Doubly lopsided mass matrices from unitary unification

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It is shown that the stratified or ''doubly lopsided'' mass matrix structure that is known to reproduce well the qualitative features of the quark and lepton masses and mixings can arise quite naturally in the context of grand unification based on the groups $SU(N)$ with $N > 5$. An $SU(8)$ example is constructed with the minimal anomaly free, three-family set of fermions, in which a realistic flavor structure results without flavor symmetry.

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I. INTRODUCTION

A still unanswered question is why the quarks and leptons of different families have different masses even though they transform in exactly the same way under the symmetries of the standard model. Most proposed answers are based on the idea that there are flavor symmetries that distinguish fermions of different families. There is another idea, however, suggested long ago [1] but much less studied, which is that there is a grand unified gauge group G, under which different families transform differently. If $G = SU(N)$, then N must be greater than 5, since under $SU(5)$ every family transforms the same way, namely, as $\mathbf{10} + \bar{\mathbf{5}}$. Under $SU(N)$, with $N > 5$, however, families or parts of families can come from multiplets of various sizes.

For instance, consider $SU(6)$ with fermion multiplets that include totally antisymmetric rank-2 and rank-3 tensors $\psi^{AB} = \mathbf{15}$ and $\psi^{ABC} = \mathbf{20}$. Both the $\mathbf{15}$ and the $\mathbf{20}$ contain a $\mathbf{10}$ of $SU(5)$ and therefore contain fermions with the quantum numbers of u_L, d_L, u_L^c, and e_L^+. Suppose further that the weak interactions were broken only by a Higgs field that is in a $\mathbf{15}$ of $SU(6)$. Then, the only mass term for the up-type quarks allowed by $SU(6)$ would be of the form $\psi^{AB}\psi^{CD}\langle H^{EF}\rangle\epsilon_{ABCDEF}$, i.e. $\mathbf{15}\ \ \mathbf{15}\ \ \langle\mathbf{15}_H\rangle$, which gives mass only to the up-type quark in the $\mathbf{15}$, but not to the up-type quark in the $\mathbf{20}$. Therefore, without any ''flavor symmetry,'' a hierarchy of fermion masses would result. ($SU(6)$ is not large enough to give interesting or realistic examples; but simple realistic examples can be constructed with $SU(N)$ groups with $N \geq 7$. A realistic $SU(8)$ example will be presented below. For models implementing a similar ''flavor without flavor symmetries'' idea using the group $SO(10)$, see [2]).

There are several ways that hierarchies can arise among the light fermion masses in such schemes. In a fermion mass matrix, some elements may arise from renormalizable Yukawa terms [like the $\mathbf{15}\ \ \mathbf{15}\ \ \mathbf{15}_H$ term in the $SU(6)$ example)], some may arise from higher-dimension operators generated by tree diagrams, and some may arise from higher-dimension operators generated by loop diagrams. Even elements that arise from operators of the same dimension and at the same loop level can still have very different magnitudes if the operators that produce them involve Higgs fields that transform differently under G.

In $SU(N)$ with the normal embedding of the standard model group, there are no exotic fermions if all the fermion multiplets are totally antisymmetric tensors. A rank p totally antisymmetric tensor will be denoted by (p) and its conjugate tensor by $\overline{[p]}$ or by $(N - p)$. If the set of fermions multiplets is anomaly free, then, as is well known, they decompose under the $SU(5)$ subgroup as some number of $\mathbf{10} + \bar{\mathbf{5}}$ families together with a vectorlike set of multiplets that can contain $\mathbf{10} + \overline{\mathbf{10}}$ pairs, $\mathbf{5} + \bar{\mathbf{5}}$ pairs, and singlets. As there is typically no symmetry to prevent it, the conjugate pairs in the vectorlike set ''mate'' with each other to acquire superheavy mass. The $\mathbf{10} + \bar{\mathbf{5}}$ families, however, being chiral, are forbidden to obtain mass and remain light. (This is Georgi's well-known ''survival hypothesis'' [3].) Therefore, the fact that the observed light fermions fit neatly into some number of $\mathbf{10} + \bar{\mathbf{5}}$ families of $SU(5)$, which is often seen as pointing to $SO(10)$ unification, has just as simple an explanation in terms of $SU(N)$ unification. Moreover, $SU(N)$ has the following theoretical advantage over $SO(10)$: In $SO(10)$, the simplest possibility is that all the $\mathbf{10} + \bar{\mathbf{5}}$ come from $\mathbf{16}$ spinor multiplets, so that the gauge group does not distinguish among the families. But for $SU(N)$, as we will see in the $SU(8)$ example described below, it can happen that even with the *simplest* anomaly free three-family set of fermion multiplets, the three light families do not transform in the same way under the $SU(N)$ group.

Before describing what happens in $SU(N)$, it will be useful to set the stage by reviewing some recent ideas for explaining the gross features of the observed patterns of quark and lepton masses and mixings in the context of $SU(5)$. It will be seen below that the $SU(5)$ structures postulated by these recent ideas emerge automatically in $SU(N)$ unification.

The recent $SU(5)$-based idea is that of ''doubly lopsided'' mass matrices. (The first paper proposing the lopsided mass matrix idea [4] actually proposed the doubly lopsided structure. Singly lopsided—or just ''lopsided''—models were independently proposed by several groups to explain the large atmospheric neutrino-mixing angle [5].

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For a review see [6]. Then doubly lopsided models were taken up again by several groups as an explanation of the fact that both the atmospheric and solar angles are large [7,8].) The doubly lopsided structure emerges naturally as follows.

Imagine that some symmetry distinguishes the three light **10**'s of quarks and leptons and prevents them from mixing strongly with each other. Let the mixing of $\mathbf{10}_1$ with $\mathbf{10}_2$ be controlled by the small parameter δ and the mixing of $\mathbf{10}_2$ with $\mathbf{10}_3$ be controlled by the small parameter ϵ. On the other hand, imagine that no symmetry distinguishes the light $\bar{\mathbf{5}}$'s from each other, so that they are allowed to mix strongly. If the masses of the third family occur directly and the others through mixing, one would expect the following structures for the three types of mass matrices (the entries in the matrices give only the order of magnitude of the elements):

$$(\mathbf{10}_1, \mathbf{10}_2, \mathbf{10}_3)\begin{pmatrix} \delta^2\epsilon^2 & \delta\epsilon^2 & \delta\epsilon \\ \delta\epsilon^2 & \epsilon^2 & \epsilon \\ \delta\epsilon & \epsilon & 1 \end{pmatrix}\begin{pmatrix} \mathbf{10}_1 \\ \mathbf{10}_2 \\ \mathbf{10}_3 \end{pmatrix}\langle \mathbf{5}_H\rangle,$$
$$(\mathbf{10}_1, \mathbf{10}_2, \mathbf{10}_3)\begin{pmatrix} \delta\epsilon & \delta\epsilon & \delta\epsilon \\ \epsilon & \epsilon & \epsilon \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \bar{\mathbf{5}}_1 \\ \bar{\mathbf{5}}_2 \\ \bar{\mathbf{5}}_3 \end{pmatrix}\langle \bar{\mathbf{5}}_H\rangle, \tag{1}$$
$$(\bar{\mathbf{5}}_1, \bar{\mathbf{5}}_2, \bar{\mathbf{5}}_3)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \bar{\mathbf{5}}_1 \\ \bar{\mathbf{5}}_2 \\ \bar{\mathbf{5}}_3 \end{pmatrix}\frac{\langle \mathbf{5}_H\rangle\langle \mathbf{5}_H\rangle}{M_R}.$$

This structure is characteristic of the kind of doubly lopsided models discussed in Refs. [4,7]. This structure would give mass matrices for the up-type quarks, down-type quarks, charged leptons, and neutrinos (denoted, respectively, by the subscripts U, D, L, and ν) of the form

$$M_U \sim \begin{pmatrix} \delta^2\epsilon^2 & \delta\epsilon^2 & \delta\epsilon \\ \delta\epsilon^2 & \epsilon^2 & \epsilon \\ \delta\epsilon & \epsilon & 1 \end{pmatrix} m,$$
$$M_D \sim \begin{pmatrix} \delta\epsilon & \delta\epsilon & \delta\epsilon \\ \epsilon & \epsilon & \epsilon \\ 1 & 1 & 1 \end{pmatrix} m',$$
$$M_L \sim \begin{pmatrix} \delta\epsilon & \epsilon & 1 \\ \delta\epsilon & \epsilon & 1 \\ \delta\epsilon & \epsilon & 1 \end{pmatrix} m' \tag{2}$$
$$M_\nu \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} m_\nu.$$

From these forms several things are immediately apparent: (a) the neutrino-mixing angles will be of order 1, (b) the quark mixing angles will be small (the 12 mixing of order δ, the 23 mixing of order ϵ, and the 13 mixing of order $\delta\epsilon$), (c) the masses of the up-type quarks will have a strong family hierarchy $(\delta\epsilon)^2$: $(\epsilon)^2$: 1, (d) the masses of the down-type quarks and charged leptons will have a weaker family hierarchy $\delta\epsilon$: ϵ: 1, and (e) the neutrino masses will have the weakest family hierarchy, since all the neutrino masses will be of roughly the same order. These five features are just exactly what is observed.

As we will see below, $SU(N)$ unification naturally leads to exactly the result that the **10**'s of fermions are distinguished from each other by symmetry—symmetries in $SU(N)/SU(5)$—whereas the $\bar{\mathbf{5}}$'s of fermions are not distinguished by symmetry.

II. AN $SU(8)$ MODEL: PARTICLE CONTENT

We shall now describe a model based on $SU(8)$ where the $SU(8)$ symmetry is sufficient to produce a nontrivial flavor structure very much like that observed in nature. [Other unification schemes based on the group $SU(8)$ have been proposed in the literature [9]. Some of these involve a unification within $SU(8)$ of a vertical $SU(5)$ group and a family $SU(3)$.]

If the number of left-handed fermion multiplets of type (p) and $\overline{[p]}$ is denoted by n_p and n_{-p}, respectively, then the condition that the $SU(8)$ anomalies cancel is $(n_1 - n_{-1}) + 4(n_2 - n_{-2}) + 5(n_3 - n_{-3}) = 0$, and the condition for three families is $(n_2 - n_{-2}) + 2(n_3 - n_{-3}) = 3$. The general solution is $(n_1 - n_{-1}) = -12 + 3p$, $(n_2 = n_{-2}) = 3 - 2p$, $(n_3 - n_{-3}) = p$. The most economical set, as measured by the total number of components, is $n_{-1} = 9$, $n_2 = 1$, $n_3 = 1$, i.e. the set $[3] + [2] + 9 \times \overline{[1]} = \mathbf{56} + \mathbf{28} + 9 \times \bar{\mathbf{8}}$. This is precisely the set of fermions that will be assumed in the model presented below.

These fermion multiplets decompose under $SU(5)$ as follows:

$$\begin{aligned} [2]_L &= \psi^{[AB]} \to \psi^{\alpha\beta} + \psi^{\alpha I} + \psi^{IJ} \\ &\quad \mathbf{28} \to \mathbf{10} + 3 \times \mathbf{5} + 3 \times \mathbf{1}, \\ [3]_L &= \psi^{[ABC]} \to \psi^{\alpha\beta\gamma} + \psi^{\alpha\beta I} + \psi^{\alpha IJ} + \psi^{IJK} \\ &\quad \mathbf{56} \to \overline{\mathbf{10}} + 3 \times \mathbf{10} + 3 \times \mathbf{5} + \mathbf{1}, \\ 9 \times \overline{[1]}_L &= \psi_{(m)A} \to \psi_{(m)\alpha} + \psi_{(m)I} \\ &\quad 9 \times \bar{\mathbf{8}} \to 9 \times \bar{\mathbf{5}} + 27 \times \mathbf{1}, \end{aligned} \tag{3}$$

The subscripts L on (p) indicate that these are left-handed fermion multplets. The indices A, B, C, etc. run from 1 to 8; the indices α, β, γ, etc. run from 1 to 5; and the indices I, J, K, etc. run from 6 to 8. All of the foregoing are $SU(8)$ gauge indices. The index $m = 1, \ldots, 9$, on the other hand, just labels the nine different antifundamental fermion multiplets. One sees from Eq. (3) that there are altogether four **10** and one $\overline{\mathbf{10}}$ of $SU(5)$, for a "net" of three **10**, and nine $\bar{\mathbf{5}}$ and six **5** of $SU(5)$, for a net of three $\bar{\mathbf{5}}$. [It should be emphasized that we refer to $SU(5)$ multiplets as a convenient way to keep track of the fermion families, even though the actual sequence of breaking of $SU(N)$ to the standard model group may not go through $SU(5)$. The sequence of breaking depends on the relative magnitudes of the superlarge vacuum expectation values (VEVs) of the model]. Which of the **10** and which of the $\bar{\mathbf{5}}$ remain light after $SU(N)$ breaks to the standard model depends on the Higgs content of the model, to which we now turn.

In the model it is assumed that the Higgs fields are in the following multiplets: $[1]_H = H^A = \mathbf{8}$, $[2]_H = H^{[AB]} = \mathbf{28}$, $[4]_H = H^{[ABCD]} = \mathbf{70}$, and $Adj_H = \Omega^A_B = \mathbf{63}$. The $[1]_H$ and $[2]_H$ are assumed to have superlarge VEVs in all the directions that leave the $SU(5)$ unbroken: i.e. H^I and H^{IJ}, $I, J = 6, 7, 8$. The $[4]_H$ has no $SU(5)$-singlet components and so must not obtain a superlarge VEV. The adjoint Higgs field has a superlarge diagonal VEV, which is needed for the breaking to the standard model. The structure of the Higgs potential and the breaking of $SU(8)$ down to the standard model gauge group are briefly discussed in Appendix A.

All three kinds of antisymmetric-tensor Higgs fields, $[1]_H$, $[2]_H$, and $[4]_H$ participate in the breaking of $SU(2)_L \times U(1)_Y$ at the weak scale via the weak doublets they contain, H^i, H^{iI}, and H^{iIJK}, where $i = 1, 2$. Of course, actually there is only one light Higgs doublet, which is a linear combination of these fields.

III. YUKAWA TERMS AND SUPERHEAVY FERMION MASSES

The renormalizable Yukawa terms that are allowed by $SU(8)$ are the following:

$$\begin{aligned}
([3]_L\overline{[1]}_L)\overline{[2]}_H &= Y_m(\psi^{[ABC]}\psi_{(m)A})H^*_{[BC]} \\
([2]_L[2]_L)\overline{[4]}_H &= Y(\psi^{[AB]}\psi^{[CD]})H^*_{[ABCD]} \\
([2]_L\overline{[1]}_L)\overline{[1]}_H &= y_m(\psi^{[AB]}\psi_{(m)A})H^*_B \\
(\overline{[1]}_L\overline{[1]}_L)[2]_H &= a_{mn}(\psi_{(m)A}\psi_{(n)B})H^{[AB]}.
\end{aligned} \tag{4}$$

A term of the form $([3]_L[3]_L)[2]_H$ vanishes by the antisymmetry of the tensors. For the same reason, the Yukawa coupling matrix a_{mn} in the fourth line of Eq. (4) is antisymmetric. Note that $H^*_{[ABCD]} = \epsilon_{[ABCDEFGH]}H^{[EFGH]}/4!$. Of course, repeated indices of all kinds are summed over throughout this paper.

The first task is to determine how the vectorlike fermion pairs "mate" to obtain superlarge mass, and which ones do, so as to identify the fermion multiplets that remain light. The "mating" of the vectorlike pairs $\mathbf{5} + \bar{\mathbf{5}}$ that gives them superheavy masses is done by terms like $y_m(\psi^{\alpha I}\psi_{(m)\alpha})\langle H_I\rangle$ and $Y_m(\psi^{\alpha IJ}\psi_{(m)\alpha})\langle H_{IJ}\rangle$. It is clear that if there is only a single $[1]_H$ the former term mates only one of the three **5**'s that are contained in the $[2]_L$, namely, the linear combination $\langle H_I\rangle\psi^{\alpha I}$. (It mates it with one of the $\bar{\mathbf{5}}$'s from among the nine $\overline{[1]}_L$, namely, the linear combination $y_m\psi_{(m)\alpha}$). In order for all three **5**'s that are contained in the $[2]_L$ to be mated by renormalizable terms,

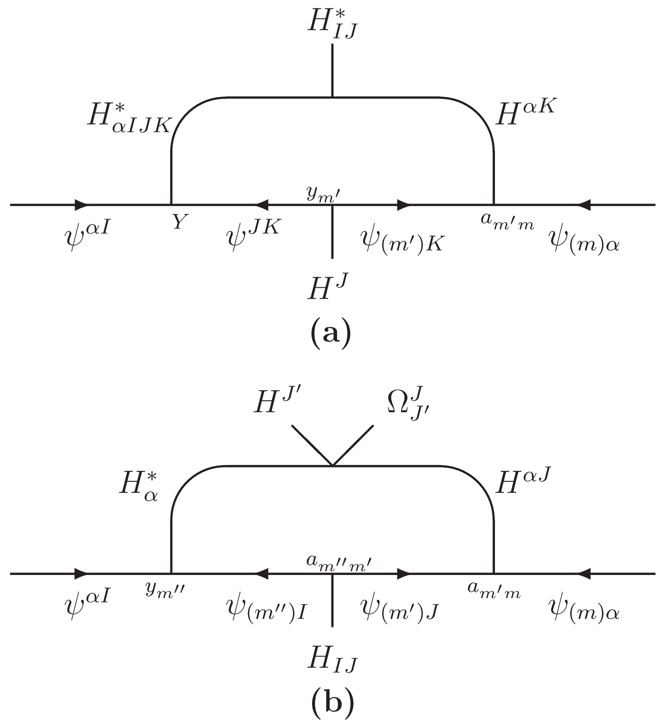

FIG. 1. Typical one-loop diagrams that "mate" fermions in **5** and $\bar{\mathbf{5}}$ multiplets of $SU(5)$ to give them superheavy mass.

there would have to be three distinct $[1]_H$ multiplets. In that case, the mass term would be written $y_{ma}(\psi^{\alpha I}\psi_{(m)\alpha}) \times \langle H_{(a)I}\rangle$, $a = 1, 2, 3$, and for each value of a one $\mathbf{5} + \bar{\mathbf{5}}$ pair would get mated. However, it is not necessary for the Higgs sector to be enlarged in that way. Even with only a single $[1]_H$ of Higgs, all the **5**'s in the $[2]_L$ get mated if higher-dimension operators induced by one-loop diagrams are taken into account. For example, the one-loop diagrams shown in Fig. 1(a) and 1(b) induce the effective operators $y_{m'}a_{m'm}(\psi^{\alpha I}\psi_{(m)\alpha})H^*_{IJ}H^J$ and $y_{m''}a_{m''m'}a_{m'm}(\psi^{\alpha I}\psi_{(m)\alpha})H^*_{IJ}\Omega^J_{J'}H^{J'}$.

In a similar way, if there is only a single $[2]_H$ Higgs multiplet, the term $y_m(\psi^{\alpha IJ}\psi_{(m)\alpha})\langle H_{IJ}\rangle$ only mates a single **5** from the $[3]_L$ with a $\bar{\mathbf{5}}$; but loop diagrams induce higher-dimension operators that mate the remaining two **5**'s from the $[3]_L$. The mating of the $\overline{\mathbf{10}}$ that is in the $[3]_L$ with a **10** is not done by any renormalizable operator, but is done by such higher-dimension operators as $\epsilon_{\alpha\beta\gamma\delta\epsilon IJK}(\psi^{\alpha\beta\gamma}\psi^{\delta\epsilon I'})\Omega^I_{I'}H^{JK}$ and $\epsilon_{\alpha\beta\gamma\delta\epsilon IJK}(\psi^{\alpha\beta\gamma}\psi^{\delta\epsilon})H^I H^{JK}$. (The adjoint Higgs in the first operator is needed to prevent it from vanishing identically by antisymmetry of indices.) These operators also come from one-loop diagrams. They mate the $\overline{\mathbf{10}}$ with some linear combination of the **10**'s from the $[3]_L$ and $[2]_L$.

IV. THE LIGHT FAMILIES AND THEIR MASSES

One sees, then, that even the small set of Higgs multiplets given above, H^A, $H^{[AB]}$, $H^{[ABCD]}$, and Ω^A_B, with one of each type, is enough to mate all of the conjugate pairs of fermion multiplets and make them superheavy. This is not surprising; indeed, it is just an illustration of the survival hypothesis [3]. Since there is no symmetry to prevent it, one expects all the vectorlike fermions to mate and get large mass. Which fermion multiplets mate determines which multiplets remain light.

The three **10**'s that remain light are linear combinations of the one that is in $[2]_L$ and the three that are in $[3]_L$. Without loss of generality, we can choose the flavor basis of the light fermions so that $\mathbf{10}_3$ comes partly from $[2]_L$, but that $\mathbf{10}_1$ and $\mathbf{10}_2$ come purely from $[3]_L$. This shows that for the **10**'s one family is automatically selected out as different by virtue of coming partly from a different $SU(8)$ multiplet than the other families. This will allow an explanation of why the t quark is so much heavier than the u and c quarks. Moreover, even though the $\mathbf{10}_1$ and $\mathbf{10}_2$ come entirely from the same $SU(8)$ multiplet, namely $[3]_L$, they come from different *components* of that multiplet. That is, they are given by $\psi^{\alpha\beta I}$ with different values of the $SU(8)/SU(5)$ index I and are thus distinguished from each other by $SU(8)$. Thus, $SU(8)$ can suppress the mixing of these **10**'s, as will be seen.

By contrast, one sees that all three light $\bar{\mathbf{5}}$'s must come from the same kind of $SU(8)$ multiplet, namely $\overline{[1]}_L$. In other words, the three light $\bar{\mathbf{5}}$'s are simply three particular linear combinations of the nine $\psi_{(m)\alpha}$. (For simplicity, we could take the basis in the space of these nine fields to be such that the light ones corresponded to the values $m = 1$, 2, 3). Since $\psi_{(m)\alpha}$ has only an $SU(5)$ index and a label (m) that has nothing to do with the gauge symmetry, the $SU(8)$ does not distinguish among the three light $\bar{\mathbf{5}}$'s in any way. One would therefore expect that these $\bar{\mathbf{5}}$'s would be able to mix strongly with each other, since such mixing is not prevented by any symmetry.

It is interesting that the large mixing among $\bar{\mathbf{5}}$'s that is an ingredient of the lopsided and doubly lopsided models emerges naturally in the context of $SU(N)$ unification with $N > 5$. The reason has to do with anomaly cancellation. The **10**'s of $SU(5)$ must come from tensors that have a rank of at least 2, which tend (for large N) to make a large positive contribution to the anomaly. In the most economical solutions of the anomaly conditions, this large contribution tends to be cancelled by large numbers of antifundamental multiplets (since fundamental and antifundamental representations have the largest anomalies per component). This, in turn, gives the result, in many cases, that the light $\bar{\mathbf{5}}$'s all come from antifundamentals, as in the present $SU(8)$ example. To take another example, in $SU(9)$ the most economical three-family solutions to the anomaly conditions are (a) $[3] + 9 \times \overline{[1]}$ (165 components) and (b) $3 \times [2] + 15 \times \overline{[1]}$ (243 components). Both of these solutions have numerous antifundamentals, and in both solutions all of the $\bar{\mathbf{5}}$ are contained in these antifundamentals.

The masses of the up-type quarks, u, c, and t come from operators that (in $SU(5)$ terms) couple $\mathbf{10}_L$ to $\mathbf{10}_L$. There is only one renormalizable operator of this type, namely,

$$\mathcal{O}_A = ([2]_L[2]_L)\overline{[4]}_H = \psi^{AB}\psi^{CD}H^*_{ABCD}, \qquad (5)$$

which contains the term $(\psi^{\alpha\beta}\psi^{\gamma\delta})H^*_{\alpha\beta\gamma\delta}$. (Note that $H^*_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta\epsilon678}H^{\epsilon678}$.) However, only one of the light $\mathbf{10}_L$'s, namely, the one that we have labeled $\mathbf{10}_3$, contains some of $[2]_L$, i.e. of $\psi^{\alpha\beta}$; the other two light **10**'s, namely, $\mathbf{10}_1$ and $\mathbf{10}_2$, are purely in $[3]_L$. Consequently, the operator $\mathcal{O}_A$ contributes only to the 33 element of M_U, the mass matrix of the up-type quarks. This element, which will be denoted A, is the only element of M_U that arises at tree level, thus explaining the relatively large magnitude of the t-quark mass.

At one-loop level, however, many higher-dimension operators are induced that contribute to the other elements of M_U. In particular, one has the following classes of operators:

$$\begin{aligned}
\mathcal{O}_\beta &= ([2]_L[3]_L)[1]_H[2]_H, \qquad ([2]_L[3]_L)\overline{[1]}_H[4]_H, \dots \\
&= \epsilon_{ABCDEFGH}(\psi^{AB}\psi^{CDE})H^F H^{GH}, \qquad \epsilon_{ABCDEFGH}(\psi^{AB}\psi^{CDI})H_I H^{EFGH}, \dots \\
\mathcal{O}_\gamma &= ([3]_L[3]_L)Adj_H[2]_H, \qquad ([3]_L[3]_L)\overline{[2]}_H[4]_H, \dots \\
&= \epsilon_{ABCDEFGH}(\psi^{ABC}\psi^{DEI})\Omega^F_I H^{GH}, \qquad \epsilon_{ABCDEFGH}(\psi^{ABC}\psi^{DEI})H_{IJ}H^{JFGH}, \dots \\
\mathcal{O}_\delta &= ([3]_L[3]_L)\overline{[1]}_H[1]_H[2]_H, \qquad ([3]_L[3]_L)\overline{[1]}_H\overline{[1]}_H[4]_H, \dots \\
&= \epsilon_{ABCDEFGH}(\psi^{ABC}\psi^{DEI})H_I H^F H^{GH}, \qquad \epsilon_{ABCDEFGH}(\psi^{ABI}\psi^{CDJ})H_I H_J H^{EFGH}, \dots
\end{aligned} \qquad (6)$$

The operators of type $\mathcal{O}_\beta$ couple $[2]_L$ to $[3]_L$, and therefore couple $\mathbf{10}_3$ to $\mathbf{10}_1$ and $\mathbf{10}_2$. These operators thus contribute to the 13 (31) and 23 (32) elements of M_U, which will be denoted β' and β, respectively. (The operators $\mathcal{O}_\beta$ will also contribute to the 33 element A.)

The operators of type $\mathcal{O}_\gamma$ couple $[3]_L$ to $[3]_L$, and therefore couple any of the $\mathbf{10}_i$ to any *other* of the $\mathbf{10}_i$. They cannot, however, contribute to any diagonal element of M_U, because of the antisymmetry of the epsilon symbol. These operators therefore contribute to the 12 (21) element of M_U, which is denoted γ, as well as to the elements β, β'.

Finally, operators of the type $\mathcal{O}_\delta$, which also couple $[3]_L$ to $[3]_L$, can contribute to any elements of M_U, including the 11 and 22 elements, which are denoted δ' and δ, respectively.

In sum, the mass matrix of the up-type quarks has the form

$$M_U = \begin{pmatrix} \delta' & \gamma & \beta' \\ \gamma & \delta & \beta \\ \beta' & \beta & A \end{pmatrix}. \tag{7}$$

There is no reason *a priori* why the different types of operators induced at one-loop level must all make contributions to M_U of the same order of magnitude. For example, the operators of type $\mathcal{O}_\delta$ are of dimension 6 or higher, whereas some of the operators of type $\mathcal{O}_\beta$ are only of dimension 5. So it could be that δ, $\delta' \ll \beta$, β'. Moreover, the superheavy VEVs of Higgs fields in different representations of $SU(8)$ could be of quite different magnitudes, so that even operators of the same dimension but involving different types of Higgs multiplets could make very different contributions.

If it were the case that γ, δ, $\delta' \ll \beta$, β', then the matrix M_U would have the observed threefold hierarchy among its eigenvalues, i.e. $m_u \ll m_c \ll m_t$.

Turning now to the masses of the down-type quarks and charged leptons, these come from operators that [in $SU(5)$ terms] couple $\mathbf{10}_L$ to $\bar{\mathbf{5}}_L$. At first glance, there seem to be dimension-4 operators that do this, namely,

$$y_m(\psi^{\alpha\beta}\psi_{(m)\alpha})H^*_\beta, \qquad Y_m(\psi^{\alpha\beta I}\psi_{(m)\alpha})H^*_{\beta I}. \tag{8}$$

However, the first of these operators is related by $SU(8)$ to the operator $y_m(\psi^{\alpha I}\psi_{(m)\alpha})H^*_I$, which mates precisely the $\bar{\mathbf{5}}_L$ that is the linear combination $y_m\psi_{(m)\alpha}$ to a $\mathbf{5}$ to make it superheavy. So that the first term in Eq. (8) is not a contribution to the light fermion mass matrices, but is a coupling of light fermions to superheavy fermions. In the same way, the second operator in Eq. (8) is related by $SU(8)$ to the operator $Y_m(\psi^{\alpha IJ}\psi_{(m)\alpha})H^*_{IJ}$, which mates precisely the $\bar{\mathbf{5}}_L$ that is the linear combination $Y_m\psi_{(m)\alpha}$ to a $\mathbf{5}$ to make it superheavy. The second term in Eq. (8) is thus also not a contribution to the mass matrices of the light fermions.

The mass matrices of the down-type quarks and charged leptons, which will be denoted M_D and M_L, respectively, do not arise until one loop. There are two kinds of operators that contribute:

$$\begin{aligned} \mathcal{O}_\epsilon &= ([2]_L\overline{[1]}_L)Adj_H\overline{[1]}_H, \qquad ([2]_L\overline{[1]}_L)\overline{[2]}_H[1]_H, \ldots \\ &= (\psi^{AB'}\psi_{(m)A})\Omega^B_{B'}H_B, \qquad (\psi^{AB'}\psi_{(m)A})H_{B'C}H^C, \ldots \\ \mathcal{O}_\zeta &= ([3]_L\overline{[1]}_L)\overline{[1]}_H Adj_H\overline{[1]}_H, \qquad ([3]_L\overline{[1]}_L)Adj_H\overline{[2]}_H, \ldots \\ &= (\psi^{ABC'}\psi_{(m)A})H_B\Omega^C_{C'}H_C, \qquad (\psi^{ABC'}\psi_{(m)A})\Omega^C_{C'}H_{BC}, \ldots \end{aligned} \tag{9}$$

The operators of type $\mathcal{O}_\epsilon$ couple $[2]_L$ to $\overline{[1]}$ and therefore $\mathbf{10}_3$ to $\bar{\mathbf{5}}_i$, $i = 1, 2, 3$. Thus, they contribute to the $3i$ elements of M_D and the $i3$ elements of M_L, which we denote ϵ_i. The operators of type $\mathcal{O}_\zeta$ couple $[3]_L$ to $\overline{[1]}$ and therefore can contribute to all the elements of the mass matrices M_D and M_L. We denote the resulting nonvanishing $2i$ elements of M_D and $i2$ elements of M_L by ζ_i, and the resulting nonvanishing $1i$ elements of M_D and $i1$ elements of M_L by ζ'_i. These matrices consequently have the form,

$$M_D = \begin{pmatrix} \zeta'_1 & \zeta'_2 & \zeta'_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix}, \qquad M_L \sim \begin{pmatrix} \zeta'_1 & \zeta_1 & \epsilon_1 \\ \zeta'_2 & \zeta_2 & \epsilon_2 \\ \zeta'_3 & \zeta_3 & \epsilon_3 \end{pmatrix}. \tag{10}$$

The matrix M_L is not exactly the transpose of M_D, because of $SU(5)$-breaking effects from the adjoint Higgs VEVs that come into the one-loop diagrams [e.g. the factors of $\Omega^B_{B'}$ in Eq. (9)]. That is why a "$\sim$" is used in the equation for M_L rather than an equal sign. These $SU(5)$-breaking effects can explain the well-known Georgi-Jarlskog factors [10], i.e. the deviations of m_s/m_μ and m_d/m_e from 1.

The notation used in writing elements of the mass matrices is as follows:

(a) Elements that come from operators of the same class are denoted by the same Greek letter. For example, β and β' in Eq. (7) both come from the operators of class $\mathcal{O}_\beta$, and ζ_1, ζ_2, ζ_3, ζ'_1, ζ'_2, and ζ'_3 all come from the operators of class $\mathcal{O}_\zeta$. Consequently, elements that are denoted by different Greek letters, since they come from entirely *different operators*, have no reason to be comparable in magnitude.

(b) Elements that are denoted by the same Greek letter but differ by a prime, such as β and β' or ζ_i and ζ'_i, come from the same operators, containing the same $SU(8)$ multiplets, but involve *different components* of those multiplets. For example, suppose that $\mathbf{10}_1 = \psi^{\alpha\beta 8}$ and $\mathbf{10}_2 = \psi^{\alpha\beta 7}$. Then the elements β and β' would both come from the operators $\mathcal{O}_\beta$, but β would come from the terms $(\psi^{\alpha\beta}\psi^{\gamma\delta 7})H^\epsilon H^{86}$, $(\psi^{\alpha\beta}\psi^{\gamma\delta 7})H_7 H^{\epsilon 678}$, etc., whereas β' would come from the terms $(\psi^{\alpha\beta}\psi^{\gamma\delta 8})H^\epsilon H^{67}$, $(\psi^{\alpha\beta}\psi^{\gamma\delta 8})H_8 H^{\epsilon 678}$, etc. Since different components of the same $SU(8)$ multiplet of Higgs fields—such as H^6, H^7, and H^8—can have vacuum expectation values that are very different from each other if there is a hierarchy of scales involved in the breaking of $SU(8)$ down to the standard model group, elements that differ by a prime can also be of very different magnitude. In other words, we see that a hierarchy among elements of a mass matrix of light fermions, i.e. a ''flavor hierarchy,'' can arise in part from a hierarchy of scales in the breaking of the grand unified group.

(c) Elements that are distinguished only by a subscript, such as ζ_2' and ζ_3', come from the same kinds of operators, and the same $SU(8)$ components of the multiplets within those operators, but involve *different antifundamental multiplets of fermions*. For example, ϵ_1, ϵ_2, and ϵ_3 all come from the same operators $\mathcal{O}_\epsilon$ [such as $\psi^{AB'}\psi_{(m)A}\Omega^B_{B'}H_B$] and with the $SU(8)$ indices taking the same values; but they involve different linear combinations of the nine antifundamental multiplets $\psi_{(m)A}$, $m = 1, \ldots, 9$. In other words, $SU(8)$ gauge symmetry in no way distinguishes among the elements ϵ_1, ϵ_2, and ϵ_3. If there are no preferred directions in the nine-dimensional space spanned by the index m—i.e. if the Yukawa couplings Y_m, y_m, and a_{mn} are ''randomly'' oriented in that space—then one expects that $\epsilon_1 \sim \epsilon_2 \sim \epsilon_3$, $\zeta_1 \sim \zeta_2 \sim \zeta_3$, and $\zeta_1' \sim \zeta_2' \sim \zeta_3'$.

We have given in Eqs. (7) and (10) the general forms of the quark and lepton mass matrices, without making specific assumptions about the pattern of $SU(8)$ breaking and the hierarchies among the VEVs. In Appendix B, a concrete example is given of the kind of mass matrix forms that can result from specific patterns of breaking.

As a consequence of the general form given in Eqs. (7) and (10), one expects the matrices M_D and M_L to have a *stratified* structure characteristic of the doubly lopsided models of Refs. [4,7]. All the elements of a row of M_D (or a column of M_L) should be comparable in magnitude; whereas the different rows of M_D (or columns of M_L) should typically be quite different in magnitude. As was explained in the Introduction, such a stratified structure leads to a situation where the mixing angles of the left-handed quarks (the CKM angles) are small, while the mixing angles of the left-handed leptons (the MNS neutrino-mixing angles) are of order one. This is clear from a direct inspection of the mass matrices: the CKM angles evidently involve ratios of elements of different rows of M_D (e.g. V_{cb} would involve $\zeta_3/\epsilon_3 \ll 1$), while the MNS angles involve elements of different rows of M_L (e.g. $U_{\mu 3} = \sin\theta_{\rm atm}$ involves the ratio $\epsilon_2/\epsilon_3 \sim 1$).

The same stratified structure of M_L and M_D typically leads to a interfamily hierarchy of masses for the charged leptons and down-type quarks. An even larger hierarchy should exist among the up-type quarks for two reasons. First, since both the left-handed and right-handed up-type quarks come from **10**'s of $SU(5)$, the hierarchy of masses in M_U arises from interfamily hierarchies among both rows and columns. Second, M_U contains one element that arises at tree level, while the others arise at one loop.

Turning to the mass matrix of the light neutrinos M_ν, it is apparent that all of its elements should be comparable, since the three light neutrinos are not distinguished in any way by $SU(8)$, but only by which antifundamental fermion multiplets they are contained in. That is, they all come from the same kind of multiplets $\psi_{(m)\alpha}$. This would imply that the ratios of neutrino masses should not exhibit a large hierarchy, which is consistent with the fact that $(\Delta m^2_{\rm sol})^{1/2}$ and $(\Delta m^2_{\rm atm})^{1/2}$ only differ by about a factor of 5.

The mass matrix M_ν comes, of course, from a seesaw mechanism involving superheavy right-handed neutrinos. In an $SU(N)$ grand unified model, the fermion multiplets typically contain large numbers of fermions that are singlets under the standard model group, and which therefore play the role of right-handed neutrinos. For example, in this $SU(8)$ model there are 31 such singlets, consisting of the following: three in the $[2]_L$ (namely, ψ^{67}, ψ^{78}, and ψ^{86}); one in the $[3]_L$ (namely, ψ^{678}); and 27 in the $\overline{[1]}_L$'s (namely, $\psi_{(m)I}$, with $m = 1, \ldots, 9$ and $I = 6, 7, 8$). It is interesting that only 24 of these 31 singlet fermions get mass at tree level. The other seven obtain mass from various one-loop diagrams. This is shown in Appendix C. The reason that some remain massless at tree level is essentially the same as the reason that some of the quarks and charged leptons remain massless at tree level: with an economical set of Higgs fields, there are simply not enough types of Yukawa terms allowed by $SU(8)$ to generate masses for all of the fermions. [Typically, in nonsupersymmetric $SU(N)$ models, there are more likely to be ''accidentally'' massless fermions at tree level than accidentally massless bosons, because the Yukawa couplings, being no more than cubic and having a certain chiral structure, are much more constrained than is the Higgs potential. For example, in this model only the four Yukawa terms in Eq. (4) are allowed by $SU(8)$, whereas 32 terms are allowed in the Higgs potential, as discussed in Appendix A.] The seven right-handed neutrinos that get mass from loops would typically have mass of order $\frac{1}{16\pi^2}M_{\rm GUT}$, while those that get mass at tree level would typically have masses of order $M_{\rm GUT}$. The ones with loop masses would therefore tend to dominate in the seesaw formula for M_ν. This may be an attractive feature of the model, because it is a way of explaining why the right-handed neutrino mass scale inferred from the seesaw formula tends to come out somewhat smaller than the GUT scale inferred from running of couplings.

Computing the mass matrix M_ν would be very messy. Even if one considered only the contributions from the right-handed neutrinos that obtain mass at one-loop level, the mass matrix of the right-handed neutrinos M_R would be 7-by-7 and the Dirac neutrino mass matrix $M_{\rm Dirac}$ would be 3-by-7. Moreover, identifying the lightest seven right-handed neutrinos in terms of the parameters of the model is complicated, as can be seen from Appendix C. Finally, both M_R and $M_{\rm Dirac}$ would have complicated forms, since they get contributions from several kinds of operators. Therefore, getting a useful quantitative prediction for the neutrino masses and mixings would be unlikely; too many parameters of the model would enter both Yukawa couplings and (through loops) couplings in the Higgs potential.

One might wonder whether the reasoning that led to the conclusion that all the elements of M_ν are naturally of the same order would be invalidated by some property of M_R, given the complexity of the right-handed neutrino sector. This seems highly unlikely, however. It is true that in some models there is a nontrivial structure in M_ν even in the absence of any symmetry that distinguishes among the left-handed neutrinos. But in the known examples this is because the right-handed neutrino sector is especially simple, for example, by M_R^{-1} being approximately rank 1, as in models with "single right-handed neutrino dominance" [11]. In that case, M_ν would tend to be approximately of rank 1 also, giving a hierarchy among the light neutrino masses. In the present case, the right-handed neutrino sector is more complicated than usually considered, with more than three right-handed neutrinos contributing importantly in the seesaw formula. There is therefore no reason to expect M_ν to have a strong hierarchy.

It also should be noted that in models of this type, because all the entries of M_ν are of the same order, there is no reason *a priori* that the neutrino angle θ_{13} should be particularly small. In this respect, these models are different from those doubly lopsided models in which the neutrino mass matrix M_ν is hierarchical and the large neutrino-mixing angles come primarily from large off-diagonal elements in the charged lepton mass matrix M_L (as in the model of [8]). In those models, θ_{13} is predicted to be small, giving naturally a "bi-large" structure. In the kind of $SU(N)$ model proposed here, however, the three neutrino angles are, in effect, all random angles that are naturally of order 1. That does not conflict with the observation that θ_{13} is less than or of order 0.2, since there is a reasonably high probability that this would happen in a random unitary matrix, as noted in [7]. However, if θ_{13} were found to be much closer to zero, it would strongly disfavor these models.

V. CONCLUSIONS

It has been shown that a realistic grand unified model can be constructed based on $SU(N)$, $N > 5$, in which the $SU(N)$ symmetry and its pattern of breaking is sufficient to create a nontrivial flavor structure for the light quarks and leptons, without there being any flavor symmetry at all. What makes the fermions of different families different from each other is the way they transform under the $SU(N)$. This is, in particular, true of the three light **10**'s of $SU(5)$, which do not all come from the same kinds of multiplets of $SU(N)$. On the other hand, in this model the three light $\bar{\mathbf{5}}$'s of $SU(5)$ do all come from the same kind of multiplet of $SU(N)$, and thus are not distinguished from each other. Since the left-handed neutrinos are all contained in the $\bar{\mathbf{5}}$'s, no fundamental symmetry distinguishes the light neutrinos from each other, and as a consequence large neutrino mixing naturally results, and the neutrino masses should not exhibit a strong hierarchy. For the mass matrices of the down-type quarks and the charged leptons a stratified or "doubly lopsided" structure results, leading to a stronger hierarchy for their masses. The strongest mass hierarchy of all is that of the up-type quarks. (In the $SU(8)$ model we present as an example, only the top quark obtains mass at tree level.)

The fact that the three light $\bar{\mathbf{5}}$'s are not distinguished by any symmetry (which is what gives the realistic stratified structure to the mass matrices) stems from the fact that they all come from antifundamental multiplets of $SU(N)$. That in turn can be traced to the requirements of anomaly cancellation. For $SU(N)$ models containing only antisymmetric-tensor multiplets of fermions, the most economical sets of fermions that have three families and are anomaly free tend to have many antifundamental multiplets, and it is usually the case that all of the $\bar{\mathbf{5}}$'s come from these multiplets.

The model described above is a nonsupersymmetric grand unified theory. It is also possible to construct models based on the same ideas that have low-energy supersymmetry. In such models, all the masses of the light families would have to come from tree-level diagrams. However, there could still be mass hierarchies, since tree diagrams can generate operators of different dimensions and of different types. Moreover, there can be a hierarchy among the scales at which $SU(N)$ breaks down to the standard model group, and this hierarchy can be reflected in the mass matrices of the light quarks and leptons, as the model presented here illustrates.

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APPENDIX A: THE HIGGS POTENTIAL AND $SU(8)$ BREAKING

The part of the Higgs potential that is relevant for the breaking of $SU(8)$ down to the standard model does not involve the rank-4 Higgs multiplet $[4]_H = H^{ABCD}$, since all components of that multiplet are nonsinglet under the standard model group and therefore cannot have superlarge vacuum expectation values. Thus, to find the $SU(8)$-breaking minimum it would be sufficient to consider only those terms not involving the $[4]_H$. Call this truncated potential V'_H. Assuming that only one Higgs multiplet exists of each of the other types ($[1]_H$, $[2]_H$, and Adj_H), one has as the most general form of V'_H consistent with $SU(8)$

$$V'_H = V'_2 + V'_3 + V'_4,$$

where

$$V'_2 = |H^A|^2 + |H^{AB}|^2 + \Omega^A_B \Omega^B_A,$$
$$V'_3 = \underbrace{H^A \Omega^B_A H^*_B}_{\equiv X_3} + \underbrace{H^{AB} \Omega^C_B H^*_{AC}}_{\equiv Y_3} + \Omega^A_B \Omega^B_C \Omega^C_A,$$
$$V'_4 = 6 \text{ products of quadratics} + \underbrace{H^A \Omega^B_A \Omega^C_B H^*_C}_{\equiv X_4}$$
$$+ \underbrace{H^{AB} \Omega^C_B \Omega^D_C H^*_{DA}}_{\equiv Y_4} + \underbrace{H^{AB} \Omega^C_A \Omega^D_B H^*_{CD}}_{\equiv Y'_4}$$
$$+ \underbrace{H^A H^*_{AB} H^{BC} H^*_C}_{\equiv Z_4} + \underbrace{(H^A H^B H^*_{AC} \Omega^C_B + \text{H.c.})}_{\equiv T_4}$$
$$+ H^{AB} H^*_{BC} H^{CD} H^*_{DA} + \Omega^A_B \Omega^B_C \Omega^C_D \Omega^D_A,$$

where the coefficients of the terms have not been shown. The terms referred to as "products of quadratics" are simply all quartic terms that can be constructed as products of the terms in V'_2. There are 19 terms in V'_H and another 13 terms in V_H that involve the field H^{ABCD}.

We do not try to minimize this potential, which would be a tedious undertaking. However, it should be obvious that with 19 parameters and nine components that can get superlarge VEVs the potential is complicated enough to allow many patterns of breaking from $SU(8)$ down to $G_{\rm SM}$. One important question that arises is whether this breaking leaves any goldstone (or pseudo-Goldstone) bosons. These, especially if colored, could lead to disasters, such as rapid proton decay.

The way a goldstone boson would arise is by having the same gauge symmetry broken by the VEVs of two scalar fields that are in multiplets that lie in sectors of the Higgs potential that do not couple to each other. Each of the two multiplets would contain a Goldstone boson corresponding to the broken generator, but only one linear combination of them would get "eaten" by the gauge boson. On the other hand, if the two sectors do couple to each other, what typically happens is that the would-be Goldstone bosons of the two sectors get mass terms with each other in such a way that only the true Goldstone boson (the one that gets eaten) remains without a mass from the Higgs potential. Another way to say this is that the couplings of the various sectors of the Higgs potential to each other prevent any accidental global symmetries from arising.

There are three sectors of Higgs in V'_H: the sectors of H^A, H^{AB}, and Ω^A_B. The fields H^A and H^{AB} are coupled to each other by the term denoted Z_4. The fields H^{AB} and Ω^A_B are coupled to each other by the terms denoted Y_3, Y_4, and Y'_4. The fields H^A and Ω^A_B are coupled to each other by the terms denoted X_3 and X_4. And finally, all three fields are coupled together by the term denoted T_4.

The generators that are broken when $SU(8)$ breaks to $SU(5)$ transform either as singlets or as **5** and $\bar{\mathbf{5}}$ of $SU(5)$. There are potential Goldstone bosons that transform as **5** in H^α, $H^{I\alpha}$, and Ω^α_I. There are potential Goldstone bosons that transform as $\bar{\mathbf{5}}$ in the conjugates of these. As an example of how these are coupled together by the terms that link the various sectors, consider the term T_4. It contains the terms $\langle H^I \rangle \langle H^J \rangle H^*_{I\alpha} \Omega^\alpha_J$, $\langle H^I \rangle H^\alpha \langle H^*_{IJ} \rangle \Omega^J_\alpha$, and $H^\alpha \langle H^I \rangle H^*_{\alpha I} \langle \Omega^I_I \rangle$.

In the discussion of possible Goldstone modes (though not in the minimization) the multiplet H^{ABCD} also has to be taken into account. It contains fields that can obtain mass with potential Goldstone bosons. However, it is not necessary to discuss these, as the foregoing makes clear that, except for the modes that are eaten by the gauge bosons, all components of the Higgs multiplets will get mass from the Higgs potential.

It should be noted that the possibility of unwanted Goldstone bosons is usually only a serious issue in supersymmetric GUTs (see, for example, the analysis in [12], since there the scalar potential has a very restricted form due to the fact that the F-term part comes from a superpotential that is only cubic and must be analytic in the chiral superfields. In nonsupersymmetric GUTs the potential is not so restricted and generally has so many terms that the problem does not usually arise, unless global symmetries are put in "by hand."

APPENDIX B: FLAVOR HIERARCHY FROM GAUGE HIERARCHY

In Sec. IV, it was shown that a nontrivial flavor structure would typically arise in the illustrative $SU(8)$ model even in the absence of any flavor symmetry. The exact pattern of quark and lepton masses depends, however, on many parameters, and, in particular, on the pattern of gauge symmetry breaking. Thus, there is no possibility in this particular model of quantitative predictions.

In this appendix, in order to illustrate in more detail the possibilities of the general scheme, a particular hierarchy among the vacuum expectation values will be assumed.

Suppose that at a scale M_{531} near $M_{P\ell}$ the group $SU(8)$ is broken to $SU(5) \times SU(3) \times U(1)$ by the adjoint Higgs VEV $\langle \Omega^A_B \rangle = \text{diag}(00\,000\,111) M_{531}$. Suppose that the group is then sequentially broken down to $SU(5)$ in these steps: (a) to $SU(5)\times SU(2)\times U(1)$ by the VEVs $\langle H^8\rangle$ and $\langle H^{67}\rangle$, (b) to $SU(5)\times U(1)$ by the VEVs $\langle H^7\rangle$ and $\langle H^{86}\rangle$, and (c) to $SU(5)$ by the VEVs $\langle H^6\rangle$ and $\langle H^{78}\rangle$. Finally, the group breaks to the standard model group at $M_{\rm SM}$.

Assume that the superheavy VEVs satisfy these conditions: $\langle H^{78}\rangle, \langle H^{68}\rangle \ll \langle H^{67}\rangle$; $\langle H^6\rangle \sim \epsilon\langle H^7\rangle$, where $\epsilon = \frac{1}{20}$, and $\langle H^7\rangle \sim \langle H^8\rangle$. The fact that $\langle H^{78}\rangle, \langle H^{68}\rangle \ll \langle H^{67}\rangle$ implies that the light **10**'s of $SU(5)$ are predominantly $\mathbf{10}_3 = \psi^{\alpha\beta}$, $\mathbf{10}_2 = \psi^{\alpha\beta 7}$, and $\mathbf{10}_1 = \psi^{\alpha\beta 6}$. (The $\psi^{\alpha\beta 8}$ gets superheavy mass with the $\psi^{\alpha\beta\gamma}$ through the coupling $\psi^{\alpha\beta\gamma}\psi^{\delta\epsilon 8}H^{67}\Omega^6_6$.)

As for the VEVs that break the weak interaction group $SU(2)_L\times U(1)_Y$, assume that $\langle H^{26}\rangle$, $\langle H^{27}\rangle$, $\langle H^{28}\rangle \ll \langle H^2\rangle \sim \epsilon^2\langle H^{2678}\rangle$, and $\langle H^{2678}\rangle \cong v/\sqrt{2}$. [Of course, at low energy there is just one Higgs doublet, which has the VEV; but this standard model doublet is a linear combination of several doublets that are in $SU(8)$ multiplets of different rank].

With these very specific assumptions, one may write for each quark and lepton mass element the operator that is expected to give the dominant contribution and the order of magnitude of that contribution. These are given in the following equation, along with the operators that are subdominant:

$$
\begin{aligned}
(M_U)_{33}O_{\rm dom} &= (\psi^{\alpha\beta}\psi^{\gamma\delta})\langle H^{2678}\rangle \sim v\\
(M_U)_{23}O_{\rm dom} &= (\psi^{\alpha\beta}\psi^{\gamma\delta 7})\langle H_7\rangle\langle H^{2678}\rangle \sim \frac{1}{16\pi^2}\frac{\langle H^7\rangle}{M_{\rm GUT}}v\\
O_{\rm subdom} &= (\psi^{\alpha\beta}\psi^{\gamma\delta 7})[\langle H^2\rangle\langle H^{68}\rangle, \langle H^6\rangle\langle H^{28}\rangle, \langle H^8\rangle\langle H^{26}\rangle]\\
(M_U)_{13}O_{\rm dom} &= (\psi^{\alpha\beta}\psi^{\gamma\delta 6})\langle H_6\rangle\langle H^{2678}\rangle \sim \frac{1}{16\pi^2}\frac{\langle H^6\rangle}{M_{\rm GUT}}v\\
O_{\rm subdom} &= (\psi^{\alpha\beta}\psi^{\gamma\delta 6})[\langle H^2\rangle\langle H^{78}\rangle, \langle H^7\rangle\langle H^{28}\rangle, \langle H^8\rangle\langle H^{27}\rangle]\\
(M_U)_{22}O_{\rm dom} &= (\psi^{\alpha\beta 7}\psi^{\gamma\delta 7})\langle H_7\rangle\langle H_7\rangle\langle H^{2678}\rangle \sim \frac{1}{16\pi^2}\frac{\langle H^7\rangle^2}{M^2_{\rm GUT}}v\\
O_{\rm subdom} &= (\psi^{\alpha\beta 7}\psi^{\gamma\delta 7})[\langle H_7\rangle\langle H^2\rangle\langle H^{68}\rangle, \langle H_7\rangle\langle H^6\rangle\langle H^{28}\rangle, \langle H_7\rangle\langle H^8\rangle\langle H^{26}\rangle]\\
(M_U)_{12}O_{\rm dom} &= (\psi^{\alpha\beta 6}\psi^{\gamma\delta 7})\langle H_6\rangle\langle H_7\rangle\langle H^{2678}\rangle \sim \frac{1}{16\pi^2}\frac{\langle H^6\rangle\langle H^7\rangle}{M^2_{\rm GUT}}v\\
O_{\rm subdom} &= (\psi^{\alpha\beta 6}\psi^{\gamma\delta 7})\langle H^{28}\rangle\langle(\Omega^6_6-\Omega^7_7)\rangle, (\psi^{\alpha\beta 6}\psi^{\gamma\delta 7})\langle\mathbf{H_{67}}\rangle\langle\mathbf{H^{2678}}\rangle\\
(M_U)_{11}O_{\rm dom} &= (\psi^{\alpha\beta 6}\psi^{\gamma\delta 6})\langle H_6\rangle\langle H_6\rangle\langle H^{2678}\rangle \sim \frac{1}{16\pi^2}\frac{\langle H^6\rangle^2}{M^2_{\rm GUT}}v\\
O_{\rm subdom} &= (\psi^{\alpha\beta 6}\psi^{\gamma\delta 6})[\langle H_6\rangle\langle H^2\rangle\langle H^{78}\rangle, \langle H_6\rangle\langle H^7\rangle\langle H^{28}\rangle, \langle H_6\rangle\langle H^8\rangle\langle H^{27}\rangle]\\
(M_D)_{3m}O_{\rm dom} &= (\psi^{\alpha 2}\psi_{(m)\alpha})\langle H_2\rangle\langle\Omega^2_2\rangle \sim \frac{1}{16\pi^2}\frac{\langle\Omega^2_2\rangle}{M_{\rm GUT}}v\\
O_{\rm subdom} &= (\psi^{\alpha 2}\psi^{(m)\alpha})[\langle H_{28}\rangle\langle H^8\rangle, \langle H_{27}\rangle\langle H^7\rangle, \langle H_{26}\rangle\langle H^6\rangle]\\
(M_D)_{2m}O_{\rm dom} &= (\psi^{\alpha 27}\psi_{(m)\alpha})\langle H_2\rangle\langle H_7\rangle\langle(\Omega^2_2-\Omega^7_7)\rangle \sim \frac{1}{16\pi^2}\frac{\langle(\Omega^2_2-\Omega^7_7)\rangle\langle H^7\rangle}{M^2_{\rm GUT}}v\\
O_{\rm subdom} &= (\psi^{\alpha 27}\psi^{(m)\alpha})\langle H_{27}\rangle\langle(\Omega^2_2-\Omega^7_7)\rangle\\
(M_D)_{1m}O_{\rm dom} &= (\psi^{\alpha 26}\psi_{(m)\alpha})\langle H_2\rangle\langle H_6\rangle\langle(\Omega^2_2-\Omega^6_6)\rangle \sim \frac{1}{16\pi^2}\frac{\langle(\Omega^2_2-\Omega^6_6)\rangle\langle H^6\rangle}{M^2_{\rm GUT}}v\\
O_{\rm subdom} &= (\psi^{\alpha 26}\psi^{(m)\alpha})\langle H_{26}\rangle\langle(\Omega^2_2-\Omega^6_6)\rangle.
\end{aligned}
$$

Note that there is an operator written in boldface that contributes to $(M_U)_{12}$. Actually, the assumed hierarchy of VEVs does not imply that this operator is subdominant. As it would lead to an unrealistically large contribution, one must assume that this operator is suppressed for some other reason: say, a smallness of a dimensionless coupling in the diagram that produces it. This operator does not contribute importantly anywhere else.

The hierarchy among the elements produced under these specific assumptions about VEVs is qualitatively realistic. However, no quantitative predictions are possible, because of the large number of parameters involved. For example, the masses denoted "$M_{\rm GUT}$" in the estimates given in the previous equation depend on the masses of the heaviest particles appearing the loops, and the spectrum of superheavy particles depends on many parameters in the Higgs potential and elsewhere.

It should also be emphasized that higher-dimension operators may arise not only from loop diagrams but also from Planck-scale physics. For the mass matrix elements that involve the third family, which are suppressed by only one power of "$M_{\rm GUT}$," it is possible that the same operators arising from Planck-scale physics (and therefore suppressed by one power of $M_{P\ell}$) may be comparable, or even slightly larger, if "$M_{\rm GUT}$" is greater than $M_{P\ell}/16\pi^2$. This would not qualitatively affect the hierarchy. On the other hand, there may be models of the type presented in this paper where operators induced by Planck-scale physics may play a dominant role. In fact, in models with low-energy supersymmetry, loop-diagrams involving GUT-scale particles would be extremely suppressed, and so Planck-scale effects would be the dominant contributions to the masses of the lighter families [13].

APPENDIX C: THE RIGHT-HANDED NEUTRINOS

In the $SU(8)$ model whose fermion content is given by Eq. (3), there are altogether 31 fermions that are singlets under $SU(5)$ and the standard model group $G_{\rm SM}$. These are the 27 fields $N_{mI} \equiv \psi_{(m)I}$, with $m = 1, \ldots, 9$ and $I = 6, 7, 8$; the 3 fields $N_I \equiv \frac{1}{2}\epsilon_{IJK}\psi^{JK}$, with $I, J, K = 6, 7, 8$; and the 1 field $N \equiv \psi^{678}$. All of these can be considered "right-handed neutrinos."

They have a mass matrix M_R that at tree level is of the form

$$(N_{mI}, N_I, N)\begin{pmatrix} M_{mI,NJ} & M_{mI,J} & M_{mI} \\ M_{I,nJ} & 0 & 0 \\ M_{nJ} & 0 & 0 \end{pmatrix}\begin{pmatrix} N_{nJ} \\ N_J \\ N \end{pmatrix}.$$

The mass submatrices in the above come from the tree-level Yukawa terms in Eq. (4) as follows:

$$Y_m(\psi^{IJK}\psi_{(m)I})H^*_{JK} \Rightarrow M_{mI} = \frac{1}{2}Y_m\epsilon^{IJK}H^*_{JK}$$

$$y_n(\psi^{JK}\psi_{(n)J})H^*_K \Rightarrow M_{I,nJ} = \frac{1}{2}y_n\epsilon^{IJK}H^*_K$$

$$a_{mn}(\psi_{(m)I}\psi_{(n)J})H^{IJ} \Rightarrow M_{mI,nJ} = a_{mn}H^{IJ}.$$

The submatrix $M_{mI,nJ}$ is 27-by-27, but it only has rank 16 as will now be shown. In determining the rank of these matrices, it is convenient to go to the following bases. For the $SU(3)$ subgroup of $SU(8)$ corresponding to the indices 6, 7, 8, go to the basis where $\langle H^{78}\rangle \neq 0$, $\langle H^{67}\rangle = 0$, $\langle H^{68}\rangle = 0$. For the nine-dimensional vectorspace spanned by the nine antifundamentals labeled by $m = 1, 2, \ldots, 9$, go to the basis where a_{mn} has the "normal form," in which $a_{2p,2p+1} = -a_{2p+1,2p} \neq 0$, where $p = 1, 2, 3, 4$, and where all other elements vanish, including a_{1m} and a_{m1}. (Since a_{mn} is a 9-by-9 antisymmetric matrix it has rank 8.) With these choices of bases, the matrix $M_{mI,nJ}$, which is given by $a_{mn}H^{IJ}$, obviously vanishes unless m and n take values in the range $2, \ldots, 9$ and I and J take values in the range 7, 8. Thus, mI and nJ each have only 16 possible values that can lead to nonzero entries for the matrix $M_{mI,nJ}$.

The 11 fields N_{mI} that have only zero entries in $M_{mI,nJ}$ are just N_{1I}, $I = 6, 7, 8$, and N_{m6}, $m = 2, \ldots, 9$. One may thus write the 31-by-31 mass matrix of the right-handed neutrinos as

$$\begin{pmatrix} 0_{11\times 11} & | & 0_{11\times 16} & | & M_{11\times 4} \\ - & | & - \;\; - \;\; - & | & - \\ & | & & | & \\ 0_{16\times 11} & | & A_{16\times 16} & | & M_{16\times 4} \\ & | & & | & \\ - & | & - \;\; - \;\; - & | & - \\ M_{4\times 11} & | & M_{4\times 16} & | & 0_{4\times 4} \end{pmatrix}.$$

The last 4 rows and columns refer to the four fields N_I and N. The remaining 27 rows and columns refer to the N_{mI} The middle 16 rows and columns refer to those fields N_{mI} that have nonzero entries in the matrix $M_{mI,nJ}$. Those nonzero entries are in the matrix that is called A. The first 11 rows and columns refer to those fields N_{mI} that have only zero entries in $M_{mI,nJ}$.

One sees that in this matrix there are submatrices that are 4×11 and 11×4 (which are transposes of each other). By a choice of basis these can obviously be made to vanish except for a 4×4 submatrix. Consequently, one sees by inspection that the full matrix has at least 7 zero eigenvalues. Unless certain Yukawa couplings vanish for no symmetry reason, there are only 7 vanishing eigenvalues.

Thus, at tree level, 7 of the 31 "right-handed neutrinos" remain massless. It can be shown that there are enough one-loop diagrams to give mass to all 7 of these states. One such diagram is shown in Fig. 2. (There are several other types of diagram.) Thus, one expects that 24 right-handed neutrinos have mass of order $M_{\rm GUT}$ and 7 have mass of order $\frac{1}{16\pi^2}M_{\rm GUT}$.

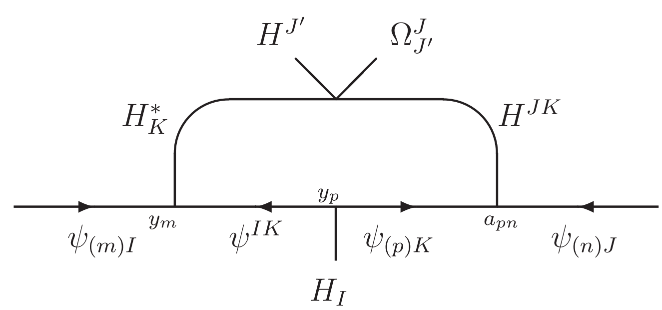

FIG. 2. A diagram that gives mass to right-handed neutrinos that do not get mass at tree level.

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