

Unique horizontal symmetry of leptons

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There is a group-theoretical connection between fermion mixing matrices and minimal horizontal symmetry groups. Applying this connection to the tribimaximal neutrino mixing matrix, we show that the minimal horizontal symmetry group for leptons is uniquely S_4 , the permutation group of four objects.

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I. INTRODUCTION

Much expectation is currently being focused on the discovery of the standard-model Higgs boson, and on possible experimental indications of what lies beyond. While we await these exciting results, we should not lose sight of the generation problem of more than 70 years, of why there are three generations of fermions and what the relations are between them, for which a certain amount of data is already available to guide us. If (horizontal) symmetry is to play a role in the generation problem, just as it does in the standard model, in supersymmetry, and in grand unified and string theories, then the generation problem may even have a direct impact on the Higgs boson being hunted for. This is so because, like other symmetries, the horizontal symmetry is expected to be spontaneously broken to yield the measured quark and lepton masses and mixings. Additional Higgs bosons introduced to break the horizontal symmetry contribute to the mass matrices and hence the fermion masses, making the fermion coupling of whichever Higgs bosons that will be first discovered no longer proportional to its mass. That means the estimated fusion production cross section of the standard-model Higgs will be off, and its expected decay branching ratio to different fermion pairs will not be as predicted. If that is indeed observed, it would be a strong support for the idea of a horizontal symmetry.

Many horizontal symmetry groups have been proposed in the literature, among them the groups Z_m [1], $Z_m \times Z_n$ [2], D_n [3,4], S_3 [5], S_4 [6], A_4 [7], T' [8], $\Delta(27)$ [9], $SO(3)$, and $SU(3)$ [10]. These models introduce a number of additional Higgs bosons to break the spontaneous symmetry, each of which has an adjustable vacuum expectation value and adjustable Yukawa coupling constants. It is the tuning of these parameters that allows the same piece of experimental data to be explained by so many different models based on many different symmetries.

This is somewhat unsatisfactory because if nature possesses a horizontal symmetry, it must be unique, but how should we determine which of these is the correct symmetry? In a previous paper [11], I suggested what seems to me

to be a natural criterion: the correct symmetry group should reveal itself experimentally without the adjustment of any dynamical parameter. It was further argued that neutrino mixing should be the piece of data used to obtain such a symmetry, whereas quark mixing and fermion masses have to be obtained in a model-dependent way from a dynamical model based on this symmetry. With this criterion, it was shown that the tribimaximal neutrino mixing matrix [12] leads to the symmetry group S_4 . It was further claimed that this group is unique, in the sense that any other viable symmetry group must contain it as a subgroup. The purpose of this paper is to show this uniqueness, as well as other details not fully discussed in [11]. Dynamical models based on S_4 to implement quark mixing and fermion masses are not unique and their discussion will not be included in this article.

Since we want to stay away from the adjustment of parameters, the symmetry must be obtained from group-theoretical arguments, not dynamical models. The group-theoretical technique needed for this purpose was developed in [4], used there and in [11] to obtain the S_4 horizontal symmetry from the tribimaximal mixing matrix. For completeness this result will be reviewed in Sec. II. The proof of the uniqueness of S_4 will be carried out in Sec. III, by eliminating all other finite subgroups of $SO(3)$ and $SU(3)$ one by one. Section IV is devoted to a brief summary and conclusions.

II. SYMMETRY AND MIXING

We wish to derive the horizontal symmetry from mixing, and vice versa. Since the mixing matrix depends only on left-handed fermions, instead of the 3×3 charged-lepton mass matrix M_e , which connects left-handed to right-handed charged leptons, it is simpler to deal with $\bar{M}_e := M_e M_e^\dagger = \bar{M}_e^\dagger$, which connects left-handed charged leptons on both sides. Assuming the active neutrinos to be Majorana, their mass matrix $M_\nu = M_\nu^T$ does not involve right-handed neutrinos and can be used as it is.

If U_e and U_ν are unitary matrices making $U_e^\dagger \bar{M}_e U_e := \bar{m}_e$ and $U_\nu^T M_\nu U_\nu := m_\nu$ diagonal, then $U = U_e^\dagger U_\nu$ is the neutrino mixing matrix. Since lepton masses are all different, these unitary matrices are essentially unique up to

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phases. To see that, suppose d_e and d_ν are unitary matrices such that $U_e d_e$ also diagonalizes \bar{M}_e and $U_\nu d_\nu$ also diagonalizes M_ν ; then $d_e^\dagger \bar{m}_e d_e = \bar{m}_e$ and $d_\nu^T m_\nu d_\nu = m_\nu$, so d_e and d_ν must both be diagonal, with $d_\nu^T d_\nu = d_e^2 = 1$. In other words, U_e is unique up to an overall phase change of each of its three columns, and U_ν is unique up to an overall sign change of each of its three columns.

If F is a unitary transformation of the left-handed charged leptons, then this transformation is a horizontal symmetry of the charged leptons iff $F^\dagger \bar{M}_e F = \bar{M}_e$. Similarly, if G is a unitary transformation of the active neutrinos, then it is a horizontal symmetry of the active neutrinos iff $G^T M_\nu G = M_\nu$. These equalities imply that $F U_e$ also diagonalizes \bar{M}_e and $G U_\nu$ also diagonalizes M_ν ; consequently, they satisfy $F U_e = U_e d_e$ and $G U_\nu = U_\nu d_\nu$. The condition $d_\nu^2 = 1$ implies $G^2 = 1$. Without loss of generality, we shall confine ourselves to the situation when $\det(G) = 1$, so that G has one $+1$ and two -1 eigenvalues. In other words, the columns of U_e are eigenvectors of F , and the columns of U_ν are eigenvectors of G with eigenvalue $+1$ or -1 . The diagonalization matrices U_e and U_ν , and hence the mixing matrix U , are thereby intimately related to the symmetry operations F and G . It is for this reason that the mixing matrix can determine the symmetry, and vice versa.

If the Hamiltonian has an unbroken symmetry, this symmetry must be simultaneously a horizontal symmetry for the charged leptons and the active neutrinos; thus $F = G$. This implies $U_e = U_\nu$ up to inconsequential phases, and hence $U = 1$. In order to have a mixing, $F = G$ is not allowed, so whatever horizontal symmetry that is present at high energies must be broken at the present energy down to the residual symmetries F for the charged leptons and G for the active neutrinos. With $F \neq G$, the minimal horizontal symmetry group \mathcal{G} at high energy is the (finite) group generated by F and G . We shall denote this by the notation $\mathcal{G} = \{F, G\}$.

From now on we choose a basis in which \bar{M}_e is diagonal. That means $U_e = 1$ and $U = U_\nu$. F is diagonal in that basis because it commutes with \bar{M}_e whose eigenvalues are all different. Conversely, if F is diagonal and nondegenerate, in the sense that all its eigenvalues are different, then \bar{M}_e must be diagonal as well. This property will be needed to recover the mixing matrix U from F and G ; hence we shall assume F to be nondegenerate from now on.

Given that $U = U_\nu$, $GU = Ud_\nu$, it is easy to construct G from the neutrino mixing matrix U . In fact, there are three solutions $G_i (i = 1, 2, 3)$, such that $G_i U = Ud_\nu^{(i)}$, with $+1$ occupying the (ii) position of $d_\nu^{(i)}$ and -1 occupying the other two diagonal entries. If the mixing matrix is chosen to have the tribimaximal form [12],

$$U = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ -1 & \sqrt{2} & -\sqrt{3} \end{pmatrix}, \quad (1)$$

a form which is well within 1 standard deviation of the experimental mixing angles, then

$$G_1 = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix},$$

$$G_2 = -\frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}, \quad G_3 = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2)$$

There is no need to consider G_1 from now on because $G_1 = G_2 G_3$.

In summary, the minimal horizontal group given by the neutrino mixing matrix U is the finite group \mathcal{G} generated by F , G_2 , and G_3 : $\mathcal{G} = \{F, G_2, G_3\}$. The matrices G_2 and G_3 are given in (2); the matrix F is unitary and diagonal but otherwise arbitrary.

Conversely, given a horizontal group \mathcal{G} for leptons, whether it can yield the tribimaximal mixing (1) without fine-tuning depends on whether we can choose three elements F , G_2 , and G_3 in \mathcal{G} to act as the residual symmetries after breaking, so that F is nondegenerate, and G_2 , G_3 are given by (2) in the basis where F is diagonal. If this can be done, then the mixing is given by (1) because $G_i U = Ud_\nu^{(i)}$ and \bar{M}_e is diagonal in that basis as remarked earlier. If no such triplets F , G_2 , G_3 can be found, then we cannot obtain (1) from \mathcal{G} without fine-tuning the dynamical parameters.

Since \mathcal{G} is assumed to be a finite group, there must be a natural integer n such that $F^n = 1$. We require $n \geq 3$ in order to keep F nondegenerate. For $n = 3$, each entry of the diagonal F must be a different cube root of unity. Let $\omega = e^{2\pi i/3}$, then there are six possible F 's, given by $F = \text{diag}(1, \omega, \omega^2) := F_1$ and the other five permutations of the three entries. Correspondingly there should be six groups, $\mathcal{G} = \{F, G_2, G_3\}$. However, since each G_i is invariant under a simultaneous permutation of its second and third columns and rows, a permutation of the (22) and (33) entries of F will not give rise to a new group. In this way we cut down the six possible \mathcal{G} to three, the other two being generated by $F_2 = \text{diag}(\omega, 1, \omega^2)$ and $F_3 = \text{diag}(\omega^2, 1, \omega)$. However, since $F_2^2 = F_3$ and $F_3^2 = F_2$, these two must generate the same group, so altogether there are only two distinct groups \mathcal{G} from the six possible F 's. These groups can be explicitly calculated by repeatedly multiplying the three generators. The result is $\mathcal{G} = \{F_1, G_2, G_3\} = S_4$, $\mathcal{G} = \{F_2, G_2, G_3\} = 3.S_4$, where S_4 is the symmetric (permutation) group of four objects with 24 elements, which is also the symmetry group of the cube and of the octahedron, and $3.S_4$ is a 72-element group consisting of S_4 , ωS_4 , and $\omega^2 S_4$. Since $3.S_4 \supset S_4$, the horizontal symmetry group for $n = 3$ is uniquely S_4 , unique in the sense that any other viable horizontal group must contain it as a subgroup.

It remains to show the uniqueness of S_4 for $n > 3$. For a given n , there are $n!/(n-3)!$ possible F 's, and there is an

infinite number of n 's; thus a straightforward calculation by direct multiplication for every case is clearly impossible. Instead, we will provide a proof in the next section in another way, by enumerating and rejecting the other relevant finite groups.

III. UNIQUENESS OF S_4

In this section we shall show that either $\mathcal{G} = \{F, G_2, G_3\}$ contains S_4 as a subgroup, or else it must have an infinite order.

To do that, we first assume $\det(F) = 1$. Since $\det(G_2) = \det(G_3) = 1$, and since F, G_2, G_3 are 3×3 unitary matrices, \mathcal{G} is clearly a finite subgroup of $SU(3)$ or $SO(3)$, or a finite subgroup of $SU(3)$ or $SU(2)$ since $SU(2)$ covers $SO(3)$ twice. The desired conclusion is reached by enumerating and considering all the finite subgroups of $SU(2)$ and $SU(3)$, which are known. At the end of the section, we will show that the conclusion remains valid when the unit-determinant assumption of F is dropped.

A. Finite subgroups of $SU(2)$ and $SU(3)$

The finite subgroups of $SU(2)$ are in one-to-one correspondence with the simply laced Lie algebras A_n, D_n, E_6, E_7, E_8 [13–15]. The two infinite series A_n and D_n correspond, respectively, to the cyclic group Z_n with n elements, and the dihedral group D_n with $2n$ elements. The three exceptional groups E_6, E_7, E_8 correspond to the three symmetry groups $\mathcal{T}, \mathcal{O}, \mathcal{I}$ of the regular polyhedrons, with \mathcal{T} the symmetry of the tetrahedron, \mathcal{O} the symmetry of the cube and the octahedron, and \mathcal{I} the symmetry of the dodecahedron and the icosahedron. They are also equal to the groups A_4, S_4, A_5 , respectively, where S_n is the symmetric group, namely, the permutation group of n objects, and A_n is the alternating group, namely, all the even permutations of n objects.

The finite subgroups of $SU(3)$ not in $SU(2)$ are also known [13,14,16–19]; the notations used below are those of [16], in which the number within the parentheses is the order of the group. There are also two infinite series, $\Delta(3n^2)$ and $\Delta(6n^2)$, and six “exceptional” ones: $\Sigma(36), \Sigma(60), \Sigma(72), \Sigma(168), \Sigma(216), \Sigma(360)$.

We shall show in the subsequent paragraphs that these groups cannot yield the tribimaximal mixing matrix (1) without parameter tuning unless they contain S_4 as a subgroup. We shall do that by dividing these groups into different categories.

B. Groups without three-dimensional irreducible representations

It was shown in [4] that the horizontal symmetry group has to possess a three-dimensional (3D) irreducible representation (IR), or else we can never recover the tribimax-

imal mixing without a tuning of parameters. This is so because of the particular form of (2), and because the left-handed fermions must belong to a 3DIR to avoid the presence of tunable parameters.

The series of groups Z_n , being Abelian, has only 1DIR, and the series of groups D_n has at most 2DIR, so both are unsuitable. The groups $\Sigma(36), \Sigma(72),$ and $\Sigma(360)$ do not have 3DIR either [16], so they must be rejected as well.

C. A_4

This subgroup of S_4 gives automatically trimaximal mixing, but it can give bimaximal mixing as well only by tuning some Yukawa coupling constants [20]. To obtain tribimaximal mixing without any tuning, we need the rest of the symmetries contained in S_4 .

D. $\Delta(3m^2)$ and $\Delta(6m^2)$

These two infinite series are discussed in [13,14,16–18,21], with their 3DIR's given in [17,18,21]. We must show that none of them are allowed unless they contain S_4 as a subgroup.

There are many 3DIR's, but every one of those matrices has only one nonzero element in each row and each column, and these nonzero elements all have absolute value 1. This is the only property that is required to rule out these two series.

There are $3! = 6$ types of matrices with this property. They are

$$\begin{aligned}
 t1 &:= \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & 0 & \sigma_1 \\ 0 & \tau_1 & 0 \end{pmatrix}, & t2 &:= \begin{pmatrix} 0 & 0 & \rho_2 \\ 0 & \sigma_2 & 0 \\ \tau_2 & 0 & 0 \end{pmatrix}, \\
 t3 &:= \begin{pmatrix} 0 & \rho_3 & 0 \\ \sigma_3 & 0 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}, & t4 &:= \begin{pmatrix} \rho_4 & 0 & 0 \\ 0 & \sigma_4 & 0 \\ 0 & 0 & \tau_4 \end{pmatrix}, \\
 t5 &:= \begin{pmatrix} 0 & \rho_5 & 0 \\ 0 & 0 & \sigma_5 \\ \tau_5 & 0 & 0 \end{pmatrix}, & t6 &:= \begin{pmatrix} 0 & 0 & \rho_6 \\ \sigma_6 & 0 & 0 \\ 0 & \tau_6 & 0 \end{pmatrix},
 \end{aligned} \tag{3}$$

with $|\rho_i| = |\sigma_i| = |\tau_i| = 1$. The question is whether we can pick from them three members g_2, g_3, f which can be turned into $G_2, G_3,$ and F by a unitary transformation V , such that F is diagonal and nondegenerate, having an order $n > 3$. $n = 1, 2$ are excluded because we need the three eigenvalues of F to be different; $n = 3$ is not considered because G_2, G_3, F simply generate S_4 . Unless three such members can be picked, there will be no new horizontal symmetry contained in these two series of groups.

G_2 and G_3 can be simultaneously diagonalized by the tribimaximal mixing matrix U of (1) so that

$$d_v^{(2)} = U^\dagger G_2 U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (4)$$

$$d_v^{(3)} = U^\dagger G_3 U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are diagonal. To find V , we first determine the unitary matrix v which simultaneously diagonalizes g_2 and g_3 into diagonal forms $g_{2d} = v^\dagger g_2 v$ and $g_{3d} = v^\dagger g_3 v$, and then identify g_{2d} with $d_v^{(2)}$ and g_{3d} with $d_v^{(3)}$. The desired matrix V so that $V g_{2,3} V^\dagger = G_{2,3}$ is then given by $V = U v^\dagger$.

Such a matrix v exists because g_2 and g_3 commute; otherwise, G_2 and G_3 would not have commuted. Let $v_a (1 \leq a \leq 6)$ be the unitary matrix which renders $t_{ad} = v_a^\dagger t_a v_a$ diagonal; then v must be one of these v_a 's.

Since $\text{Tr}(G_2) = \text{Tr}(G_3) = -1$, we must require $\text{Tr}(g_2) = \text{Tr}(g_3) = -1$. Since $n > 3$ and the three eigenvalues of F are required to be different, we also have $\text{Tr}(F) \neq 0$, and hence $\text{Tr}(f) \neq 0$. The trace of any matrix in t_5 and t_6 is zero; thus we know that g_2, g_3, f must all come from t_i for $1 \leq i \leq 4$. To have the right trace, we need to have $\rho_1 = \sigma_2 = \tau_3 = \rho_4 + \sigma_4 + \tau_4 = -1$. To have the unit determinant which G_2 and G_3 possess, we also require $\sigma_1 \tau_1 = \rho_2 \tau_2 = \rho_3 \sigma_3 = -1$.

Incidentally, this analysis excludes $\Delta(24) = S_4$, which has $n = 3$. In that case $\text{Tr}(f) = 0$ and its f belongs to type t_5 or t_6 .

The unitary diagonalization matrices v_a of the first four types can be taken to be

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & w_1 & -w_1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} w_2 & 0 & -w_2 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad (5)$$

$$v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} w_3 & -w_3 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix},$$

with $|w_i| = 1$. v_4 can be taken to be the identity matrix.

It is easy to check from (3) that the vanishing of the commutators between g_2 and g_3 requires either (i) both of them to belong to the same type t_i , or (ii) one of them to belong to type t_4 . For $j = 1, 2, 3$, the (jj) matrix element of t_j and t_{jd} is -1 , so we see from (4) that when (i) occurs, both g_2 and g_3 must belong to type t_1 or type t_4 . Let us consider the different cases separately.

i (ia) *Both g_2 and g_3 belong to type t_1 .*

In order for $t_{1d} = v_1^\dagger t_1 v_1$ to be diagonal, $w_1 \tau_1 = w_1^* \sigma_1$ is required, in which case

$$t_{1d} = \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & \tau_1 w_1 & 0 \\ 0 & 0 & -\tau_1 w_1 \end{pmatrix}. \quad (6)$$

This can be identified with $d_v^{(2)}$ and $d_v^{(3)}$ in (4) by setting $\rho_1 = -1$ and $w_1 = \pm 1/\tau_1$.

Now that we know $V = U v^\dagger$, we can compute $V f V^\dagger$ to see whether it can be identified with F , which is a diagonal nondegenerate matrix with order $n > 3$. Since f has to be in one of the four types t_1, t_2, t_3, t_4 , we can calculate them all. The result is that this can never happen no matter what parameters we choose.

(ib) *Both g_2 and g_3 belong to type t_4 .*

In this case v is the identity and $V = U$. Again we can verify that none of $U t_i U^\dagger$ for $i \leq 4$ can be the desired F .

ii *One of g_2 and g_3 belongs to t_4 and the other to t_i .*

We shall deal with the case when g_2 is of type t_4 . The other case, when g_3 is in t_4 , is very similar.

Comparing with (4), we see that $\rho_4 = -1 = -\sigma_4 = \tau_4$. If g_3 belongs to type t_i , its diagonalization matrix v_i must diagonalize g_2 as well; hence $i = 2$. As in case (ia) above, $V = U v^\dagger = U v_2^\dagger$ can then be computed and we can check whether $V f V^\dagger$ can lead to a desired F . The result is the same as before: no matter which one of the four types t_i that f belongs to, it can never give rise to the desired F .

This concludes the proof that $\Delta(3m^2)$ and $\Delta(6m^2)$ cannot give rise to a new horizontal symmetry not containing S_4 .

E. $A_5, \Sigma(168), \Sigma(216)$

The remaining three groups, the icosahedral group $A_5 = \Sigma(60)$, the Klein group $\Sigma(168) = L_2(7)$ [19], and the Hessian group $\Sigma(216)$, can all be ruled out by using their class structures and character tables. To explain how this is done let us denote any of these three groups by \mathcal{G} .

The question is whether an equivalent representation can be chosen to enable three members $f, g_2, g_3 \in \mathcal{G}$ to be picked out to be equal to F, G_2, G_3 , respectively. If so, \mathcal{G} is capable of giving the tribimaximal mixing. If not, \mathcal{G} can be ruled out. Such an identification requires f to have a finite order $n \geq 3$, and g_2, g_3 to have order 2, and be given by the formula (2).

Since the proof relies only on characters and eigenvalues, which are the same for all equivalent representations, we do not have to worry about which equivalent representation we choose.

We need not consider those f with $n = 3$ because they would never give us anything new. If we can find $g_{2,3}$ that are equal to $G_{2,3}$, then \mathcal{G} must contain S_4 as a subgroup, so S_4 remains the only minimal horizontal group compatible with tribimaximal mixing without tuning. If not, then \mathcal{G} can never give rise to tribimaximal mixing, so it is ruled out.

For $n > 3$, we rely on the following strategy to rule out these groups \mathcal{G} . It is shown below how the class structure and character table can be used to determine the eigenvalues of $f = F$. This tells us what the diagonal forms of F are. If $g_{2,3}$ can be found to be equal to $G_{2,3}$ in this representation of diagonal F , then the group generated by $\{F, G_2, G_3\}$ must be a subgroup of \mathcal{G} . However, we shall show that in each case when $n > 3$, the element FG_2 and/or the element FG_3 has an order larger than the order of the group \mathcal{G} ; hence this group cannot be a subgroup of these groups \mathcal{G} , and there can be no $g_{2,3}$ that can be identified with $G_{2,3}$. This is how these groups \mathcal{G} are ruled out.

I. A_5

The character table from [16] is given in Table I, where $b_{5+} = (-1 + \sqrt{5})/2$ and $b_{5-} = (-1 - \sqrt{5})/2$. The first row of the table names the five conjugacy classes; the second row gives the permutation structure of each class in cycle notations of S_5 . For example, 1^23 consists of two 1-cycles and one 3-cycle. The third row tells us the order of the elements, for example, class C_4 consists of elements of order 5 and their fourth powers, and C_5 consists of the second and third powers of elements of order 5. E is the identity element. The next five rows are the characters of the irreducible representations of each class, with the boldface numerals in the first column giving the dimension of the representation. The last two rows will be explained later.

In principle, F can be taken to be any of the 60 elements of the group, but to get anything other than S_4 , we merely have to concentrate on those elements with order $n > 3$. This leaves elements in class C_4 or C_5 , both of order $n = 5$. To determine the eigenvalues in each class, we list in the last two rows of the table what class the square and the cube of every C_i belong to. Once this is known, the eigenvalues of every element F can be deduced from $\text{Tr}(F)$, $\text{Tr}(F^2)$, and $\text{Tr}(F^3)$, namely, from the character table.

Let us illustrate how the last two rows are obtained. For example, class C_4 consists of elements of the type C_5 and C_5^4 ; hence the square of any element $f \in C_4$ is of the form C_5^2 and C_5^3 , so $f^2 \in C_5$. Similarly, if $f \in C_5$, then $f^2 \in C_4$.

We will now proceed to determine the eigenvalues of F from the characters.

Let $\omega_n = \exp(2\pi i/n)$. Then it is well known that

$$\sum_{i=0}^{n-1} \omega_n^i = 0. \quad (7)$$

Using that for $n = 5$, it will now be shown that $b_{5+} = \omega_5 + \omega_5^4$, and $b_{5-} = \omega_5^2 + \omega_5^3$. To start with, let $x = \omega_5 + \omega_5^4$ and $y = \omega_5^2 + \omega_5^3$. Then (7) tells us that $x + y = -1$. Now $x^2 = y + 2 = -x + 1$ and $y^2 = x + 2 = -y + 1$. Thus both x and y satisfy the quadratic equation $z^2 + z = 1$, whose two solutions are $z = (-1 \pm \sqrt{5})/2 = b_{5\pm}$, so one must be x and the other must be y . To determine which is which, note that ω_5 is in the first quadrant of the complex plane and ω_5^4 is in the fourth quadrant, so x must have a positive real part. Thus $x = b_{5+}$ and $y = b_{5-}$.

Knowing that, it is now easy to verify that if $F \in C_4$, then its eigenvalues in the $\mathbf{3}$ representation are $1, \omega_5^2, \omega_5^3$. These are also the eigenvalues of F in the $\mathbf{3}'$ representation if $F \in C_5$. Similarly, the eigenvalues for any $F \in C_5$ in the $\mathbf{3}$ representation and any $F \in C_4$ in the $\mathbf{3}'$ representation are $1, \omega_5, \omega_5^4$.

Given three distinct eigenvalues, $3! = 6$ diagonal F 's can be produced, depending on where the eigenvalues are put. In order to show that $h = FG_2$ or $h = FG_3$ has an order larger than 60, the order of A_5 , we compute its three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of h numerically. Let us assume $|\lambda_i| = 1$; otherwise h cannot have a finite order. In that case, $\lambda_i = \exp(2\pi i\theta_i)$, and the numerical value for each θ_i is approximated by a rational number k_i/m_i . Then h would have an order larger than 60 if $m_i > 60$, and that turns out to be true in every case.

2. $\Sigma(168)$

This group is studied in great detail in [19], but for our present purpose, it is sufficient just to use the character table taken from [16], which is shown in Table II below.

This table is listed in the same way as the table for A_5 , with $b_{7\pm} = (-1 \pm i\sqrt{7})/2$. From this table we see that candidates for F with $n > 3$ should come from classes

TABLE I. Character table for A_5 .

Classes	C_1	C_2	C_3	C_4	C_5
Permutation type	1^5	1^23	12^2	5	5
Element type	E	(C_3, C_3^2)	C_2	(C_5, C_5^4)	(C_5^2, C_5^3)
1	1	1	1	1	1
3	3	0	-1	$-b_{5-}$	$-b_{5+}$
3'	3	0	-1	$-b_{5+}$	$-b_{5-}$
4	4	1	0	-1	-1
5	5	-1	1	0	0
C^2	C_1	C_2	C_1	C_5	C_4
C^3	C_1	C_1	C_3	C_4	C_5

TABLE II. Character table for $\Sigma(168)$.

Classes	C_1	C_2	C_3	C_4	C_5	C_6
Permutation type	1^7	$1^3 2^2$	124	13^2	7	7
Element type	E	C_2	(C_4, C_4^3)	(C_3, C_3^2)	(C_7, C_7^2, C_7^4)	(C_7^3, C_7^5, C_7^6)
1	1	1	1	1	1	1
3	3	-1	1	0	b_{7+}	b_{7-}
3*	3	-1	1	0	b_{7-}	b_{7+}
6	6	2	0	0	-1	-1
7	7	-1	-1	1	0	0
8	8	0	0	-1	1	1
C^2	C_1	C_1	C_2	C_4	C_5	C_6
C^3	C_1	C_2	C_3	C_1	C_6	C_5

C_3 , C_5 , or C_6 . Therefore we need to figure out the eigenvalues of elements in these classes.

For C_3 , since the order of its elements is $n = 4$, the three eigenvalues of F have to be chosen from the four values $\pm 1, \pm i$. Both the **3** and **3*** characters of C_3 are 1, and the C_2 characters are -1 . Thus if $F \in C_3$, then $\text{Tr}(F) = \text{Tr}(F^3) = 1$ and $\text{Tr}(F^2) = -1$, so the eigenvalues are $1, +i, -i$.

The character of C_5 is b_{7+} in **3** and b_{7-} in **3***. It is the reverse for C_6 . Let us now prove that $b_{7+} = \omega_7 + \omega_7^2 + \omega_7^4$ and $b_{7-} = \omega_7^3 + \omega_7^5 + \omega_7^6 = b_{7+}^*$. The proof is similar to the case of $b_{5\pm}$. Letting $x = \omega_7 + \omega_7^2 + \omega_7^4$ and $y = \omega_7^3 + \omega_7^5 + \omega_7^6$, it follows that both x and y satisfy the quadratic equation $z^2 + z + 2 = 0$, whose solutions are $b_{7\pm}$. Moreover, x should be identified with b_{7+} because their imaginary parts are both positive.

With this relation we can now determine the eigenvalues of F in C_5 and C_6 , both for **3** and **3***. For C_5 and **3** or C_6 and **3***, $\text{Tr}(F) = \text{Tr}(F^2) = b_{7+} = \omega_7 + \omega_7^2 + \omega_7^4$ and $\text{Tr}(F^3) = b_{7-} = \omega_7^3 + \omega_7^5 + \omega_7^6$. Hence the eigenvalues of F are $\omega_7, \omega_7^2, \omega_7^4$. Similarly, for $F \in C_6$ and **3** or $F \in C_5$ and **3***, the eigenvalues are $\omega_7^3, \omega_7^5, \omega_7^6$. For each set of eigenvalues, $3! = 6$ diagonal matrices F can be produced.

As in the case of A_5 , $h = FG_2$ or $h = FG_3$ has an order larger than 168 in every case, thereby ruling out $\Sigma(168)$.

3. $\Sigma(216)$

The character table taken from [16] is shown in Table III, where $\omega = \omega_3$. To have $n > 3$, F must come from C_4, C_5 , or C_6 . From the **3**-representation row of the character table, we find that if $F \in C_4$, then $\text{Tr}(F) = \text{Tr}(F^3) = -1$ and $\text{Tr}(F^2) = 3$. The eigenvalues of F are then $1, -1, -1$. Since two of the three eigenvalues are identical, we must reject this case. If $F \in C_5$ or C_6 , then $\text{Tr}(F) = \text{Tr}(F^2) = 0$ and $\text{Tr}(F^3) = 3$. The eigenvalues are then $1, \omega, \omega^2$. In spite of having order $n = 6$ for elements in C_5 , their 3DIR is identical to an F with $n = 3$, whose answer is already known. Hence $\Sigma(216)$ cannot produce anything new.

F. General F

We will now relax the condition $\det(F) = 1$. Since $F^n = 1$ for some n , $\eta := \det(F)$ is an n th root of unity, and the

TABLE III. Character table for $\Sigma(216)$.

Classes	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
Permutation type	1^9	$1^3 3^2$	$1^3 3^2$	14^2	126	126	12^4	3^3	3^3	3^3
Element type	E	C_3	C_3^2	(C_4, C_4^3)	C_6	C_6^5	C_2	(C_3, C_3^2)	C_3''	$C_3''^2$
1	1	1	1	1	1	1	1	1	1	1
1'	1	ω	ω^2	1	ω	ω^2	1	1	ω	ω^2
1''*	1	ω^2	ω	1	ω^2	ω	1	1	ω^2	ω
2	2	-1	-1	0	1	1	-2	2	-1	-1
2'	2	$-\omega$	$-\omega^2$	0	ω	ω^2	-2	2	$-\omega$	$-\omega^2$
2''*	2	$-\omega^2$	$-\omega$	0	ω^2	ω	-2	2	$-\omega^2$	$-\omega$
3	3	0	0	-1	0	0	3	3	0	0
8	8	2	2	0	0	0	0	-1	-1	-1
8'	8	2ω	$2\omega^2$	0	0	0	0	-1	$-\omega$	$-\omega^2$
8''*	8	$2\omega^2$	2ω	0	0	0	0	-1	$-\omega^2$	$-\omega$
C^2	C_1	C_3	C_2	C_7	C_2	C_3	C_1	C_8	C_{10}	C_9
C^3	C_1	C_1	C_1	C_4	C_7	C_7	C_7	C_1	C_1	C_1

3×3 unitary matrix F can be written $F = \eta^{1/3} F'$, where $\det(F') = 1$.

If $\mathcal{G} = \{F, G_2, G_3\}$ forms a finite group, then its presentation [22,23] is defined by a number of relations $R_a(F, G_2, G_3) = 1$, where R_a are monomials of F, G_2 , and G_3 . Three of these relations are $R_1 = F^n$, $R_2 = G_2^2$, and $R_3 = G_3^2$, but there must be others relating F, G_2 , and G_3 . If F appears k times in R_a , then by taking the determinant of the relation on both sides, we see that $\eta^k = 1$.

Every group element g_i of \mathcal{G} can be written as a monomial of F, G_2, G_3 : $g_i = g_i(F, G_2, G_3)$. This monomial is not unique because we can always insert a number of R_a in it. Nevertheless, if $g_1 g_2 = g_3$, the sum of powers of F appearing in g_1 and g_2 must be equal to the power of F appearing in g_3 , modulo m , where m is the smallest integer such that $\eta^m = 1$. This conclusion can be reached by taking the determinant on both sides of the equation.

Now consider the group $\mathcal{G}' = \{F', G_2, G_3\}$ generated by F', G_2 , and G_3 . Since $\eta^k = 1$, $R_a(F', G_2, G_3) = 1$; hence a relation of \mathcal{G} is also a relation of \mathcal{G}' . Moreover, the mapping $g_i \rightarrow g'_i := g_i(F', G_2, G_3)$ is a homomorphism from \mathcal{G} to \mathcal{G}' , preserving multiplication relations. The kernel of the mapping is a subgroup Z of \mathcal{G} , consisting of all elements $z(F, G_2, G_3)$ for which $z' := z(F', G_2, G_3) = 1$. If the power of F in the monomial z is d , then it follows that $z = \eta^d 1$. In other words, $Z = \{\eta^d 1\}$ is isomorphic to a subgroup of the cyclic group Z_n .

Thus \mathcal{G} is a central extension of \mathcal{G}' , consisting of elements of the form $z g'$, with $z \in Z$ and $g' \in \mathcal{G}'$. More-

over, \mathcal{G}' is a finite subgroup of $SU(3)$. Since the only group \mathcal{G}' that can naturally lead to the tribimaximal mixing is a group containing S_4 , the same is true for $\mathcal{G} = \{F, G_2, G_3\}$, so the uniqueness of S_4 is established even if $\det(F) \neq 1$.

IV. CONCLUSION

We have shown that the horizontal group for leptons is uniquely S_4 , or any group containing it. To reach this conclusion, we have used the criterion that a horizontal group should be obtained from the neutrino mixing matrix and vice versa without parameter tuning. To implement this criterion, a purely group-theoretical link between neutrino mixing matrices and horizontal symmetry groups is established. When this link is applied to the tribimaximal neutrino mixing matrix, S_4 emerges as a possible horizontal group. Other finite groups not containing S_4 are all ruled out by studying the finite subgroups of $SU(2)$ and $SU(3)$.

Quark mixing and fermion masses are obtained from S_4 -invariant dynamical models. They are model dependent and their discussion will be postponed to a future publication.

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