Semileptonic b decay at intermediate recoil

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We compute the $\mathcal{O}(\alpha_s^2)$ corrections to the differential rate of the semileptonic decay $b \to c\ell\nu_\ell$ at the ''intermediate recoil'' point, where the c-quark mass and the invariant mass of the leptons are equal. The calculation is based on an expansion around two opposite limits of the quark masses $m_{b,c}$: $m_c \simeq m_b$ and $m_c \ll m_b$. The former case was previously studied; we correct and extend that result. The latter case is new. The smooth matching of both expansions provides a check of both. We clarify the discrepancy between the recent determinations of the full NNLO QCD correction to the semileptonic $b \to c$ rate, and its earlier estimate.

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I. INTRODUCTION

Approximately one out of five decays of the b-quark produces a c-quark accompanied by leptons. Those semileptonic decays provide information about quark masses, the Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{cb}, as well as properties of hadrons containing heavy quarks. In order to access that information, measurements of the decay probability and distributions are compared with theoretical predictions that account for radiative corrections, quark masses, and nonperturbative effects of strong interactions. Given that the strong coupling constant at the mass-scale of the b-quark is quite large, $\alpha_s(q^2 = m_b^2) \simeq 0.2$, and that the present uncertainty in V_{cb} approaches the one-percent level [1], it is important to determine the second-order effects, $\mathcal{O}(\alpha_s^2)$.

The full $\mathcal{O}(\alpha_s^2)$ correction to the decay rate was first calculated in the limit of a massless produced quark (relevant for the decay $b \to u\ell\nu_\ell$) [2]. Effects of the c-quark mass were known, until recently, only in the so called Brodsky-Lepage-Mackenzie (BLM) approximation [3,4], estimating the largest part of the second-order corrections using the running of α_s. The remaining, non-BLM corrections, are usually smaller and much more difficult to determine. They were known only in three special points of lepton kinematics: the zero recoil, where the leptons are emitted back-to-back and the produced quark remains at rest [5–7]; the maximum recoil, with the vanishing invariant mass of the leptons [8,9]; and the intermediate recoil, where the invariant mass of the leptons equals that of the c-quark [10]. In the latter study, the information from all three points was used to estimate the $\mathcal{O}(\alpha_s^2)$ correction to the total decay rate with a massive c-quark.

Very recently, two independent studies determined the full mass dependence of the non-BLM corrections: in [11], the calculation was performed numerically for arbitrary quark masses, and in [12] an expansion around small m_c/m_b was obtained analytically. The two methods are very different, with the former being more accurate at large, and the latter at small m_c, but they agree very well in the physically interesting region of $m_c = (0.25\ldots0.30)m_b$. Unfortunately, the resulting non-BLM correction disagrees almost by a factor of 2 with the estimate found in [10].

The goal of the present paper is twofold. First, we want to check the intermediate-recoil expansion presented in [10]. Among the three kinematical points on which the estimate [10] of the total correction was based, the intermediate-recoil is the only one not checked by an independent calculation. An expansion is constructed from the opposite limit than in [10]: whereas there the expansion was around the zero-recoil limit, here we start from the vanishing m_c, as shown in Fig. 1. In addition, the old expansion around the zero-recoil limit is repeated and extended to higher orders. Our second goal is to clarify the source of the disagreement between the three-point estimate and the recent explicit calculations.

Figure 1 puts the present expansion in the context of the possible kinematics of a heavy to light quark decay, $Q \to q + (W^* \to \ell\nu_\ell)$. Along the diagonal, the mass of the virtual W^* is equal to the light-quark mass. The arrow originating from the zero-recoil line corresponds to the expansion done in [10] (and repeated in the present paper),

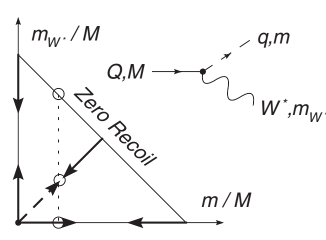


FIG. 1. The kinematic region where the decay $Q \to qW^*$ is allowed. The solid arrows show known expansions while the dashed arrow shows the expansion presented here. The decay width is also known analytically along the whole zero recoil line. The dotted line corresponds to the decay width $Q \to q\ell\nu_\ell$. The three circles along this line show known values coming from the different expansions. In the case considered in this paper, $M = m_b$ and $m = m_c$.

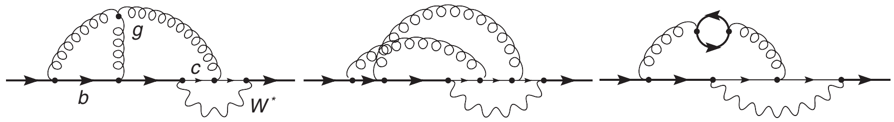


FIG. 2. A sample of the diagrams needed for the intermediate-recoil expansion presented here.

while the dashed arrow coming from the zero mass point corresponds to the expansion presented here for the first time. Ultimately, these expansions should give a consistent value for the decay width $\Gamma(b \to cW^*)$. It is related to the differential semileptonic width,

$$\frac{d\Gamma(b \to c\ell\nu_\ell)}{dq^2} = \frac{G_F}{6\pi^2\sqrt{2}M_W^2} q^2\Gamma(b \to cW^*)|_{m^2(W^*)=q^2}, \tag{1}$$

where q^2 is the invariant mass squared of the leptons, and Fermi constant is $G_F = \frac{\sqrt{2}g_w^2}{8M_W^2}$.

II. EXPANSION FROM ZERO MASS POINT

Using the intermediate-recoil relation $m_{W^*} = m_c$, we here calculate the width as a series in $\rho \equiv \frac{m_c}{m_b} \ll 1$, and α_s,

$$\Gamma(b \to cW^*) = \Gamma_0\left[X_0 + C_F\frac{\alpha_s}{\pi}X_1 + C_F\left(\frac{\alpha_s}{\pi}\right)^2 X_2 + \mathcal{O}(\alpha_s^3)\right], \tag{2}$$

where

$$\Gamma_0 = \frac{g_w^2|V_{cb}|^2m_b^3}{64\pi m^2(W^*)}. \tag{3}$$

The tree-level and first-order results $X_{0,1}$ are known exactly [13,14], and the present approach, described below, has been tested with them up to $\mathcal{O}(\rho^{10})$,

$$X_0 = (1-\rho^2)\sqrt{1-4\rho^2}, \tag{4}$$

$$X_1 = \frac{5}{4} - \frac{\pi^2}{3} + \rho^2\left(\frac{\pi^2}{3} - \frac{5}{4} - 9\ln\rho\right) + \rho^4\left(9\ln\rho - \frac{15}{4}\right) + \ldots. \tag{5}$$

To evaluate the $\mathcal{O}(\alpha_s^2)$ corrections, we considered the imaginary parts of 39 three-loop self-energy diagrams with massive propagators, such as in Fig. 2, and used the optical theorem to calculate the decay width. To deal with the two scales, m_b and m_c, we used the method of asymptotic expansion [15,16]. As an example of how this asymptotic expansion is done, consider the left hand diagram in Fig. 2. We consider "regions" where each loop momentum is either hard, $\sim m_b$, or soft, $\sim m_c$, and Taylor expand the propagators so that in the end we only have to deal with single scale diagrams as shown in Fig. 3. This method produced as many as 11 regions for a single topology. Expansions to the desired order $\mathcal{O}(\rho^{10})$ required algorithm [17] for the unfactorized three-loop regions, (e.g. Region 1 in Fig. 3).

The second-order results can be separated into a sum of gauge invariant parts, each with a different color factor,

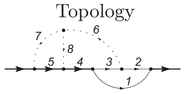
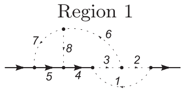
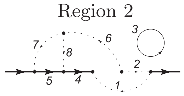
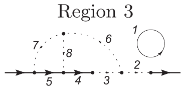
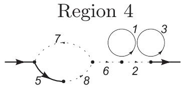
Topology 	$[1]=k_1^2+m_c^2$ $[2]=k_2^2+m_c^2$ $[3]=k_3^2+m_c^2$ $[4]=(k_3-k_1)^2+m_b^2$ $[5]=k_4^2+m_b^2$ $[6]=(k_3-k_2)^2$ $[7]=(k_4-p)^2$ $[8]=(p+k_3-k_2-k_4)^2$	
Region 1 	$\lvert k_1\rvert \gg m_c$, $\lvert k_2\rvert \gg m_c$, $\lvert k_3\rvert \gg m_c$	$[1]\to k_1^2$, $[2]\to k_2^2$, $[3]\to k_3^2$
Region 2 	$\lvert k_1\rvert \gg m_c$, $\lvert k_2\rvert \gg m_c$, $\lvert k_3\rvert \sim m_c$	$[1]\to k_1^2$, $[2]\to k_2^2$, $[4]\to k_1^2+m_b^2$ $[6]\to k_2^2=[2]$, $[8]\to(p-k_2-k_4)^2$
Region 3 	$\lvert k_1\rvert \sim m_c$, $\lvert k_2\rvert \gg m_c$, $\lvert k_3\rvert \gg m_c$	$[2]\to k_2^2$, $[3]\to k_3^2$, $[4]\to k_3^2+m_b^2$
Region 4 	$\lvert k_1\rvert \sim m_c$, $\lvert k_2\rvert \gg m_c$, $\lvert k_3\rvert \sim m_c$	$[2]\to k_2^2$, $[4]\to m_b^2$, $[6]\to k_2^2$, $[8]\to k_4^2$

FIG. 3. The asymptotic expansion of a two-scale diagram used to integrate the left-hand diagram in Fig. 2. The dashed, thin, and thick lines correspond to massless, soft-scale massive, and hard-scale massive propagators, respectively.

$$X_2 = T_R(N_L X_L + N_S X_S + N_H X_H) + C_F X_F + C_A X_A. \tag{6}$$

In this expression, the X_i's are the gauge invariant parts in terms of ρ, N_L is the number of quarks lighter than a c-quark, N_S and N_H serve as markers to separate the c-quark and b-quark loop contributions. $C_F = \frac{4}{3}$, $C_A = 3$, and $T_R = \frac{1}{2}$, are the appropriate color factors in $SU(3)$. The contributions from a top quark loop are not considered here because they are suppressed by the ratio m_b/m_t and are negligible. Terms up to $\mathcal{O}(\rho^{10})$ have been calculated completely analytically. Here we present the formulas for terms up to ρ^4 to save space, Eqs. (7)–(11), while the numerical coefficients of all terms are given in Table I.

$$X_H = \frac{12\,991}{1296} - \frac{\zeta_3}{3} - \frac{53\pi^2}{54} + \rho^2\left[\frac{89\pi^2}{54} - \frac{137\,567}{6480} + \frac{13\zeta_3}{3}\right] + \rho^4\left[\frac{4\pi^2}{3} - \frac{10\,081\,601}{705\,600} - \frac{23}{840}\ln\rho\right], \tag{7}$$

$$X_S = \zeta_3 - \frac{4}{9} + \frac{23\pi^2}{108} - \rho\frac{3\pi^2}{4} + \rho^2\left[\frac{4}{9} + \frac{13}{2}\ln\rho - 3\ln^2\rho - \zeta_3 + \frac{157\pi^2}{108}\right] - \rho^3\frac{25\pi^2}{18} + \rho^4\left[\frac{1193}{36} - \frac{61}{3}\ln\rho + 9\ln^2\rho - \frac{16\pi^2}{3}\right], \tag{8}$$

$$X_L = \zeta_3 - \frac{4}{9} + \frac{23\pi^2}{108} + \rho^2\left[\frac{13}{2}\ln\rho - \frac{1}{18} - 3\ln^2\rho - \zeta_3 - \frac{77\pi^2}{108}\right] + \rho^4\left[\frac{865}{72} - \frac{34}{3}\ln\rho + 6\ln^2\rho\right], \tag{9}$$

$$X_A = \frac{521}{576} + \frac{9\zeta_3}{16} + \frac{505\pi^2}{864} - \frac{19\pi^2}{8}\ln 2 + \frac{11\pi^4}{1440} - \rho^2\left[\frac{1223}{576} + \frac{185}{8}\ln\rho - \frac{33}{4}\ln^2\rho + \frac{107\zeta_3}{16} + \frac{145\pi^2}{864} - \frac{57\pi^2}{8}\ln 2 + \frac{161\pi^4}{720}\right] + \rho^3\frac{2\pi^2}{3} + \rho^4\left[\ln\rho\left(\frac{2027}{48} - \frac{23\pi^2}{8}\right) - \frac{13\,391}{288} - \frac{33}{2}\ln^2\rho - \frac{403\zeta_3}{64} + \frac{27\pi^2}{8} - \frac{201\pi^2}{32}\ln 2 - \frac{31\pi^4}{720}\right], \tag{10}$$

$$X_F = 5 - \frac{53\zeta_3}{8} - \frac{119\pi^2}{48} + \frac{19\pi^2}{4}\ln 2 - \frac{11\pi^4}{720} + \rho^2\left[\frac{151\zeta_3}{8} + \frac{743\pi^2}{48} + \ln\rho\left(\pi^2 - \frac{75}{8}\right) - \frac{27}{2}\ln^2\rho - \frac{97}{2} - \frac{57\pi^2}{4}\ln 2 - \frac{127\pi^4}{360}\right] - \rho^3\frac{4\pi^2}{3} + \rho^4\left[\frac{7145}{288} + \ln\rho\left(\frac{25\pi^2}{12} - \frac{329}{24}\right) + 18\ln^2\rho + \frac{547\zeta_3}{32} - \frac{83\pi^2}{12} + \frac{201\pi^2}{16}\ln 2 + \frac{19\pi^4}{72}\right]. \tag{11}$$

For this expansion, we have used the $\overline{\mathrm{MS}}$ definition of α_s normalized at the pole mass m_b.

TABLE I. Numerical coefficients of the expansion presented here to all orders calculated.

	ρ^0	ρ^1	ρ^2	$\rho^2\ln\rho$	$\rho^2\ln^2\rho$	ρ^3	ρ^4	$\rho^4\ln\rho$	$\rho^4\ln^2\rho$	ρ^5	ρ^6
X_A	−8.154	0	15.14	−23.12	8.25	6.580	−67.92	13.85	−16.5	77.64	−124.2
X_F	3.575	0	−4.887	0.4946	−13.5	−13.16	88.74	6.853	18	−155.3	262.4
X_L	2.859	0	−8.294	6.5	−3	0	12.01	−11.33	6	0	−12.45
X_S	2.859	−7.402	13.59	6.5	−3	−13.71	−19.50	−20.33	9	30.98	−13.50
X_H	−0.06360	0	0.2460	0	0	0	−1.129	−0.02738	0	0	1.656

	$\rho^6\ln\rho$	$\rho^6\ln^2\rho$	ρ^7	ρ^8	$\rho^8\ln\rho$	$\rho^8\ln^2\rho$	ρ^9	ρ^{10}	$\rho^{10}\ln\rho$	$\rho^{10}\ln^2\rho$
X_A	−96.19	14.17	270.6	−666.7	−235.6	40.01	973.0	−2327.3	−705.7	48.98
X_F	38.03	−66.89	−541.3	1127.6	−41.98	−245.5	−1945.9	3771.9	−516.1	−733.0
X_L	19.30	−6	0	18.39	35.59	−18	0	80.68	101.0	−76
X_S	15.77	−12	64.15	−33.67	44.76	8	0	−0.5973	151.8	34
X_H	−0.8866	0	0	1.984	−0.2800	0	0	4.494	0.5912	0

III. EXPANSION FROM THE ZERO-RECOIL LINE

An alternative way to compute at the intermediate recoil is to expand around the zero-recoil limit where $m_c = m_{W^*} = \frac{m_b}{2}$. The decay width parametrization in Eqs. (1) and (2), as well as the decomposition of the second-order correction into color parts, Eq. (6), are still valid. For the purpose of the expansion around the zero-recoil limit, it is convenient to parameterize the dependence on the quark variable in terms of a new variable, $\beta = 1 - 4\rho^2$, and pull out its square root, thus defining new functions Δ_i,

$$X_i(\rho) = \sqrt{\beta}\Delta_i(\beta), \qquad i = 0, 1, 2, A, F, L, S, H.$$

The expansion around $\beta = 0$ was first carried out in [10]. Our purpose in this section is to repeat that calculation, extend it to higher powers in β, and make sure that the results match the expansion around the zero-mass point, $\rho = 0$, presented in Sec. II. The one-loop correction in the β expansion reads

$$\Delta_1 = \tfrac{27}{8}\ln 2 - 3 + \beta\left(\tfrac{25}{8}\ln 2 + \tfrac{1}{2}\ln\beta - \tfrac{95}{48}\right) + \beta^2\left(\tfrac{28}{15}\ln 2 + \tfrac{7}{15}\ln\beta - \tfrac{13\,483}{7200}\right) + \beta^3\left(\tfrac{44}{35}\ln 2 + \tfrac{11}{35}\ln\beta - \tfrac{143\,263}{117\,600}\right).$$

In [10] the strong coupling constant was normalized at the geometrical mean of the quark masses, $\alpha_s(\sqrt{m_b m_c})$, while here we use $\alpha_s(m_b)$, in order to be able to match with the expansion around $\rho = 0$. Also, in [10], the c-quark and b-quark loop contributions were added together and denoted Δ_H, while here we separate them. The b-quark loop contribution is denoted by Δ_H and the c-quark by Δ_S. For reference, the Δ_S and Δ_H terms are given in Eqs. (12) and (13) up to order β^2 [both normalized with $\alpha_s(m_b)$],

$$\Delta_H = \frac{509}{48} + \frac{999}{32}R_2 + \frac{87}{16}\ln 2 + \frac{337}{64}\ln^2 2 + \frac{75\pi^2}{128} + \beta\left(\frac{7937}{864} + \frac{1449}{32}R_2 + \frac{275}{144}\ln 2 + \frac{767}{64}\ln^2 2 + \frac{655\pi^2}{384}\right) + \beta^2\left(\frac{610\,309}{51\,840} + \frac{204\,969}{2560}R_2 + \frac{59\,519}{17\,280}\ln 2 + \frac{973\,327}{46\,080}\ln^2 2 + \frac{317\,957\pi^2}{92\,160}\right), \tag{12}$$

$$\Delta_S = \frac{361}{96} - \frac{621}{256}R_2 - \frac{25}{64}\ln 2 - \frac{531}{512}\ln^2 2 - \frac{1445\pi^2}{3072} + \beta\left(\frac{433\pi^2}{3072} - \frac{757}{864} - \frac{207}{256}R_2 - \frac{91}{576}\ln 2 - \frac{1}{3}\ln 2\ln\beta - \frac{2579}{1536}\ln^2 2\right) + \beta^2\left[\frac{287\,639}{414\,720} - \frac{3243}{20\,480}R_2 + \frac{1\,120\,967}{691\,200}\ln 2 - \frac{85\,913}{73\,728}\ln^2 2 - \frac{51\,907\pi^2}{442\,368} - \left(\frac{1}{6} + \frac{14}{45}\ln 2\right)\ln\beta\right], \tag{13}$$

where R_2 is obtained from [5] and has a numerical value of $R_2 \approx -0.729\,64$.

While these changes were carried out, an error was noticed in the charge renormalization used in [10]. In that paper, α_s was normalized at $\sqrt{m_b m_c}$. The error consisted in using five quark flavors to run α_s down to that scale, instead of excluding the b-quark in the range between m_b and $\sqrt{m_b m_c}$. This error originates in [6]. We have corrected for this in Eq. (12) and Table II.

To have proper matching between the expansion in [10] and the expansion presented here, we also found that the old expansion needed more terms than could be obtained with the available computing resources in 1998. We have updated the expansion to include analytical terms up to β^8

TABLE II. Numerical coefficients to all orders calculated for the updated expansion from the zero-recoil limit. The values have been calculated using $\alpha_s(m_b)$.

	β^0	β^1	$\beta^1\ln\beta$	$\beta^1\ln^2\beta$	β^2	$\beta^2\ln\beta$	$\beta^2\ln^2\beta$	β^3	$\beta^3\ln\beta$	$\beta^3\ln^2\beta$	β^4	$\beta^4\ln\beta$	$\beta^4\ln^2\beta$
Δ_A	−1.849	0.420	2.421	−0.458	−1.960	2.109	−0.428	−1.411	1.702	−0.288	−0.730	1.311	−0.218
Δ_F	1.762	−0.854	−0.440	0	0.015	−0.140	0.167	0.208	−0.650	0.256	0.442	−0.642	0.270
Δ_L	0.419	0.086	−0.982	0.167	0.576	−0.804	0.156	0.568	−0.712	0.105	0.257	−0.512	0.079
Δ_S	0.118	0.189	−0.231	0	0.215	−0.382	0	0.323	−0.384	0	0.257	−0.348	0
Δ_H	−0.087	0.072	0	0	−0.045	0	0	−0.002	0	0	-5×10^{-4}	0	0

	β^5	$\beta^5\ln\beta$	$\beta^5\ln^2\beta$	β^6	$\beta^6\ln\beta$	$\beta^6\ln^2\beta$	β^7	$\beta^7\ln\beta$	$\beta^7\ln^2\beta$	β^8	$\beta^8\ln\beta$	$\beta^8\ln^2\beta$
Δ_A	−0.429	1.032	−0.176	−0.306	0.883	−0.147	−0.223	0.766	−0.127	−0.174	0.682	−0.111
Δ_F	0.298	−0.670	0.264	0.189	−0.494	0.254	0.177	−0.493	0.242	0.122	−0.410	0.230
Δ_L	0.159	−0.413	0.064	0.110	−0.349	0.054	0.081	−0.304	0.0462	0.063	−0.269	0.041
Δ_S	0.191	−0.314	0	0.147	−0.285	0	0.117	−0.261	0	0.096	−0.241	0
Δ_H	-1×10^{-4}	0	0	-5×10^{-5}	0	0	-2×10^{-5}	0	0	-1×10^{-5}	0	0

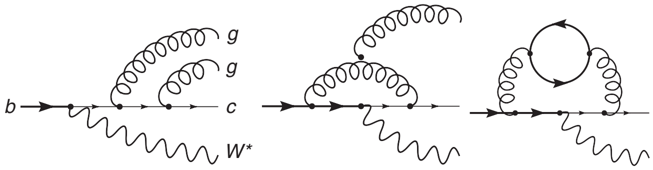


FIG. 4. A sample of the diagrams needed to calculate the expansion presented in [10] and updated here.

as compared to β^4 previously. To carry out this calculation, we used the same methods as the authors of [10]. Instead of calculating the corrections using self-energy diagrams and the optical theorem, we calculated each second-order decay diagram separately. This required the calculation of 73 diagrams with zero, one or two loops and up to four-particle phase space integrations, Fig. 4.

In this expansion, the loops have been integrated using the same methods described earlier. This lead to the calculation of 14 regions with only one having an eikonal propagator [18]. The numerical coefficients of all terms that have been calculated here are shown in Table II.

Figure 5 shows how the updated expansion differs from the previous one and clearly displays the need for the higher-order terms, at and below the physical value $\rho \sim 0.3$.

In an attempt to account for the higher-order terms, the authors of [10] added a term to approximate the remainder of the series equal to the product of highest order term and $\frac{\beta}{2(1-\beta)}$. This also gave an estimate of the error in their calculation. For a value of $\rho = 0.3$ ($\beta = 0.64$) they found,

$$\sqrt{\beta}\Delta_2 = -4.72(14). \tag{14}$$

With the extra terms we have calculated here and the corrections to the charge renormalization, this changes to

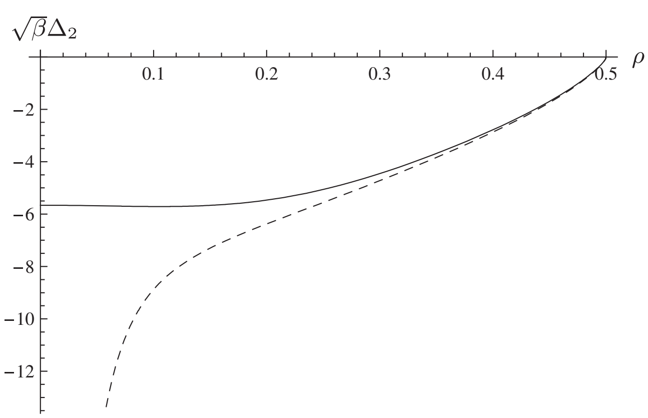


FIG. 5. Second-order corrections expanded from the zero recoil line. The dashed line shows the expansion up to $\mathcal{O}(\beta^4)$ while the solid line shows the expansion up to $\mathcal{O}(\beta^8)$. For the purpose of comparing with [10], this plot is made assuming $\alpha_s(\sqrt{m_b m_c})$ is used in the NLO correction.

$$\sqrt{\beta}\Delta_2 = -4.45(1), \tag{15}$$

where we have used the same method of estimating the error. With an error of $\approx 0.2\%$, we have sufficient accuracy for computing the full decay width $\Gamma(b \to c\ell\nu_\ell)$ in the next section.

Comparing the two expansions, around $\rho = 0$ and around $\rho = \frac{1}{2}$, as shown in Fig. 6, one can now see that

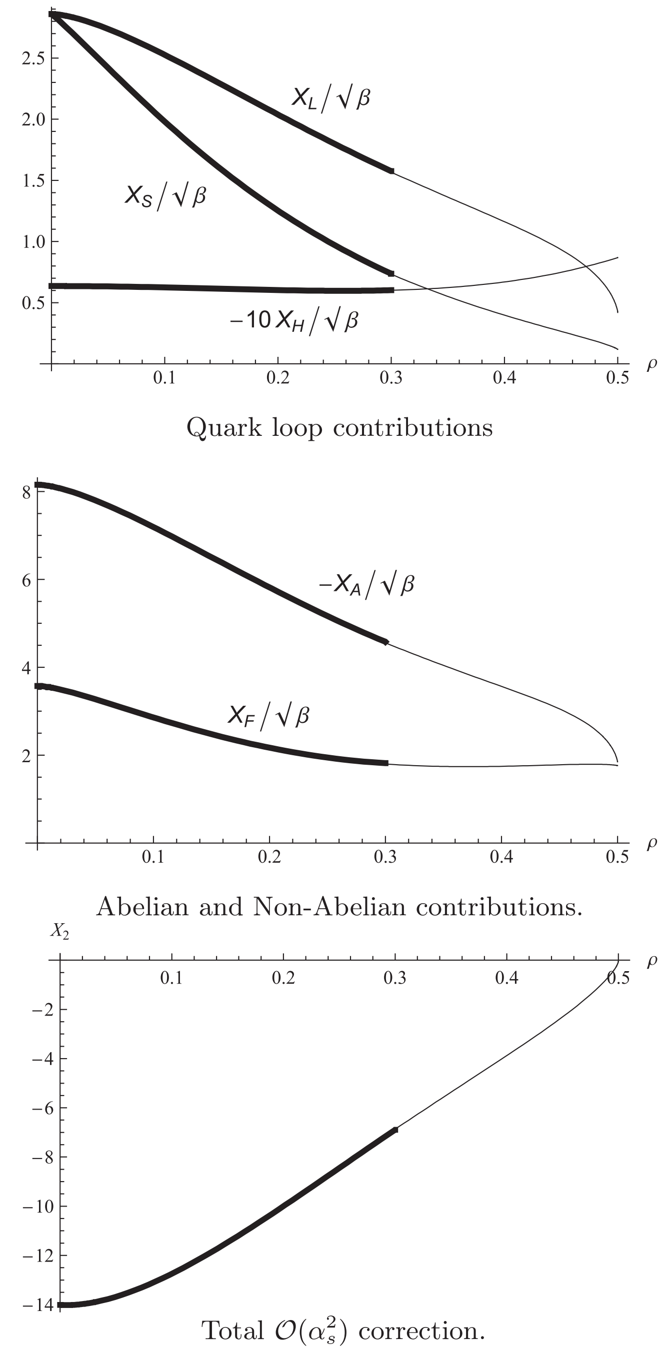


FIG. 6. The matching between the different color contributions and total X_2 contribution. The thick line corresponds to the expansions presented here and the thin line corresponds to the updated expansion from the zero recoil line.

all of the different color contributions and thus the full α_s^2 corrections agree very well.

IV. ESTIMATE OF THE FULL CORRECTION TO THE SEMILEPTONIC DECAY RATE

A. Notation

So far in this paper we have been concerned with the decay of a b-quark into a c-quark and a virtual W-boson, at the intermediate recoil where the masses of c and W^* are equal. We now want to use the results we have obtained, together with previously obtained values at zero- and maximum-recoil to fit the corrections to the decay $b \to c\ell\nu_\ell$. We follow the notation of [10],

$$\frac{d\Gamma(b \to c\ell\nu_\ell)}{dq^2} = \frac{G_F^2 m_b^3 |V_{cb}|^2}{96\pi^3}\left[A_{\rm Born} + \frac{\alpha_s(\sqrt{m_b m_c})}{\pi} C_F A_1 + \left(\frac{\alpha_s}{\pi}\right)^2 C_F A_2\right], \tag{16}$$

$$A_{\rm Born} = \sqrt{(1-\rho^2-q^2)^2 - 4\rho^2 q^2} \times [(1-\rho^2)^2 + (1+\rho^2)q^2 - 2q^4].$$

In addition, we define the corrections for the integrated decay rate,

$$\Gamma(b \to c\ell\nu_\ell) = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} F(\rho)\left[1 + \frac{\alpha_s(\sqrt{m_b m_c})}{\pi} B_1 + \left(\frac{\alpha_s}{\pi}\right)^2 B_2\right], \tag{17}$$

$$F(\rho) \equiv 1 - 8\rho^2 - 24\rho^4 \ln\rho + 8\rho^6 - \rho^8.$$

As we have already mentioned in the Introduction, the NNLO corrections A_2 and B_2 can be divided into the BLM and the non-BLM parts,

$$A_2 = T_R(N_L A_L + N_S A_S + N_H A_H) + C_F A_F + C_A A_A \equiv A^{\rm BLM} + A^{\rm nBLM}, \tag{18}$$

$$A^{\rm BLM} \equiv A_L[T_R(N_L + N_S) - \tfrac{11}{4} C_A],$$

and similarly for the integrated corrections B. All the functions A in the Eq. (16) depend on two variables: the quark-mass ratio ρ and the invariant mass of the leptons $\sqrt{q^2}$. The full dependence on these variables is not yet known. The expansions such as described in this paper and earlier studies determine A's along the sides and the bisector of the triangle shown in Fig. 1. Of particular interest are their values along the vertical line corresponding to the physical value of $\rho \simeq 0.3$, describing the differential decay rate $d\Gamma(b \to c\ell\nu_\ell)/dq^2$. Reference [10] used the three known points along that line to fit a polynomial and integrate Eq. (1) over q^2, thus providing an estimate of the second-order non-BLM corrections to the full semileptonic decay width

$$B_{\rm fit}^{\rm nBLM} = 0.9(3). \tag{19}$$

This numerical value is quoted from [11], where the author fixed another mistake in [10], related to using the maximum-recoil result. Recently, however, two calculations of the full decay width $\Gamma(b \to c\ell\nu_\ell)$ [11,12] gave

$$B^{\rm nBLM} = 1.73(4). \tag{20}$$

This section is devoted to clarifying the discrepancy between these results.

For comparison purposes we use $\alpha_s(\sqrt{m_b m_c})$, $N_f = 4$, and $\rho = 0.3$ as in [10], where N_f refers to the number of light quarks used to calculate the BLM contributions. The authors of [11,12] use a different set of parameters, to which we will return in Sec. IV D.

B. Effect of corrected coupling constant normalization

A large part of the discrepancy between Eqs. (19) and (20) is due to the incorrect charge renormalization, as discussed in Sec. III. We have corrected this and recalculated the non-BLM contributions using the same fitting method described in [10]. Analogously to Eq. (8) in [10], we introduce a new function of the lepton invariant mass q^2 at fixed quark-mass ratio ρ (we use $\rho = 0.3$). It is denoted $\xi(q^2)$ and defined as

$$\xi(q^2) = \frac{A_2(q^2) - A_2^{\rm BLM}(q^2)}{A_{\rm Born}(q^2)}, \tag{21}$$

The three available values of $\xi(q^2)$ at $q^2 = 0$, m_c^2, and $q^2_{\rm max} = (m_b - m_c)^2$ are

$$\xi(0) = 1.26, \qquad \xi(m_c^2) = 1.27, \qquad \xi(q^2_{\rm max}) = 0.19. \tag{22}$$

These values have been obtained using results from [6,9,10], with the b-quark charge renormalization terms from [6,10] corrected. Fitting these values to a function defined by

$$\xi(q^2) = a_1 q^4 + a_2 q^2 + a_3, \tag{23}$$

we integrate over q^2 to find a value for the non-BLM corrections. The values quoted in Eq. (22) are normalized to the Born rate, $A_{\rm Born}$, so the integral needed is analogous to Eq. (9) in [10],

$$B_{\rm fit}^{\rm nBLM} = C_F \frac{\int_0^{q^2_{\rm max}} dq^2 A_{\rm Born}(q^2)\xi(q^2)}{\int_0^{q^2_{\rm max}} dq^2 A_{\rm Born}(q^2)}. \tag{24}$$

This integration, with the input from Eq. (22), gives $B_{\rm fit}^{\rm nBLM} = 1.4(2)$. This agrees with Eq. (20) much better than the value given in Eq. (19). The error is estimated by performing the analogous fit of the BLM corrections and comparing the result to the exact value [4].

C. Effect of extending the expansion to higher orders in β

In Sec. IV B we have merely corrected the renormalization in the old results. In addition, using the results of the two expansions in the present paper, we can obtain a more accurate input along the intermediate recoil line. Instead of the value 1.27 in Eq. (22), this gives $\xi(m_c^2) = 1.33$. This is related to the shift induced by the higher-order terms, illustrated in Fig. 5. There we see that the full correction is less negative than previously assumed, thus the difference with the BLM correction is more positive (larger). Since the zero- and the intermediate-recoil points are close to each other, even a small shift of the value at one of them may be amplified in the integral of the fitted function.

After the integration in Eq. (24), this change leads to the new value $B_{\rm fit}^{\rm nBLM} = 1.5(2)$ which now agrees with Eq. (20). The error estimated by comparing with the BLM case is about 12 per cent. By correcting the mistake in renormalization and including more terms in the expansion from zero recoil, we have brought the disagreement from about a factor of 2 down to ≈ 10 percent, within the quoted error margins.

D. A better fitting method

Further improvement is possible using a better method of fitting the polynomial. In Eq. (21), we normalized the points to the tree-level result $A_{\rm Born}$. We find that, if this normalization is not done, i.e. instead we use

$$\zeta(q^2) = A_2(q^2) - A_2^{\rm BLM}(q^2), \tag{25}$$

the polynomial fit gives a much better estimate of the exact result. With this adjustment of the fitting procedure, we end up with a final non-BLM estimate of $B_2^{\rm nBLM} = 1.76(4)$, a significant improvement. Without knowing the shape of the $d\Gamma(b \to c\ell\nu_\ell)/dq^2$ curve, we cannot say whether this is a numerical coincidence. We have also performed this fitting for the $\mathcal{O}(\alpha_s)$ corrections and BLM approximation. Both estimates give results that are within ≈ 3 per cent of the exact known result and are more accurate than using an analog of Eq. (21) for the fit.

In the more recent papers [11,12], the authors use a different set of parameters for their calculations, namely $\alpha_s(m_b)$, $N_f = 3$, and $\rho = 0.25$. For easy comparison, we have also calculated the non-BLM corrections with this set of parameters. Using Eq. (25) for the fitting procedure and the expansion about $\rho = 0$ presented here, we find $B_{\rm fit}^{\rm nBLM} = 3.37(15)$, as compared with the result of $B^{\rm nBLM} = 3.40(7)$ from [11].

V. SUMMARY

To summarize: we have corrected an error in the strong coupling constant normalization in the previous intermediate-recoil expansion. We have extended that expansion to several higher orders in the parameter β, describing the difference between m_c and $m_b/2$. We have confirmed the correctness of that expansion by constructing a new one, also along the intermediate-recoil diagonal but around its other end, corresponding to $m_c/m_b \to 0$. This analysis allowed us to reevaluate the fit of the $d\Gamma(b \to c\ell\nu_\ell)/dq^2$ curve based on the three kinematical points, and remove the disagreement between the correction to the total rate $\Gamma(b \to c\ell\nu_\ell)$ obtained from such a fit, and that obtained from the direct four-loop calculations [11,12]. With this result, the full massive calculation of the $\mathcal{O}(\alpha_s^2)$ corrections to the semileptonic b-quark decay rate is confirmed.

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APPENDIX A: CONTRIBUTIONS FROM $b \to c\bar{c}cW^*$

The expansions used to obtain the maximum recoil point for our polynomial fit were calculated in [8,9]. The two expansions agree very well except for the region with $m_c < \frac{m_b}{3}$. This discrepancy can be attributed to the omission of the amplitude of $b \to c\bar{c}cW^*$ from the expansion in [8]. For completeness, we have calculated this contribution for both maximum recoil and intermediate recoil, as this expansion was also not included in [10]. These expansions are calculated as threshold expansions in terms of δ given by

$$m_c = \frac{m_b}{3(4)}(1 - \delta), \tag{A1}$$

where the 3(4) indicates the factor used when calculating the maximum recoil, 3, or intermediate recoil, 4 expansion. The methods used for both expansions are discussed in [19]. This calculation relies on the ability to reduce the four-particle phase space integrals needed, into a product of two particle phase spaces.

For the maximum recoil case, the expansion has been calculated up to δ^{14}, with the first four terms given here,

$$\Gamma(b \to c\bar{c}cW^*)_{m_{W^*}=0} = \frac{\Gamma_0 \alpha_s^2 \sqrt{3}\delta^6 C_F T_R}{5\pi} \times \left(1 + \frac{83}{56}\delta + \frac{7}{64}\delta^2 + \frac{55}{896}\delta^3 + \frac{753}{896}\delta^4 + \dots\right). \tag{A2}$$

For intermediate recoil, the expansion has been calculated up to $\delta^9\sqrt{\delta}$,

$$\Gamma(b \to c\bar{c}cW^*)_{m_{W^*}=m_c} = \frac{3\Gamma_0 \alpha_s^2 \delta^3 \sqrt{\delta} C_F T_R}{140\sqrt{2}\pi} \times \left(1 + \frac{535}{108}\delta - \frac{137\,045}{85\,536}\delta^2 + \frac{175\,277\,863}{13\,343\,616}\delta^3 + \dots\right). \tag{A3}$$

[1] C. Amsler *et al.*, Phys. Lett. B **667**, 1 (2008).
[2] T. van Ritbergen, Phys. Lett. B **454**, 353 (1999); M. Steinhauser and T. Seidensticker, Phys. Lett. B **467**, 271 (1999).
[3] S. J. Brodsky, G. P. Lepage, and P. B. Mackenzie, Phys. Rev. D **28**, 228 (1983).
[4] M. E. Luke, M. J. Savage, and M. B. Wise, Phys. Lett. B **345**, 301 (1995).
[5] A. Czarnecki and K. Melnikov, Nucl. Phys. **B505**, 65 (1997).
[6] A. Czarnecki, Phys. Rev. Lett. **76**, 4124 (1996).
[7] J. Franzkowski, and J. B. Tausk, Eur. Phys. J. C **5**, 517 (1998).
[8] A. Czarnecki and K. Melnikov, Phys. Rev. Lett. **78**, 3630 (1997).
[9] A. Pak, I. R. Blokland, and A. Czarnecki, Phys. Rev. D **73**, 114009 (2006).
[10] A. Czarnecki and K. Melnikov, Phys. Rev. D **59**, 014036 (R) (1998).
[11] K. Melnikov, Phys. Lett. B **666**, 336 (2008).
[12] A. Pak and A. Czarnecki, Phys. Rev. Lett. **100**, 241807 (2008).
[13] M. Jezabek and J. H. Kuhn, Nucl. Phys. **B314**, 1 (1989).
[14] A. Czarnecki, Phys. Lett. B **252**, 467 (1990).
[15] F. V. Tkachev, Sov. J. Part. Nucl. **25**, 649 (1994).
[16] V. A. Smirnov, Springer Tracts Mod. Phys. **177**, 1 (2002).
[17] S. Laporta, Int. J. Mod. Phys. A **15**, 5087 (2000).
[18] A. Czarnecki and V. A. Smirnov, Phys. Lett. B **394**, 211 (1997).
[19] A. Czarnecki and K. Melnikov, Nucl. Phys. **B544**, 520 (1999).