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Volume dependence of spectral weights for unstable particles in a solvable model

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Volume dependence of the spectral weight is usually used as a simple criteria to distinguish single-particle states from multiparticle states in lattice QCD calculations. Within a solvable model, the Lee model, we show that this criteria is in principle only valid for a stable particle or a narrow resonance. If the resonance being studied is broad, then the volume dependence of the corresponding spectral weight resembles that of a multiparticle state instead of a single-particle one. For an unstable V particle in the Lee model, the transition from single-particle to multiparticle volume dependence is governed by the ratio of its physical width to the typical level spacing in the finite volume. We estimate this ratio for practical lattice QCD simulations and find that, for most cases, the resonance studied in lattice QCD simulations still resembles the single-particle behavior.

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I. INTRODUCTION

Quantum chromodynamics (QCD) is believed to be the underlying theory of strong interactions. Because of its nonperturbative nature, low-energy properties of strong interaction should be studied with a nonperturbative method. Typical problems include light hadron spectrum and low-energy hadron-hadron scattering. Lattice QCD provides a genuine nonperturbative framework in which nonperturbative problems can be tackled using numerical simulations. In a typical lattice calculation for hadron spectrum, energy eigenvalues of the QCD Hamiltonian are measured numerically, with different quantum numbers that are conserved by the strong interaction.[1] People tend to interpret these energy eigenvalues as mass values of corresponding particles. This seems to provide a nonperturbative definition for the mass value of a hadron. The width of a hadronic resonance is a more complex issue. Using Lüscher's formula, scattering phase shifts can also be calculated from the two-particle energy eigenvalues [1–12]. However, a direct, model-independent and nonperturbative calculation of the width parameter remains a difficult task [13–16].

Phenomenologically, a resonance is characterized by its mass parameter M and the width parameter Γ. A common theoretical definition for these two physical parameters refers to the pole of the S matrix on the second sheet of the complex energy plane:

$$z = M^{(p)} - i\Gamma^{(p)}/2, \qquad S(z) \to \infty. \tag{1}$$

However, experimentalists prefer more tractable definitions such as the scattering phase shift, or the total cross sections which are in principle measurable physical quantities in the scattering experiment. For example, the position for the mass of a resonance can be defined to be the position where total cross section reaches its maximum or the corresponding phase shift passing $\pi/2$.[2] The definition of the width in this case is somewhat ambiguous except for narrow resonances. In the case of infinitely narrow resonance, when $E_{\text{c.m.}} = M^{(\delta)} \pm \Gamma^{(\delta)}/2$, the phase shift exactly passes through $\pi/4$ and $3\pi/4$, respectively. In terms of total cross section, this also corresponds to the position where cross section has dropped to half its peak value. For wide resonances, the peak is usually not symmetric with respect to the peak position and we may choose to define the width by demanding the phase shift to be exactly $\pi/4$ at $M^{(\delta)} - \Gamma^{(\delta)}/2$. So, using the phase shift, one possible definition goes

$$\delta(M^{(\delta)}) = \pi/2, \qquad \delta(M^{(\delta)} - \Gamma^{(\delta)}/2) = \pi/4. \tag{2}$$

It is a well-known fact that, the above-mentioned definitions for the spectral parameters of a resonance, namely the energy eigenvalues measured in lattice calculations, the S-matrix pole definition and the phase-shift definition, do not coincide with one another in general. One expects that they only agree when the resonance becomes infinitely narrow. Although it is difficult to show this for a general theory nonperturbatively, we will show this explicitly in a totally solvable field theoretical model, the Lee model.

Numerical simulations in lattice QCD are performed within a finite volume. All energy levels in this finite box are discrete. Therefore, it is an important and nontrivial question for the lattice calculations to properly identify single-particle states and multiparticle states which might mix within a particular symmetry channel. A typical example is the ρ meson, which is a single resonance that mixes with two-pion scattering states. To distinguish the

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[1]Strictly speaking, only the eigenvalues at finite lattice spacing are measured. To obtain the continuum eigenvalues, one has to perform the continuum extrapolation of lattice results.

[2]In this discussion, we have neglected all other background contributions. We also assume that in the energy range we are investigating, there is only one resonance.

single-particle states from multiparticle states, it is suggested that spectral weights of the eigenstates are to be measured. One expects that spectral weights of single and multiparticle eigenstates show different volume dependence and thus can be utilized to disentangle the single-particle states from the multiparticle states. Typically, the spectral weight of a single-particle state has little volume dependence if the volume is not too small while the spectral weight of two-particle states will exhibit a typical $1/\Omega$ dependence which can be captured by performing the simulation on two different Ω which is the three-volume of the lattice. This strategy has been used in Ref. [17] where the authors show that the so-called pentaquark states measured in their lattice calculations are in fact kaon-nucleon two-particle scattering states. However, this conclusion is not so settled even in the first-principle lattice QCD calculations [18–20].

Although well suited for a stable particle or a narrow resonance, one also expects the above-mentioned criteria to be modified when the resonance becomes broad. The reason for this is quite clear. For a broad resonance, the scattering states themselves form a complete set in the corresponding Hilbert space and one thus expects the spectral weight of a broad resonance to behave more like that of multiparticle scattering states. Although this sounds reasonable, the transition of the spectral weight from the single-particle behavior to the multiparticle behavior has never been shown explicitly in the literature. In this paper, we demonstrate this scenario within the Lee model where everything can be computed explicitly. This result suggests that, using the volume dependence of spectral weight as *the criteria* to distinguish single particle from the multiparticle states is only valid when the single particle is either stable or unstable but narrow. In other words, it cannot be applied to broad resonances without further careful considerations. Moreover, within the Lee model we can also verify that the transition from single-particle to multiparticle behavior is governed by the ratio of the physical width to the typical level spacing in the finite box. Assuming this ratio is also the relevant quantity in lattice QCD, we estimate this ratio for some typical lattice calculations and find that, in most cases, the resonance studied still resembles the single-particle behavior for recent lattice simulations and conclude.

This paper is organized as follows. In Sec. II, we introduce the Lee model and summarize its main results. The scattering phase shifts and the S-matrix element are also calculated and various definitions for the spectral parameters of a resonance is compared directly. In Sec. III, we focus on the Euclidean correlation functions that are measured in lattice simulations. Spectral weights for the eigenstates are computed and the volume dependence of the spectral weights are analyzed. As the resonance becomes broader, the transition from the single-particle behavior to the multiparticle behavior is explicitly shown. In Sec. IV, we will discuss the possible impact of our results by estimating the ratio in recent lattice simulations on pion-pion scattering.

II. THE ENERGY EIGENSTATES AND THE PHASE SHIFT IN THE LEE MODEL

The Lee model [21] is a completely solvable field theoretical model proposed by Lee a long time ago. The model involves three types of "particles": the so-called V particle, the N particle, and the θ particle. The Hamiltonian of the model is given by

$$\begin{aligned} H &= H_0 + H_1, \\ H_0 &= m_V \sum_{\mathbf{p}} V^\dagger_{\mathbf{p}} V_{\mathbf{p}} + m_N \sum_{\mathbf{p}} N^\dagger_{\mathbf{p}} N_{\mathbf{p}} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}, \\ H_1 &= -\frac{g_0}{\sqrt{\Omega}} \sum_{\mathbf{p},\mathbf{k}} \frac{f(\omega_{\mathbf{k}})}{\sqrt{2\omega_{\mathbf{k}}}} [V^\dagger_{\mathbf{p}+\mathbf{k}} + N_{\mathbf{p}} a_{\mathbf{k}} + V_{\mathbf{p}+\mathbf{k}} + N^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{k}}]. \end{aligned} \tag{3}$$

Here Ω is a large but finite volume of the system; g_0 is the bare coupling constant. $V^\dagger_{\mathbf{p}}(V_{\mathbf{p}})$, $N^\dagger_{\mathbf{p}}(N_{\mathbf{p}})$, and $a^\dagger_{\mathbf{p}}(a_{\mathbf{p}})$ correspond to the creation (annihilation) operators of the V, N, and θ particles, respectively. They satisfy the usual commutation relations:[3]

$$[V_{\mathbf{p}}, V^\dagger_{\mathbf{k}}] = [N_{\mathbf{p}}, N^\dagger_{\mathbf{k}}] = [a_{\mathbf{p}}, a^\dagger_{\mathbf{k}}] = \delta_{\mathbf{pk}}. \tag{4}$$

We will call m_V, m_N in (3) bare mass of the V and N particle. The energy of the θ particle is given by $\omega_{\mathbf{k}} = \sqrt{\mu^2 + \mathbf{k}^2}$, where μ is the mass of the θ particle. To make the whole system well defined, we have enclosed the system in a three-volume of size Ω and introduced a form factor $f(\omega)$ to regularize the possible ultraviolet divergences. It is easy to see that the free one N-particle and one θ-particle states remain eigenstates of the full Hamiltonian. However, the free one V-particle states are not since it is coupled to the $N\theta$ pair states. We will restrict our discussion in the sector of one V-particle and $N\theta$ pair states.

We are concerned with the following properties of the Lee model: the exact eigenstates and eigenvalues of the Hamiltonian, which is what is measured in lattice simulations; the exact S-matrix element and scattering phase shift, which will be utilized to locate possible resonances in the model, and the spectral weight from the Euclidean correlation function whose volume dependence is our major concern in this paper. In this section, we will first summarize the results for the eigenstates, eigenvalues, and S-matrix elements. Spectral weights will be dealt with in the next section.

[3]In the original Lee model, the V particles and N particles are fermions while θ particles are bosons. In this paper, we assume they are all bosons. This does not change the results.

In the V-$N\theta$ sector, the exact eigenstates $|n\rangle_{\mathbf{p}}$ of the full Hamiltonian (with eigenvalue E_n) can be obtained as [22]

$$|n\rangle_{\mathbf{p}} = Z_n^{-1/2}\bigg[|V_{\mathbf{p}}\rangle + \frac{g_0}{\sqrt{\Omega}}\sum_{\mathbf{k}}\frac{f(\omega_{\mathbf{k}})}{\sqrt{2\omega_{\mathbf{k}}}} \times \frac{1}{m_N+\omega_{\mathbf{k}}-E_n}|N_{\mathbf{p}-\mathbf{k}}-\theta_{\mathbf{k}}\rangle\bigg], \tag{5}$$

$$m_V - E_n = F(E_n - m_N), \qquad F(x) \equiv \frac{g_0^2}{\Omega}\sum_{\mathbf{k}}\frac{f^2(\omega_{\mathbf{k}})}{2\omega_{\mathbf{k}}}\left(\frac{1}{\omega_{\mathbf{k}}-x}\right), \tag{6}$$

and the normalization factor Z_n is also found to be

$$Z_n(E_n) = 1 + F'(E_n - m_N) = 1 + \frac{g_0^2}{\Omega}\sum_{\mathbf{k}}\frac{f^2(\omega_{\mathbf{k}})}{2\omega_{\mathbf{k}}}\left(\frac{1}{\omega_{\mathbf{k}}+m_N-E_n}\right)^2. \tag{7}$$

The function $F(x - m_N)$ has simple poles at each $x = m_N + \omega_{\mathbf{k}}$ and one obtains a series of eigenvalues E_n from Eq. (6). It is also seen that there is always a root E_n satisfying $E_n < m_N + \mu$. However, one cannot draw the conclusion that the V particle is always stable. The fact is that, if m_V is small enough, then a stable V particle exists. If m_V is too large, then no stable V particle exists. The precise condition in the infinite volume limit is [22]

$$m_V - m_N > \mu + \phi(\mu), \tag{8}$$

where $\phi(x)$ is the principle-valued integral counterpart of the function F defined in Eq. (6):

$$\phi(x) = g_0^2\int\frac{d^3k}{(2\pi)^3}\frac{f^2(\omega_{\mathbf{k}})}{2\omega_{\mathbf{k}}}\left(\frac{1}{\omega_{\mathbf{k}}-x}\right). \tag{9}$$

Note that the function $\phi(x)$ is a monotonically increasing function of x. Therefore, if condition (8) is satisfied, V particles are unstable and they decay into $N\theta$ particle pairs.

At this stage, it is useful to point out the following fact. If we were to replace the eigenvalues E_n in Eq. (5) by $E + i\lambda$ with both E and λ being real but λ is small, we can construct an *approximate* eigenstate of the Hamiltonian. We find that the value of E has to be one of those E_n values. However, it is easy to see that the equation for the imaginary part λ can never be satisfied exactly for nonvanishing λ. This is due to the fact that the Hamiltonian is Hermitian and the eigenvalues thus have to be real. However, if we take E to be one of those E_n values but λ being small enough, we indeed obtain an approximate eigenstate of the Hamiltonian. The approximation becomes better and better as $\lambda \to 0$. It is well known that a narrow resonance in scattering theory in fact corresponds to such a scenario. Nevertheless, one should keep in mind that such a description is in fact only meaningful when the resonance is narrow.

Not only the exact energy eigenstates can be obtained, the scattering phase shifts can also be calculated within this sector of the Lee model. It can be shown that, when the V particle is unstable, the $N\theta$ scattering states form a complete set in the Hilbert space. These scattering states are the solution of the corresponding Lippmann-Schwinger equation:

$$|N_{\mathbf{q}}\theta_{\mathbf{k}}\rangle_{\pm} = |N_{\mathbf{q}}\theta_{\mathbf{k}}\rangle + \frac{1}{m_N+\omega_{\mathbf{k}}-H\pm i\epsilon}H_1|N_{\mathbf{q}}\theta_{\mathbf{k}}\rangle. \tag{10}$$

The states $|N_{\mathbf{q}}\theta_{\mathbf{k}}\rangle_{+/-}$ corresponds to well-prescribed incoming/outgoing waves in the infinite past/future, respectively. These states are also eigenstates of the full Hamiltonian with eigenvalues $m_N + \omega_{\mathbf{k}}$. It can be shown explicitly that both $|N_{\mathbf{q}}\theta_{\mathbf{k}}\rangle_+$ and $|N_{\mathbf{q}}\theta_{\mathbf{k}}\rangle_-$ form a complete set in this particular Hilbert subspace. They also form an orthonormal basis:

$$_{\pm}\langle N_{\mathbf{q}'}\theta_{\mathbf{k}'}|N_{\mathbf{q}}\theta_{\mathbf{k}}\rangle_{\pm} = \delta_{\mathbf{q}\mathbf{q}'}\delta_{\mathbf{k}\mathbf{k}'}. \tag{11}$$

The unitary matrix which relates these two sets of orthonormal states is nothing but the S matrix whose matrix elements is defined via

$$S_{\mathbf{q}'\mathbf{k}';\mathbf{q}\mathbf{k}} \equiv {}_{-}\langle N_{\mathbf{q}'}\theta_{\mathbf{k}'}|N_{\mathbf{q}}\theta_{\mathbf{k}}\rangle_{+}. \tag{12}$$

For the Lee model, the Lippmann-Schwinger states defined in Eq. (10) can be computed exactly with the result [22,23]:

$$|N_{\mathbf{q}}\theta_{\mathbf{k}}\rangle_{\pm} = |N_{\mathbf{q}}\theta_{\mathbf{k}}\rangle - \frac{g_0 f(\omega_{\mathbf{k}})}{\sqrt{2\Omega\omega_{\mathbf{k}}}} \times \frac{1}{m_N+\omega_{\mathbf{k}}-m_V\pm i\epsilon+F(\omega_{\mathbf{k}}\pm i\epsilon)} \times\bigg[|V_{\mathbf{k}+\mathbf{q}}\rangle - \frac{g_0}{\sqrt{\Omega}}\sum_{\mathbf{p}}\frac{f(\omega_{\mathbf{p}})}{\sqrt{2\omega_{\mathbf{p}}}} \times\frac{1}{\omega_{\mathbf{k}}-\omega_{\mathbf{p}}\pm i\epsilon}|N_{\mathbf{q}+\mathbf{k}-\mathbf{p}}\theta_{\mathbf{p}}\rangle\bigg]. \tag{13}$$

From this result, one gets the scattering phase shift for the $N\theta$ scattering. It turns out that there is only s-wave scattering in this sector of the Lee model and the corresponding phase shift satisfies the following equation:

$$e^{2i\delta(k)} = \frac{m_N+\omega-m_V+\phi(\omega)-i\Gamma(\omega)/2}{m_N+\omega-m_V+\phi(\omega)+i\Gamma(\omega)/2}, \qquad \tan\delta(k) = -\frac{\Gamma(\omega)/2}{m_N+\omega-m_V+\phi(\omega)}, \tag{14}$$

where $\omega = \sqrt{k^2+\mu^2}$ and the function $\phi(\omega)$ is given by Eq. (9). If the parameters of the theory is chosen such that $m_N + \mu - m_V + \phi(\mu) < 0$ at the threshold, which is the same condition for the V particle becoming unstable, it is seen that the real part of the denominator in the S matrix will vanish at some energy $M^{(\delta)}$ above the threshold. At this particular energy, the model exhibits a typical resonance. However, the (bare) width of the resonance is in general not a constant but energy dependent:

$$\Gamma(\omega) = \frac{g_0^2}{2\pi} f^2(\omega)\sqrt{\omega^2 - \mu^2}\theta(\omega - \mu). \quad (15)$$

The total energy at which the phase shift passes through $\pi/2$ is given by

$$M^{(\delta)} - m_V + \phi(M^{(\delta)} - m_N) = 0, \quad (16)$$

while the definition for the width gives

$$M^{(\delta)} - \frac{\Gamma^{(\delta)}}{2} - m_V + \phi\left(M^{(\delta)} - \frac{\Gamma^{(\delta)}}{2} - m_N\right) + \frac{1}{2}\Gamma\left(M^{(\delta)} - \frac{\Gamma^{(\delta)}}{2} - m_N\right) = 0. \quad (17)$$

The pole mass and the corresponding width are given by

$$z^{(p)} = M^{(p)} - i\frac{\Gamma^{(p)}}{2}, \qquad z^{(p)} - m_V + \mathcal{F}(z^{(p)} - m_N) = 0, \quad (18)$$

where the function $\mathcal{F}(z)$ is given by

$$\mathcal{F}(z) = \int_\mu^\infty \frac{\Gamma(\omega)}{2\pi}\frac{d\omega}{\omega - z}, \quad (19)$$

with the understanding that this pole position should be solved on the second sheet.[4] Comparison of the above explicit formulas shows that they are generally different if the width of the resonance is not narrow. It is also evident from the above formulas that, when the width is becoming infinitely narrow, $M^{(\delta)}$ coincides with $M^{(p)}$ while $\Gamma^{(\delta)}$ coincides with $\Gamma^{(p)}$.

III. THE EUCLIDEAN CORRELATION FUNCTIONS AND THE SPECTRAL WEIGHTS

In this section, we discuss the mass values and the corresponding spectral weights measured in a lattice Monte Carlo simulation. In such a calculation, by measuring appropriate Euclidean correlation functions, the eigenvalues (typically a few lowest) of the Hamiltonian are obtained.[5] In the Lee model, these eigenvalues are precisely those E_n values given by Eq. (6). It is then clear that E_n in principle is different from any of $M^{(p)}$ or $M^{(\delta)}$ defined by the S-matrix pole or the phase shift. But if the resonance is narrow enough, E_n coincides with $M^{(p)}$ or $M^{(\delta)}$. As pointed out in the Introduction, it is suggested that one can distinguish the single-particle, two-particle, and multiparticle states by inspecting the volume dependence of the so-called spectral weights for the states [17]. Here we would like to investigate this possibility within the Lee model where the eigenstates and the corresponding eigenvalues are explicitly known.

We will first look at an interpolating field $V(\mathbf{x})$. The correlation function that we are interested in is

$$\sum_{\mathbf{x}} \langle 0|V(\mathbf{x}, t)V^\dagger(0)|0\rangle, \quad (20)$$

where we have assumed that the fields are now defined on a lattice with the lattice spacing being set to unity. Inserting the complete set of states we have

$$\sum_{\mathbf{x}} \langle 0|V(\mathbf{x}, t)V^\dagger(0)|0\rangle \propto \sum_n Z_n^{-1}(E_n)e^{-E_n t}, \quad (21)$$

where E_n and $Z_n(E_n)$ are given by Eqs. (6) and (7) respectively. Therefore, the spectral weight function W_n for each eigenstate $|n\rangle_{\mathbf{k}}$ is simply

$$W_n = Z_n^{-1}(E_n) = \left(1 + \frac{g_0^2}{\Omega}\sum_{\mathbf{k}}\frac{f^2(\omega_{\mathbf{k}})}{2\omega_{\mathbf{k}}}\left(\frac{1}{\omega_{\mathbf{k}} + m_N - E_n}\right)^2\right)^{-1}. \quad (22)$$

At first sight, the spectral weights in Eqs. (21) and (22) do not seem to show the expected volume dependence at all. However, we will show below that if the V particle is stable or if the width of the unstable V particle is small, Eq. (22) does provide the expected volume dependence for single- and two-particle states, respectively.

In general, the volume dependence of W_n is quite complicated for a finite (not necessarily large) volume Ω. However, if the volume Ω is sufficiently large, the volume dependence can be estimated. As Eq. (22) shows, one is led to consider the function $F(x)$ defined in Eq. (6). The spectral weight of a particular energy eigenvalue is simply related to the derivative of this function evaluated at the exact energy eigenvalue:

$$W_n = 1/(1 + F'(E_n - m_N)). \quad (23)$$

The behavior of the function $F(x)$ is drastically different for values of x below the threshold and above the threshold. If x is below the threshold, i.e. $x < \mu$, the contribution to be summed is bounded in the large volume limit and the function $F(x)$ goes over to its integration counterpart $\phi(x)$ smoothly as the volume goes to infinity. However, if x is above the threshold ($x > \mu$), there exist values of $\omega_{\mathbf{k}}$ which are sufficiently close to x in the large volume limit and therefore some contributions are unbounded. We will discuss this situation in the following.

For large enough three-volume Ω, a typical spacing between adjacent energy levels, which we denote as $\Delta\omega$, can be estimated as follows:

[4] It can be shown that the above equation can never be satisfied on the first sheet where the solutions correspond to stable bound states on the real axis.

[5] In reality, lattice data still contain lattice artifacts caused by the finite lattice spacing. In this paper, we assume that these finite lattice spacing errors have already been subtracted, namely the continuum limit is already taken.

$$\frac{\Omega}{(2\pi)^3} g(\omega)\Delta\omega = 1, \qquad \mapsto \Delta\omega = \frac{(2\pi)^3}{\Omega}\frac{1}{g(\omega)}, \quad (24)$$

where $g(\omega) = 4\pi\sqrt{\omega^2 - \mu^2}\omega$ is the density of states for the $N\theta$ pairs. Therefore, in the infinite volume limit, the level spacing is proportional to $1/\Omega$.

In the definition of $F(x)$, the function to be summed over factorizes into two parts: the fast-changing part $1/(\omega - x)$ and the slow-changing part $f^2(\omega)/(2\omega)$. Here the term slow-changing refers to the fact that when x changes an amount of the order of $\Delta\omega$, the function changes little (and likewise for the definition of fast-changing). Note that this factorization is meaningful only when the volume is large and hence $\Delta\omega$ is small. Assuming that we are in such a situation, then the summation for the function $F(x)$ may be separated into two parts:

$$F(x) = \frac{g_0^2}{\Omega}\left(\sum_{\mathbf{k},|\omega_{\mathbf{k}}-x|\geq\epsilon} + \sum_{\mathbf{k},|\omega_{\mathbf{k}}-x|<\epsilon}\right)\frac{f^2(\omega_{\mathbf{k}})}{2\omega_{\mathbf{k}}}\left(\frac{1}{\omega_{\mathbf{k}} - x}\right), \quad (25)$$

where ϵ is a small positive number within which the function $f^2(\omega)/(2\omega)$ is almost a constant, but $\epsilon \gg \Delta\omega$. The first summation in the above expression is nothing but the principle-valued integral $\phi(x)$ once the volume is going to infinity and the parameter ϵ is going to zero. We will denote it as $\phi_\epsilon(x)$. In the second summation, since the function $f^2(\omega)/(2\omega)$ can be viewed as a constant, we have

$$F(x) = \phi_\epsilon(x) + \frac{g_0^2}{\Omega}\frac{f^2(x)}{2x}\sum_{\mathbf{k},|\omega_{\mathbf{k}}-x|<\epsilon}\frac{1}{\omega_{\mathbf{k}} - x}. \quad (26)$$

Now that the density of state function $g(\omega)$ is also a slow-changing function of the energy, therefore, within the interval $|\omega_{\mathbf{k}} - x| < \epsilon$, the energy levels can be viewed as almost equally spaced with the level spacing given by Eq. (24). Denoting the level closest to x by ω^*, we have

$$\sum_{\mathbf{k},|\omega_{\mathbf{k}}-x|<\epsilon}\frac{1}{\omega_{\mathbf{k}} - x} \simeq \sum_{n=-\infty}^{\infty}\frac{1}{\omega^* + n\Delta\omega - x}, \quad (27)$$

where we have extended the summation to infinity. Now the summation can be computed exactly and using the relation in Eq. (24) and the definition (15) we finally have

$$F(x) = \phi(x), \qquad x < \mu, \quad (28)$$

$$F(x) = \phi(x) - \frac{\Gamma(x)}{2}\cot\left[\pi\left(\frac{x - \omega^*}{\Delta\omega}\right)\right], \qquad x \geq \mu. \quad (28')$$

This expression is a good estimate for the function $F(x)$ in the large volume limit for $x > \mu$. Note that, if we set $x - (m_V - m_N) + F(x) = 0$, which is nothing but the eigenvalue equation (6), we would obtain all the energy eigenvalues: $x = E_n - m_N$. Using the estimate (28′) and the result for the scattering phase shift (14), we thus arrive at a relation between the phase shift and the corresponding energy shift:

$$E - (m_N + \omega^*) = -\frac{1}{\pi}\delta(\omega^*)\Delta\omega, \quad (29)$$

where E is the exact energy eigenvalue perturbed from $(m_N + \omega^*)$. This result was first obtained by DeWitt a long time ago [24]. It is in fact a quite general result which can be derived from formal scattering theory.

Let us now come to the discussion of the spectral weights. According to Eq. (22), the spectral weights are related to the derivative of the function $F(x)$ evaluated at $x = E_n - m_N$. Taking the derivative of Eqs. (28) and (28′) and using DeWitt's relation (29), we get

$$F'(x) = \phi'(x), \qquad x < \mu, \quad (30)$$

$$F'(x) = \phi'(x) - \frac{\Gamma'(x)}{2}\cot\delta(x) + \frac{\pi}{2}\frac{\Gamma(x)}{\Delta\omega}\csc^2\delta(x), \quad (30')$$
$$x \geq \mu.$$

It then becomes clear that, for eigenvalues below the threshold, the spectral weight will contain almost no volume dependence when the volume is sufficiently large:

$$W_n \simeq \frac{1}{1 + \phi'(E_n - m_N)}. \quad (31)$$

In the Lee model, this can only happen when the V particle is below the threshold and thus is stable. For eigenvalues above the threshold, however, the last term in Eq. (30′) is clearly proportional to the volume Ω. As a consequence, the corresponding spectral weight is proportional to $1/\Omega$, provided the energy level is above the threshold. It is interesting to note that, if we take x to be at the location of the resonance, i.e. $x = E_n - m_N = M^{(\delta)} - m_N$, the spectral weight is

$$W \simeq \frac{1}{1 + \phi'(x) + \frac{\pi}{2}\frac{\Gamma(x)}{\Delta\omega}} \propto \frac{1}{1 + \frac{\pi}{2}\frac{\Gamma_R(x)}{\Delta\omega}}, \quad (32)$$

where $\Gamma_R(x) \equiv \Gamma(x)/[1 + \phi'(x)]$ is the *renormalized* (physical) width of the resonance [23]. As was pointed out at the beginning of this section, all of the above discussion assumes that the volume is large enough. To be more precise, Eq. (30′) shows that this requires $\Gamma_R(x)/\Delta\omega(x) \gg 1$. Another equivalent form for this condition is, combining Eq. (24),

$$\Gamma_R(x)g(x)\Omega \gg 1. \quad (33)$$

A resonance satisfying this inequality is called a *broad* resonance. If this condition is satisfied, then the spectral weight (32) behaves like $W(x) \simeq 1/(\Gamma_R(x)g(x)\Omega)$. The physical meaning of the condition $\Gamma_R(x)/\Delta\omega(x) \gg 1$ is very clear. A resonance can be considered as broad if its width is much larger than the typical level spacing in the finite box. That is to say, if the volume is such that within the peak of the resonance there are many available scattering states that the resonance can decay into, then the resonance is a broad one and the corresponding spectral weight for this resonance will exhibit typical two-particle state behavior, namely it is proportional to $1/\Omega$. In the opposite limit,

$$\Gamma_R(x)/\Delta\omega(x) \ll 1, \quad \text{or: } \Gamma_R(x)g(x)\Omega \ll 1. \tag{34}$$

The spectral weight (32) behaves like that of a stable single particle. A resonance satisfying this inequality is therefore called *infinitely narrow*. Only in this limit does a resonance look like a single particle as far as the volume dependence for the spectral weight is concerned. If the width and the volume are such that

$$\Gamma_R(x)g(x)\Omega \sim 1, \tag{35}$$

then the resonance is neither broad nor infinitely narrow and the spectral weight (32) for the resonance will also be different from both single- and two-particle spectral weights.

IV. DISCUSSIONS AND CONCLUSIONS

In this paper, we have studied the volume dependence of spectral weight of an unstable particle within the Lee model. It is shown that if the V particle is stable or unstable but narrow, the volume dependence of the particle indeed behaves like a single particle, namely it is almost volume independent. However, if the V particle is unstable and the width is large, then the volume dependence of its spectral weight exhibits two-particle properties, i.e. proportional to $1/\Omega$, reflecting the fact that all asymptotic states are two-particle scattering states. Thus, when the resonance changes from narrow to broad, the volume dependence of its spectral weight also undergoes a transition from a single-particle behavior to a multiparticle behavior. This transition can be computed exactly within the Lee model. The condition for a broad (or a infinitely narrow) resonance is also given. In real lattice QCD calculations, the criteria for a resonance being regarded as narrow or broad will depend on the specific problem being studied although the qualitative feature should remain the same.

As an example, let us estimate the ratio $\Gamma_R/\Delta\omega$ in lattice calculations on low-energy pion-pion scattering. We use this as an example because pion-pion scattering exhibits both a broad resonance in the scalar channel and a relatively narrow resonance in the vector channel. Recently, the CP-PACS collaboration has computed the width of the ρ resonance in the vector channel [16] using $N_f = 2$ dynamical Wilson fermion lattices of size $12^3 \times 24$ with the lattice spacing given by $1/a = 0.92$ GeV. The simulation was done at $m_\pi/m_\rho = 0.41$ which translates into pion mass of about 0.32 GeV in physical unit. It is then estimated that the first and second $\Delta\omega$ to be about 0.5 and 0.35 GeV which is larger than the ρ meson width (about 0.15 GeV). Therefore, we expect that in this scenario, the ρ meson behaves more like a narrow resonance. Indeed, the authors in Ref. [16] have found a consistent result for the mass of the ρ meson using two different methods: one using the naive vector meson time correlation function, the other by fitting the phase shifts near the resonance. Note that the typical level spacing depends on the physical size of the volume. Since the largest physical size used in present lattice simulations are in the range of a few Fermi, we expect that the typical level spacing is usually larger than the width of the hadron in most cases. An exceptional case might be the very broad σ resonance in two-pion systems in the scalar channel. In Ref. [25], a quenched studied is performed and a single-particle behavior is found for a scalar state in this channel. They used $16^3 \times 28$ lattices with $a = 0.2$ fm and lowest pion mass is around 0.182 GeV. The first two level spacings for the two-pion states in this calculation are estimated to be 0.45 and 0.27 GeV which are comparable (or somewhat smaller) than the expected physical width of the σ. Of course, it is difficult to draw definite conclusions by this naive estimate. Further studies have to be carried out to clarify the situation.

To conclude, by studying the volume dependence of the spectral weight in a simple model, we show how the volume dependence of the spectral weight changes from single particle to multiparticle behavior as the width of the resonance is getting broad. It is found that the ratio of its physical width Γ_R to the typical level spacing $\Delta\omega$ in the finite box controls this transition. Note that this ratio usually can be estimated before the simulation is actually performed, assuming the physical width of the resonance is known. We also demonstrate this by estimating this ratio for the case of pion-pion scattering in recent lattice calculations. Although studied in a simple model, we think that the lessons learned from the model are also relevant and helpful for realistic lattice simulations on unstable particles in QCD.

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