac{\zeta''(-1)}{\zeta(-1)} - 1 - \frac{\pi^2}{12} - 4\gamma_E + 8\gamma_1\right]. \end{aligned} \tag{D28}$$

The last term in Eq. (D11) is

$$\begin{aligned} I_3^{\rm c} &= 2T\int_p \frac{1}{p^2}\Pi_s^0(p_0 = 0, p)\Pi^T(p_0 = 0, p) \\ &= 2T\int_p \frac{1}{p^2}\Pi_s^0(p_0 = 0, p) \times [\Pi(p_0 = 0, p) - \Pi^0(p_0 = 0, p)]. \end{aligned} \tag{D29}$$

The second term vanishes in dimensional regularization since there is no mass scale in the integral, i.e.

$$2T\int_p \frac{1}{p^2}\Pi_s^0(p_0 = 0, p)\Pi^0(p_0 = 0, p) = 0. \tag{D30}$$

In order to evaluate the first term in Eq. (D29), we must calculate $\Pi(p_0 = 0, p)$. Using Feynman parameters, we obtain

$$\begin{aligned} \Pi(p_0 = 0, p) &= \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2(\mathbf{p} + Q)^2} \\ &= T\left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^\epsilon\frac{\Gamma(1/2 + \epsilon)}{(4\pi)^{(3/2)-\epsilon}} \times \sum_{q_0}\int_0^1 \frac{dx}{[x(1-x)p^2 + q_0^2]^{(1/2)+\epsilon}}. \end{aligned} \tag{D31}$$

Inserting the expression for $\Pi_s^0(p_0 = 0, p)$ and $\Pi(p_0 = 0, p)$, we obtain

$$\begin{aligned} I_3^{\rm c} &= \frac{2T^2}{(4\pi)^2}\left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^\epsilon\frac{\Gamma(1/2 + \epsilon)}{(4\pi)^{(3/2)-\epsilon}}\int_p \frac{1}{p^2} \\ &\quad \times \left[\left(\frac{e^{\gamma_E}\mu^2}{p^2}\right)^\epsilon\frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} - \frac{1}{\epsilon}\right] \\ &\quad \times \sum_{q_0}\int_0^1 \frac{dx}{[x(1-x)p^2 + q_0^2]^{(1/2)+\epsilon}}. \end{aligned} \tag{D32}$$

$$\begin{aligned} I_3^{\rm c} &= \frac{2T^2}{(4\pi)^4}\frac{(e^\gamma\mu^2)^{2\epsilon}}{2\pi}\frac{\Gamma(\frac{1}{2} + \epsilon)}{\Gamma(\frac{3}{2} - \epsilon)}\left[(e^\gamma\mu^2)^\epsilon\frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\int_0^\infty dp\frac{p^{-4\epsilon}}{(p^2+1)^{(1/2)+\epsilon}}\int_0^1 dx[x(1-x)]^{-(1/2)+2\epsilon}\sum_{q_0}{}'\frac{1}{|q_0|^{6\epsilon}} \right. \\ &\quad \left. - \frac{1}{\epsilon}\int_0^\infty dp\frac{p^{-2\epsilon}}{(p^2+1)^{(1/2)+\epsilon}}\int_0^1 dx[x(1-x)]^{-(1/2)+\epsilon}\sum_{q_0}{}'\frac{1}{|q_0|^{4\epsilon}}\right] \\ &= \frac{2T^2}{(4\pi)^4}\left(\frac{e^\gamma\mu^2}{4\pi^2T^2}\right)^{2\epsilon}\frac{\Gamma(\frac{1}{2} + \epsilon)}{\Gamma(\frac{3}{2} - \epsilon)}\left[\left(\frac{e^\gamma\mu^2}{4\pi^2T^2}\right)^\epsilon\frac{1}{2^{1+4\epsilon}\sqrt{\pi}}\frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)\Gamma(\frac{1}{2} - 2\epsilon)\Gamma(3\epsilon)\Gamma(\frac{1}{2} + 2\epsilon)}{\Gamma(2-2\epsilon)\Gamma(\frac{1}{2} + \epsilon)\Gamma(1+2\epsilon)}\zeta(6\epsilon) \right. \\ &\quad \left. - \frac{1}{4\pi\epsilon}\frac{\Gamma(\frac{1}{2} - \epsilon)\Gamma(\epsilon)\Gamma(\frac{1}{2} + \epsilon)}{\Gamma(1+\epsilon)}\zeta(4\epsilon)\right], \end{aligned} \tag{D33}$$

where the prime indicates that we have omitted the $p_0 = 0$ mode from the sum. Expanding Eq. (D33) in powers of ϵ, we obtain

$$\begin{aligned} I_3^{\rm c} &= \frac{T^2}{6(4\pi)^4}\left[\frac{1}{\epsilon^2} - \frac{2}{\epsilon} - 12 - \frac{11\pi^2}{3} - 24\log(2\pi) - 12\log^2(2\pi) - 24\log\frac{\mu}{4\pi T} - 12\log^2\frac{\mu}{4\pi T} \right. \\ &\quad \left. - 24\log(2\pi)\log\frac{\mu}{4\pi T} + 12\gamma_E^2 + 24\gamma_1\right] + \mathcal{O}(\epsilon). \end{aligned} \tag{D34}$$

The last term in Eq. (D8) is

$$I_4 = \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2}[\Pi^T(P)]^2. \tag{D35}$$

Since the UV behavior of $\Pi^T(P)$ is $1/P^2$, the sum-integral in Eq. (D35) is UV finite. However, $\Pi_T(P)$ has a logarithmic infrared divergence for the $p_0 = 0$ mode. This implies that the sum-integral I_4 has linear and logarithmic IR divergences. The linear divergence is set to zero in dimensional regularization while the logarithmic is not. In order to isolate these divergences, we rewrite the sum-integral as

$$I_4 = \sum\!\!\!\!\!\!\!\int_P' \frac{1}{P^2}[\Pi^T(P)]^2 + T\int_p \frac{1}{p^2}[\Pi^T(p_0=0,p)]^2, \tag{D36}$$

where the prime indicates that we have omitted the $p_0 = 0$ mode from the sum. The primed sum-integral in Eq. (D36) is finite both in the ultraviolet and in the infrared. Using the three-dimensional representation of the $\Pi_T(P)$, Eq. (D6), the first term in Eq. (D36) can be written as

$$\begin{aligned} I_4^{\rm a} &= \sum\!\!\!\!\!\!\!\int_P' \frac{1}{P^2}[\Pi^T(P)]^2 \\ &= \frac{T^3}{(4\pi)^4}\sum_{p_0}{}' \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3r}{r^2}\frac{d^3r'}{(r')^2}\frac{1}{p_0^2+p^2}\left(\coth\bar{r}-\frac{1}{\bar{r}}\right) \\ &\quad\times\left(\coth\bar{r}'-\frac{1}{\bar{r}'}\right)e^{i\mathbf{p}\cdot(\mathbf{r}+\mathbf{r}')}e^{-|p_0|(r+r')}. \end{aligned} \tag{D37}$$

The integral over three-momentum p corresponds to a Fourier transform of a massive propagator and so gives rise to a Yukawa potential. The sum over nonzero Matsubara frequencies can also be done analytically and we obtain

$$\begin{aligned} I_4^{\rm a} &= \frac{2T^3}{(4\pi)^5}\int\frac{d^3r}{r^2}\frac{d^3r'}{(r')^2}\frac{1}{|\mathbf{r}+\mathbf{r}'|}\left(\coth\bar{r}-\frac{1}{\bar{r}}\right)\left(\coth\bar{r}'-\frac{1}{\bar{r}'}\right) \\ &\quad\times\frac{1}{e^{\bar{r}+\bar{r}'+|\bar{\mathbf{r}}+\bar{\mathbf{r}}'|}-1}. \end{aligned} \tag{D38}$$

Averaging over angles, one finds

$$\begin{aligned} I_4^{\rm a} &= \frac{2T^2}{(4\pi)^4}\int_0^\infty\frac{d\bar{r}d\bar{r}'}{\bar{r}\bar{r}'}\left(\coth\bar{r}-\frac{1}{\bar{r}}\right)\left(\coth\bar{r}'-\frac{1}{\bar{r}'}\right) \\ &\quad\times[\log(e^{2(\bar{r}+\bar{r}')}-1)-\log(e^{\bar{r}+\bar{r}'+|\bar{r}-\bar{r}'|}-1) \\ &\quad+|\bar{r}-\bar{r}'|-\bar{r}-\bar{r}']. \end{aligned} \tag{D39}$$

The remaining integrals over $\bar{r}$ and $\bar{r}'$ must be done numerically and we obtain

$$I_4^{\rm a} = \frac{T^2}{(4\pi)^4}[0.058\,739\,2]. \tag{D40}$$

The second term in Eq. (D36) is rewritten as

$$\begin{aligned} I_4^{\rm b} &= T\int_p\frac{1}{p^2}[\Pi^T(p_0=0,p)]^2 \\ &= T\int_p\frac{1}{p^2}\{[\Pi^T(p_0=0,p)-\Pi^T_{\rm IR}(p)]^2 \\ &\quad+2\Pi^T(p_0=0,p)\Pi^T_{\rm IR}(p)-[\Pi^T_{\rm IR}(p)]^2\}, \end{aligned} \tag{D41}$$

where $\Pi_{\rm IR}(p)$ is given by the $q_0 = 0$ term in Eq. (D31):

$$\begin{aligned} \Pi^T_{\rm IR}(p) &= T\int_q\frac{1}{q^2(\mathbf{p}+\mathbf{q})^2} \\ &= T\left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^\epsilon\frac{4^\epsilon\sqrt{\pi}}{(4\pi)^{(3/2)-\epsilon}} \\ &\quad\times\frac{\Gamma(1/2+\epsilon)\Gamma(1/2-\epsilon)}{\Gamma(1-\epsilon)}p^{-1-2\epsilon}. \end{aligned} \tag{D42}$$

The first integral in Eq. (D41) is now well-behaved in both the ultraviolet and the infrared. It can be evaluated numerically using the representation of $\Pi^T(p_0=0,p)$ in three dimensions. The subtracted terms are infrared divergent and are calculated with dimensional regularization. The first integral can be calculated directly in three dimensions. In this case, $\Pi^T_{\rm IR}(p)$ reduces to

$$\Pi^T_{\rm IR}(p) = \frac{T}{8p}. \tag{D43}$$

Using the three-dimensional representation (D6) for $\Pi^T(P)$ with $p_0 = 0$ and Eq. (D43), we get

$$\begin{aligned} I_4^{\rm b1} &= T\int_p\frac{1}{p^2}[\Pi^T(p_0=0,p)-\Pi^T_{\rm IR}(p)]^2 \\ &= T^3\int_p\frac{1}{p^2}\left[\frac{1}{(4\pi)^4}\int\frac{d^3r}{r^2}\frac{d^3r'}{(r')^2}e^{i\mathbf{p}\cdot(\mathbf{r}+\mathbf{r}')}\left(\coth r-\frac{1}{r}\right)\right. \\ &\quad\times\left(\coth\bar{r}'-\frac{1}{\bar{r}'}\right)-\frac{1}{4(4\pi)^2p}\int\frac{d^3r}{r^2}e^{i\mathbf{p}\cdot\mathbf{r}}\left(\coth\bar{r}-\frac{1}{\bar{r}}\right) \\ &\quad\left.+\frac{1}{64p^2}\right]. \end{aligned} \tag{D44}$$

The averages over the angles between $\mathbf{p}$ and $\mathbf{r}$, and between $\mathbf{p}$ and $\mathbf{r}'$ can be done analytically and we obtain

$$\begin{aligned} I_4^{\rm b1} &= T^3\int_p\frac{1}{p^2}\left[\frac{1}{(4\pi)^2}\int_0^\infty dr\int_0^\infty dr'\frac{\sin pr}{pr}\frac{\sin pr'}{pr'}\right. \\ &\quad\times\left(\coth r-\frac{1}{r}\right)\left(\coth\bar{r}'-\frac{1}{\bar{r}'}\right) \\ &\quad\left.-\frac{1}{4(4\pi)p}\int_0^\infty dr\frac{\sin pr}{pr}\left(\coth\bar{r}-\frac{1}{\bar{r}}\right)+\frac{1}{64p^2}\right]. \end{aligned} \tag{D45}$$

The integrals over r, r', and p must be done numerically. The result is

$$I_4^{\rm b1} = \frac{T^2}{(4\pi)^4}[9.5763]. \tag{D46}$$

The first subtraction term in Eq. (D41) is

$$
\begin{aligned}
I_4^{b2} &= 2T\int_p \frac{1}{p^2}\Pi_T(p_0=0,p)\Pi_{IR}^T(p)\\
&= 2T\int_p \frac{1}{p^2}[\Pi(p_0=0,p)-\Pi^0(p_0=0,p)]\Pi_{IR}^T(p)\\
&= 2T\int_p \frac{1}{p^2}\Pi(p_0=0,p)\Pi_{IR}^T(p),
\end{aligned}
\tag{D47}
$$

where we have used the fact that the second term vanishes in dimensional regularization. This term is logarithmically divergent both in the infrared and in the ultraviolet. If we use the same scale for the regularization of ultraviolet and infrared divergences, the integral vanishes [6].

Inserting the expressions for $\Pi^0(p_0=0,p)$ and $\Pi_{IR}^T(p)$ into Eq. (D47), we obtain

$$
\begin{aligned}
I_4^{b2} &= \frac{T^3}{(4\pi)^{4-3\epsilon}}\left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{3\epsilon}2^{1+2\epsilon}\frac{\Gamma^2(\frac{1}{2}+\epsilon)\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(\frac{3}{2}-\epsilon)\Gamma(1-\epsilon)}\int_0^\infty dp\int_0^1 dx\sum_{q_0}\frac{p^{-1-4\epsilon}}{[x(1-x)p^2+q_0^2]^{(1/2)+\epsilon}}\\
&= \frac{T^3}{(4\pi)^{4-3\epsilon}}\left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{3\epsilon}2^{1+2\epsilon}\frac{\Gamma^2(\frac{1}{2}+\epsilon)\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(\frac{3}{2}-\epsilon)\Gamma(1-\epsilon)}\int_0^\infty dp\frac{p^{-1-4\epsilon}}{(p^2+1)^{(1/2)+\epsilon}}\int_0^1 dx[x(1-x)]^{2\epsilon}{\sum_{q_0}}'\frac{1}{|q_0|^{1+6\epsilon}}\\
&= \frac{T^2}{(4\pi)^4}\left(\frac{e^{\gamma_E}\mu^2}{4\pi^2T^2}\right)^{3\epsilon}\frac{4^\epsilon}{\pi}\frac{\Gamma(\frac{1}{2}+\epsilon)\Gamma(\frac{1}{2}-\epsilon)\Gamma(-2\epsilon)\Gamma(\frac{1}{2}+3\epsilon)\Gamma^2(1+2\epsilon)}{\Gamma(\frac{3}{2}-\epsilon)\Gamma(1-\epsilon)\Gamma(2+4\epsilon)}\zeta(1+6\epsilon).
\end{aligned}
\tag{D48}
$$

The prime on the sum in the second line indicates that we have excluded the zero mode $q_0=0$ from the sum. This mode gives rise to an integral that is linearly divergent in the infrared. Since there is no mass scale in this integral, it vanishes. Note also that the integral over p is logarithmically divergent in the infrared and this divergence is *not* set to zero in dimensional regularization [13]. Expanding Eq. (D48) in powers of ϵ, we obtain

$$
\begin{aligned}
I_4^{b2} = &-\frac{T^2}{6(4\pi)^4}\left[\frac{1}{\epsilon^2}+\left(6\log\frac{\mu}{4\pi T}+6\gamma_E-2\right)\frac{1}{\epsilon}+12+\frac{25}{12}\pi^2-12\log\frac{\mu}{4\pi T}+18\log^2\frac{\mu}{4\pi T}+36\gamma_E\log\frac{\mu}{4\pi T}\right.\\
&\left.-12\gamma_E-36\gamma_1\right]+\mathcal{O}(\epsilon).
\end{aligned}
\tag{D49}
$$

Finally, we consider the last subtraction term in Eq. (D41). Since $\Pi_{IR}^T(p_0=0,p)$ goes like $1/p$ for small p, the integrand has a linear infrared divergence. This divergence is set to zero in dimensional regularization. In fact, since there is no mass scale in the integral, it vanishes:

$$
T\int_p\frac{1}{p^2}[\Pi_{IR}^T(p)]^2=0.
\tag{D50}
$$

Adding Eqs. (D9), (D10), (D26), (D28), (D34), (D40), (D46), and (D49), we can write I in the following form:

$$
\begin{aligned}
I = &-\frac{T^2}{4(4\pi)^4}\left(\frac{\mu}{4\pi T}\right)^{6\epsilon}\left\{\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left[\frac{4}{3}+2\frac{\zeta'(-1)}{\zeta(-1)}+4\gamma_E\right]\right.\\
&+\frac{1}{3}\left[46-8\gamma_E-16\gamma_E^2-104\gamma_1-24\gamma_E\log(2\pi)+24\log^2(2\pi)+\frac{45\pi^2}{4}+24\frac{\zeta'(-1)}{\zeta(-1)}+2\frac{\zeta''(-1)}{\zeta(-1)}+16\gamma_E\frac{\zeta'(-1)}{\zeta(-1)}\right]\\
&\left.-38.5309+\mathcal{O}(\epsilon)\right\}.
\end{aligned}
\tag{D51}
$$

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