

High-statistics study of the $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ decay

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We report a high-statistics measurement of the branching fraction for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ and the invariant mass spectrum of the produced $\pi^- \pi^0$ system using 72.2 fb^{-1} of data recorded with the Belle detector at the KEKB asymmetric-energy $e^+ e^-$ collider. The branching fraction obtained is $(25.24 \pm 0.01 \pm 0.39)\%$, where the first error is statistical and the second is systematic. The unfolded $\pi^- \pi^0$ mass spectrum is used to determine resonance parameters for the $\rho(770)$, $\rho'(1450)$, and $\rho''(1700)$ mesons. We also use this spectrum to estimate the hadronic (2π) contribution to the anomalous magnetic moment of the muon ($a_\mu^{\pi\pi}$). Our result for $a_\mu^{\pi\pi}$ integrated over the mass range $\sqrt{s} = 2m_\pi - 1.8 \text{ GeV}/c^2$ is $a_\mu^{\pi\pi} = (523.5 \pm 1.5(\text{exp}) \pm 2.6(\text{Br}) \pm 2.5(\text{isospin})) \times 10^{-10}$, where the first error is due to the experimental uncertainties, the second is due to the uncertainties in the branching fractions, and the third is due to the uncertainties in the isospin-violating corrections.

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I. INTRODUCTION

Hadronic decays of the τ lepton provide a clean environment for studying the dynamics of hadronic states with various quantum numbers. Among the decay channels of the τ lepton, $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ has the largest branching fraction [1]. The decay is dominated by intermediate resonances and thus can be used to extract information on the properties of the $\rho(770)$, $\rho'(1450)$, and $\rho''(1700)$ mesons and their mutual interference.

From the conservation of vector current (CVC) theorem, the $\pi^- \pi^0$ mass spectrum in this range can be related to the cross section for the process $e^+ e^- \rightarrow \pi^+ \pi^-$ and thus used to improve the theoretical error on the anomalous magnetic moment of the muon $a_\mu = (g_\mu - 2)/2$. Recent reviews of calculations of a_μ are given in Refs. [2–4]. It is known that the theoretical error on a_μ is dominated by the contribution from the leading-order hadronic vacuum polarization $a_\mu^{\text{had,LO}}$. This contribution cannot be derived within the framework of perturbative QCD and is usually evaluated using dispersion relations and the experimental cross section for $e^+ e^-$ annihilation to hadrons [5–8]. Alternatively, CVC relates the properties of the $\pi^+ \pi^-$ system produced in $e^+ e^- \rightarrow \pi^+ \pi^-$ to those of the $\pi^- \pi^0$ system produced in $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ decay; thus, using CVC and correcting for isospin-violating effects, τ data have also been used to obtain a more precise prediction for $a_\mu^{\text{had,LO}}$ [5,6,8,9].

Recently, new precise data on $e^+ e^- \rightarrow \pi^+ \pi^-$ have become available from the CMD-2, KLOE, and SND ex-

periments [10–17]. ALEPH [18,19], CLEO [20,21], and OPAL [22,23] measured both the 2π spectral function and the branching fraction for the $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ decay; the latter was also determined by L3 [24] and DELPHI [25]. Recent evaluations of the hadronic contribution to a_μ using $e^+ e^-$ data result in $a_\mu^{\text{exp}} - a_\mu^{\text{th}} = (27.5 \pm 8.4) \times 10^{-10}$ [26,27], while that using the τ lepton data where applicable gives $a_\mu^{\text{exp}} - a_\mu^{\text{th}} = (9.4 \pm 10.5) \times 10^{-10}$ [5], where the experimental value a_μ^{exp} is dominated by the BNL E821 measurement [28] $(11\,659\,208.0 \pm 6.3) \times 10^{-10}$. These differences correspond to 3.3 and 0.9 standard deviations, respectively. For the evaluation based on the $e^+ e^-$ data, a deviation of similar size that corresponds to a 3.4σ discrepancy is claimed in Ref. [29]. To clarify these differences between the $e^+ e^-$ -based and τ -based predictions, more data on $e^+ e^- \rightarrow \pi^- \pi^+$ and $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ decays are needed. In this paper we present a high-statistics measurement of the $\pi^- \pi^0$ mass spectrum produced in $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ decays using data collected with the Belle experiment at the KEKB asymmetric-energy $e^+ e^-$ collider operating at a center-of-mass (CM) energy of 10.6 GeV. The data sample is about 50 times larger than those of previous experiments.

II. BASIC FORMULAS

The differential decay rate for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ can be expressed as [30]

$$\frac{d\Gamma(\tau^- \rightarrow \pi^- \pi^0 \nu_\tau)}{ds} = \Gamma_e^0 \cdot \frac{6\pi |V_{ud}|^2 S_{EW}^{\pi\pi}}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \times \left(1 + \frac{2s}{m_\tau^2}\right) v_-(s), \quad (1)$$

with

$$\Gamma_e^0 = \frac{G_F^2 m_\tau^5}{192\pi^3}. \quad (2)$$

Here s is the invariant mass squared of the $\pi^- \pi^0$ system, $v_-(s)$ is the weak spectral function characterizing the $\pi^- \pi^0$ system, G_F is the Fermi coupling constant, $|V_{ud}| = 0.97377 \pm 0.00027$ [31] is the Cabibbo-Kobayashi-Maskawa (CKM) matrix element [32], $m_\tau = 1776.99_{-0.26}^{+0.29}$ MeV/ c^2 [31] is the τ lepton mass, and $S_{EW}^{\pi\pi}$ accounts for short-distance electroweak radiative corrections for the $\pi^- \pi^0$ system. The measured electron decay rate of the τ lepton is related to Γ_e^0 by

$$\Gamma(\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau) \equiv \Gamma_e^0 S_{EW}^e = \Gamma_e^0 \left[1 + \frac{\alpha(m_\tau)}{2\pi} \left(\frac{25}{4} - \pi^2\right)\right], \quad (3)$$

where S_{EW}^e is the electroweak radiative correction for the decay $\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau$.

The corresponding $\pi^+ \pi^-$ spectral function $v_0(s)$ can be obtained from the $e^+ e^- \rightarrow \pi^+ \pi^-$ cross section

$$\sigma(e^+ e^- \rightarrow \pi^+ \pi^-) = \frac{4\pi^2 \alpha_0^2}{s} v_0(s), \quad (4)$$

where s is the $e^+ e^-$ CM energy squared and α_0 is the fine-structure constant at $s = 0$. Up to isospin-violating effects, CVC allows one to relate the spectral function from τ decays to the isovector part of the $e^+ e^-$ spectral function [33]:

$$v_-(s) = v_0^{I=1}(s). \quad (5)$$

Alternatively, the mass spectrum of the two-pion system can be expressed in terms of pion form factors, which are useful for comparing resonance shapes in the charged and neutral two-pion systems. The spectral function $v_j(s)$ ($j = -, 0$) is related to the form factor $F_\pi^j(s)$ via

$$v_j(s) = \frac{\beta_j^3(s)}{12\pi} |F_\pi^j(s)|^2, \quad (6)$$

where $\beta_-(s)$ ($\beta_0(s)$) is the pion velocity in the $\pi^- \pi^0$ ($\pi^+ \pi^-$) rest system. The velocities $\beta_j(s)$ are explicitly given by $\beta_-(s) = \lambda^{1/2}(1, m_\pi^2/s, m_\pi^2/s)$ [34] and $\beta_0(s) = \lambda^{1/2}(1, m_\pi^2/s, m_\pi^2/s) = [1 - 4m_\pi^2/s]$, with $\lambda(x, y, z) = [x - (\sqrt{y} + \sqrt{z})^2][x - (\sqrt{y} - \sqrt{z})^2]$.

The hadronic physics is contained within $v_j(s)$ or, equivalently, in $F_\pi^j(s)$. One goal of this analysis is to provide a high-precision determination of the weak form factor $|F_\pi^-(s)|$ using $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ data, so that a comparison

with $|F_\pi^0(s)|$ from the $e^+ e^-$ data can be used to test CVC. From Eqs. (1), (3), and (6), one can obtain the basic formula that expresses the form factor $F_\pi^-(s)$ in terms of the observables:

$$|F_\pi^-(s)|^2 = \frac{2m_\tau^2}{|V_{ud}|^2 (1 - \frac{s}{m_\tau^2})^2 (1 + \frac{2s}{m_\tau^2}) S_{EW}} \frac{1}{\beta_-^3} \left(\frac{\mathcal{B}_{\pi\pi}}{\mathcal{B}_e}\right) \times \left(\frac{1}{N_{\pi\pi}} \frac{dN_{\pi\pi}}{ds}\right), \quad (7)$$

where $\mathcal{B}_{\pi\pi}$ is the branching fraction, $(1/N_{\pi\pi}) (dN_{\pi\pi}/ds)$ is the normalized invariant mass-squared distribution for the $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ decay, \mathcal{B}_e is the branching fraction for $\tau^- \rightarrow e^- \nu_\tau \bar{\nu}_e$, and $S_{EW} = S_{EW}^{\pi\pi}/S_{EW}^e$.

In this paper, we report new measurements for both the branching fraction $\mathcal{B}_{\pi\pi}$ and the normalized mass spectrum $(1/N_{\pi\pi}) (dN_{\pi\pi}/ds)$. These results are used to provide a new evaluation of the hadronic contribution to the muon anomalous magnetic moment from the 2π channel.

III. DATA SAMPLE AND SELECTION CRITERIA

The data sample used was collected with the Belle detector at the KEKB asymmetric-energy $e^+ e^-$ collider [35]. It is based on an integrated luminosity of 72.2 fb $^{-1}$ recorded at a CM energy of 10.58 GeV. The Belle detector is a large-solid-angle magnetic spectrometer consisting of several detector components. Charged track coordinates near the collision point are measured by a three-layer silicon-vertex detector (SVD) that surrounds a 2 cm radius beryllium beam pipe. Track trajectory coordinates are reconstructed in a 50-layer central drift chamber (CDC), and momentum measurements are made together with the SVD. An array of 1188 silica-aerogel Cherenkov counters (ACC), a barrel-like arrangement of time-of-flight scintillation counters (TOF), and specific ionization measurements (dE/dx) in the CDC provide a capability for the identification of charged particles. Photon detection and energy measurement of the photons and electrons are provided by an electromagnetic calorimeter (ECL) consisting of an array of 8736 CsI (TI) crystals all pointing toward the interaction point. These detector components are located in a magnetic field of 1.5 T provided by a superconducting solenoid. An iron flux-return located outside the coil is instrumented to identify muons and to detect K_L^0 mesons (KLM). A comprehensive description of the detector is given in Ref. [36].

To study backgrounds and determine selection criteria, we perform Monte Carlo (MC) simulation studies for various processes. Signal and background $\tau^+ \tau^-$ -pair events are simulated using the KKMC generator [37]. The τ decays are modeled with the TAUOLA program [38,39] in which the values of the branching fractions are updated to more recent values [40]. The cross section for $e^+ e^- \rightarrow \tau^+ \tau^- (\gamma)$ is also updated to the recent measurement re-

ported in Ref. [41]. The radiative corrections to the τ -hadronic decays are simulated by the PHOTOS program [42]. The QQ generator [43] is used for $\bar{B}B$ and $\bar{q}q$ continuum processes, the BHLUMI [44] program for radiative Bhabha events, the KKMC [37] program for radiative $\mu^+\mu^-$ -pair events, and the AAFH [45] program for two-photon processes. The BHLUMI and KKMC programs include higher-order radiative corrections and are among the most accurate programs available. The detector response is simulated by a GEANT3-based program [46]. In order to simulate beam-induced background realistically, detector hits taken from randomly triggered data are added to wire hits in the CDC and to energy deposits in the ECL.

Uncertainties due to imperfections in the Monte Carlo generators and detector simulation are discussed in the later sections.

A. $\tau^+\tau^-$ pair selection

The event selection consists of two steps. Initially, a sample of generic $e^+e^- \rightarrow \tau^+\tau^-(\gamma)$ events is selected with relatively loose criteria. From this sample $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$ decays are identified. The number of generic $\tau^+\tau^-$ events is used to determine the $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$ branching fraction.

Generic $\tau^+\tau^-$ events are selected by requiring that the number of charged tracks in an event be two or four with zero net charge; that each track have a momentum transverse to the beam axis (p_T) of greater than 0.1 GeV/ c to avoid tracks reentering the CDC; and that each track extrapolate to the interaction point (IP) within ± 1 cm transversely and within ± 5 cm along the beam direction to suppress tracks that originate from beam-particle interactions with the residual gas in the vacuum chamber. To suppress background from Bhabha and $\mu^+\mu^-$ events, the reconstructed CM energies and the sum of the momenta of the first and the second highest momentum tracks are required to be less than 9.0 GeV/ c . The maximum p_T among the tracks is required to be greater than 0.5 GeV/ c . Beam-related background is rejected by requiring that the position of the reconstructed event vertex be less than 0.5 cm from the IP in the transverse direction and less than 2.5 cm from the IP along the beam direction. The polar angle of the leading particle with respect to the beam axis (θ^*) in the CM frame is required to be in the fiducial region of the detector: $35^\circ < \theta^* < 145^\circ$.

To reduce the remaining background from Bhabha, $\mu^+\mu^-(\gamma)$, and two-photon events, a requirement is imposed in the plane of the missing mass M_{miss} and the direction of missing momentum in CM θ_{miss}^* , where M_{miss} is evaluated from the four-momenta of the measured tracks and photons: $(M_{\text{miss}})^2 = (p_{\text{in}} - p_{\text{tr}} - p_\gamma)^2$. In this expression p_{in} is the four-momentum of the initial e^+e^- system, while p_{tr} and p_γ are the sum of the momenta of measured tracks and photons, respectively. A pion mass is

assumed for the charged tracks if they are not identified as electrons or muons. Each photon (reconstructed from clusters in the calorimeter) must be separated from the nearest track projection by at least 20 cm and have an energy greater than 0.05 GeV in the barrel region ($-0.63 \leq \cos\theta < 0.85$), and greater than 0.1 GeV in the endcap region ($-0.90 \leq \cos\theta < -0.62$ and $0.85 \leq \cos\theta < 0.95$). Photons near the edge of the detector fiducial volume are rejected. Scatter plots of M_{miss} versus θ_{miss} for data, the τ -signal MC, the Bhabha and the two-photon MC are shown in Figs. 1(a)–1(d), respectively. The band of events in data at $M_{\text{miss}} \approx 0$ is due to backgrounds from Bhabha and $\mu^+\mu^-(\gamma)$ processes. Small vertical bands at $\theta_{\text{miss}}^* = 45^\circ$ and $= 150^\circ$ are Bhabha events where the energy of one of the final state electron/positron is poorly measured because it has scattered in the material at the boundary of the barrel and end cap calorimeters. The events in the high- M_{miss} region (≥ 7 GeV/ c^2) are from the two-photon processes.

Events within the octagonal region are selected as $\tau^+\tau^-$ candidates to avoid the tail from background processes.

To display the τ -pair and background contribution quantitatively, we divide the scatter plots of M_{miss} vs θ_{miss} into three vertical and three horizontal slices as shown in Fig. 2. Projections for the six slices are shown in Fig. 3, where each process shows a characteristic shape: the $\tau^+\tau^-$ candidates dominate in the central region in M_{miss} and θ_{miss}^* . Both Bhabha and $\mu^+\mu^-$ show a prominent peak at $M_{\text{miss}} \approx 0$, but the width for the Bhabha is slightly wider than that of $\mu^+\mu^-$. We use the events in the region $|M_{\text{miss}}| < 0.5$ GeV/ c^2 to determine the normalization for the Bhabha and $\mu^+\mu^-$. As two-photon processes dominate in the high- M_{miss} region, the normalization for the two-photon processes is determined using the events at $|M_{\text{miss}}| > 8.0$ GeV/ c^2 . The arrows with solid (dotted) lines indicate the narrowest (widest) areas used to select $\tau^+\tau^-$ -pairs by the octagonal selection. Although overall features of the data are modeled reasonably well by MC, some discrepancies are seen, for example, in the regions $M_{\text{miss}} \approx 0$ and > 6 GeV/ c^2 in Fig. 4(c), which are taken into account as the systematic error on the background estimation.

Candidate events are divided into two hemispheres in the CM frame by the plane perpendicular to the highest momentum particle, and the remaining background from e^+e^- annihilation is suppressed by selecting events with low multiplicity as characterized by the quantity $X_{\text{part}} \equiv (n_{\text{tr}} + n_\gamma)_1 \times (n_{\text{tr}} + n_\gamma)_2$, where $n_{\text{tr},j}$ and $n_{\gamma,j}$ are the numbers of tracks and photons in hemisphere j . We require $X_{\text{part}} \leq 25$. Finally, in order to eliminate Bhabha events in which one or both electrons produce a shower in material near the interaction region, the acoplanarity angle ξ between the first and second highest momentum tracks is required to be $\xi > 1^\circ$, where $\xi \equiv |\phi_1 - \phi_2| - \pi$ is defined as the two-track acollinearity in azimuth.

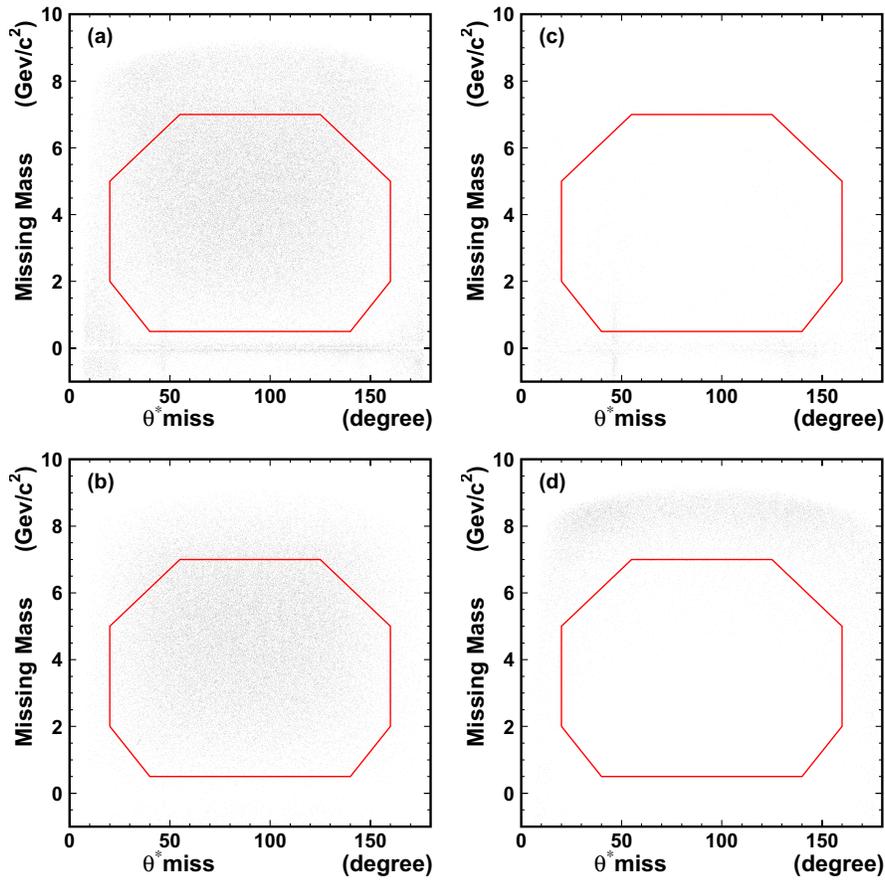


FIG. 1 (color online). Missing mass (M_{miss}) versus the polar angle direction of the missing momentum (θ_{miss}^*) for (a) the data, (b) MC $e^+e^- \rightarrow \tau^+\tau^-$ events, (c) MC Bhabha and $\mu^+\mu^-(\gamma)$ events, and (d) two-photon processes. Events inside the octagonal region are selected as $\tau^+\tau^-$ -pair candidates.

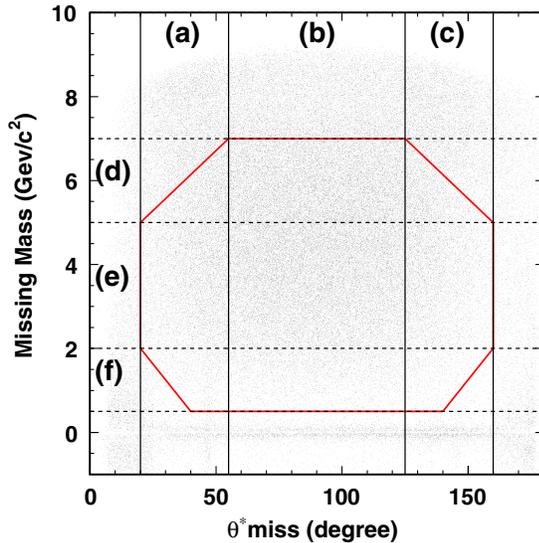


FIG. 2 (color online). Missing mass (M_{miss}) versus the polar angle direction of the missing momentum (θ_{miss}^*) for data. The solid (dashed) lines show three vertical (horizontal) slices that are used to present the projections in Fig. 3. The coordinates of the vertical (horizontal) lines are $\theta_{\text{miss}}^* = 20^\circ, 55^\circ, 125^\circ, 160^\circ$ ($M_{\text{miss}} = 0.5 \text{ GeV}/c^2, 2.0 \text{ GeV}/c^2, 5.0 \text{ GeV}/c^2, 7.0 \text{ GeV}/c^2$).

The X_{part} and ξ distributions after applying all selection criteria except for the quantity in question are shown in Figs. 4(c) and 4(d), respectively. The selection boundary is shown by the arrows.

After applying all selection criteria, 22.83×10^6 $\tau^+\tau^-$ -pairs survive.

The M_{miss} and θ_{miss} distributions for the surviving events, shown in Figs. 4(a) and 4(b), respectively, demonstrate a low level of the background and an overall good agreement between the data and the MC model.

For surviving events, the dominant background is from the $e^+e^- \rightarrow q\bar{q}$ ($q = u, d, s, c$) continuum and amounts to $(5.30 \pm 0.53)\%$ of the total number of events. The systematic error for the $q\bar{q}$ background is determined from the uncertainty of the normalization of the events in the region $25 < X_{\text{part}} < 30$, where $q\bar{q}$ processes dominate. The background from $e^+e^- \rightarrow Y(4S) \rightarrow B\bar{B}$ is small (0.1%). Backgrounds from Bhabha, $\mu^+\mu^-$, two-photon leptonic and hadronic events are to be $0.92 \pm 0.09\%$, $0.28 \pm 0.01\%$, $0.62 \pm 0.03\%$ and $0.60 \pm 0.09\%$, respectively. Here the systematic errors for each background is determined from the uncertainty of the normalization of the events in the background enhanced region mentioned before.

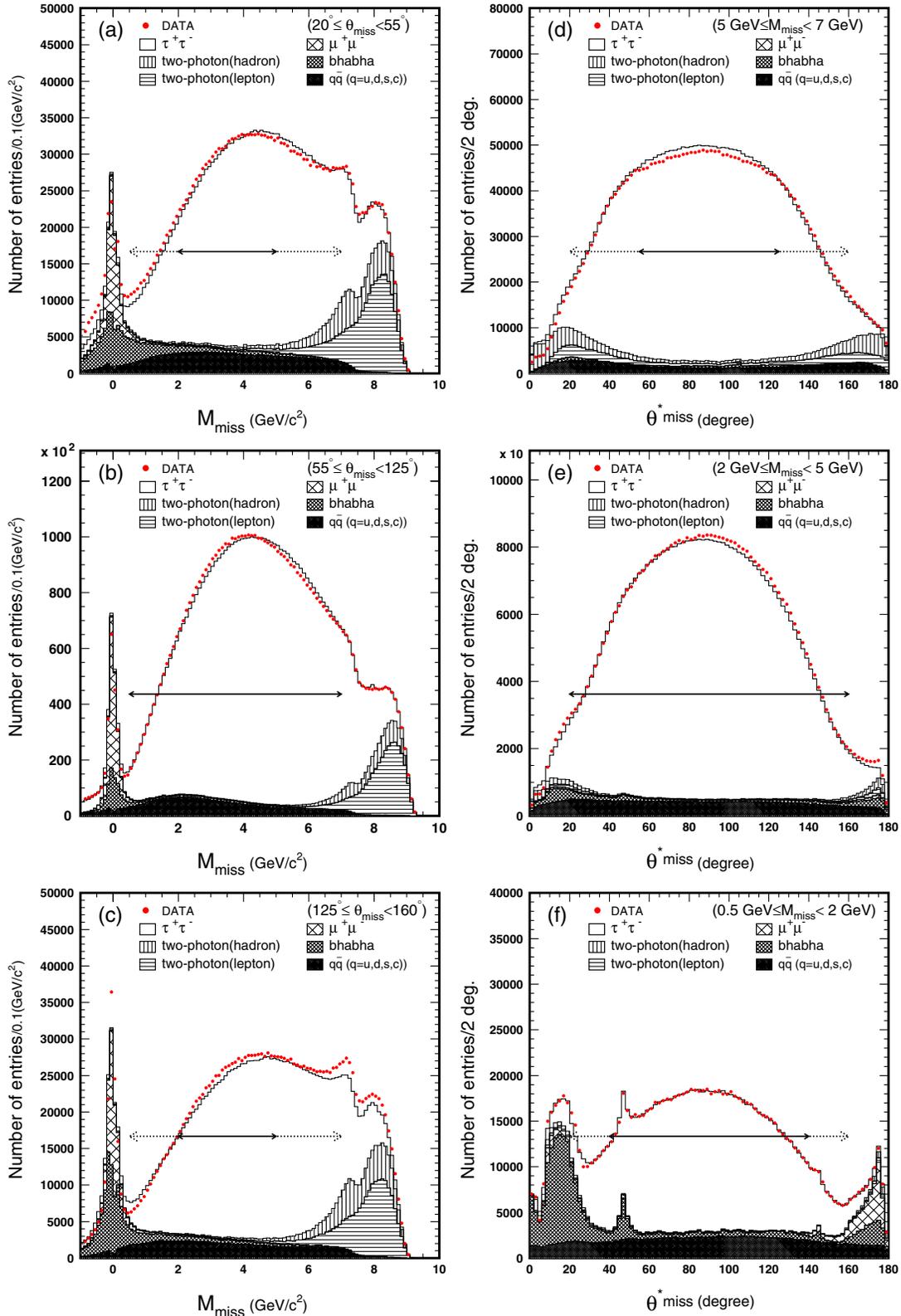


FIG. 3 (color online). Projections to the missing mass (M_{miss}) and the missing direction (θ_{miss}^*): (a)–(c) correspond to the vertical slices from left to right in Fig. 2. (d)–(f) correspond to the horizontal slices from top to bottom. The solid circles represent the data, and the histogram represents MC simulation (signal + background). The open histogram shows the contribution from the $\tau^+\tau^-$ -pair process, the vertical (horizontal) striped area shows that from two-photon leptonic (hadronic) processes; the wide (narrow) hatched area shows that from the Bhabha ($\mu^+\mu^-$) process; and the shaded area shows that from the $q\bar{q}$ continuum processes. The arrows with solid (dotted) lines indicate the widest (narrowest) region corresponding to the octagonal boundary shown in Fig. 2.

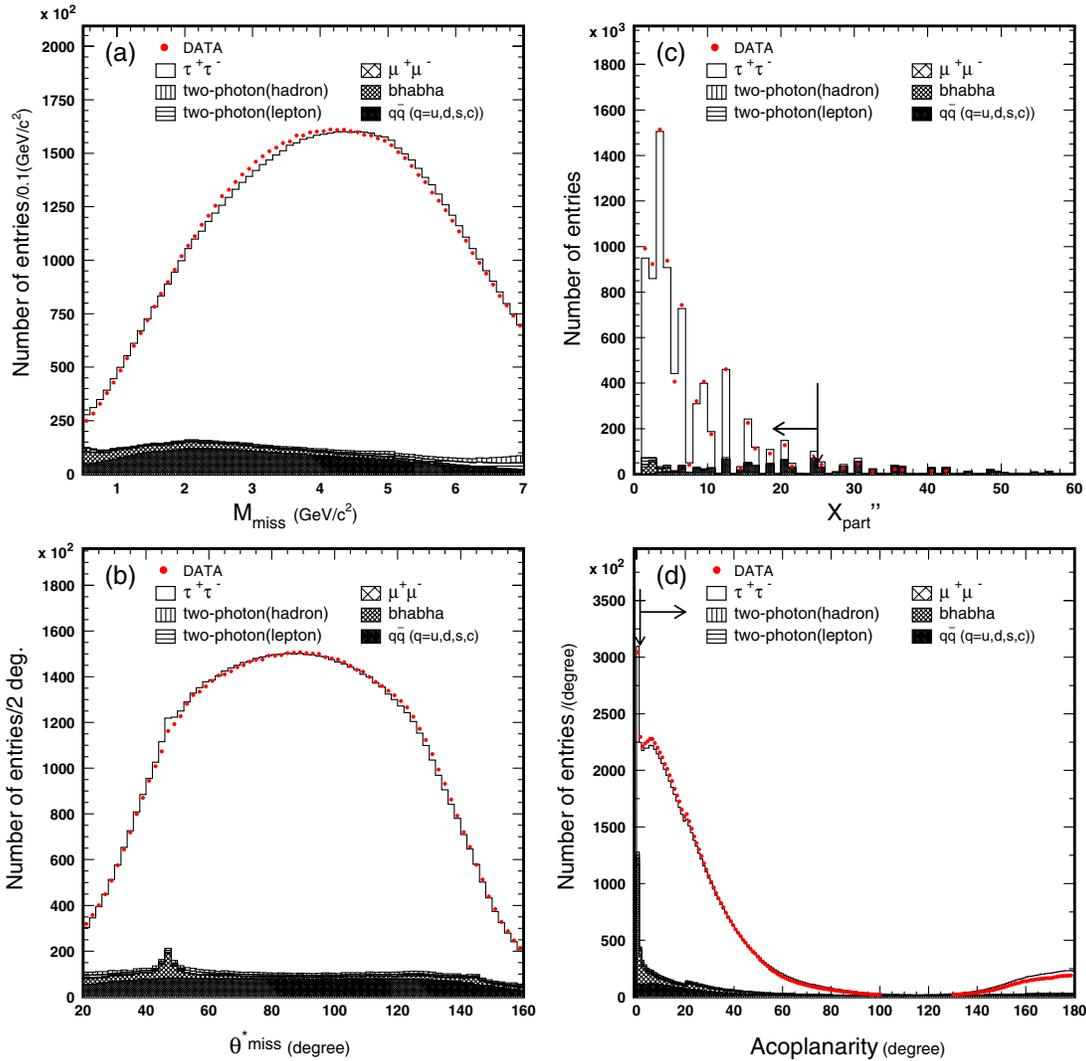


FIG. 4 (color online). Characteristic distributions for surviving $\tau^+\tau^-$ candidates: (a) M_{miss} , (b) θ_{miss} , (c) particle multiplicity $X_{\text{part}} \equiv (n_{\text{tr}} + n_{\gamma})_1 \times (n_{\text{tr}} + n_{\gamma})_2$, (d) acoplanarity angle ξ . The points indicate the data, the open histogram shows the τ -pair MC and the hatched histogram shows the background from $e^+e^- \rightarrow q\bar{q}$ and other sources. All selection criteria are applied for (a) and (b). All criteria except for the quantity in question are applied for (c) and (d). The arrows in (c) and (d) indicate the boundary used to select a τ -pair sample.

B. $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ selection

Within the $\tau^+\tau^-$ -pair sample, $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ decays are reconstructed by requiring that there be both one charged track and one π^0 in a single hemisphere. The π^0 candidate is selected based on the normalized invariant mass $S_{\gamma\gamma} \equiv (m_{\gamma\gamma} - m_{\pi^0})/\sigma_{\gamma\gamma}$, where $\sigma_{\gamma\gamma}$ is the mass resolution of the $\gamma\gamma$ system. The value of $\sigma_{\gamma\gamma}$ ranges from 0.005 GeV/c^2 to 0.008 GeV/c^2 , depending on the π^0 momentum and polar angle. Pairs of photons with $|S_{\gamma\gamma}| < 9$ are considered as π^0 candidates. To keep beam-related background at a negligible level, we require that the CM momentum of the π^0 be greater than 0.25 GeV/c and the photon CM energy be greater than 0.08 GeV .

The distribution of $S_{\gamma\gamma}$ for the selected $\pi^- \pi^0$ sample, with one charged track and one π^0 candidate in a single hemisphere, is shown in Fig. 5. The lower-side tail of the $S_{\gamma\gamma}$ distribution is primarily due to rear and transverse leakage of electromagnetic showers out of the CsI (TI) crystals and the conversion of photons in the material located in front of the crystals. Good agreement between data (points) and MC (open histogram) indicates that these effects are properly modeled by the MC simulation. We define the interval $-6 < S_{\gamma\gamma} < 5$ as the π^0 signal region. Spurious π^0 background is small and estimated from the sideband regions $7 < |S_{\gamma\gamma}| < 9$. To reduce feed-down background from multi- π^0 decays such as $\tau^- \rightarrow \pi^-(n\pi^0)\nu_\tau$ ($n \geq 2$), signal candidates are rejected if there

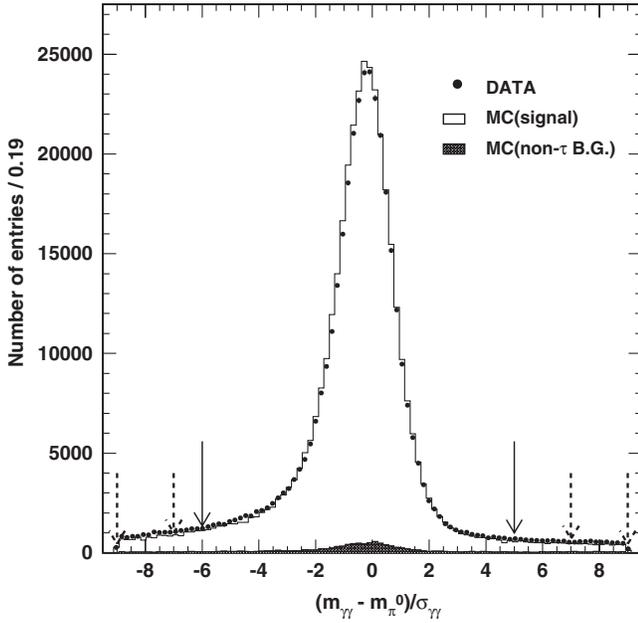


FIG. 5. Normalized $\gamma\gamma$ invariant mass ($S_{\gamma\gamma}$) spectrum for data (points) and the $\tau^- \rightarrow h^- \pi^0 \nu_\tau$ signal MC (open histogram), for the sample described in the text. The data plotted here correspond to 6% of the full data sample used in this analysis. The arrows indicate the signal region $-6 < S_{\gamma\gamma} < 5$ and the sideband regions $7 < |S_{\gamma\gamma}| < 9$. The sideband regions are used to subtract fake- π^0 background. The shaded histogram shows the non- τ background determined from MC simulation.

are additional γ 's in the same hemisphere with energy greater than 0.2 GeV.

The $\pi^- \pi^0$ invariant-mass-squared ($M_{\pi\pi^0}^2$) spectrum is obtained assuming the pion mass for the charged track; it is shown in Fig. 6 along with the MC prediction. To improve the π^0 energy resolution, a π^0 mass constraint is imposed. The spurious π^0 background level depends on the $M_{\pi\pi^0}^2$ region, varying from 4% to 7%. (This is subtracted using $S_{\gamma\gamma}$ sidebands.) The final sample contains 5.43×10^6 $\tau^- \rightarrow h^- \pi^0 \nu_\tau$ candidates after the π^0 background subtraction, where h^- denotes π^- or K^- . This sample is 50 times larger than those of previous studies.

The spectrum is dominated by the $\rho(770)$ peak and a shoulder due to the $\rho'(1450)$. A small but clear structure from the $\rho''(1700)$ is visible at $M_{\pi\pi^0}^2 \sim 2.7$ (GeV/c^2)².

There are two sources of background: feed-down from other τ decay modes and the $q\bar{q}$ -continuum. Feed-down background arises mainly from multi- π^0 modes such as $\tau^- \rightarrow h^- (n\pi^0) \nu_\tau$ ($6.02 \pm 0.08\%$), $\tau^- \rightarrow K_L h^- \pi^0 \nu_\tau$ ($0.48 \pm 0.04\%$), and $\tau^- \rightarrow \omega \pi^- \nu_\tau$ ($\omega \rightarrow \pi^0 \gamma$) ($0.10 \pm 0.01\%$). Here h^- denotes either π^- or K^- . After all modes are included, the total feed-down background level is $(7.02 \pm 0.08)\%$. The error given here includes a MC statistical uncertainty as well as the uncertainty on relevant branching fractions. The contribution of these feed-down backgrounds dominates at low values of $M_{\pi\pi^0}^2$ (Fig. 6).

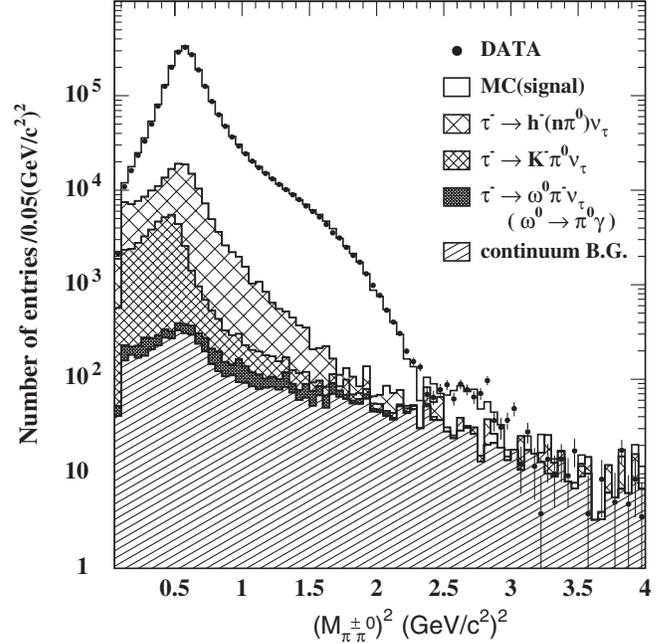


FIG. 6. Invariant-mass-squared ($M_{\pi\pi^0}^2$) distribution for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ after imposing tight tag-side requirements. The solid circles with error bars represent the data, and the histogram represents the MC simulation (signal + background). The open histogram shows the contribution from $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$; the narrow cross-hatched area shows that from $\tau^- \rightarrow K^- \pi^0 \nu_\tau$; the wide cross-hatched area shows that from $\tau^- \rightarrow h^- (n\pi^0) \nu_\tau$; and the striped area shows that from the $q\bar{q}$ continuum and other non- τ processes.

The $q\bar{q}$ -continuum background level is $(2.22 \pm 0.05)\%$ in total, and is concentrated mostly in the high $M_{\pi\pi^0}^2$ region above 2.0 (GeV/c^2)². Since the reduction of this high-mass background is essential in the measurement of the mass spectrum, we impose the stringent requirement that the tag side contain only one charged track and no photons. This requirement improves the signal-to-noise ratio in the high-mass region $M_{\pi\pi^0}^2 \geq 2.0$ (GeV/c^2)² by a factor of 3, although the total size of the $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ sample is reduced by a factor of 2.5. The normalization of the continuum MC is validated using data in the mass region above the τ lepton mass: $M_{\pi\pi^0}^2 > m_\tau^2$. Background from the other non- τ processes, such as $B\bar{B}$, Bhabha and $\mu^+ \mu^- \gamma$ in the final sample is negligible ($< 0.1\%$).

IV. MEASUREMENT OF THE BRANCHING FRACTION

A. Basic method

The branching fraction for $\tau^- \rightarrow h^- \pi^0 \nu_\tau$ ($\mathcal{B}_{h\pi^0}$) is determined by dividing the signal yield $N_{h\pi^0}$ by the total number of selected τ leptons $2N_{\tau\tau}$ taking into account various efficiencies and background corrections:

TABLE I. Values of parameters used for the branching fraction measurement along with MC statistical errors.

Parameter	Value
$\epsilon_{\tau\tau}$	$32.59 \pm 0.05\%$
$\epsilon_{h\pi^0}^\tau$	$36.24 \pm 0.07\%$
$f_b = \frac{\epsilon_{h\pi^0}^\tau}{\epsilon_{\tau\tau}}$	1.112 ± 0.003
$\epsilon_{h\pi^0}^{ID}$	$41.01 \pm 0.13\%$
$b_{\tau\tau}$	$7.80 \pm 0.03\%$
$b_{h\pi^0}^{\text{feed-down}}$	$7.02 \pm 0.08\%$
$b_{h\pi^0}^{\text{non-}\tau}$	$2.22 \pm 0.05\%$

$$\mathcal{B}_{h\pi^0} = \frac{N_{h\pi^0}}{2N_{\tau\tau}} \times \frac{(1 - b^{\text{feed-down}} - b^{\text{non-}\tau})}{(1 - b_{\tau\tau})} \times \left(\frac{\epsilon_{\tau\tau}}{\epsilon_{h\pi^0}^\tau} \right) \times \frac{1}{\epsilon_{h\pi^0}^{ID}}. \quad (8)$$

In this formula, $b_{\tau\tau}$ is the background fraction in the $\tau^+\tau^-$ sample, $\epsilon_{\tau\tau}$ is the efficiency of the $\tau^+\tau^-$ -pair selection, $\epsilon_{h\pi^0}^\tau$ is the efficiency for $\tau^- \rightarrow h^-\pi^0\nu$ decays to pass the $\tau^+\tau^-$ -pair selection, and $\epsilon_{h\pi^0}^{ID}$ is the efficiency for $\tau^- \rightarrow h^-\pi^0\nu$ decays satisfying the $\tau^+\tau^-$ -pair selection to pass the $h^-\pi^0$ selection. The product $\epsilon_{h\pi^0}^\tau \cdot \epsilon_{h\pi^0}^{ID}$ is the overall detection efficiency for the $h^-\pi^0\nu$ final state. The parameter $b^{\text{feed-down}}$ is the fraction of $h^-\pi^0\nu$ candidates coming from other τ decay modes, and $b^{\text{non-}\tau}$ is the fraction coming from non- τ processes. In this formula, several common uncertainties such as that on the luminosity, on the cross section for $\tau^+\tau^-$ -pair production, on the trigger efficiency, and on the $\tau^+\tau^-$ selection efficiency cancel in the ratio. In the measurement of the branching fraction, the stringent tag-side condition is not imposed to avoid any possible bias that it might introduce. The values for all factors are listed in Table I along with the MC statistical error.

B. Systematic uncertainties

The sources of systematic uncertainties on $\mathcal{B}_{h\pi^0}$ are listed in Table II. The uncertainty on the tracking efficiency is estimated using $D^{*+} \rightarrow D^0\pi^+ \rightarrow K^-\pi^+\pi^+$ decays to

be 1% per track. A large part of this uncertainty cancels in the ratio of Eq. (8); the resulting relative uncertainty from this source is $\Delta\mathcal{B}/\mathcal{B} = 0.47\%$.

The systematic error on the π^0 detection efficiency has two components: one is the uncertainty coming from the π^0 selection criteria and the other is that from the absolute efficiency calibration. For the uncertainty coming from the π^0 selection, we check the uncertainty by changing the definition of the signal and background region, by taking into account the uncertainty in the resolution function and by changing the π^0 threshold momentum. For example, the relative branching fraction changes by only $\Delta\mathcal{B}/\mathcal{B} = 0.1\%$ if the signal region is changed from the nominal one to $-7 < S_{\gamma\gamma} < 7$. Also the uncertainty is $\Delta\mathcal{B}/\mathcal{B} = \pm 0.2\%$ for the changes of the π^0 threshold ± 0.05 GeV from the nominal value.

In order to make an absolute efficiency calibration independently of the signal process, we use the $\eta \rightarrow \gamma\gamma$ and $\eta \rightarrow \pi^0\pi^0\pi^0$ signals, whose branching fractions are known rather precisely. Combining the PDG world average [31] for the $\eta \rightarrow \gamma\gamma$ and $\eta \rightarrow \pi^0\pi^0\pi^0$ branching fractions and the recent measurement from the CLEO collaboration [47], we obtain the ratio of the branching fractions of

$$\frac{\mathcal{B}(\eta \rightarrow \gamma\gamma)}{\mathcal{B}(\eta \rightarrow \pi^0\pi^0\pi^0)} = 0.829 \pm 0.007,$$

which has 0.84% relative accuracy.

By comparing the signal ratio $R_i \equiv N(\eta \rightarrow \pi^0\pi^0\pi^0)/N(\eta \rightarrow \gamma\gamma)$ for the data ($i = 1$) and the MC ($i = 2$), the correction factor for the detection efficiency of one π^0 , η_{cor} , is determined to be $\eta_{\text{cor}} = \sqrt{R_{\text{data}}/R_{\text{MC}}} = 0.950 \pm 0.012$, where the error includes the uncertainties in the η signal measurement (1.2%) and the errors on the η decay branching fractions (0.4%).

This correction factor is also confirmed by a study of electron/positron tracks from photon conversions (i.e. $\gamma \rightarrow e^+e^-$) in the SVD region. It is found that the E/P distribution for those tracks is simulated correctly above 1.0 GeV, but requires some tuning below 1.0 GeV. This imperfection of the MC primarily leads to a difference in

TABLE II. Systematic uncertainties for the $\tau^- \rightarrow h^-\pi^0\nu_\tau$ branching fraction.

Source of uncertainty	$\Delta\mathcal{B}_{h\pi^0}$ (%)	$(\Delta\mathcal{B}/\mathcal{B})$ (%)
Tracking efficiency	0.12	0.47
π^0 efficiency	0.32	1.27
Background for $\tau^+\tau^-$	0.15	0.59
Feed-down background for $\tau^- \rightarrow h^-\pi^0\nu_\tau$	0.04	0.16
Non- τ background for $\tau^- \rightarrow h^-\pi^0\nu_\tau$	0.05	0.20
γ veto	0.05	0.20
Trigger	0.08	0.32
MC statistics	0.02	0.08
Total	0.39	1.52

the π^0 signal shape and an efficiency difference between data and MC.

The non- τ background is dominated by $q\bar{q}$ continuum processes; this is estimated by using the events above the τ mass: $M_{\pi\pi^0}^2 > m_\tau^2$. The statistics of the data and MC sample determine the error.

The uncertainty on the feed-down background $\Delta b_{h\pi^0}^{\text{feed-down}}$ comes from the MC statistics and the uncertainty on the branching fractions for $\tau^- \rightarrow h^-(n\pi^0)\nu_\tau$, $\tau^- \rightarrow K^-\pi^0\nu_\tau$, and $\tau^- \rightarrow \omega\pi^-\nu_\tau$ ($\omega \rightarrow \pi^0\gamma$).

The veto of additional γ 's is required in the event selection to reduce background from multi- π^0 decay channels. However, this veto can reject signal itself if photons are radiated in the initial or final state and those photons are detected within the detector fiducial volume. In addition, photon candidates can also appear due to electromagnetic shower fragments and/or misreconstructed electrons. Therefore a precise simulation of the photon radiation as well as the shower simulation are important. The uncertainty from these sources is estimated by changing the veto threshold by ± 0.1 GeV around the nominal value of 0.2 GeV; the resulting relative change in $\mathcal{B}_{h\pi^0}$ is only $\pm 0.20\%$. Signal events are flagged by several trigger conditions that require two or more CDC tracks with associated TOF hits, ECL clusters, or a significant sum of energy in the ECL. This redundancy allows one to monitor the efficiency of each trigger requirement. The uncertainty arising from the trigger is estimated by assuming that there is a $\pm 3\%$ uncertainty on the track and energy trigger efficiencies, which is the maximum variation measured during experimental running. The resulting relative uncertainty is small (0.32%) since the $\tau^+\tau^-$ trigger efficiency is high (97%).

C. Results

Inserting all values into Eq. (8) we obtain

$$\mathcal{B}_{h\pi^0} = (25.67 \pm 0.01 \pm 0.39)\%, \quad (9)$$

where the first error is statistical and the second is systematic. This result is in good agreement with previous measurements, as shown in Table III. Our statistical error is significantly lower than those of the other measurements;

TABLE III. Branching fractions for $\tau^- \rightarrow h^-\pi^0\nu_\tau$ measured by different experiments.

Experiment	$\mathcal{B}_{h\pi^0}(\%)$	Reference
CLEO	$25.87 \pm 0.12 \pm 0.42$	[21]
L3	$25.05 \pm 0.35 \pm 0.50$	[24]
OPAL	$25.89 \pm 0.17 \pm 0.29$	[23]
ALEPH	$25.924 \pm 0.097 \pm 0.085$	[19]
DELPHI	$25.740 \pm 0.201 \pm 0.138$	[25]
This work	$25.67 \pm 0.01 \pm 0.39$	

our systematic error is similar to those of CLEO, L3, and OPAL, and larger than those of ALEPH and DELPHI.

We combine the PDG world average for the $\tau^- \rightarrow K^-\pi^0\nu_\tau$ branching fraction [31] with a recent BABAR measurement [48] to obtain the result $\mathcal{B}_{K^-\pi^0} = (0.428 \pm 0.015)\%$. Subtracting this from our $\tau^- \rightarrow h^-\pi^0\nu_\tau$ result gives a $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$ branching fraction of

$$\mathcal{B}_{\pi\pi^0} = (25.24 \pm 0.01 \pm 0.39)\%, \quad (10)$$

which is consistent with the previous measurements from CLEO [21] and ALEPH [19].

V. MEASUREMENT OF THE MASS SPECTRUM

In order to obtain the true $\pi^-\pi^0$ mass spectrum, one must apply corrections for: (1) background, (2) smearing due to finite resolution and radiative effects, and (3) mass-dependent acceptance.

A. Background correction

As noted earlier, there are three sources of the background that enter the $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$ sample: (1) fake π^0 background, (2) feed-down background from other τ decay channels, and (3) the background from the $q\bar{q}$ continuum.

The total magnitude of these background contributions is about 7% in the $\rho(770)$ peak region, but the fraction of the background varies strongly with $M_{\pi\pi^0}^2$; there is approximately a 4-order-of-magnitude difference between the signal level in the $\rho(770)$ peak region and that in the high $M_{\pi\pi^0}^2$ region above 2.5 (GeV/ c^2)². Thus a reliable estimation of the background is important for the measurement of the mass spectrum.

The sidebands of the $M_{\gamma\gamma}$ distribution are used to estimate the fake π^0 contribution. This background dominates at values of $M_{\pi\pi^0}^2$ less than about 0.25 (GeV/ c^2)².

In the $\pi^-\pi^0$ system, the feed-down background dominates at similarly low values of $M_{\pi\pi^0}^2$ while the $q\bar{q}$ -continuum background dominates at high values of $M_{\pi\pi^0}^2$ (see Fig. 6). These backgrounds are subtracted bin-by-bin.

B. Acceptance correction

The acceptance determined from MC simulation as a function of the generated $\pi^-\pi^0$ mass squared is shown in Fig. 7. The acceptance varies smoothly, and its average value is 7%. This acceptance includes a factor for the tag-side branching fractions $\mathcal{B}(\tau^- \rightarrow \ell^-\bar{\nu}_\ell\nu_\tau)$ ($\ell^- = \mu^-, e^-$), $\mathcal{B}(\tau^- \rightarrow h^-\nu_\tau)$ ($h^- = \pi^-, K^-$), which does not affect the shape of the mass spectrum. The acceptance decreases at low values of M_{gen}^2 due to the overlap of γ clusters with the π^- track in the calorimeter.

The detector effects include $M_{\pi\pi^0}^2$ -dependent acceptance and bin-by-bin migration caused by the finite mass resolution. The radiative decay $\tau^- \rightarrow \pi^-\pi^0\gamma\nu_\tau$ also

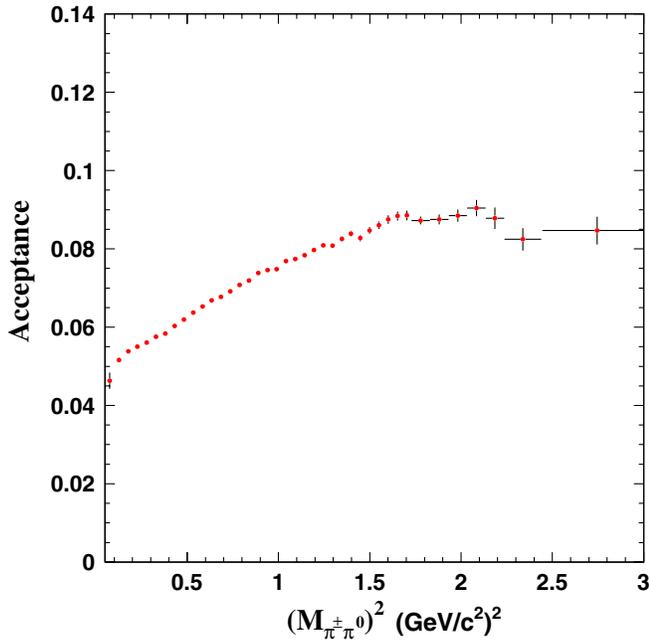


FIG. 7 (color online). The acceptance determined from MC simulation as a function of the $\pi^-\pi^0$ mass squared.

causes some bin migration. We correct for these effects using an unfolding procedure that makes use of the MC to characterize the acceptance and the bin migration. These effects can be characterized by the acceptance matrix A defined by

$$\mathbf{b} = A\mathbf{x},$$

where \mathbf{x} is the vector containing the generated $\pi^-\pi^0(\gamma)$ mass-squared spectrum and \mathbf{b} is the reconstructed one. It is possible to apply the inverse of A to the spectrum observed in the data to obtain an unfolded spectrum. However, this procedure is not robust with respect to the statistical fluctuations entering the determination of A , and can yield unphysically large point-by-point fluctuations. To cure this problem, we use an unfolding program employed in the ALEPH experiment [49]. In this program, the unfolding is based on the singular-value-decomposition (SVD) method, in which the acceptance matrix is inverted by constraining the number of singular values to only those elements that are statistically significant.

The acceptance matrix is determined iteratively using a signal MC based on the KKMC/TAUOLA program. In the second iteration, the $\rho''(1700)$ resonance is included in the MC based on our measurement. Final state radiation in τ hadronic decays is simulated by the PHOTOS program. In order to take into account the effects of γ radiation in the decay $\tau^- \rightarrow \pi^-\pi^0\gamma\nu_\tau$, the invariant mass squared of the $\pi^-\pi^0\gamma$ system is taken as the generated quantity.

The output of the program is the unfolded distribution and its covariance matrix. The correlation between the generated quantity and the measured one is shown in

Fig. 8. The figure shows a clear correlation between the measured and generated values. The resolution in $M_{\pi^-\pi^0}^2$ is $0.005 (\text{GeV}/c^2)^2$ in the low-mass region and $0.030 (\text{GeV}/c^2)^2$ in the high-mass region; thus by choosing the bin size to be $\Delta M^2 = 0.050 (\text{GeV}/c^2)^2$, the off-diagonal components of the acceptance matrix are small.

C. Systematic uncertainties

The sources of systematic errors associated with the unfolded mass spectrum $(1/N) (dn/ds)$ are subdivided into several classes according to their origin, which are the unfolding procedure (UNF), the background subtraction (BKG), the acceptance correction (ACC), and the energy scale (ENG). These contributions are summarized in Table IV for each $M_{\pi^-\pi^0}^2$ region and are described below.

The systematic error due to the unfolding procedure is determined from MC by comparing the true and the unfolded results (UNF1). Another estimate of the uncertainty of the unfolding is made by changing the value of the unfolding parameter that determines the optimum number of the singular values of the acceptance matrix (UNF2).

The uncertainty of the background subtraction is estimated for each source. BKG1 is from continuum processes. Its uncertainty is estimated using the control sample in the mass region higher than the τ mass. The statistics of the data and MC sample determine its error. BKG2 is the feed-down background. Its uncertainty is estimated by varying the branching fraction values [31] used in the MC by $\pm 1\sigma$. BKG3 is the non- π^0 background. The uncertainty of the non- π^0 background is estimated by changing the π^0 sideband region. This uncertainty dominates in the threshold region but is negligible elsewhere.

The acceptance uncertainty is dominated by the uncertainty of the π^0 efficiency.

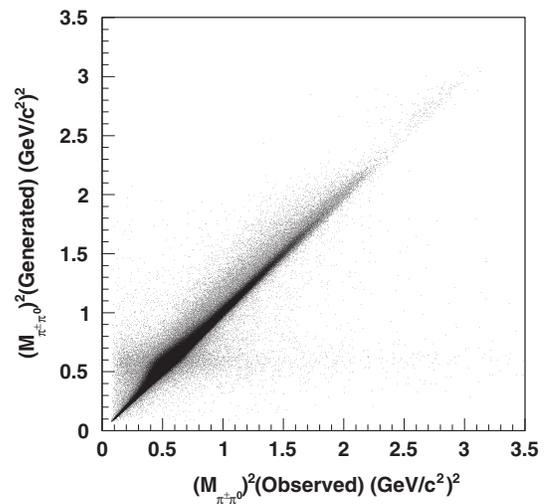


FIG. 8. Correlation between the generated and observed invariant masses squared of the $\pi^-\pi^0$ system in the $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$ decay.

TABLE IV. Relative systematic errors (in %) of the unfolded spectrum for each $M_{\pi\pi^0}^2$ region for the different sources of uncertainty: unfolding procedure (UNF1, UNF2), the background subtraction (BKG1, BKG2, BKG3), the acceptance correction (ACC), and the photon energy scale (PES). See the text for a more detailed description.

$M_{\pi\pi^0}^2$ region ($(\text{GeV}/c^2)^2$)	First bin (0.08)	Threshold region (0.2–0.3)	ρ region (0.55–0.60)	ρ' region (1.0–1.2)	ρ'' region (1.9–2.0)	ρ'' region (2.5–2.7)
UNF1	2.50	0.79	0.31	0.85	1.50	1.50
UNF2	2.60	0.53	0.09	0.27	0.58	9.19
BKG1	1.13	0.09	0.01	0.04	0.52	5.76
BKG2	4.90	0.65	0.10	0.10	...	0.50
BKG3	25.21	4.80
ACC	5.36	1.44	0.03	0.15	0.15	0.40
PES	1.24	1.08	0.59	0.99	0.05	0.50
Total	26.5	5.3	0.7	1.5	1.8	11.4

This is estimated by changing the measured values of the photon efficiency by 1 standard deviation. In addition, the effect of requiring that photons be isolated from charged tracks is checked by changing the isolation criteria from 20 cm (default) to 30 cm.

The uncertainty of the photon energy scale (PES) is estimated from the π^0 peak position to be $\pm 0.2\%$. This uncertainty is important for the peak position of the resonances. The uncertainty in the charged track momentum scale is negligible compared to that of the photon energy scale.

Individual components of the uncertainty are added in quadrature to obtain total systematic errors of 5.3% in the threshold region, 0.7% near the $\rho(770)$ peak, and 1.8% in the vicinity of the $\rho''(1450)$ (see Table IV).

D. Results

The unfolded $s = M_{(\pi\pi^0_{\text{unf}})}^2$ spectrum $dN_{\pi\pi}/ds$ is shown in Fig. 9. The error bars in the figure include both statistical and systematic errors added in quadrature, and in most cases they are smaller than the size of the data points shown by closed circles. The results are also presented in terms of the normalized unfolded spectrum ($1/N_{\pi\pi}$) ($dN_{\pi\pi}/ds$) in Table V and in terms of the pion form factor in Table VI. In these tables, the statistical and systematic errors are given separately. The statistical errors in the figure and the table are the square roots of the diagonal components of the covariance matrix.

In Fig. 9, the ρ peak and a shoulder due to the $\rho'(1450)$ are clearly visible. The dip at $s \approx 2.5$ (GeV/c^2)² is caused by destructive interference between the $\rho'(1450)$ and $\rho''(1700)$ resonances.

To determine the parameters of the ρ , ρ' , and ρ'' resonances, a χ^2 fit using Breit-Wigner (BW) functions is performed. The pion form factor is parametrized with Breit-Wigner functions corresponding to the ρ , $\rho'(1450)$, and $\rho''(1700)$ resonances:

$$F_{\pi}(s) = \frac{1}{1 + \beta + \gamma} (\text{BW}_{\rho} + \beta \cdot \text{BW}_{\rho'} + \gamma \cdot \text{BW}_{\rho''}), \quad (11)$$

where the parameters β and γ (denoting the relative magnitude of the two resonances) are in general complex. We use the Gounaris-Sakurai (GS) model [50] for the Breit-Wigner shape:

$$\text{BW}_i^{\text{GS}} = \frac{M_i^2 + d \cdot M_i \Gamma_i(s)}{(M_i^2 - s) + f(s) - i\sqrt{s}\Gamma_i(s)}, \quad (12)$$

with an energy-dependent width

$$\Gamma_i(s) = \Gamma_i \left(\frac{M_i^2}{s} \right) \left(\frac{k(s)}{k(M_i^2)} \right)^3. \quad (13)$$

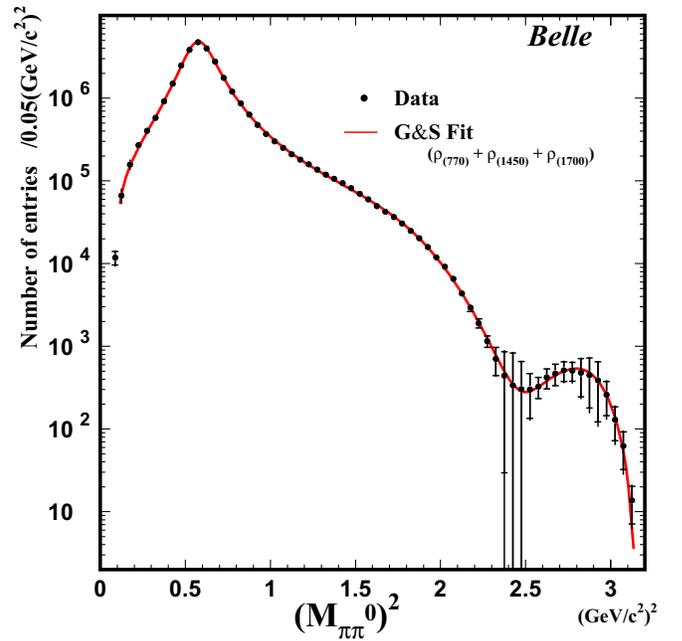


FIG. 9 (color online). The fully corrected $M_{\pi\pi^0}^2$ distribution for $\tau^- \rightarrow \pi^- \pi^0 \nu_{\tau}$. The solid curve is the result of a fit to the Gounaris-Sakurai model with the $\rho(770)$, $\rho'(1450)$, and $\rho''(1700)$ resonances. All resonance parameters (mass, width, phase, and the normalization factor $|F_{\pi}(0)|^2$) are allowed to float.

TABLE V. The unfolded normalized spectrum ($1/N_{\pi\pi}$) ($dN_{\pi\pi}/ds$) as a function of the invariant mass squared $s = M_{\pi\pi}^2$. The square of the diagonal element of the error matrix is taken for the statistical errors. Note that the scale is different for the left and right sides.

Bin No.	$M_{\pi\pi}^2$ (GeV/c ²) ²	$\frac{1}{N} \frac{dN}{ds} \times 10^{-3}$ GeV ⁻² c ⁴	Stat. $\times 10^{-3}$ GeV ⁻² c ⁴	Syst. $\times 10^{-3}$ GeV ⁻² c ⁴	Bin No.	$M_{\pi\pi}^2$ (GeV/c ²) ²	$\frac{1}{N} \frac{dN}{ds} \times 10^{-4}$ GeV ⁻² c ⁴	Stat. $\times 10^{-4}$ GeV ⁻² c ⁴	Syst. $\times 10^{-4}$ GeV ⁻² c ⁴
1	0.088	8.1	3.1	2.1	32	1.625	341.47	5.91	5.83
2	0.125	45.8	2.4	9.6	33	1.675	290.94	5.56	5.39
3	0.175	108.4	2.4	15.5	34	1.725	250.39	4.99	4.32
4	0.225	185.1	2.6	14.3	35	1.775	210.22	4.86	3.40
5	0.275	278.0	2.8	8.3	36	1.825	170.63	4.36	2.75
6	0.325	396.0	3.3	4.1	37	1.875	139.72	3.89	2.26
7	0.375	628.6	4.2	5.0	38	1.925	109.26	3.82	1.85
8	0.425	1024.2	5.5	14.7	39	1.975	81.85	3.20	1.39
9	0.475	1710.8	7.2	21.5	40	2.025	63.08	2.84	1.25
10	0.525	2643.0	9.0	22.5	41	2.075	45.02	2.72	0.89
11	0.575	3268.0	9.7	21.4	42	2.125	29.89	2.22	0.65
12	0.625	2755.5	9.0	19.4	43	2.175	20.06	1.94	0.61
13	0.675	1907.2	7.4	17.0	44	2.225	13.08	1.68	0.42
14	0.725	1214.0	5.8	8.4	45	2.275	7.93	1.25	0.42
15	0.775	826.4	4.6	10.7	46	2.325	4.85	1.82	0.51
16	0.825	592.8	3.6	8.6	47	2.375	3.04	2.84	1.39
17	0.875	435.6	2.9	3.5	48	2.425	2.32	3.36	1.17
18	0.925	327.5	2.4	4.6	49	2.475	2.09	2.39	0.76
19	0.975	253.7	2.1	3.7	50	2.525	2.07	1.14	0.37
20	1.025	206.3	1.8	2.7	51	2.575	2.24	0.64	0.19
21	1.075	172.5	1.5	3.6	52	2.625	2.88	0.78	0.17
22	1.125	144.0	1.3	1.4	53	2.675	3.22	0.94	0.33
23	1.175	124.0	1.2	3.7	54	2.725	3.52	0.95	0.51
24	1.225	108.7	1.1	3.3	55	2.775	3.49	0.89	0.57
25	1.275	94.0	1.0	1.0	56	2.825	3.29	1.60	0.62
26	1.325	82.0	0.9	2.4	57	2.875	3.09	1.86	0.80
27	1.375	72.6	0.9	1.3	58	2.925	2.65	1.81	0.65
28	1.425	64.5	0.8	1.8	59	2.975	1.79	0.79	0.43
29	1.475	56.1	0.8	1.8	60	3.025	0.88	0.39	0.22
30	1.525	48.0	0.7	1.4	61	3.075	0.43	0.20	0.12
31	1.575	41.1	0.6	0.7	62	3.125	0.09	0.04	0.03

Here, $k(s) = \frac{1}{2}\sqrt{s}\beta_{-}(s)$ is the pion momentum in the $\pi^{-}\pi^0$ rest frame. The functions $f(s)$ and $h(s)$ are defined as

$$f(s) = \Gamma_i \frac{M_i^2}{k^3(M_i^2)} \left[k^2(s)(h(s) - h(M_i^2)) + (M_i^2 - s)k^2(M_i^2) \frac{dh}{ds} \Big|_{s=M_i^2} \right], \quad (14)$$

$$h(s) = \frac{2}{\pi} \frac{k(s)}{\sqrt{s}} \ln \left(\frac{\sqrt{s} + 2k(s)}{2m_{\pi}} \right), \quad (15)$$

with $dh/ds|_{M_i^2} = h(M_i^2)[(8k^2(M_i^2))^{-1} - (2M_i^2)^{-1}] + (2\pi M_i^2)^{-1}$ and

$$d = \frac{3}{\pi} \frac{m_{\pi}^2}{k^2(M_i^2)} \ln \left(\frac{M_i + 2k(M_i^2)}{2m_{\pi}} \right) + \frac{M_i}{2\pi k(M_i^2)} - \frac{m_{\pi}^2 M_i}{\pi k^3(M_i^2)}. \quad (16)$$

Note that the function d is chosen so that the BW^{GS} function is unity at $s = 0$ [50].

Since the unfolded mass spectrum has bin-by-bin correlations, the off-diagonal components of the covariance matrix X are included in the χ^2 evaluation:

$$\chi^2 = \sum_{i,j} (y_i - f(s_i; \alpha))(X^{-1})_{ij}(y_j - f(s_j; \alpha)), \quad (17)$$

where y_i is the measured value at the i th bin, $f(s; \alpha)$ is the value of the function for parameters α , and $(X^{-1})_{ij}$ is the inverse of the covariance matrix.

TABLE VI. The form factor squared $|F_{\pi}^{-}(s)|^2$ as a function of the invariant mass squared s . The results are obtained by inserting the measured value $(1/N_{\pi\pi})(dN_{\pi\pi}/ds)$ into Eq. (7). Note that the short-distance radiative correction is already applied, where the value of S_{EW} is taken to be 1.0235 ± 0.0003 (see the discussion in the appendix).

Bin No.	$M_{\pi\pi}^2$ (GeV/c ²) ²	$ F_{\pi}^{-} ^2$	Statistical	Systematic	Bin No.	$M_{\pi\pi}^2$ (GeV/c ²) ²	$ F_{\pi}^{-} ^2$	Statistical	Systematic
1	0.088	1.434	0.549	0.377	32	1.625	0.711	0.012	0.012
2	0.125	1.707	0.091	0.358	33	1.675	0.636	0.012	0.012
3	0.175	2.362	0.053	0.337	34	1.725	0.576	0.011	0.010
4	0.225	3.211	0.045	0.248	35	1.775	0.511	0.012	0.008
5	0.275	4.260	0.042	0.127	36	1.825	0.439	0.011	0.007
6	0.325	5.622	0.046	0.058	37	1.875	0.382	0.011	0.006
7	0.375	8.492	0.057	0.067	38	1.925	0.318	0.011	0.005
8	0.425	13.392	0.072	0.193	39	1.975	0.255	0.010	0.004
9	0.475	21.894	0.093	0.275	40	2.025	0.211	0.009	0.004
10	0.525	33.384	0.113	0.284	41	2.075	0.162	0.010	0.003
11	0.575	40.996	0.122	0.269	42	2.125	0.117	0.009	0.003
12	0.625	34.503	0.112	0.243	43	2.175	0.085	0.008	0.003
13	0.675	23.936	0.093	0.214	44	2.225	0.060	0.008	0.002
14	0.725	15.324	0.074	0.106	45	2.275	0.040	0.006	0.002
15	0.775	10.525	0.058	0.137	46	2.325	0.027	0.010	0.003
16	0.825	7.637	0.047	0.111	47	2.375	0.019	0.018	0.009
17	0.875	5.693	0.038	0.046	48	2.425	0.017	0.024	0.008
18	0.925	4.350	0.032	0.061	49	2.475	0.017	0.019	0.006
19	0.975	3.435	0.028	0.050	50	2.525	0.019	0.011	0.003
20	1.025	2.851	0.024	0.037	51	2.575	0.024	0.007	0.002
21	1.075	2.439	0.022	0.052	52	2.625	0.036	0.010	0.002
22	1.125	2.087	0.019	0.020	53	2.675	0.050	0.014	0.005
23	1.175	1.847	0.018	0.056	54	2.725	0.066	0.018	0.010
24	1.225	1.667	0.017	0.050	55	2.775	0.083	0.021	0.013
25	1.275	1.486	0.016	0.015	56	2.825	0.102	0.050	0.019
26	1.325	1.339	0.015	0.039	57	2.875	0.132	0.079	0.034
27	1.375	1.229	0.015	0.022	58	2.925	0.165	0.113	0.040
28	1.425	1.132	0.014	0.031	59	2.975	0.178	0.079	0.043
29	1.475	1.025	0.014	0.032	60	3.025	0.165	0.073	0.041
30	1.525	0.913	0.013	0.027	61	3.075	0.203	0.098	0.056
31	1.575	0.818	0.013	0.014	62	3.125	0.287	0.138	0.081

There are 10 parameters in this formula: the masses (M_i) and widths (Γ_i) for the ρ , ρ' , and ρ'' resonances, their relative amplitudes $|\beta|$, $|\gamma|$, and phases ϕ_{β} and ϕ_{γ} . In addition, as an overall normalization factor, we introduce $|F_{\pi}(0)|^2$ as an additional parameter. In the BW form, this value should be unity. However, in order to take into account a possible deviation from the form of the fitting function, two kinds of fits, in which this parameter is either fixed or floated, are carried out. The other 10 parameters are floated in the fit.

The results of the fit are shown as the solid line in Fig. 9 for the $M_{\pi\pi}^2$ distribution as well as in Figs. 10, 11(a), and 11(b), where the results are compared directly to the weak form factor squared $|F_{\pi}^{-}(s)|^2$, derived bin-by-bin from Eq. (7). In Eq. (7), we use the world average value (including our measurement) for the branching fraction $\mathcal{B}_{\pi\pi} = (25.24 \pm 0.10)\%$ and for the CKM matrix element $V_{ud} = 0.97377 \pm 0.00027$ [31]. For the short-distance radiative correction $S_{EW} = S_{EW}^{\pi\pi}/S_{EW}^e$, we take the value 1.0235 ± 0.0003 , to be consistent with the isospin breaking correc-

tion discussed in Refs. [5,30] (see the appendix for more details).

The fitted results are summarized in Table VII for the cases when $|F_{\pi}(0)|^2$ is fixed to unity (the second column) and is allowed to float (the third column). In the table, the first error is statistical and the second is systematic. The value of the χ^2 per degree of freedom (NDF) is 80/52 for the fixed and 65/51 for the floated cases. It is found that $|F_{\pi}(0)|^2$ is close to unity ($|F_{\pi}(0)|^2 = 1.02 \pm 0.01 \pm 0.04$) even when it is allowed to float. It should be noted that the data can be fitted using BW resonances only, without any additional background terms. The fit quality for the fixed case is slightly worse than that for the floated one. The curves shown in Figs. 9, 10, 11(a), and 11(b) correspond to the case when the parameter $|F_{\pi}(0)|^2$ is floated, but the differences between the floated and the fixed cases are small.

The significance of the $\rho''(1700)$ signal is given in the last row of Table VII. The significance is determined from the change in the χ^2 when the signal and its associated

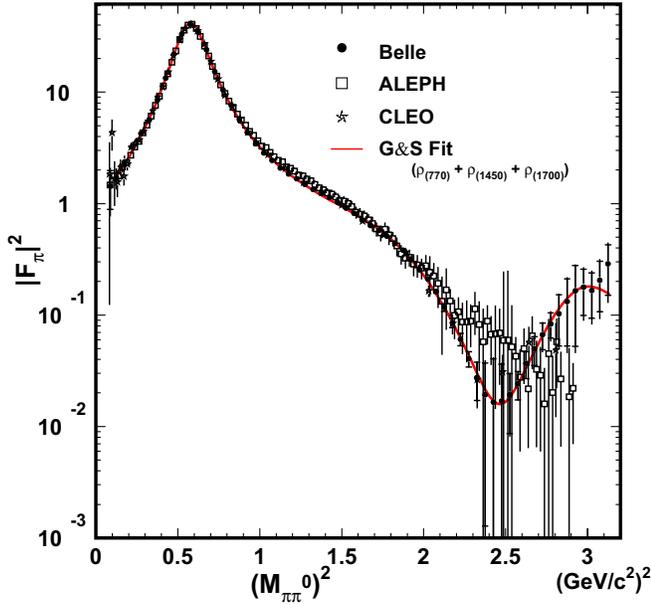


FIG. 10 (color online). Pion form factor for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$. The solid circles are the Belle result while the squares and stars show the result of ALEPH [19] and CLEO [20], respectively. The error bars for the Belle data include both statistical and systematic errors added in quadrature. The solid curve is the result of a fit to the Gounaris-Sakurai model with the $\rho(770)$, $\rho'(1450)$, and $\rho''(1700)$ resonances, where all parameters are floated.

TABLE VII. Results of fitting the $M_{\pi\pi^0}^2$ distribution for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ to the Gounaris-Sakurai model with the $\rho(770)$, $\rho'(1450)$, and $\rho''(1700)$ resonances. The results are shown for two cases, fixed $|F_\pi(0)|^2 = 1$ (the second column) and all parameters are allowed to float (the third column). The first error is statistical and the second one is systematic. The systematic errors include the uncertainty in the backgrounds, unfolding as well as the uncertainty of the photon energy scale. The last row gives the significance of the $\rho''(1700)$ signal.

Parameter	Fit result (fixed $ F(0) ^2$)	Fit result (all free)
M_ρ , MeV/ c^2	$774.6 \pm 0.2 \pm 0.5$	$774.9 \pm 0.3 \pm 0.5$
Γ_ρ , MeV	$148.1 \pm 0.4 \pm 1.7$	$148.6 \pm 0.5 \pm 1.7$
$M_{\rho'}$, MeV/ c^2	$1446 \pm 7 \pm 28$	$1428 \pm 15 \pm 26$
$\Gamma_{\rho'}$, MeV	$434 \pm 16 \pm 60$	$413 \pm 12 \pm 57$
$ \beta $	$0.15 \pm 0.05^{+0.15}_{-0.04}$	$0.13 \pm 0.01^{+0.16}_{-0.04}$
ϕ_β , degree	$202 \pm 4^{+41}_{-8}$	$197 \pm 9^{+50}_{-5}$
$M_{\rho''}$, MeV/ c^2	$1728 \pm 17 \pm 89$	$1694 \pm 41 \pm 89$
$\Gamma_{\rho''}$, MeV	$164 \pm 21^{+89}_{-26}$	$135 \pm 36^{+50}_{-26}$
$ \gamma $	$0.037 \pm 0.006^{+0.065}_{-0.009}$	$0.028 \pm 0.020^{+0.059}_{-0.009}$
ϕ_γ , degree	$24 \pm 9^{+118}_{-28}$	$-3 \pm 13^{+136}_{-29}$
$ F(0) ^2$	[1.0]	$1.02 \pm 0.01 \pm 0.04$
χ^2/NDF	80/52	65/51
$\rho''(1700)$ signif., σ	6.5	7.0

degree of freedom are removed from the fit. If the $\rho''(1700)$ signal is excluded from the fit, the χ^2 for the fit increases by 55 (60) units, in the case that $|F(0)|^2$ is fixed to unity (allowed to float). This increase in the χ^2 for the fit, with the joint estimation of four removed parameters (mass,

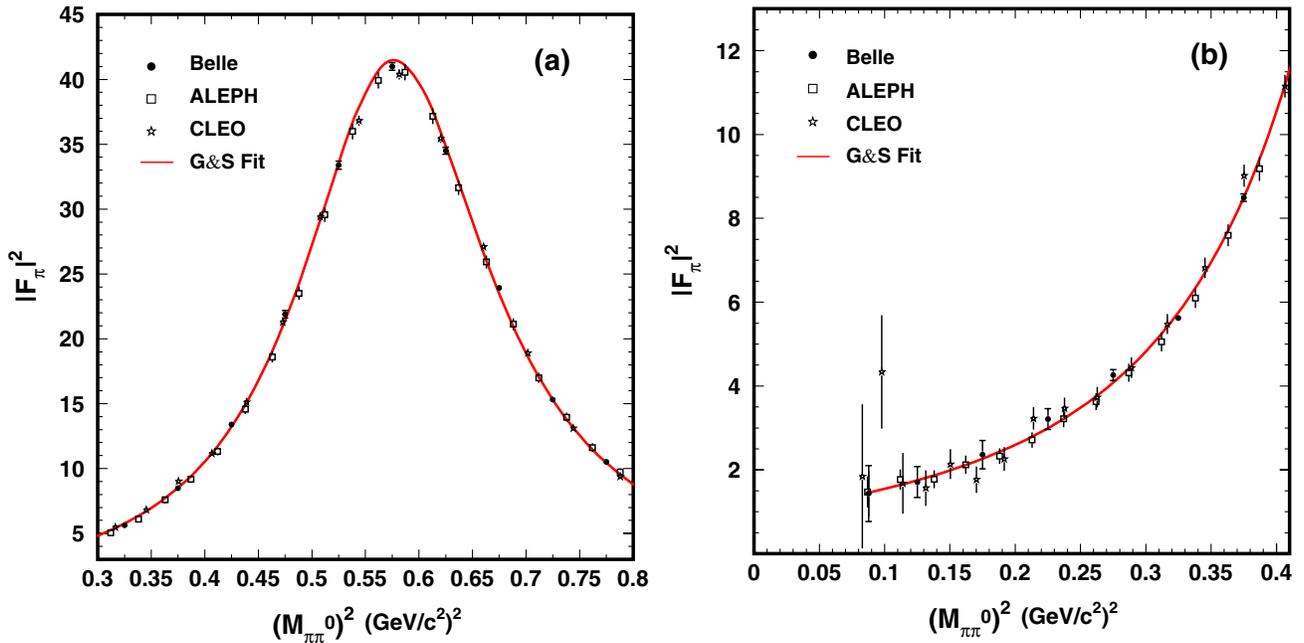


FIG. 11 (color online). (a) Pion form factor for $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ in the $\rho(770)$ mass region and (b) in the threshold region. The convention of the plots is the same as in Fig. 10. The error bars for the Belle data include both statistical and systematic errors added in quadrature. The solid curve is the result of a fit to the Gounaris-Sakurai model, where all parameters are floated. See the text for details.

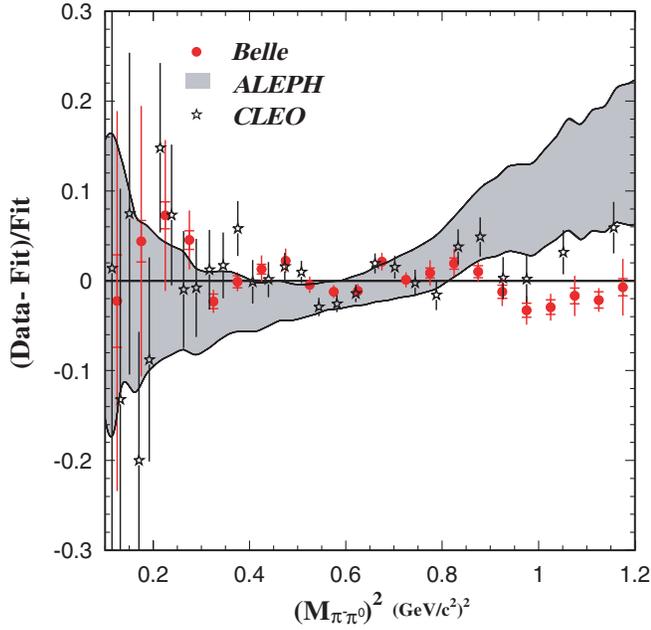


FIG. 12 (color online). Comparison of the pion form factor squared $|F_\pi(s)|^2$ measured with the Belle detector to results from the CLEO [20] and ALEPH [19] experiments in the $\rho(770)$ and $\rho'(1450)$ mass region. The difference from the fit of the Belle $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ data divided by the fit value is plotted. The solid circles show the Belle result, the open stars show the CLEO result [20], and the hatched band shows the ALEPH result [19]. Both Belle and ALEPH results include the systematic errors. The inner error bars in the Belle data indicate the statistical errors.

The value of $a_\mu^{\text{had,LO}}$ is related to the e^+e^- annihilation cross section via the dispersion integral

$$a_\mu^{\text{had,LO}} = \frac{1}{4\pi^3} \int_{4m_\pi^2}^{\infty} \sigma_{\text{had}}^0(s) \left(\frac{m_\mu^2}{3s} \right) \hat{K}(s) ds, \quad (19)$$

where σ_{had}^0 is the total cross section for $e^+e^- \rightarrow \text{hadrons}$ at the center-of-mass energy \sqrt{s} . The superscript in σ_{had}^0 denotes the “bare” hadronic cross section, which is defined as the measured cross section, corrected for QED radiative corrections such as initial state radiation, electron-vertex correction, and the vacuum polarization in the photon propagator. However, the final state radiation (FSR) photons coming from the process $\sigma^0(e^+e^- \rightarrow V\gamma \rightarrow \pi^+\pi^-\gamma)$ should be included in the σ_{had}^0 term (See the detailed discussion on page 12 of Ref. [5]). The kernel function $\hat{K}(s)$ is given by

$$\hat{K}(s) = \frac{3s}{m_\mu^2} \left\{ x^2 \left(1 - \frac{x^2}{2} \right) + (1+x)^2 \left(1 + \frac{1}{x^2} \right) \left(\ln(1+x) - x + \frac{x^2}{2} \right) + \left(\frac{1+x}{1-x} \right) x^2 \ln x \right\},$$

with $x = (1 - \beta_\mu)/(1 + \beta_\mu)$ and $\beta_\mu \equiv \sqrt{1 - 4m_\mu^2/s}$. $\hat{K}(s)$ is a smooth function increasing from 0.63 at the

threshold $s = 4m_\pi^2$ to unity at $s = \infty$. Because of the $1/s$ dependence of $\sigma_{\text{had}}^0(s)$ and the additional $1/s$ factor in the integral in Eq. (19), low-mass hadronic final states dominate the contribution to $a_\mu^{\text{had,LO}}$; in fact about 70% of $a_\mu^{\text{had,LO}}$ is due to the two-pion final state with $4m_\pi^2 \leq s \leq 0.8$ (GeV/c²)². Consequently, the 2π mass spectrum from τ data is useful for obtaining predictions for $a_\mu^{\text{had,LO}}$ using CVC.

B. Results

Details of our determination of $a_\mu^{\pi\pi}$, the 2π contribution to $a_\mu^{\text{had,LO}}$, are given in the appendix, where the basic formulas, the corrections applied for the isospin-violating effects, and discussions on the error estimation are presented.

Our result on $a_\mu^{\pi\pi}$ over the mass range $\sqrt{s} = 2m_\pi - 1.8$ GeV/c² is

$$a_\mu^{\pi\pi}[2m_\pi, 1.8 \text{ GeV}/c^2] = (523.5 \pm 1.5(\text{exp.}) \pm 2.6(\text{Br.}) \pm 2.5(\text{isospin})) \times 10^{-10},$$

where the first error is due to the experimental uncertainties, i.e. the statistical error (1.1×10^{-10}) and experimental systematic error (1.0×10^{-10}) added in quadrature. The second error comes from the uncertainties in the branching fractions. The third one is the error on the isospin-violating corrections. These sources of error are discussed in the appendix.

This result can be compared to those from previous ALEPH, CLEO, and OPAL τ data. The combined result given in Ref. [5] is

$$a_\mu^{\pi\pi}[2m_\pi, 1.80 \text{ GeV}/c^2] = (520.1 \pm 2.4(\text{exp.}) \pm 2.7(\text{Br.}) \pm 2.5(\text{isospin})) \times 10^{-10} \quad (\tau).$$

In terms of the experimental error (i.e. the first uncertainty), our result improves the previous combined result by 40%. A detailed comparison from our results and those of ALEPH [19] and CLEO [20] is given in Table IX for the $a_\mu^{\pi\pi}$ contribution. As seen in Table IX, the contribution from the mass-squared region 0.8 (GeV/c²)² $< s < 1.25$ (GeV/c²)², where one observes a deviation between the Belle and ALEPH data in Fig. 9, is only 4.6% of the total 2π contribution. In this region, the contribution from the ALEPH measurements is higher than that from Belle but it is compensated by the opposite tendency in the region $s < 0.50$ (GeV/c²)². Moreover, since the contribution from CLEO is between Belle and ALEPH, the difference between our result and the combined previous τ result becomes smaller. Consequently, our result agrees well with that of the combined result given from the previous τ data within $< 1\sigma$ of the experimental error.

On the other hand, the value of $a_\mu^{\pi\pi}$ in the same \sqrt{s} region evaluated from the e^+e^- cross section measurements is [27]

TABLE IX. Summary of the $a_{\mu}^{\pi\pi}$ contribution from τ data from Belle, ALEPH [19], and CLEO [20] experiments. The errors are only shown for the Belle data. In this table, the isospin breaking correction is not applied except for the short-distance radiative correction given by the term S_{EW} . The errors in the Belle data are only statistical. The errors for other experiments are similar or slightly worse than the Belle value in the same mass range.

$M_{\pi\pi}^2$ range ($(\text{GeV}/c^2)^2$)	$a_{\mu}^{\pi\pi}(10^{-10})$			Integrated $a_{\mu}^{\pi\pi}(10^{-10})$		
	Belle	ALEPH	CLEO ^a	Belle	ALEPH	CLEO ^a
0.075–0.200	39.55 ± 0.97	38.20	43.81	39.55 ± 0.97	38.20	43.81
0.200–0.350	70.62 ± 0.46	66.84	75.71	110.3 ± 1.07	105.0	119.5
0.350–0.500	123.25 ± 0.28	119.10	106.10	233.5 ± 1.11	224.1	225.6
0.500–0.650	196.78 ± 0.23	194.00	197.80	430.3 ± 1.13	418.2	423.3
0.650–0.800	62.35 ± 0.10	62.35	73.03	492.6 ± 1.14	480.5	496.3
0.800–0.950	15.64 ± 0.04	16.40	14.31	508.3 ± 1.14	496.9	510.6
0.950–1.100	5.74 ± 0.02	6.50	6.01	514.0 ± 1.14	503.4	516.6
1.100–1.250	2.86 ± 0.01	3.27	2.32	516.9 ± 1.14	506.7	519.0
1.250–1.400	1.65 ± 0.01	1.89	2.65	518.5 ± 1.14	508.6	521.6
1.400–1.550	1.03 ± 0.01	1.13	0.77	519.5 ± 1.14	509.7	522.4
1.550–1.700	0.60 ± 0.00	0.66	0.49	520.2 ± 1.14	510.3	522.9
1.700–1.850	0.36 ± 0.00	0.37	0.33	520.5 ± 1.14	510.7	523.2
1.850–2.000	0.19 ± 0.00	0.19	0.21	520.7 ± 1.14	510.9	523.4
2.000–2.600	0.13 ± 0.00	0.21	0.15	521.8 ± 1.14	511.1	523.6
2.600–3.200	0.13 ± 0.00	0.21	0.04	522.0 ± 1.14	511.1	523.6

^aFor the CLEO data, the boundary of the mass range is slightly different from the one shown in the first column, since \sqrt{s} bins are used.

$$a_{\mu}^{\pi\pi}[2m_{\pi}, 1.80 \text{ GeV}/c^2] = (504.6 \pm 3.1(\text{exp.}) \pm 0.9(\text{rad.})) \times 10^{-10}$$

(e^+e^- : CMD2, SND),

where the first error includes both statistical and experimental systematic errors added in quadrature. The second error is due to radiative corrections.

Our τ result is noticeably higher than the e^+e^- result. This confirms the longstanding difference between the spectral functions of the 2π systems produced in τ -decay and e^+e^- annihilation [52].

In summary, we have studied the decay $\tau^- \rightarrow \pi^- \pi^0 \nu_{\tau}$ using a high-statistics data sample taken with the Belle detector at the KEKB e^+e^- collider. The branching fraction is measured with 1.5% accuracy. From Table III we can calculate the accuracy of the previous experiments: CLEO 1.7%, L3 2.4%, OPAL 1.3%, ALEPH 0.5%, DELPHI 0.9%. These comparison shows that the accuracy of the Belle result is better than CLEO and L3, similar to OPAL and worse than ALEPH and DELPHI. The result is in good agreement with previous measurements. In the unfolded $\pi^- \pi^0$ mass spectrum, in addition to the $\rho(770)$ and $\rho'(1450)$ mesons, the production of the $\rho''(1700)$ in τ^- decays has been unambiguously demonstrated and its parameters determined. The unfolded spectrum is used to evaluate the 2π contribution to the muon anomalous magnetic moment $a_{\mu}^{\pi\pi}$ in the region $\sqrt{s} = 2m_{\pi} - 1.80 \text{ GeV}/c^2$. Our results agree well with the previous τ based results but are higher than those from e^+e^- annihilation.

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APPENDIX: DETERMINATION OF $a_{\mu}^{\pi\pi}$ FROM BELLE DATA

In this appendix, we list the details of our determination of $a_{\mu}^{\pi\pi}$ over the range $\sqrt{s} = 2m_{\pi} - 1.8 \text{ GeV}/c^2$. We discuss the basic formula, the integration procedure, the corrections for the isospin-violating effects, and the evaluation of the experimental errors.

1. Basic formula

In the isospin symmetry limit, CVC relates the quantities in $\tau^- \rightarrow \pi^- \pi^0 \nu_{\tau}$ decay to the cross section $e^+ e^- \rightarrow \pi^+ \pi^-$ ($\sigma_{\pi\pi}^0$) through the relation [30]

$$\begin{aligned} \sigma_{\pi\pi}^0|_{\text{CVC}} &= \frac{1}{\mathcal{N}(s)\Gamma_e^0} \frac{d\Gamma(\tau^- \rightarrow \pi^- \pi^0 \nu_{\tau})}{ds} \\ &= \frac{1}{\mathcal{N}(s)} \times \left(\frac{\mathcal{B}_{\pi\pi}}{B_e}\right) \times \left(\frac{1}{N_{\pi\pi}} \frac{dN_{\pi\pi}}{ds}\right), \end{aligned} \quad (\text{A1})$$

where $\mathcal{N}(s)$ is given by

$$\mathcal{N}(s) = \frac{3|V_{ud}|^2}{2\pi\alpha_0^2 m_{\tau}^2} s \left(1 - \frac{s}{m_{\tau}^2}\right)^2 \left(1 + \frac{2s}{m_{\tau}^2}\right). \quad (\text{A2})$$

A more precise link between hadronic spectral functions from τ decays and the $e^+ e^-$ hadronic cross section requires a calculation of radiative corrections as well as the inclusion of the isospin breaking effects (both of kinematic and dynamic origin). These effects have recently been discussed by Cirigliano *et al.* [30] and by Flores-Tlalpa *et al.* [53]. According to them, the formula in Eq. (A1) is modified to

$$\sigma_{\pi^+ \pi^-}^0|_{\text{CVC}} = \frac{1}{\mathcal{N}(s)} \times \left(\frac{\mathcal{B}_{\pi\pi}}{B_e}\right) \times \left(\frac{1}{N_{\pi\pi}} \frac{dN_{\pi\pi}}{ds}\right) \left(\frac{R_{\text{IB}}(s)}{S_{\text{EW}}}\right), \quad (\text{A3})$$

with

$$R_{\text{IB}}(s) = \frac{1}{G_{\text{EM}}(s)} \frac{\beta_0^3(s)}{\beta_-(s)} \left| \frac{F_0(s)}{F_-(s)} \right|^2, \quad (\text{A4})$$

where S_{EW} is the dominant short-distance electroweak correction and $R_{\text{IB}}(s)$ takes care of the isospin-violating corrections. $R_{\text{IB}}(s)$ includes the long-distance QED correction $G_{\text{EM}}(s)$, the phase space correction factor $\beta_0^3(s)/\beta_-(s)$, and the ratio of the pion form factors $|F_0(s)/F_-(s)|^2$.

2. Integration procedure

Using the measured distribution $(1/N_{\pi\pi})$, $(dN_{\pi\pi}/ds)$, the moment $a_{\mu}^{\pi\pi}$ can be obtained by inserting the bare cross section Eq. (A3) to Eq. (19) and integrating over s . The integration in Eq. (19) is carried out numerically by taking the sum of the integrand evaluated at the center of each bin. The statistical error on $a_{\mu}^{\pi\pi}$ is calculated including the off-diagonal elements of the covariance error matrix X_{ij} :

$$\Delta a_{\mu}^{\pi\pi} = \sum_{i,j} \left(\frac{\partial a_{\mu}}{\partial \alpha_i}\right) X_{ij} \left(\frac{\partial a_{\mu}}{\partial \alpha_j}\right). \quad (\text{A5})$$

There are several external parameters in these equations; the values used for them are listed in Table X. For m_{τ} , V_{ud} , and B_e , PDG [31] values are used. For the $\pi^- \pi^0$ branching fraction, our measurement is consistent with the world average given in Ref. [31]. Including our result and the recent ALEPH $B_{\pi\pi^0}$ measurement [19], the new world average is

$$B_{\pi\pi^0} = (25.42 \pm 0.10)\%. \quad (\text{A6})$$

We use this new world average for the evaluation of $a_{\mu}^{\pi\pi}$.

The errors on $a_{\mu}^{\pi\pi}$ arising from external parameters are summarized in Table X; the total systematic error from these sources is $\pm 2.7 \times 10^{-10}$ (dominated by $\Delta B_{\pi\pi^0}$).

3. Experimental systematic uncertainty

The systematic errors on $a_{\mu}^{\pi\pi}$ arising from internal sources (specific to this measurement) are listed in Table XI and discussed below. There are two sources of background in the $\pi^- \pi^0$ sample: (i) feed-down from $\tau^- \rightarrow h^- (n\pi^0)\nu_{\tau}$, $\tau^- \rightarrow K^- \pi^0 \nu_{\tau}$, $\tau^- \rightarrow \omega \pi^- \nu_{\tau}$ ($\omega \rightarrow \pi^0 \gamma$) and (ii) non- τ background. In the first case, MC statistics and the uncertainty on the branching fraction are used to estimate the error. In the second case, the uncertainty on the background as estimated from the control samples is assigned as an error. As mentioned earlier, the fake- π^0 background is subtracted using sideband events and the uncertainty is determined by varying the signal and sideband regions.

It is found that the shape of the mass spectrum is insensitive to uncertainties in the π^0 efficiency, as it is only at a few % level. The uncertainty of the integration procedure comes from the binning effects. Adding all

TABLE X. Values of the external parameters and systematic errors on $a_{\mu}^{\pi\pi}$ arising from these sources.

Source	Value	Relative error (%)	$\Delta a_{\mu}^{\pi\pi}$ (10^{-10})	Reference
V_{ud}	0.97377 ± 0.00027	0.027	± 0.26	[31]
B_e	$(17.84 \pm 0.05)\%$	0.28	± 1.45	[31]
$B_{\pi\pi^0}$	$(25.42 \pm 0.10)\%$	0.41	± 2.13	
Total external			± 2.6	

TABLE XI. Systematic errors on $a_{\mu}^{\pi\pi}$ arising from internal sources (specific to this measurement).

Source	$\Delta a_{\mu}^{\pi\pi} \times 10^{10}$
Background:	
Non- τ ($e^+e^- \rightarrow \bar{q}q$)	± 0.11
Feed-down $h(n\pi^0)\nu$	± 0.09
Feed-down $K^-\pi^0\nu$	± 0.15
Energy scale	± 0.10
π^0/γ selection	± 0.24
γ veto	± 0.93
Efficiency:	
π^0/γ	± 0.35
Charged track	< 0.10
Integration procedure	< 0.10
Total internal	± 1.04

individual errors in quadrature we obtain a total error on $a_{\mu}^{\pi\pi}$ arising from internal sources of $\pm 1.0 \times 10^{-10}$.

To check the stability of $a_{\mu}^{\pi\pi}$, we perform the following tests:

- (1) The sample is divided into subsamples based on the tag-side topology, i.e. one electron, one-prong, or three-prong. The values of $a_{\mu}^{\pi\pi}$ obtained from these subsamples are consistent within the statistical errors.
- (2) The sample is divided into subsamples based on the running period. Again, the values of $a_{\mu}^{\pi\pi}$ obtained are consistent within the statistical errors.
- (3) The sample might be sensitive to the requirement on the overlap region between the charged track and γ clusters. To estimate this sensitivity, we select events with a tighter isolation requirement on γ 's and on the track extrapolation: 30 cm instead of 20 cm.

The resulting variation in $a_{\mu}^{\pi\pi}$ is small and is included as an additional systematic error.

4. Isospin-violating corrections for $a_{\mu}^{\pi\pi}$

Three identifiable sources of isospin breaking corrections are the mass difference of the charged and neutral

TABLE XII. Sources of the isospin violation between the e^+e^- and τ spectral function in the 2π channel and the corrections to $a_{\mu}^{\pi\pi}$. The correction is based on the procedure given in Refs. [30,53].

Source of isospin violation	Correction to $a_{\mu}^{\pi\pi}(10^{-10})$	Uncertainty on $a_{\mu}^{\pi\pi}(10^{-10})$	References
Short-distance rad. cor. (S_{EW})	-12.0	± 0.2	[5,54–56]
Long-distance rad. cor. ($G_{EM}(s)$)	-1.0	\dots	[30,53]
$m_{\pi^-} \neq m_{\pi^0}$ (β^3 in phase space)	-7.0	\dots	
$\rho - \omega$ interference	+3.5	± 0.6	
$m_{\pi^-} \neq m_{\pi^0}$ (β^3 in the decay width)	+4.2	\dots	
Electromagnetic decay modes	-1.4	± 1.4	
$m_{\rho^0} \neq m_{\rho^-}$	\dots	± 2.0	
Total	-13.7	± 2.5	

pions, $\rho - \omega$ interference effects, and the radiative corrections, which are included in the factor $R_{IB}(s)/S_{EW}$ in Eq. (A1). The size of the isospin-violating corrections from these sources and the possible uncertainties from other sources are summarized in Table XII.

- (i) The dominant contribution from electroweak radiative corrections comes from the short-distance corrections. In the leading logarithmic order, the short-distance radiative corrections to the decays $\tau^- \rightarrow (d\bar{u})\nu_{\tau}$ are enhanced by the factor [54,55],

$$S(m_{\tau}, m_Z) = 1 + \frac{3\alpha(m_{\tau})}{4\pi} (1 + 2\bar{Q}) \ln \frac{m_Z^2}{m_{\tau}^2} = 1.01878, \quad (\text{A7})$$

where m_Z is the Z boson mass, and $\alpha(m_{\tau}) = 1/133.50(2)$ is the QED coupling at the τ lepton mass. \bar{Q} is the average quark-doublet charge. Therefore, $\bar{Q} = \frac{1}{2}(\frac{2}{3} - \frac{1}{3}) = \frac{1}{6}$ for the semileptonic decays, $\tau \rightarrow \nu_{\tau}\bar{u}d$. Since $\bar{Q} = -\frac{1}{2}$ for leptons, there are no leading logarithmic corrections for leptonic decays.

We can go further and sum up all short-distance logarithms of the $\alpha^n \ln^n m_Z$ via the renormalization group. This procedure replaces Eq. (A7) by

$$S(m_{\tau}, m_Z) = \left[\frac{\alpha(m_b)}{\alpha(m_{\tau})} \right]^{9/19} \left[\frac{\alpha(m_W)}{\alpha(m_{m_b})} \right]^{9/20} \times \left[\frac{\alpha(m_Z)}{\alpha(m_W)} \right]^{36/17} \times \left[\frac{\alpha_s(m_b)}{\alpha_s(m_{\tau})} \right]^{(3/25)(\alpha(m_{\tau})/\pi)} \times \left[\frac{\alpha_s(m_Z)}{\alpha_s(m_b)} \right]^{(3/23)(\alpha(m_b)/\pi)} = 1.01907, \quad (\text{A8})$$

where the last two terms are the short-distance QCD corrections [56].

Taking into account the subleading correction for the leptonic decay, the short-distance electroweak correction S_{EW} is given by

$$S_{EW} = S(m_\tau, m_Z) \frac{1}{S_{EW}^{e,\text{sub}}} = 1.0235 \pm 0.0003, \quad (\text{A9})$$

where $S_{EW}^{e,\text{sub}} = (1 + \frac{\alpha(m_\tau)}{2\pi} (\frac{25}{4} - \pi^2)) = 0.9957$. The difference between the resummed value (1.01907) and the lowest-order estimate (1.01878) is taken as the error on S_{EW} and the resummed S_{EW} value is used throughout this paper. The shift of $a_\mu^{\pi\pi}$ from this correction is $\Delta a_\mu^{\pi\pi} = (12.0 \pm 0.2) \times 10^{-10}$. Note that this correction is already applied in Eq. (7) and the pion form factor given in Table VI.

- (ii) The long-distance QED radiative correction $G_{EM}(s)$ was computed in the framework of chiral perturbation theory by Cirigliano *et al.* in Ref. [30]. Recently, it was reevaluated based on a meson dominance model by Flores-Tlalpa *et al.* in Ref. [53]. It is found that the predictions of both models coincide if the contribution of the $\omega(782)$ intermediate state $\tau \rightarrow \omega \pi^- \nu_\tau (\omega \rightarrow \pi^0 \gamma)$, is excluded in the latter model calculation. Since in our data this intermediate $\omega(782)$ contribution is already subtracted at the analysis level, we use the s -dependent correction factor provided by Flores-Tlalpa, which does not include the ω contribution. This correction produces a shift of $\Delta a_\mu^{\pi\pi} = -1.0 \times 10^{-10}$,
- (iii) The π^- and π^0 mass difference in the ratio of the phase space β_0^3/β_-^3 results in a shift of $\Delta a_\mu^{\pi\pi} = -7.0 \times 10^{-10}$.
- (iv) The π^- and π^0 mass difference also affects the β^3 factor in the energy-dependent decay width (Eq. (13)), which provides a positive shift $\Delta a_\mu^{\pi\pi} = +4.2 \times 10^{-10}$.
- (v) The effect of the $\rho - \omega$ interference is estimated using the interference amplitude parametrized in the following form [30]:

$$F_\pi^0(s)_{\rho-\omega} = -\frac{\theta_{\rho\omega}}{3m_\rho^2} \frac{s}{m_\omega^2 - s - im_\omega\Gamma_\omega}. \quad (\text{A10})$$

For the numerical evaluation, we take $\theta_{\rho\omega} =$

$(-3.3 \pm 0.7) \times 10^{-3} \text{ GeV}^2$, $m_\omega = 0.783 \text{ GeV}$ and $\Gamma_\omega = 0.00844 \text{ GeV}$. The net effect is $\Delta a_\mu^{\pi\pi} = (+3.5 \pm 0.6) \times 10^{-10}$.

- (vi) The largest source of the uncertainty for the isospin-violating effects is from the ρ^- and ρ^0 mass difference. The mass difference is consistent with zero within about 1 MeV, which gives an uncertainty on $a_\mu^{\pi\pi}$ of $\pm 2.0 \times 10^{-10}$.
- (vii) Finally, electromagnetic decay of the ρ meson is the source of the isospin violation. The decay $\pi\pi\gamma$ deserves particular attention. The shift and its uncertainty are estimated to be $\Delta a_\mu^{\text{had}} = (-1.4 \pm 1.4) \times 10^{-10}$ from the width difference $\Gamma(\rho^0 \rightarrow \pi^+ \pi^- \gamma) - \Gamma(\rho^+ \rightarrow \pi^+ \pi^0 \gamma) = (0.45 \pm 0.45) \text{ MeV}$ [30].

Summing all these corrections, the overall isospin-violating correction and its uncertainty are estimated to be $(-13.7 \pm 2.5) \times 10^{-10}$.

These corrections relate the τ spectral function to the “pure” $e^+e^- \rightarrow \pi^+ \pi^-$ with all QED corrections switched off. Since the σ_{had}^0 term in Eq. (19) must include processes with the final state radiation (FSR) photons coming from the process $e^+e^- \rightarrow V\gamma \rightarrow \pi^+ \pi^- \gamma$, we must reintroduce the FSR contributions [5]. After including those contributions ($+4.2 \times 10^{-10}$) [5], the total correction becomes $(-9.5 \pm 2.5) \times 10^{-10}$.

With the external parameters in Table X and corrections discussed above, we obtain $a_\mu^{\pi\pi}$ over the range $\sqrt{s} = 2m_\pi - 1.80 \text{ GeV}/c^2$

$$a_\mu^{\pi\pi}[2m_\pi, 1.80 \text{ GeV}/c^2] = (523.5 \pm 1.1(\text{stat.}) \pm 1.0(\text{sys.}) \pm 2.6(\text{Br.}) \pm 2.5(\text{isospin})) \times 10^{-10},$$

where the first error is statistical, the second is the experimental systematic (Table XI), the third comes from the uncertainties on the branching fractions (Table X), and the fourth is from isospin-violating corrections (Table XII).

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