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Singularities in geodesic surface congruence

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In the stringy cosmology, we investigate singularities in geodesic surface congruences for the timelike and null strings to yield the Raychaudhuri type equations possessing correction terms associated with the novel features owing to the strings. Assuming the stringy strong energy condition, we have a Hawking-Penrose type inequality equation. If the initial expansion is negative so that the congruence is converging, we show that the expansion must pass through the singularity within a proper time. We observe that the stringy strong energy conditions of both the timelike and null string congruences produce the same inequality equation.

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The recent experimental data of the accelerating expansion of the universe has suggested a small, positive vacuum expectation value of the cosmological constant [1]. Vacua of string theory was constructed to yield a positive cosmological constant by introducing supersymmetry breaking quantum corrections [2]. A supersymmetric partition function for a four-dimensional Bogomolny-Prasad-Sommerfield black hole in a Calabi-Yau manifold [3,4] compactification of superstring was conjectured to be related to a second quantized topological string partition function associated with the black hole charges [5]. The string theory [6,7] has given us a better understanding of the universe and the black holes in cosmology and may provide an analytical tool for studying the nature of the Hawking-Penrose big bang singularity theory [8].

Recently, we studied the variation of the surface spanned by strings in a spacetime manifold [9]. Using the Nambu-Goto string action [10,11], we produced the geodesic surface equation and the geodesic surface deviation equation which yields a Jacobi field. Exploiting symplectic cut-and-gluing formulas of the relative Gromov-Witten invariants, one of us obtained a recursive formula for the Hurwitz number of triple ramified geodesic surface coverings of a Riemann surface by a Riemann surface [12].

We consider a fibration π: $M \to N$ over a spacetime four manifold N with a D-dimensional total manifold M associated with the metric g_{ab} and a Calabi-Yau manifold F as a fiber space. In analogy of the relativistic action of a point particle in N, the action for a string is proportional to the area of the surface spanned in the total manifold M by the evolution along the time direction of the string in F. We first introduce a smooth congruence of timelike geodesic surfaces in M. We parametrize the surface generated by the evolution of a timelike string by two world sheet coordinates τ and σ, and then we have the corresponding vector fields $\xi^a = (\partial/\partial\tau)^a$ and $\zeta^a = (\partial/\partial\sigma)^a$. Since we have gauge degrees of freedom, we can choose the orthonormal gauge [6,7,13] $\xi \cdot \zeta = 0$ and $\xi \cdot \xi + \zeta \cdot \zeta = 0$. In the orthonormal gauge, we introduce tensor fields B_{ab} and $\bar{B}_{ab}$ defined as

$$B_{ab} = \nabla_b \xi_a, \qquad \bar{B}_{ab} = \nabla_b \zeta_a, \tag{1}$$

which satisfy the identities $B_{ab}\xi^a = \bar{B}_{ab}\zeta^a = 0$ and $-B_{ab}\xi^b + \bar{B}_{ab}\zeta^b = 0$. Here we have used the geodesic surface equation $-\xi^a\nabla_a\xi^b + \zeta^a\nabla_a\zeta^b = 0$ [9,13]. If the timelike curves of the geodesic surfaces are geodesic, then the string curves are also geodesic.

We introduce the deviation vector field $\eta^a = (\partial/\partial\alpha)^a$ which represents the displacement to an infinitesimally nearby world sheet, and we consider the three-dimensional submanifold spanned by the world sheets. We then may choose τ, σ, and α as coordinates of the submanifold to yield the commutator relations $\pounds_\xi \eta^a = \pounds_\zeta \eta^a = \pounds_\xi \zeta^a = 0$. Using the above relations, we obtain $\xi^a\nabla_a\eta^b - \zeta^a\nabla_a\eta^b = (B^b{}_a - \bar{B}^b{}_a)\eta^a$. Next we define the metrics h_{ab} and $\bar{h}_{ab}$:

$$h_{ab} = g_{ab} + \xi_a\xi_b, \qquad \bar{h}_{ab} = g_{ab} - \zeta_a\zeta_b. \tag{2}$$

Here one notes that h_{ab} and $\bar{h}_{ab}$ are the metrics on the hypersurfaces orthogonal to ξ^a and ζ^a, respectively. We split B_{ab} into three pieces

$$B_{ab} = \frac{1}{D-1}\theta h_{ab} + \sigma_{ab} + \omega_{ab}, \tag{3}$$

where the expansion, shear, and twist [8,14,15] of the stringy congruence along the time direction are defined as $\theta = B^{ab}h_{ab}$, $\sigma_{ab} = B_{(ab)} - \frac{1}{D-1}\theta h_{ab}$ and $\omega_{ab} = B_{[ab]}$.

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Similarly, as B_{ab} in (3) we can decompose $\bar{B}_{ab}$ into three parts; the expansion, shear, and twist of the stringy congruence along the string coordinate σ-direction which are defined as $\bar{\theta} = \bar{B}^{ab}\bar{h}_{ab}$, $\bar{\sigma}_{ab} = \bar{B}_{(ab)} - \frac{1}{D-1}\bar{\theta}\bar{h}_{ab}$ and $\bar{\omega}_{ab} = \bar{B}_{[ab]}$, respectively.

Taking an *ansatz* that the expansion $\bar{\theta}$ is constant along the σ-direction, one obtains a Raychaudhuri type equation, namely, an evolution equation for the expansion

$$\frac{d\theta}{d\tau} = -\frac{1}{D-1}(\theta^2 - \bar{\theta}^2) - \sigma_{ab}\sigma^{ab} + \bar{\sigma}_{ab}\bar{\sigma}^{ab} + \omega_{ab}\omega^{ab} - \bar{\omega}_{ab}\bar{\omega}^{ab} - R_{ab}(\xi^a\xi^b - \zeta^a\zeta^b). \quad (4)$$

We now assume $\omega_{ab} = \bar{\omega}_{ab}$, $\sigma_{ab}\sigma^{ab} \gg \bar{\sigma}_{ab}\bar{\sigma}^{ab}$ and a stringy strong energy condition $R_{ab}(\xi^a\xi^b - \zeta^a\zeta^b) \geq 0$ where

$$R_{ab}(\xi^a\xi^b - \zeta^a\zeta^b) = 8\pi\left(T_{ab}(\xi^a\xi^b - \zeta^a\zeta^b) + \frac{2}{D-2}T\right), \quad (5)$$

and T_{ab} and T are the energy-momentum tensor and its trace, respectively. The Raychaudhuri type equation (4) then has a solution of the form

$$\frac{1}{\theta} \geq \frac{1}{\theta_0} + \frac{1}{D-1}\left(\tau - \int_0^\tau d\tau\left(\frac{\bar{\theta}}{\theta}\right)^2\right), \quad (6)$$

where θ_0 is the initial value of θ at $\tau = 0$. We assume that θ_0 is negative so that the congruence is initially converging as in the point particle case shown below. The inequality (6) implies that θ must pass through the singularity within a proper time

$$\tau \leq \frac{D-1}{|\theta_0|} + \int_0^\tau d\tau\left(\frac{\bar{\theta}}{\theta}\right)^2. \quad (7)$$

For a perfect fluid, the energy-momentum tensor given by $T_{ab} = \rho u_a u_a + P(g_{ab} + u_a u_b)$ where ρ and P are the mass-energy density and pressure of the fluid as measured in its rest frame, respectively, and u^a is the timelike D-velocity in its rest frame [14,16], the stringy strong energy condition (5) yields only one inequality equation

$$\frac{D-4}{D-2}\rho + \frac{D}{D-2}P \geq 0. \quad (8)$$

Here one notes that, if the fiber space F in our fibration π: $M \to N$ is a point, then the total space M is the same as the base spacetime four manifold N. In this case, the geodesic surfaces are geodesic in N, the congruence of timelike geodesic surfaces is a congruence of timelike geodesics, and so $\bar{B}_{ab} = \bar{\theta} = \bar{\sigma}_{ab} = \bar{\omega}_{ab} = 0$. If the congruence is hypersurface orthogonal, then we have $\omega_{ab} = 0$. Suppose that the strong energy condition $R_{ab}\xi^a\xi^b \geq 0$ is satisfied to yield two inequalities [8,14,15]

$$\rho + 3P \geq 0, \qquad \rho + P \geq 0. \quad (9)$$

We then have the differential inequality equation $\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0$, which has a solution in the following form: $\frac{1}{\theta} \geq \frac{1}{\theta_0} + \frac{1}{3}\tau$. If we assume that θ_0 is negative, the expansion θ must go to the negative infinity along that geodesic within a proper time $\tau \leq \frac{3}{|\theta_0|}$. This consequence coincides with the one of Hawking and Penrose [8].

Next, we investigate the congruence of the null strings, where the tangent vector of a null curve is normal to itself. See Refs. [17–19] for the proper definition and propagation of the classical null strings. We consider the evolution of vectors in a $(D-2)$-dimensional subspace of spatial vectors normal to the null tangent vector field $k^a = (\partial/\partial\lambda)^a$, where λ is the affine parameter, and to an auxiliary null vector l^a which points in the opposite spatial direction to k^a, normalized by $l^a k_a = -1$ [15] and is parallel transported, namely, $k^a\nabla_a l^b = 0$. The spatial vectors in the $(D-2)$-dimensional subspace are then orthogonal to both k^a and l^a.

We now introduce the metrics n_{ab} defined below and $\bar{h}_{ab}$ defined in (2),

$$n_{ab} = g_{ab} + k_a l_b + l_a k_b. \quad (10)$$

Similar to the timelike case, we introduce tensor fields $B_{ab} = \nabla_b k_a$ and $\bar{B}_{ab}$ in (1) satisfying the identities $B_{ab}k^a = \bar{B}_{ab}\zeta^a = 0$ and $-B_{ab}k^b + \bar{B}_{ab}\zeta^b = 0$. We also define the deviation vector $\eta^a = (\partial/\partial\alpha)^a$ representing the displacement to an infinitesimally nearby world sheet so that we can choose λ, σ, and α as coordinates of the three-dimensional submanifold spanned by the world sheets. We then have the commutator relations $\pounds_k\eta^a = \pounds_\zeta\eta^a = \pounds_k\zeta^a = 0$ and $k^a\nabla_a\eta^b - \zeta^a\nabla_a\eta^b = (B^b{}_a - \bar{B}^b{}_a)\eta^a$.

We decompose B_{ab} into three pieces

$$B_{ab} = \frac{1}{D-2}\theta n_{ab} + \sigma_{ab} + \omega_{ab}, \quad (11)$$

where the expansion, shear, and twist of the stringy congruence along the affine direction are defined as $\theta = B^{ab}n_{ab}$, $\sigma_{ab} = B_{(ab)} - \frac{1}{D-2}\theta n_{ab}$ and $\omega_{ab} = B_{[ab]}$. It is noteworthy that even though we have the same notations for B_{ab}, θ, σ_{ab}, and ω_{ab} in (3) and (11), the differences of these notations among the timelike sting cases and null string cases are understood in the context. Similarly, we decompose $\bar{B}_{ab}$ into three parts as in the timelike case. Taking the *ansatz* that the expansion $\bar{\theta}$ is constant along the σ-direction as in the timelike case, we have another Raychaudhuri type equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{D-2}\theta^2 + \frac{1}{D-1}\bar{\theta}^2 - \sigma_{ab}\sigma^{ab} + \bar{\sigma}_{ab}\bar{\sigma}^{ab} + \omega_{ab}\omega^{ab} - \bar{\omega}_{ab}\bar{\omega}^{ab} - R_{ab}(k^a k^b - \zeta^a\zeta^b). \quad (12)$$

Assuming $\omega_{ab} = \bar{\omega}_{ab}$, $\sigma_{ab}\sigma^{ab} \gg \bar{\sigma}_{ab}\bar{\sigma}^{ab}$ and a stringy strong energy condition for null case $R_{ab}(k^a k^b - \zeta^a\zeta^b) \geq 0$ and exploiting the energy-momentum tensor of the perfect fluid, we reproduce the inequality (8) in the timelike congruence of strings. The Raychaudhuri type equation (12) for the null strings then has a solution in the following form:

$$\frac{1}{\theta} \geq \frac{1}{\theta_0} + \frac{1}{D-2}\left(\lambda - \frac{D-2}{D-1}\int_0^\lambda d\lambda \left(\frac{\bar{\theta}}{\theta}\right)^2\right), \quad (13)$$

where θ_0 is the initial value of θ at $\lambda = 0$. We assume again that θ_0 is negative. The inequality (13) then implies that θ must pass through the singularity within an affine length

$$\lambda \leq \frac{D-2}{|\theta_0|} + \frac{D-2}{D-1}\int_0^\lambda d\lambda \left(\frac{\bar{\theta}}{\theta}\right)^2. \quad (14)$$

In the point particle limit with $\bar{B}_{ab} = \bar{\theta} = \bar{\sigma}_{ab} = \bar{\omega}_{ab} = 0$ and $\omega_{ab} = 0$, we assume that the strong energy condition $R_{ab}k^a k^b \geq 0$ is satisfied to yield the second inequality of (9) [8,14,15]. If we assume that the initial value is negative, the expansion θ must go to the negative infinity along that geodesic within a finite affine length [8].

We now have several comments to address. In (6), (7), (13), and (14), one notes that the correction terms associated with $(\bar{\theta}/\theta)^2$ are the novel features of the stringy congruence. Moreover, taking the *ansatz* $\bar{\theta}^2 \ll \theta^2$, which indicates that the internal expansion $\bar{\theta}$ along the σ-direction is negligibly small compared to the expansion θ along the time (or affine) direction, the results (6) and (13) reduce into the point particle results in the $D = 4$ limit, respectively. This observation does not contradict the fact that the internal size of the string remains much less than the Planck length $l_P = 1.61 \times 10^{-33}$ cm, as believed in the string theory [6,7].

Moreover, the $D = 4$ limit of the stringy strong energy condition (8) for both the timelike and null cases does not reduce to the well-known strong energy conditions in (9) for the $D = 4$ point particle cases since the stringy strong energy conditions have the additional contributions $R_{ab}(-\zeta^a \zeta^b)$ originating from the stringy degrees of freedom. It is also remarkable to see that the stringy strong energy conditions of both the timelike and null string congruences produce for the perfect fluid the same result in (8), in contrast to the fact that the corresponding strong energy conditions of the point particle congruences have different forms, as shown in (9). We, thus, conclude that, in the higher D-dimensional stringy congruence cosmology, both the massless gauge particles, such as photons and gravitons, and the massive particles can be created in the same cosmological environment described by the stringy strong energy condition in (8). In the point particle standard cosmology in four-dimensions, it is well known that the radiation-to-matter transition exists and the radiation-dominated phase precedes the matter-dominated one. However, if one follows the above conclusion originating from the D-dimensional stringy congruence theory, this stringy cosmology could not demand such radiation-to-matter transition, and, thus, the radiation and the matter can coexist in the same epoch along the evolution of the universe after the big bang without any preference of the dominated phases.

It is also interesting to see that, in the limit of $D = 6$, the stringy strong energy condition (8) becomes the first one in (9) as for the massive pointlike particle. This result from the extra 2-dimensional cosmology could be an accidental phenomenon without any physical reminiscence. However, it is an open problem whether this extra 2-dimensional model could be a relevant new physical one which seems to be related to the point particle physics.

In the higher D-dimensional stringy congruence theory, one can have the condition $\omega_{ab} = \bar{\omega}_{ab}$ associated with the additional $\bar{\omega}_{ab}$. First, in the case of $\omega_{ab} = \bar{\omega}_{ab} = 0$, we can have the Hawking and Penrose limit with $\omega_{ab} = 0$ in the $D = 4$ point particle congruence cosmology [8]. Second, we can additionally have the $\omega_{ab} = \bar{\omega}_{ab} \neq 0$. In this case, one can have the nonvanishing ω_{ab} initiate the desirable rotational degrees of freedom encountered in the universe such as the rotational motions of galaxies, stars, and planets. Moreover, the nonvanishing $\bar{\omega}_{ab}$ could explain the rotational degrees of freedom of the strings themselves [6,7,13].

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