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Tachyon condensation and quark mass in the modified Sakai-Sugimoto model

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This paper continues the investigation of the modified Sakai-Sugimoto model proposed previously [J. High Energy Phys. 01 (2008) 055]. Here we discuss in detail numerical solutions to the classical equations for the brane profile and the tachyon condensate. An ultraviolet cutoff turns out to be essential because the numerical solutions tend to rapidly diverge from the desired asymptotic solutions, beyond a sufficiently large value of the holographic coordinate. The required cutoff is determined by the non-normalizable part of the tachyon and is parametrically far smaller than that dictated by consistency of a description in terms of ten-dimensional bulk gravity. We had argued [J. High Energy Phys. 01 (2008) 055] that the solution in which the tachyon field goes to infinity at the point where the brane and antibrane meet has only one free parameter, which may be taken to be the asymptotic brane-antibrane separation. Here we present numerical evidence in favor of this observation. We also present evidence that the non-normalizable part of the asymptotic tachyon solution, which is identified with quark mass in the QCD-like boundary theory, is determined by this parameter. We show that the normalizable part of the asymptotic tachyon solution determines the quark condensate, but this requires holographic renormalization of the on-shell boundary brane action because of the presence of infinite cutoff-dependent terms. Our renormalization scheme gives an exponential dependence on the cutoff to the quark mass. We also discuss meson spectra in detail and show that the pion mass is nonzero and satisfies the Gell-Mann-Oakes-Renner relation when a small quark mass is switched on.

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I. INTRODUCTION

The model of Sakai and Sugimoto (SS) [1] has been very successful in reproducing many of the qualitative features of non-Abelian chiral symmetry breaking in QCD. In this model, the "color" Yang-Mills fields are provided by the massless open string fluctuations of a stack of a large number N_c of $D4$-branes, which are extended along the four space-time directions and in addition wrap a circle [2]. In the strong coupling limit, this stack of $D4$-branes has a dual description in terms of a classical gravity theory. Flavor degrees of freedom are introduced in the probe approximation as fermionic open string fluctuations between the color branes and an additional set of "flavor" branes [3–8], which are provided by pairs of $D8$- and $\overline{D8}$-branes. In this setting, chiral symmetry breaking has a nice geometrical picture. In the ultraviolet, chiral symmetry arises on flavor $D8$-branes and $\overline{D8}$-branes, which are located at well-separated points on the circle, while they are extended along the remaining eight spatial directions, including the holographic radial direction. Chiral symmetry breaking in the infrared is signaled by a smooth joining of the flavor branes and antibranes at some point in the bulk.

Despite its many qualitative and some quantitative successes [1,9–17], the SS model has some deficiencies: (i) It does not have parameters associated with quark mass and the chiral condensate. On the other hand, there is a parameter, the asymptotic separation between the flavor branes and antibranes, which, within the SS scenario, finds no counterpart in QCD. (ii) The SS model also ignores the open string tachyon between flavor $D8$-branes and $\overline{D8}$-branes, which may be reasonable in the ultraviolet, where the branes and antibranes are well-separated, but is not so at the place in the bulk where the branes join. It is often argued that in the curved background of the wrapped $D4$-branes the geometry forces flavor branes to join in the interior. While this is true of flavor branes and antibranes that are well-separated asymptotically (separation of the order of the antipodal distance), it cannot be the reason when the separation is small and the branes and antibranes meet far away from the central region. For small separation, the effective radius of the direction on which the $D4$-branes are wrapped is very large, and so one would expect tachyon condensation to be the primary reason for branes and antibranes meeting, as in the extremal $D4$-brane metric. Since the tachyon field takes an infinitely large value in the true ground state,[1] the perturbative stability argument given in [1], valid for small fluctuations of the tachyon field near the local minimum at the origin, does not apply.

It has recently been suggested in [19–21] that tachyon condensation on a brane-antibrane system describes the physics of chiral symmetry breaking in a better and more complete way. If the brane and antibrane are well-separated [20,21], then one also retains the nice geometric picture of the SS model for non-Abelian chiral symmetry

[1]For a recent review of this subject, see [18].

breaking. The purpose of the present work is to complete the investigations started in [21]. Here we give detailed numerical solutions to the classical equations for the brane profile and the tachyon. We show that the solution in which the tachyon diverges at the point in the bulk where the brane and antibrane meet has only one free parameter, which may be taken to be the asymptotic separation between the flavor brane and the antibrane. We present numerical evidence that the non-normalizable part of the asymptotic tachyon solution is determined by this parameter. Thus, by the usual dictionary of anti–de Sitter/conformal field theory (AdS/CFT) [22–25], this parameter determines quark mass in the boundary theory [16,26]. The parameter for the asymptotic brane-antibrane separation is present in the SS model also, but in that setting it cannot be explained as a parameter in QCD. Thus this parameter, which seems mysterious in the SS setting, finds a natural explanation in our model. The presence of a non-normalizable part in the tachyon solution necessitates introduction of an ultraviolet cutoff. This is because in this case the numerical solutions tend to rapidly diverge from the desired asymptotic solutions, beyond a sufficiently large value of the radial coordinate, determined by the magnitude of the non-normalizable part. This cutoff is parametrically far smaller than the cutoff of order $N^{4/3}$ expected because of the breakdown of description in terms of a ten-dimensional gravity theory. Removing the cutoff, therefore, necessarily involves tuning the non-normalizable part to zero. We discuss how this should be done appropriately. We also discuss the chiral condensate and its determination by the normalizable part of the asymptotic tachyon solution. This determination is subtle for two reasons. One is the fact that the space-time-independent classical solutions are described by a single parameter and hence the non-normalizable part of the tachyon cannot be varied independently of the other parameters. The resolution of this issue requires us to consider more general solutions by incorporating space-time dependence. But for this one has to go beyond the expansion in small space-time-dependent fluctuations around space-time-independent solutions, basically because this expansion is singular for the tachyon solution in the infrared. An exact space-time-dependent action is needed, which we derive. The other subtlety has to do with the necessity of an ultraviolet cutoff. To extract cutoff-independent physics, we add counterterms to the $D8$-brane action to remove terms in the boundary action which are divergent as the cutoff is formally allowed to go to infinity. With an appropriate choice of the counterterms we get a finite value for the chiral condensate. Finally, we discuss meson spectra in detail and show that the pion mass is nonzero in the presence of a non-normalizable part of the tachyon and that it satisfies the Gell-Mann-Oakes-Renner (GOR) relation when the quark mass is small.

The organization of this paper is as follows. In the next section we will briefly review the essential features of the modified SS model with the tachyon present. This section also includes a more detailed discussion of the cutoff and its implications than given in [21]. In Sec. III we describe in detail the numerical solutions for the brane profile and the tachyon. This section also contains a discussion of the parameters of the solutions and their determination in terms of a single parameter, namely, the asymptotic brane-antibrane separation. In Sec. IV we discuss the subtleties involved in deriving an expression for the chiral condensate in terms of the parameters of the solutions. We derive the exact five-dimensional action in which the tachyon and brane-antibrane separation fields have dependence on space-time as well as the holographic coordinate and discuss solutions to the equations derived from this action. We also discuss the counterterms required to make the chiral condensate finite as the cutoff is formally removed to infinity. In Sec. V we analyze small fluctuations around the classical solution for the meson spectra. We show that the existence of a massless pion is guaranteed if the non-normalizable part of the tachyon solution vanishes. For a nonvanishing non-normalizable part of the tachyon solution, we obtain an expression for the pion mass and derive the GOR relation for it. We end with a summary and discussion in Sec. VI. Appendixes A, B, and C contain details of some calculations.

As this work was nearing completion, the works in [27,28] appeared which have discussed the problem of quark mass in the SS model using different methods.

II. MODIFIED SAKAI-SUGIMOTO MODEL WITH TACHYON

The Yang-Mills part of the SS model is provided by the near horizon limit of a set of N_c overlapping $D4$-branes, filling the $(3 + 1)$-dimensional space-time directions x^μ ($\mu = 1, 2, 3$ and 0) and wrapping a circle in the x^4 direction of radius R_k. An antiperiodic boundary condition for fermions on this circle gives masses to all fermions at the tree level (and scalars at the one-loop level) and breaks all supersymmetries. At low energies compared to l_s^{-1}, the theory on the $D4$-branes is $(4 + 1)$-dimensional pure Yang-Mills with 't Hooft coupling $\lambda_5 = (2\pi)^2 g_s l_s N_c$, of length dimension. At energies lower than the Kaluza-Klein mass scale R_k^{-1}, this reduces to pure Yang-Mills in $(3 + 1)$ dimensions. This is true in the weak coupling regime $\lambda_5 \ll R_k$, in which the dimensionally transmuted scale developed in the effective Yang-Mills theory in $(3 + 1)$ dimensions is much smaller than the Kaluza-Klein mass scale, which is the high energy cutoff for the effective theory. In the strong coupling regime $\lambda_5 \gg R_k$, in which the dual gravity description is reliable, these two scales are similar. Therefore in this regime there is no separation between the masses of glueballs and Kaluza-Klein states. This is one of the reasons why the gravity regime does not describe real QCD, but the belief is that qualitative features of QCD-like confinement and chiral symmetry breaking, which are easy to study in the strong coupling regime using dual geometry, survive tuning of the dimensionless parameter λ_5/R_k to low values.

Flavors are introduced in this setting by placing a stack of N_f overlapping $D8$-branes at the point x^4_L and N_f $\overline{D8}$-branes at the point x^4_R on the thermal circle. Massless open strings between $D4$-branes and $D8$-branes, which are confined to the $(3+1)$-dimensional space-time intersection of the branes, provide N_f left-handed flavors. Similarly, massless open strings between $D4$-branes and $\overline{D8}$-branes provide an equal number of right-handed flavors, leading to a local $U(N_f)_L \times U(N_f)_R$ chiral gauge symmetry on the flavor $D8$- and $\overline{D8}$-branes. This chiral gauge symmetry is seen in the boundary theory as a global chiral symmetry.

In the large N_c and strong coupling limit, the appropriate description of the wrapped $D4$-branes is given by the dual background geometry. This background solution can be obtained from the Euclidean type IIA supergravity solution for nonextremal $D4$-branes by a wick rotation of one of the four noncompact directions which the $D4$-branes fill, in addition to wrapping the compact (temperature) direction. In the near horizon limit, it is given by [2,29]

$$\begin{aligned} ds^2 &= \left(\frac{U}{R}\right)^{3/2}(\eta_{\mu\nu}dx^\mu dx^\nu + f(U)(dx^4)^2) \\ &\quad + \left(\frac{R}{U}\right)^{3/2}\left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right), \\ e^\phi &= g_s\left(\frac{U}{R}\right)^{3/4}, \\ F_4 &= \frac{2\pi N_c}{V_4}\epsilon_4, \\ f(U) &= 1 - \frac{U_k^3}{U^3}, \end{aligned} \tag{1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ and U_k is a constant parameter of the solution.[2] R is related to the 5D Yang-Mills coupling λ_5, which is kept fixed in the decoupling limit, by $R^3 = \frac{\lambda_5\alpha'}{4\pi}$. Also, $d\Omega_4$, ϵ_4 and $V_4 = 8\pi^2/3$ are, respectively, the line element, the volume form and the volume of a unit S^4.

The above metric has a conical singularity at $U = U_k$ in the $U - x^4$ subspace which can be avoided only if x^4 has a specific periodicity. This condition relates the radius of the circle in the x^4 direction to the parameters of the background by

$$R_k = \frac{2}{3}\left(\frac{R^3}{U_k}\right)^{1/2}. \tag{2}$$

[2]Note that U has dimension of length and is related to the energy scale $\tilde{U}$, which is kept fixed in the decoupling limit, by $U = \tilde{U}\alpha'$.

For $\lambda_5 \gg R_k$ the curvature is small everywhere, and so the approximation to a classical gravity background is reliable. As discussed in [29], at very large values of U, the string coupling becomes large, and one has to lift the background over to the eleven-dimensional M-theory description.

A. Brane-antibrane pair with tachyon

The effective field theory describing the dynamics of a brane-antibrane pair in a background geometry[3] with the tachyon included has been discussed in [31,32]. The simplest case occurs when the brane and antibrane are on top of each other since in this case all of the transverse scalars are set to zero. This is the situation considered in [19]. However, in this configuration one loses the nice geometrical picture of chiral symmetry breaking of the SS model. The geometrical picture is retained in the case considered in [20,21] where the brane and antibrane are separated in the compact x^4 direction. This requires construction of an effective tachyon action on a brane-antibrane pair, taking into account the transverse scalars. Such an effective action with the brane and antibrane separated along a noncompact direction has been proposed in [31,32].[4] A generalization of this action to the case when the brane and antibrane are separated along a periodic direction is not known. However, for small separation $l(U)$ compared to the radius R_k of the circle, the action in [32] should provide a reasonable approximation to the compact case. In the following we will assume this to be so. Then, the effective low-energy tachyon action for a $D8$- and $\overline{D8}$-brane pair for $l(U) \ll R_k$ is given, in the above background, by[5]

[3]For simplicity, we will discuss the case of a single flavor, namely, one brane-antibrane pair. Generalization to the multi-flavor case can be done using the symmmetrized trace prescription of [30].

[4]Also see [33].

[5]There are two important caveats for the validity of this action. One is that this action was derived in [32] under the condition that the brane and antibrane are separated along a noncompact direction. Therefore, strictly speaking, it can be used only for the case when $R_k \to \infty$, i.e. for $f(U) = 1$. However, as we shall see later, in the classical solutions for small asymptotic brane-antibrane separation, the brane and antibrane meet far away from the $U = U_k$ region. For such configurations, $f(U) = 1$ to a good approximation. Thus this error can be minimized by restricting to configurations with small asymptotic brane-antibrane separation. The other caveat, as discussed in [16], is that even with $f(U) = 1$ the background geometry is such that the lowest energy configuration for a fundamental string stretched between the flavor brane and the antibrane has a much smaller length than the naive straight string, which is what the expansion of this action for small T gives. A correction for this in the action is likely to make the tachyon even "more tachyonic" in the region where the brane and antibrane meet, making our argument about tachyon condensation in this region even better. Thus, no qualitative change in the physics discussed in this paper is expected from this correction.

$$S = -\int d^9\sigma V(T,l)e^{-\phi}(\sqrt{-\det A_L} + \sqrt{-\det A_R}),$$

$$(A_i)_{ab} = \left(g_{MN} - \frac{T^2 l^2}{2\pi\alpha' Q} g_{M4}g_{4N}\right)\partial_a x_i^M \partial_b x_i^N + 2\pi\alpha' F^i_{ab} + \frac{1}{2Q}(2\pi\alpha'(D_a\tau(D_b\tau)^* + (D_a\tau)^* D_b\tau) + il(g_{a4} + \partial_a x_i^4 g_{44})(\tau(D_b\tau)^* - \tau^* D_b\tau) + il(\tau(D_a\tau)^* - \tau^* D_a\tau)(g_{4b} - \partial_b x_i^4 g_{44})), \tag{3}$$

where

$$Q = 1 + \frac{T^2 l^2}{2\pi\alpha'} g_{44},$$
$$D_a\tau = \partial_a\tau - i(A_{L,a} - A_{R,a})\tau, \tag{4}$$
$$V(T,l) = g_s V(T)\sqrt{Q}.$$

$T = |\tau|$, $i = L, R$ and we have used the fact that the background does not depend on x^4. The complete action also includes terms involving Chern-Simons (CS) couplings of the gauge fields and the tachyon to the Ramond-Ramond background sourced by the $D4$-branes. These will not be needed in the following analysis and hence have not been included here.

The potential $V(\tau)$ depends only on the modulus T of the complex tachyon τ. It is believed that $V(\tau)$ satisfies the following general properties [18]:

(i) $V(T)$ has a maximum at $T = 0$ and a minimum at $T = \infty$ where it vanishes.

(ii) The normalization of $V(T)$ is fixed by the requirement that the vortex solution on the brane-antibrane system should produce the correct relation between Dp and $D(p-2)$-brane tensions. In the present case this means $V(0) = \mathcal{T}_8 = 1/(2\pi)^8 l_s^9 g_s$, the $D8$-brane tension.

(iii) In flat space, the expansion of $V(T)$ around $T = 0$ up to terms quadratic in T gives rise to a tachyon with mass squared equal to $-1/2\alpha'$.

There are several proposals for $V(T)$ which satisfy these requirements [18], although no rigorous derivation exists. Examples are (i) the potential used in [34–36] for calculation of decay of unstable D-branes in two-dimensional string theory

$$V(T) = \mathcal{T}_8 \operatorname{sech}\sqrt{\pi}T \tag{5}$$

and (ii) the potential obtained using boundary string field theory computation [37–40]

$$V(T) = \mathcal{T}_8 e^{-(\pi/2)T^2}. \tag{6}$$

Both of these potentials satisfy the properties listed above. Note that the asymptotic form of the potential in (5) for large T is $\sim e^{-\sqrt{\pi}T}$. The linear growth of the exponent with T should be contrasted with the quadratic growth for the potential in (6). This difference will turn out to be important for the background tachyon solutions, which are discussed next.

We end this subsection with the following observation. It can be easily seen that in the decoupling limit all factors of α' scale out of the entire action, without requiring any scaling of the transverse scalar l or the tachyon τ. In fact, the entire action can be rewritten in terms of λ_5 and $\tilde{U}$, quantities that are kept fixed in the scaling limit. Henceforth, we will use the convention $2\pi\alpha' = 1$.

B. Classical equations for brane profile and tachyon

We will now look for an appropriate classical solution of the brane-antibrane-tachyon system. Let us set the gauge fields and all but the derivatives with respect to U of T and x_i^4 to zero. Moreover, we choose $x_L^4 = l/2$ and $x_R^4 = -l/2$ so that the separation between the brane and antibrane is l. In this case, in the static gauge the action (3) simplifies to[6]

$$S = -V_4 \int d^4x \int dU V(T)\left(\frac{U}{R}\right)^{-3/4} U^4(\sqrt{D_{L,T}} + \sqrt{D_{R,T}}), \tag{7}$$

where $D_{L,T} = D_{R,T} \equiv D_T$ and

$$D_T = f(U)^{-1}\left(\frac{U}{R}\right)^{-3/2} + f(U)\left(\frac{U}{R}\right)^{3/2}\frac{l'(U)^2}{4} + T'(U)^2 + T(U)^2 l(U)^2. \tag{8}$$

It is convenient to remove the dependence on R (except for an overall factor in the action) through a redefinition of variables:

$$U = u/R^3, \qquad l(U) = R^3 h(u), \qquad U_k = u_k/R^3. \tag{9}$$

In terms of the new variables, we get

$$S = -V_4 R^{-9}\int d^4x \int du u^{13/4} V(T)(\sqrt{d_{L,T}} + \sqrt{d_{R,T}}), \tag{10}$$

where

$$d_{L,T} = d_{R,T} \equiv d_T = f(u)^{-1}u^{-3/2} + f(u)u^{3/2}\frac{h'(u)^2}{4} + T'(u)^2 + T(u)^2 h(u)^2, \tag{11}$$

with $f(u) = (1 - u_k^3/u^3)$.

The effective potential for the tachyon can be obtained from this action by setting $T' = h' = 0$. It is

$$V_{\text{eff}}(T,l) \sim \operatorname{sech}\sqrt{\pi}T\sqrt{1 + u^{3/2}T^2h^2}. \tag{12}$$

In Fig. 1 we have plotted V_{eff} as a function of T for various values of u. We see that a perturbatively stable minimum at

[6] The CS term in the action does not contribute for such configurations.

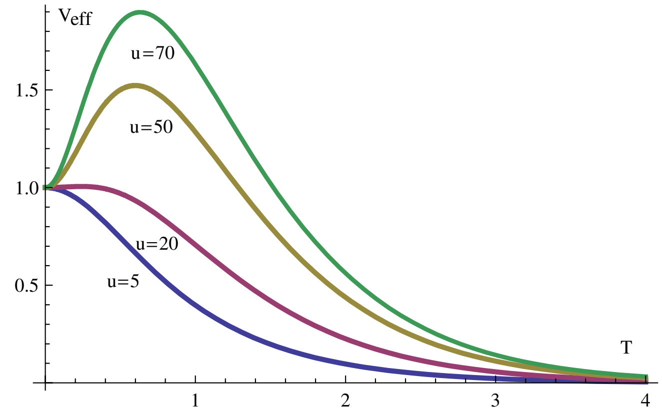

FIG. 1 (color online). The effective potential $V_{\rm eff}$ as a function of T for different values of u for a fixed nonzero value of h.

$T = 0$ for large values of u turns into an unstable maximum at a sufficiently small value of u. This is true for any fixed, nonzero value of h. Moreover, the value of u at which there is an unstable maximum at $T = 0$ increases as h decreases. Similar potentials were considered earlier in [41] in flat space in the context of dynamical decay of a brane-antibrane pair with a finite separation.

The equations of motion obtained from the action (10) are

$$\left(\frac{u^{13/4}}{\sqrt{d_T}}T'(u)\right)' = \frac{u^{13/4}}{\sqrt{d_T}}\left[T(u)h(u)^2 + \frac{V'(T)}{V(T)}(d_T - T'(u)^2)\right], \tag{13}$$

$$\left(\frac{u^{13/4}}{\sqrt{d_T}}\frac{f(u)}{4}u^{3/2}h'(u)\right)' = \frac{u^{13/4}}{\sqrt{d_T}}\left[T(u)^2h(u) - \frac{V'(T)}{V(T)} \times \frac{f(u)}{4}u^{3/2}h'(u)T'(u)\right]. \tag{14}$$

Note that the "prime" on $V(T)$ denotes a derivative with respect to its argument T and not a derivative with respect to u.

This is a complicated set of coupled nonlinear differential equations which can be solved completely only numerically. To get some insight into the kind of solutions that are possible, however, we had analyzed these equations in [21] for large u and for u near the brane-antibrane joining point in the bulk. For these values of u the equations simplify and can be treated analytically. For the sake of completeness, we will summarize the results of this analysis here before proceeding to describe numerical solutions to these equations. As in the case without the tachyon, we are looking for solutions in which the brane and antibrane have a given asymptotic separation h_0, i.e. $h(u) \rightarrow h_0$ as $u \rightarrow \infty$, and they join at some interior point in the bulk, i.e. $h(u) \rightarrow 0$ at $u = u_0 \geq u_k$.[7] Moreover, we want the tachyon (i) to vanish as $u \rightarrow \infty$ so that the chiral symmetry is intact in the ultraviolet region and (ii) to go to infinity as u approaches u_0 so that the QCD chiral anomalies are reproduced correctly [19].

[7]The inequality results from the lower bound on u.

C. Solution for large u

Here we seek a solution in which $h(u)$ approaches a constant h_0 and T becomes small as $u \rightarrow \infty$. For small T one can approximate $V'/V \sim -\pi T$.[8] If T and h' go to zero sufficiently fast as $u \rightarrow \infty$ such that to the leading order one might approximate $d_T \sim u^{-3/2}$, then (13) can be approximated to

$$(u^4T'(u))' = h_0^2u^4T. \tag{15}$$

The general solution of this equation is

$$T(u) = \frac{1}{u^2}(T_+e^{-h_0u} + T_-e^{h_0u}). \tag{16}$$

In writing this solution we have ignored a higher order term in $1/u$ for consistency with other terms in Eq. (13) that we have neglected at large u. We will discuss consistency of this solution below. Let us first discuss the solution for $h(u)$.

The fact that the tachyon takes small values for large u makes it irrelevant for the leading asymptotic behavior of h, which can be extracted from (14) by setting the right-hand side to zero. The resulting equation is

$$(u^{11/2}h'(u))' = 0, \tag{17}$$

which has the solution

$$h(u) = h_0 - h_1u^{-9/2}. \tag{18}$$

Here h_1 is restricted to positive values so that the branes come together in the bulk. For a SS model without the tachyon, $h_1 = \frac{4}{9}u_0^4f_0^{1/2}$, where $f_0 = f(u_0)$, u_0 being the value of u where the branes meet in the bulk.

It is easy to convince oneself that the only solution to Eqs. (13) and (14) in which T vanishes asymptotically and h goes to a constant is (16) with $T_- = 0$. In particular, for example, these equations have no solutions in which T vanishes asymptotically as a power law.

D. Quark mass and the ultraviolet cutoff

In the tachyon solution (16), the exponentially falling part satisfies the approximations under which (15) was derived for any large value of u. The exponentially rising part will, however, eventually become large and cannot be self-consistently used. This is because for sufficiently large u, there is no consistent solution for T which grows exponentially or even as a power law to the original Eqs. (13) and (14), if we impose the restriction that $h(u)$ should go to

[8]This follows from the general properties of the potential discussed in Sec. II A.

a constant asymptotically. This puts a restriction on the value of u beyond which the generic solution (16) cannot be used. The most restrictive condition comes from the approximation $d_T \sim u^{-3/2}$. This requires the maximum value $u_{\max}$ to satisfy the condition

$$T_+^2 e^{-2h_0 u_{\max}} + T_-^2 e^{2h_0 u_{\max}} \ll \frac{u_{\max}^{5/2}}{2h_0^2}. \tag{19}$$

For generic values of $|T_\pm|$ and h_0, this inequality determines a range of values of $u_{\max}$ for which the solution (16) can be trusted. The value $T_- = 0$ is special since in this case there is no upper limit on $u_{\max}$, except the cutoff that comes from the fact that the ten-dimensional description of the background geometry breaks down beyond some very large value ($\sim N_c^{4/3}$) of u. However, as is clear from (19), for nonzero $|T_-|$ one needs to choose a much smaller value of $u_{\max}$. Numerical calculations reported in the next section bear out this expectation.

It is important to emphasize that the ultraviolet cutoff we are talking about here does not merely play the usual role of a cutoff needed in any example of AdS/CFT with a non-normalizable part present in a solution to the bulk equations. The point is that there is no growing solution to the tachyon equation in the ultraviolet which is consistent with a brane profile that goes to a finite asymptotic brane-antibrane separation. This constraint limits the value of u up to which the asymptotic solutions (16) and (18) can be trusted.

One way to think about the inequality (19) is the following. Suppose for given values of $|T_\pm|$ we have chosen the largest value of $u_{\max}$ consistent with (19). Increasing $u_{\max}$ further would then be possible only if $|T_-|$ is decreased appropriately, while $|T_+|$ can be kept fixed, as $u_{\max}$ is increased. To be concrete, let us keep $|T_+|$ and $|T_-|e^{h_0 u_{\max}}$ fixed as $u_{\max}$ is increased. The process of "removing the cutoff" can then be understood as increasing $u_{\max}$ and simultaneously decreasing $|T_-|$ while keeping $|T_+|$ and the combination $|T_-|e^{h_0 u_{\max}}$ fixed. In this process, at some point $|T_+|e^{-h_0 u_{\max}}$ would become much smaller than $|T_-|e^{h_0 u_{\max}}$. As we shall see in the next section, however, limitations due to numerical accuracy prevent us from tuning $|T_-|$ to very small values or equivalently tuning $u_{\max}$ to be very large. Thus we are numerically restricted to rather small values of $u_{\max}$. For values of u larger than $u_{\max}$, the inequality (19) breaks down and consequently the asymptotic solution (16) is not applicable. Clear evidence for this breakdown is seen in the numerical calculations reported in the next section.

It is natural to associate T_- with the quark mass since this parameter comes with the growing solution. Evidence for this will be given in Sec. V where we will show that for a small nonzero value of this parameter the pion mass is nonzero and proportional to it. It is also natural to associate T_+ with the chiral condensate because it comes with the normalizable solution. It turns out that this association too is consistent, though this part of the story is somewhat more complicated, as we shall see in Sec. IV.

It is interesting to mention here that keeping the combination $|T_-|e^{h_0 u_{\max}} = \rho$ fixed as the cutoff becomes large implies an exponential dependence of $|T_-|$ on the $u_{\max}$, i.e. $|T_-| = \rho e^{-h_0 u_{\max}}$. A similar dependence of the quark mass on the cutoff has been observed in [27,28], though the methods used for computing quark mass in these works are quite different from ours. In [28] the cutoff arises from the location of a $D6$-brane, which is additionally present in that model, thereby giving a physical meaning to the cutoff.

E. Solution for $u \sim u_0$

Here we are looking for a solution in which $h \to 0$ and $T \to \infty$ as $u \to u_0$. Let us assume a power-law ansatz, namely,

$$h(u) \sim (u - u_0)^\alpha, \qquad T(u) \sim (u - u_0)^{-\beta}. \tag{20}$$

For a smooth joining of the brane and antibrane at u_0, the derivative of h must diverge at this point, which is ensured if $\alpha < 1$. Since for this ansatz T'^2 is the largest quantity for $u \to u_0$, we can approximate $d_T \sim T'(u)^2$. We will also need the asymptotic form of the potential $V(T)$ for large T, which depends on the specific potential being used. From the asymptotic form of the potential in (5), we get $V'(T)/V(T) \sim -\sqrt{\pi}$, while for the potential in (6), we get $V'(T)/V(T) \sim -\pi T$. Putting all of this in (13) and (14), it is easy to verify that these equations cannot be satisfied by the ansatz (20) for the potential (6). They are, however, satisfied for the potential in (5).[9] In fact, in this case the powers as well as the coefficients all get fixed[10]:

$$h(u) = \sqrt{\frac{26}{\pi u_0 f_0}} u_0^{-3/4} (u - u_0)^{1/2} + \cdots, \tag{21}$$

$$T(u) = \frac{\sqrt{\pi}}{4} f_0 u_0^{3/2} (u - u_0)^{-2} + \cdots. \tag{22}$$

An important feature of the above solution is that it depends only on a single parameter, namely, the value of u_0. We have checked that this feature persists in the next few higher orders in a power series expansion in $(u - u_0)$. This is in sharp contrast to the asymptotic solution (16) and

[9] A direct comparison with the potential in (6), obtained in boundary string field theory calculations, may, however, not be quite appropriate since the corresponding actions could be related by a complicated field redefinition, which would also change the kinetic (and higher derivative) terms.

[10] In [20] the power of $(u - u_0)$ with which the brane-antibrane separation falls off in the bulk has been left undetermined. This power is actually determined by (13) and (14), as can be easily checked by consistently expanding these equations on both sides and going beyond the leading order in powers of $(u - u_0)$. We have also verified this power by numerical calculations reported in the next section.

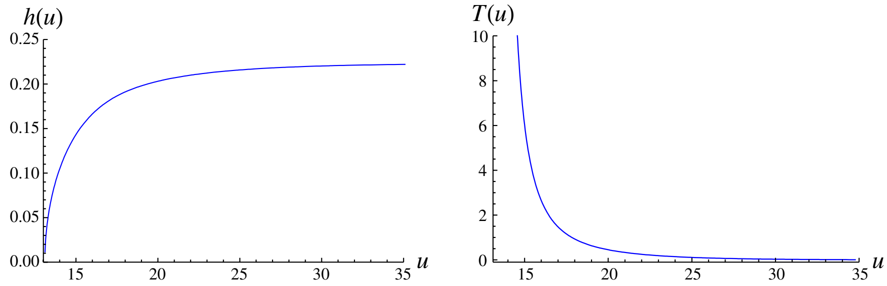

FIG. 2 (color online). The brane profile and the tachyon solution for $u_0 = 12.7$.

(18) which depends on all four expected parameters: T_+, T_-, h_0, and h_1. This reduction in the number of parameters is similar to what happens in the SS model where the solution for $u \sim u_0$ depends only on one parameter, although the asymptotic solution depends on two parameters. In the present case the reduction in the number of parameters is even more severe; the solution for $u \sim u_0$ matches with only a one-parameter subspace of the four-parameter space of asymptotic solutions. As we will discuss later, this one-parameter freedom of the classical solution turns out to be analogous to the freedom to add a bare quark mass in QCD.

For completeness, we note that there exists another solution in which T does not diverge but goes to a nonzero constant as $u \rightarrow u_0$. In this case we can approximate $d_T \sim f(u)u^{3/2}h'(u)^2/4$. Substituting in (13) we see that the left-hand side diverges as $(u - u_0)^{-\alpha}$. The first term on the right-hand side vanishes as a positive power, but the second term diverges as $(u - u_0)^{\alpha-1}$, since $\alpha < 1$. For consistency we must have $\alpha = 1/2$. The resulting solution

$$h(u) = \frac{4}{u_0}(f_0(5f_0 + 3))^{-1/2}(u - u_0)^{1/2} + \cdots, \tag{23}$$

$$T(u) = t_0 + \frac{2u_0^{-1/2}}{(5f_0 + 3)} \frac{V'(u_0)}{V(u_0)}(u - u_0) + \cdots \tag{24}$$

also satisfies (14). Note that no special condition was required for the tachyon potential to get this solution; this solution exists for any potential.

III. NUMERICAL SOLUTIONS

Equations (13) and (14) cannot be solved analytically. One needs to use numerical tools to get a solution. We have made use of MATHEMATICA for this. Also, for numerical calculations we have chosen the potential (5), since there is no diverging solution for $T(u)$ for $u \sim u_0$ for the potential (6), as discussed above.

The numerical calculations are easier to do if we start from the $u = u_0$ end and evolve towards the large u end. This avoids the fine-tuning one would have to do if one were to start from large values of u, where the general solution has four parameters, and end on a one-parameter subspace for $u \sim u_0$. We must also satisfy the requirement of working in the parameter region of the background geometry corresponding to the strong coupling. In addition, we need to ensure that the asymptotic separation between flavor branes and antibranes is small compared to the radius of the x^4 circle. Mathematically, these requirements are $\lambda_5 = 8\pi^2 R^3 \gg 2\pi R_k$ and $l_0 \ll \pi R_k$. Using (2) and (9), one gets $R^3 = \frac{3}{2}R_k\sqrt{u_k}$. Then, these requirements become $\frac{1}{36\pi^2} \ll u_k \ll \frac{4\pi^2}{9h_0^2}$. Throughout our numerical calculations we will work with $u_k = 1$, which satisfies the first condition easily, while it requires from the second that $h_0 \ll \frac{2\pi}{3}$. This condition is also easily satisfied by choosing $u_0 \gg u_k = 1$.[11] For such values of u_0, $f(u) \sim 1$ for all $u \geq u_0$.

The boundary conditions are imposed using (21) and (22) at a point $u = u_1$ which we choose as close to u_0 as allowed by numerics. Generally we were able to reduce $(u_1 - u_0)$ down to about 0.1% of the value of u_0. Starting from the values of $T(u_1)$, $T'(u_1)$, $h(u_1)$ and $h'(u_1)$ obtained from (21) and (22) at $u = u_1$, the system was allowed to evolve to larger values of u. Figure 2 shows an example for $u_0 = 12.7$. Solutions for both $h(u)$ and $T(u)$ are shown.

A. Verification of the UV and IR analytic solutions

From the numerical solutions one can verify that $h(u)$ and $T(u)$ are given by the forms (21) and (22), for $u \sim u_0$. Figure 3 shows the impressive fits between the numerical data and the analytical expectations for the powers of $(u - u_0)$ for $h(u)$ and $T(u)$. We have plotted $h(u)/h'(u)$ and $T(u)/T'(u)$, calculated from the numerical solutions, as functions of u. The numerical data are plotted in dashed lines, while the theoretical solutions are plotted in solid lines. As one can see, these graphs are linear at the IR end, and their slopes turn out to be close to the expected values 0.5 and -2, respectively. In fact, the numerical and the

[11] As we shall see below, the asymptotic separation decreases with an increasing value of u_0, as is the case for the SS model.

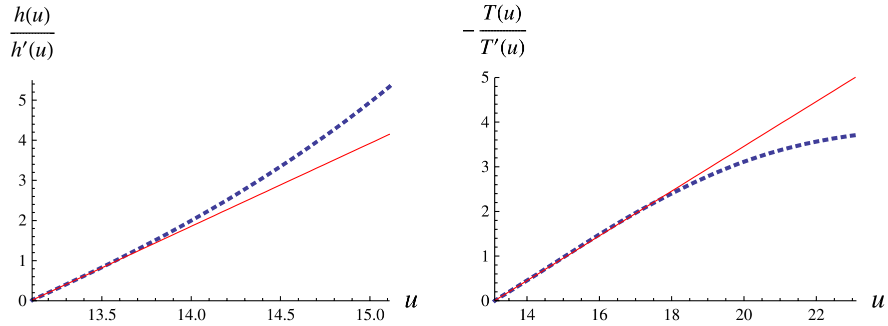

FIG. 3 (color online). Numerical verification of exponents in the IR behavior of the brane profile and the tachyon. The fits give the two exponents, respectively, to be 0.50 and -2.07 for $u_0 = 13.1$.

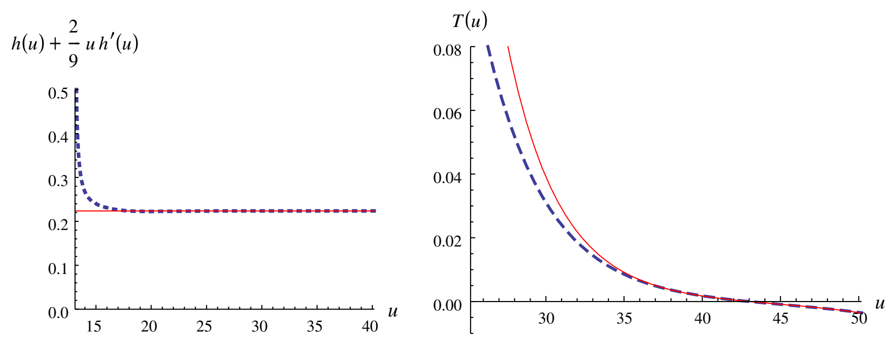

FIG. 4 (color online). Numerical verification of the asymptotic form of the brane profile and the tachyon.

theoretical curves entirely overlap in the IR region of u, as shown in Fig. 3. At the other end also, namely, for large u, one can verify that the numerical solutions have the analytic forms (16) and (18). The goodness of the fits of these analytic forms to numerical data is shown in Fig. 4 where again the two curves overlap in the asymptotic region of u. The fits yield values of the four parameters: $h_0 = 0.224$, $h_1 = -16068$, $T_+ = 29\,194.5$, and $T_- = -1.25 \times 10^{-4}$ for $u_0 = 13.1$.

B. Behavior of the non-normalizable part

For $T_- \neq 0$, extending numerical calculations much beyond the values of u shown in Fig. 2 meets with a difficulty. It turns out that, for small u_0, T_- is positive. Since T_- is the coefficient of the rising exponential in $T(u)$, for a sufficiently large value of u this term dominates, and so $T(u)$ begins to rise.[12] Eventually, T becomes so large that the conditions under which the asymptotic solutions (16) and (18) were obtained no longer apply. Figure 5 illustrates this; it shows the solutions for $u_0 = 12.7$ for two different large values of u. In Fig. 5(a), after falling very fast, T rises and then falls again. Almost simultaneous with this is a rapid rise of h from one nearly constant value to a higher constant value. Evidently, this behavior continues indefinitely with u, as can be seen in Fig. 5(b).[13]

The value of T_- decreases with increasing u_0. This can be easily deduced from the fact that the maximum value of u up to which the asymptotic solutions (16) and (18) apply, namely, before the oscillations begin, increases with increasing u_0. Figure 6 illustrates this by showing the solutions for increasing values of u_0, close to where T_- is small. As one can see, increasing the value of u_0 by a very small amount, from $u_0 = 13$ to $u_0 = 13.0878$, dramatically increases the threshold for oscillatory behavior of T from $u \sim 50$ to $u \sim 120$. As u_0 increases further, T_- decreases, becomes zero[14] and eventually is negative. Since we want to interpret T_- as the bare quark mass parameter, negative values for it are allowed. However, a

[12]We would like to thank Matt Headrick for a discussion on this point and some other aspects of our numerical calculations.

[13]In [20], the authors claim that this effect is due to sensitivity of the solutions to the boundary conditions at the infrared end at $u = u_1$, which must necessarily be chosen slightly away from the actual value u_0. We have not found any evidence for this sensitivity. On the other hand, it is clear that the approximation made in deriving the asymptotic solution (16) and (18) must break down for sufficiently large u, for any nonzero value of T_-. We see convincing numerical evidence for this. Further evidence of this follows.

[14]We have found that $T_- = 1.92 \times 10^{-9}$ at $u_0 \sim 13.087\,778\,1$. Fine-tuning u_0 such that T_- is precisely zero is hard. This requires numerical methods which are beyond the scope of those used here. However, the trend is clear from Figs. 6 and 7.

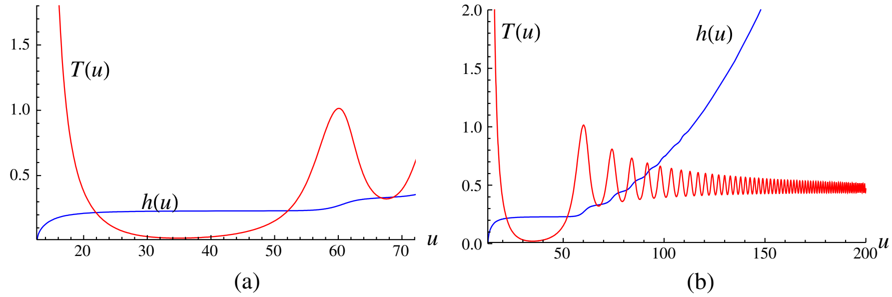

FIG. 5 (color online). Solutions for two different large values of u.

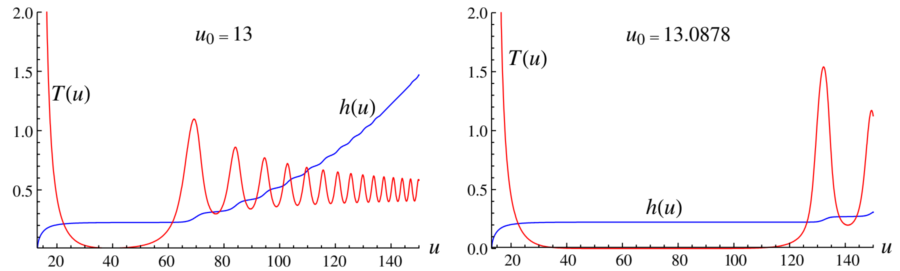

FIG. 6 (color online). Numerical solutions for increasing values of u_0 for positive T_-.

large value for $|T_-|$ will eventually again make T large in magnitude for large enough u. So once again we expect that, at some sufficiently large u, T will become so large that the conditions under which the asymptotic solutions (16) and (18) were obtained no longer apply. So, as before, one should find oscillations in $T(u)$, which now start at smaller and smaller u as u_0 grows. This is indeed seen to be the case, as is evident in Fig. 7. This happens because $|T_-|$ grows with u_0, beyond the value at which it becomes zero. Figure 8 shows the change of T_- with u_0. We see that T_- vanishes at $u_0 \sim 13.0878$ and $|T_-|$ grows on both sides away from this value. It is hard to understand what is special about this value of u_0. One might have thought that the role of zero mass would be played by the antipodal configuration, which has $u_0 = u_k$ and is beyond our approximation. It is possible that this is an artifact of using the approximate action (3), valid for a noncompact x^4 coordinate, although the value $u_0 \sim 13.0878$ is fairly large and seems to be within the validity of our approximation. We also note that, for negative T_-, negative $T(u)$ can be avoided by imposing a suitable cutoff on u. As we have already discussed, the cutoff is in any case required to fulfil the condition (19) so that the asymptotic solutions (16) and (18) may apply.

C. Behavior of the asymptotic brane-antibrane separation

Another interesting quantity is the asymptotic brane-antibrane separation h_0 as a function of u_0. This quantity has been plotted in Fig. 9. We see that h_0 steadily decreases through the special value $u_0 \sim 13.0878$. Although we do not have an analytical formula for the dependence of h_0 on u_0 for large values of the latter, the trend in Fig. 9 seems to indicate that it decreases to zero as u_0 becomes large. Presumably the brane-antibrane pair overlap and disappear as u_0 goes to infinity. This is consistent with the trend of increasing bare quark mass for increasing values of u_0 (far beyond $u_0 \sim 13.0878$) which we have seen in Fig. 8. Therefore, unlike in the Sakai-Sugimoto model, the disappearance of the brane-antibrane pair for $u_0 = \infty$ can be understood in the present setup as the infinite bare quark mass limit.

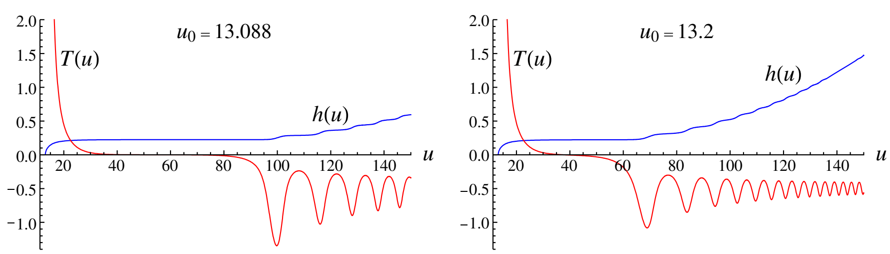

FIG. 7 (color online). Numerical solutions for increasing values of u_0 for negative T_-.

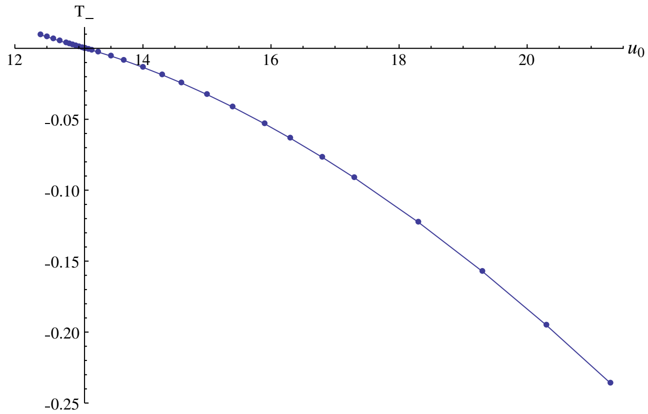

FIG. 8 (color online). T_- as a function of u_0.

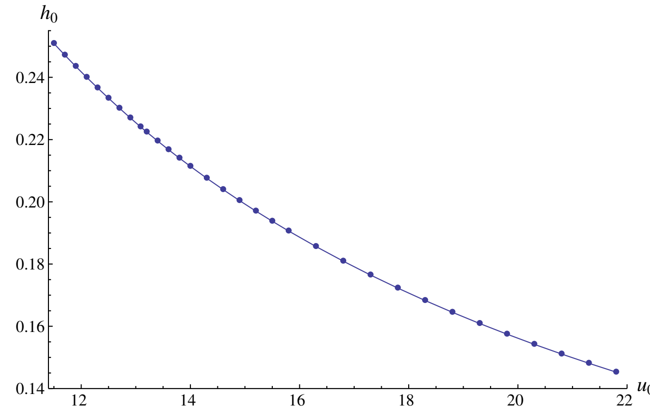

FIG. 9 (color online). h_0 as a function of u_0.

It should be clear from the above discussion that the limit $h_0 \to 0$ does not reduce to the case of overlapping $D8$-branes and $\overline{D8}$-branes considered in [19]. For this case, one must begin afresh with $x_i^4 = 0$, $l = 0$ in the action (3). However, the classical equation for T can be obtained from Eq. (13) by setting $h = 0$ in it. As above, we find that solutions which are divergent in the IR depend on only one free parameter. For further details about the tachyon solutions in this case, we refer the interested reader to Appendix A.

D. Comparison with the Sakai-Sugimoto solution

Finally, we must ensure that the solution with the tachyon has lower energy compared to the SS model. The energy density in the modified model is given by

$$E_{\rm T} = 2V_4R^9V(0)\int_{u_0}^{u_{\rm max}} du E_{\rm T}(u),$$
$$E_{\rm T}(u) = u^{13/4}\frac{V(T)}{V(0)} \times \sqrt{u^{-3/2} + \frac{1}{4}u^{3/2}h'(u)^2 + T'(u)^2 + T(u)^2h(u)^2}, \tag{25}$$

while for the SS model it is given by

$$E_{\rm SS} = 2V_4R^9V(0)\int_{u_0}^{u_{\rm max}} du E_{\rm SS}(u),$$
$$E_{\rm SS}(u) = u^{13/4}\sqrt{u^{-3/2} + \frac{1}{4}u^{3/2}h'_{\rm SS}(u)^2}. \tag{26}$$

To get these expressions for energy density, we have set $f(u)$ to unity, which is a good approximation for large u_0.[15] Also, in the SS model one must use the solution of the tachyon free equation $h'_{\rm SS}(u) = 2u_0^4u^{-3/2}(u^8 - u_0^8)^{-1/2}$.

Close to u_0, in the IR, the exponentially vanishing tachyon potential suppresses contribution to $E_{\rm T}$ compared to $E_{\rm SS}$. Since the UV solutions for the two models are almost identical,[16] one might argue that the energy for the modified model must be lower than that for the SS model. However, for $u \gtrsim u_0$ there is a competition between the exponentially vanishing tachyon potential and the power-law increase of the square-root factor coming from $|T'|$ in the integrand $E_{\rm T}(u)$ in (25). This results in a local maximum in $E_{\rm T}(u)$ at some value of u, which can be easily estimated analytically. The relevant quantity

$$e^{-(\pi/4)u_0^{3/2}(u-u_0)^{-2}}(u - u_0)^{-3}$$

has a maximum at $u = u_0 + (\frac{\pi}{6})^{1/2}u_0^{3/4}$. For small u_0, the position of the maximum is close to u_0, so in this case the argument about the IR behavior of the integrand in (25) is not very clean, except in the very deep IR. But since the position of the maximum grows with increasing u_0 as $u_0^{3/4}$, our argument should hold for large values of u_0, which is precisely where the action for the modified model can be trusted. However, the expression used for estimating the position of the local maximum breaks down if it is too far away from u_0. So, in practice we need to do a numerical calculation to see what the real story is. As we will see in the numerical plots given below, what really happens is that for relatively large values of u_0 the integrand $E_{\rm T}(u)$ increases rapidly at first, then slows down almost to a constant and finally settles into an asymptotic power-law increase similar to that of the integrand $E_{\rm SS}(u)$ for the SS model. Moreover, the place where the rapid increase begins

[15] As in the previous calculations, we have set $u_k = 1$ for the calculation done here. Then, for $u \geq 10$ we get $0.999 \leq f(u) \leq 1$. We have checked that in this region the small deviation of $f(u)$ from unity does not make any appreciable change in the calculated values of h_0 and $T_\pm$.

[16] There is a caveat here. Strictly speaking this is true only when the coefficient of the non-normalizable term T_- in the asymptotic tachyon solution (16) vanishes. As we have discussed, when T_- is nonzero, one must introduce a cutoff $u_{\rm max}$, chosen carefully such that the asymptotic solution is satisfied. In particular, one must ensure T is positive in the region below $u_{\rm max}$. In the calculations reported here and earlier in this section, this is what we have done.

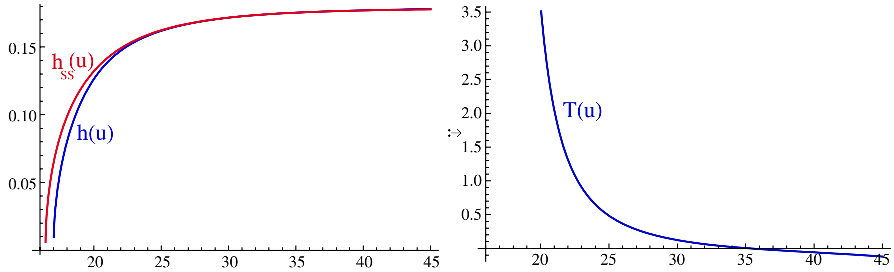

FIG. 10 (color online). $h(u)$ and $T(u)$ profiles for $u_0 = 17$. For comparison, the h_{SS} profile has also been plotted after adjusting the value of u_0 to 16.4 for it since this value of u_0 produces the same asymptotic brane-antibrane separation.

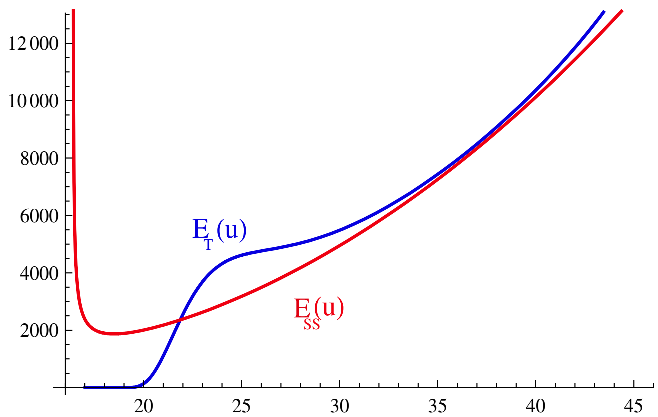

FIG. 11 (color online). The energy density integrands $E_{SS}(u)$ and $E_T(u)$. The rapid rise of the latter in the IR is clearly seen. The divergence between the two curves in the asymptotic region $u \gtrsim u_{max}$ is due to a nonzero T_-.

shifts to larger values of u as u_0 increases, in accordance with the above expectation.

We have numerically evaluated the integrals in (25) and (26). Because the relation between u_0 and the asymptotic brane-antibrane separation is different in the two models, a given value of u_0 corresponds to two different values of the latter and vice versa. We have chosen to do the comparison for the same value of the asymptotic brane-antibrane separation in the two models, but similar conclusions are expected with the other choice as well. In Fig. 10 we have plotted numerical solutions for $h(u)$ and $T(u)$ for $u_0 = 17$. For comparison with the SS model, we have also plotted h_{SS} after adjusting the value of u_0 for it to produce the same value of the asymptotic brane-antibrane separation. The required value turns out to be $u_0 = 16.4$. The corresponding energy density integrands $E_T(u)$ and $E_{SS}(u)$ have been plotted in Fig. 11. We can clearly see the rapid rise of $E(u)$ in the IR, the subsequent flattening out and finally the power-law rise in the asymptotic region. Using $u_{max} = 35.316$,[17] numerical evaluation of the integrals gives $(E_T - E_{SS})/2V_4R^9V(0) = -521.3$. Similar behavior is seen[18] for values of $u_0 > 13.5$. For $u_0 < 13.5$, however, the energy difference becomes very small and is sensitive to the choice of u_1, the value of u where the IR boundary conditions are imposed on the numerical solutions, the UV cutoff u_{max} and the deviation of $f(u)$ from unity. In this regime one may have to worry about the caveats discussed in footnote 5. For this reason we prefer to restrict ourselves to the region $u_0 > 13.5$.

We conclude that there is strong numerical evidence that the solution with the tachyon taken into account corresponds to a lower energy state.

IV. THE CHIRAL CONDENSATE

By the standard dictionary of AdS/CFT [22–25], once we have identified T_- with the quark mass parameter, we should identify T_+ with the chiral condensate. However, it is not clear that the standard rules apply to the present case of a boundary theory which is not a CFT and has a scale. Moreover, the fact that there is no known lift of $D8$-brane to 11 dimensions forces an essential cutoff in the theory with flavors. In fact, for a nonzero value of T_-, the real cutoff is much lower, as we have seen from numerical computations in the last section. Despite these difficulties, we will assume that the identification of sources in the boundary theory with boundary values of bulk fields holds in the theory with a cutoff.

There is an additional difficulty in the present case. As we have seen above, the desired solutions have only one independent parameter, which we take to be T_-. The other

[17]This is the value at which $T(u)$ vanishes. The asymptotic form (16) fits the numerically computed $T(u)$ in the range $33 \leq u \leq u_{max}$ to better than a percent with the parameter values $h_0 = 0.175$, $T_+ = 28\,911.3$, and $T_- = -0.0937$.

[18]For accurate numerical calculations, it is crucial to impose the IR boundary conditions on $h_{SS}(u)$ at a value $u = u_1$ which is as close to u_0 as possible. This is because $E_{SS}(u)$ is large at the IR end, as can be seen in Fig. 11, and so the value of E_{SS} increases considerably as the difference $(u_1 - u_0)$ decreases. On the other hand, the value of E_T is insensitive to the precise value of u_1 because $E_T(u)$ vanishes at the IR end. Happily, it turns out that the IR boundary conditions on the numerical solution for $h_{SS}(u)$ can be imposed at a value of u_1 which is much closer to u_0 than is allowed by the numerics in the case with the tachyon present. Typically in our calculations we are able to take $(u_1 - u_0)$ of order 10^{-8} in the SS model but only of order 0.01 when the tachyon is present.

three parameters T_+, h_0 and h_1 should then be considered to be functions of T_-. Thus, the chiral condensate cannot be computed naively by varying the on-shell flavor brane action with respect to T_-, since this would also include contributions from the variation of the other three parameters with T_-. The one-parameter solutions that we have found constitute the most general class of space-time-independent solutions with the specified boundary conditions.[19] Therefore, if we want only to make a variation of T_- only, we must go out of the present one-parameter class of solutions to more general solutions, which are space-time-dependent, in addition to being dependent on u, and have enough parameters. These solutions to (u, x)-dependent equations should have the same singularities at $u = u_0$ as the solutions in (21) and (22). Moreover, the asymptotic solutions should have the form of (16) and (18) with x-dependent coefficients. If solutions satisfying these conditions exist and have enough parameters, then we can make the required variation of T_- only and identify T_+ as the condensate in a coherent state formed from fluctuations of T and h (scalar mesons) around the ground state with broken chiral symmetry. Specializing to the x-independent case, after varying the on-shell action, then, gives us the condensate in the vacuum state. What we, therefore, need to do is to analyze the x-dependent case to see if the required solutions exist. This is what we will do next.

A. Action for (u, x)-dependent T and h

The full (u, x)-dependent action for tachyon and brane-antibrane separation is given by

$$S = -\frac{2V_4}{R^9}\int d^4x \int du u^{13/4} V(T)\sqrt{d_T}\sqrt{\det(1+K)}, \quad (27)$$

where K is the matrix with the elements

$$\begin{aligned} K^\mu{}_\nu &= \frac{f}{4Q}\partial^\mu h \partial_\nu h + \frac{u^{-3/2}}{Q}\partial^\mu T \partial_\nu T, \\ K^\mu{}_u &= \frac{f}{4Q} h' \partial^\mu h + \frac{u^{-3/2}}{Q} T' \partial^\mu T, \\ K^u{}_\mu &= \frac{f u^{3/2}}{4d_T} h' \partial_\mu h + \frac{1}{d_T} T' \partial_\mu T, \qquad K^u{}_u = 0. \end{aligned} \quad (28)$$

To look for a generalization of the x-independent solutions for equations of motion derived from this action, the most obvious thing to do is to generalize the earlier solutions by making all parameters functions of x. In particular, this means making u_0, the place where the flavor brane and antibrane meet, a function of x. For $u \sim u_0$, expansion of this solution around a constant u_0 is singular, since it involves arbitrary higher powers of $1/(u - u_0)$. Therefore, we do not expect analysis of (27) by expanding in small fluctuations around the x-independent solution to work for u close to u_0. This is confirmed by explicit fluctuation calculations in Appendix B. We need to go beyond small fluctuations analysis of (27), and this requires us to get an exact expression for the determinant in terms of space-time derivatives of T and h.

A direct calculation of $\det(1 + K)$ is tedious, but the calculation can be simplified using a trick which has been described in Appendix C, where a rather simple expression for the determinant has been obtained. The complete five-dimensional action then reads

$$S = -\frac{2V_4}{R^9}\int d^4x \int du u^{13/4} V(T)\sqrt{\Delta_T}, \quad (29)$$

where $\Delta_T = d_T\Delta$ and we have defined

$$\begin{aligned} \Delta \equiv 1 &+ \beta_1(\partial T)^2 + \beta_2(\partial h)^2 + 2\beta_3(\partial h.\partial T) \\ &+ \beta_4[(\partial T)^2(\partial h)^2 - (\partial h.\partial T)^2]. \end{aligned} \quad (30)$$

The β's are given by

$$\beta_1 = \frac{u^{-3/2}}{Q}\left(1 - \frac{T'^2}{d_T}\right), \qquad \beta_2 = \frac{f}{4Q}\left(1 - \frac{f u^{3/2} h'^2}{4d_T}\right),$$
$$\beta_3 = -\frac{f h' T'}{4Q d_T}, \qquad \beta_4 = \beta_1\beta_2 - \beta_3^2. \quad (31)$$

As a check on the action (29), we note that it reduces to the action (10) if T and h are x-independent. Also, it correctly reproduces the action (B1) which retains only terms that are quadratic in space-time derivatives of T and h. This latter action was derived independently by expanding $\det(1 + K)$ in powers of K and retaining only the first nontrivial correction.

The equations of motion that follow from the action (29) are rather complicated and have been derived in Appendix C, (C13) and (C14). As we did in the x-independent case, we will solve these equations in the two limiting cases of large u and $u \sim u_0$.

$u \to u_{\max}$: In this limit, $h(u, x)$ goes to a fixed value $h_0(x)$, which is assumed to be a slowly varying function of x. We will also assume that T and all its derivatives are small in this limit. Then Eqs. (C13) and (C14) can be approximated as

$$-(u^4 T'(u, x))' + (h_0(x))^2 u^4 T(u, x) = 0, \quad (32)$$

$$(u^{11/2} h'(u, x))' = 0. \quad (33)$$

The space-time derivatives are comparatively suppressed by powers of $1/u$ and hence have been ignored. These equations are identical to (15) and (17) and so have solutions similar to (16) and (18), but now with parameters that are functions of x:

[19] These boundary conditions are (i) vanishing tachyon and fixed brane-antibrane separation asymptotically and (ii) divergent tachyon and vanishing brane-antibrane separation at some point in the bulk.

$$T(u,x) = \frac{1}{u^2}(T_+(x)e^{-h_0(x)u} + T_-(x)e^{h_0(x)u}),$$
$$h(u,x) = h_0(x) - h_1(x)u^{-9/2}. \tag{34}$$

$u \to u_0$: The analysis in this limit is somewhat more involved. We assume an ansatz similar to the solutions (21) and (22), but now with x-dependent u_0 and coefficients:

$$h(u,x) = \rho_0(x)(u-u_0(x))^{1/2} + \rho_1(x)(u-u_0(x))^{3/2} + \cdots,$$
$$T(u,x) = \sigma_0(x)(u-u_0(x))^{-2} + \sigma_1(x)(u-u_0(x))^{-1} + \cdots. \tag{35}$$

As consequence of this ansatz, one can show that

$$\partial_\mu h = -h'\left[\partial_\mu u_0 - \frac{2\partial_\mu \rho_0}{\rho_0}(u-u_0) + \cdots\right], \tag{36}$$

$$\partial_\mu T = -T'\left[\partial_\mu u_0 + \frac{\partial_\mu \sigma_0}{2\sigma_0}(u-u_0) + \cdots\right]. \tag{37}$$

These relations are correct to the order shown. Putting all of this in the equation of motion for T, (C13), we see that this equation is satisfied to the leading order provided the following condition holds:

$$\frac{13}{4u_0} - \frac{\sqrt{\pi}}{2}\sigma_0\rho_0^2 = u_0^{-3/2}\partial_\mu(u_0^{-3/2}\partial^\mu u_0) - \frac{1}{2}u_0^{-3}\partial^\mu u_0 \frac{\partial_\mu(u_0^{-3}(\partial u_0)^2)}{1 + u_0^{-3}(\partial u_0)^2}. \tag{38}$$

In obtaining this we have set $f_0 = 1$. Similarly, from (C14) one gets the condition

$$\sigma_0 = \frac{\sqrt{\pi}}{4}(u_0^{3/2} + u_0^{-3/2}(\partial u_0)^2). \tag{39}$$

If u_0 is a constant independent of x, then from Eqs. (38) and (39) one gets

$$\sigma_0 = \frac{\sqrt{\pi}}{4}u_0^{3/2}, \qquad \rho_0 = \sqrt{\frac{26}{\pi u_0}}u_0^{-3/4}. \tag{40}$$

These reproduce the x-independent solutions in (21) and (22), remembering that we have set $f_0 = 1$. Let us now consider a small fluctuation around this constant solution. Linearizing Eqs. (38) and (39) in fluctuations, we get

$$\delta\sigma_0(x) = \frac{3\sqrt{\pi}}{8}u_0^{1/2}\delta u_0(x),$$
$$\delta\rho_0(x) = -\frac{4u_0^{-13/4}}{\sqrt{26\pi}}\left(\partial^2 + \frac{65}{8}u_0\right)\delta u_0. \tag{41}$$

Now, clearly we could choose the fluctuation $\delta u_0(x)$ to be such that $\delta\rho_0(x)$ vanishes. Under such an infinitesimal change of u_0, σ_0 would change, but not ρ_0. It is this kind of greater freedom in independently varying the parameters of the solution that we have wanted. Presumably in higher orders the situation gets better because there are more terms in the ansatz (35), and for each coefficient there is some freedom because of the space-time dependence. It would be nice to analyze the higher order terms, but that is beyond the scope of this work. Here we will assume that the introduction of space-time dependence as above can give us the required freedom to do the calculation of the condensate as follows.

Finally, let us compare the solution (40) and (41) with the solution obtained by the singular perturbation expansion in Appendix B, (B31). Expanding (20) around the constant u_0 solution to the lowest nontrivial order in $\epsilon \equiv (u-u_0)$ and comparing with (B28), we get the relations

$$\varphi_0(x) = 2\delta u_0(x),$$
$$\varphi_1(x) = \frac{1}{\sigma_0}(\delta\sigma_0(x) + \sigma_1\delta u_0(x)),$$
$$\vartheta_0(x) = -\frac{1}{2}\delta u_0(x),$$
$$\vartheta_1(x) = \frac{1}{\rho_0}\left(\delta\rho_0(x) - \frac{3}{2}\rho_1\delta u_0(x)\right). \tag{42}$$

These relations involve not only the leading order parameters (40) of the constant solution but also the nonleading parameters σ_1 and ρ_1, which are given by

$$\sigma_1 = \frac{\sigma_0}{6u_0}, \qquad \rho_1 = -\frac{5\rho_0}{8u_0}. \tag{43}$$

Using (40)–(43), one can show that the equations in (B31) are satisfied. This equivalence is, however, only formal. As we have argued above, the method given in this section is the correct one to use since it does not involve a singular expansion in arbitrarily high powers of $1/(u-u_0)$.

B. Condensate in terms of the tachyon solution

To derive an expression for the condensate, we calculate the variation of the action in (29) under a general variation of T and use the equation of motion (C13) to reduce it to a boundary term:

$$\delta S = -\frac{2V_4}{R^9}\int d^4x \frac{V(T)u^{13/4}}{\sqrt{d_T}}T'(u,x)\delta T(u,x)|_{u=u_{\max}}. \tag{44}$$

We have ignored terms with space-time derivatives because from now on we will be specializing to the x-independent case, except in the variation δT, so these terms will drop out. Only the UV boundary contributes to the on-shell action; there is no IR contribution because the tachyon potential vanishes exponentially for the diverging tachyon in the IR. We are interested only in retaining the variation $\delta T_-(x)$, so we set $\delta T_+(x)$ to zero. Doing this and using (34) in (44), we get the leading contribution for large $u_{\max}$:

$$\delta S \approx \frac{2h_0 V_4 V(0)}{R^9}(T_+ - T_- e^{2h_0 u_{\max}}) \int d^4x \delta T_-(x). \quad (45)$$

On-shell brane actions have UV divergences which need to be removed by the holographic renormalization procedure[20] to get finite answers for physical quantities. One adds boundary counterterms to the brane action to remove the divergences, following a procedure described in [44]. Our on-shell action (45) diverges as the cutoff is removed. This is because, as discussed in Sec. II D, we are keeping T_+ and $T_- e^{h_0 u_{\max}}$ fixed as the cutoff is removed and the last term in (45) diverges as $e^{h_0 u_{\max}}$ in this limit. The holographic renormalization procedure has been developed for examples with CFT boundary theories. Since, with the $D8$-branes present, there is no eleven-dimensional description available to us, it is not clear that the procedure described in [44] is applicable to the present case. We will proceed on the assumption that this is the case. Therefore, to subtract the UV divergent term in (45), we will add the following counterterm to the boundary action:

$$S_{\rm ct} = \frac{V_4 V(0)}{R^9} \int d^4x \sqrt{-\gamma} h(u,x) T^2(u,x)|_{u_{\max}}, \quad (46)$$

where $\gamma = -u_{\max}^8$ is the determinant of the metric on the eight-dimensional boundary orthogonal to the slice at $u = u_{\max}$. Note that the counterterms must be even in powers of the tachyon because of gauge symmetry. Using the solution (34) and retaining only the parameter $T_-(x)$, we find that the variation of the counterterm action is

$$\delta S_{\rm ct} = \frac{2h_0 V_4 V(0)}{R^9}(T_+ + T_- e^{2h_0 u_{\max}}) \int d^4x \delta T_-(x). \quad (47)$$

Adding to (45), the divergent term drops out, and we get the variation of the renormalized action

$$\delta S_{\rm renorm} \approx \frac{4h_0 V_4 V(0)}{R^9} T_+ \int d^4x \delta T_-(x). \quad (48)$$

Note that the variation of the renormalized action is twice as large as it would have been if we had simply dropped the divergent term[21] in (45).

We are now ready to calculate an expression for the chiral condensate in terms of the parameters of the tachyon solution. The parameters $T_\pm$ are dimensionless. To construct a parameter of dimension mass from T_-, we introduce a scale μ and define $m_q = \mu|T_-|$. Then, identifying the chiral condensate $\chi \equiv \langle \bar{q}_L q_R \rangle$, with

[20]For reviews, see [42,43].

[21]In (45), it is inconsistent to drop the term proportional to T_- in the limit of a large cutoff, holding T_+ and $T_- e^{h_0 u_{\max}}$ fixed. In fact, it is the T_- term that dominates in the action (45) in this limit. Taking a different limit that allows one to simply drop this term creates difficulties in the calculation of the pion mass; see Sec. V C. Consistency with the chiral condensate calculation then demands that the term proportional to $T_+ T_-$ be dropped in the pion mass calculation since it is smaller than the T_+^2 term.

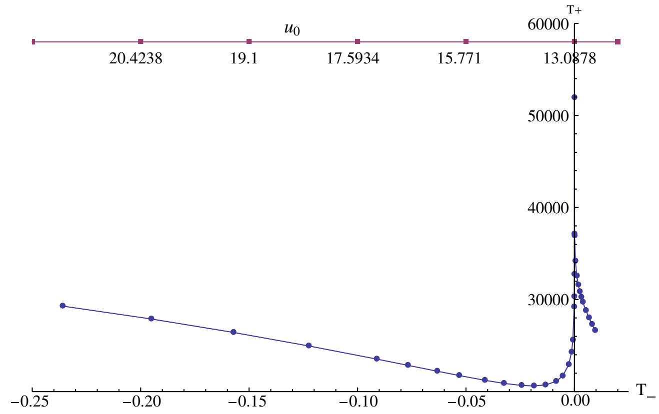

FIG. 12 (color online). T_+ as a function of T_-.

$\delta S_{\rm renorm}/\mu \delta T_-(x)$, we get

$$\chi \approx \frac{4h_0 V_4 V(0)}{\mu R^9} T_+. \quad (49)$$

We see that the parameter T_+ determines the condensate. Figure 12 shows a plot of T_+ as a function of T_- for $T_- \sim 0$. T_+ seems to attain a maximum value at $T_- = 0$ and drops off rapidly, at least for small values of $|T_-|$.

V. THE MESON SPECTRA

In this section we will discuss the spectra for various low spin mesons which are described by the fluctuations of the flavor branes around the classical solution.[22] The action for the fluctuations of the gauge fields can be computed from (3). Parametrizing the complex tachyon τ in terms of its magnitude and phase $\tau = Te^{i\theta}$, we get the following action, correct to second order in the fluctuations:

$$\Delta S_{\rm gauge} = -\int d^4x du [a(u) A_u^2 + b(u) A_\mu^2 + c(u)((F_{\mu\nu}^V)^2 + (F_{\mu\nu}^A)^2) + e(u) F_{\mu u}^A A^\mu + d(u)((F_{\mu u}^V)^2 + (F_{\mu u}^A)^2)], \quad (50)$$

$$a(u) = R^{-15} V_4 V(T) u^{13/4} \frac{T^2}{\sqrt{d_T}}, \quad (51)$$

$$b(u) = R^{-3} V_4 V(T) u^{7/4} \sqrt{d_T} \frac{T^2}{Q}\left(1 + \frac{f^2 T^2 h^2 h'^2}{4 d_T} u^3\right), \quad (52)$$

$$c(u) = \frac{R^3}{8} V_4 V(T) u^{1/4} \sqrt{d_T}, \quad (53)$$

[22]For a general review of mesons in gauge/gravity duals, see [45].

$$d(u) = R^{-9}V_4V(T)u^{7/4}\frac{Q}{4\sqrt{d_T}}, \tag{54}$$

$$e(u) = R^{-6}V_4V(T)u^{13/4}\frac{fT^2hh'}{2\sqrt{d_T}}. \tag{55}$$

Here $F^V_{\mu\nu}$ is the usual field strength for the vector gauge field $V = (A_1 + A_2)$, and $F^A_{\mu\nu}$ is the field strength for the gauge-invariant combination of the axial-vector field and the phase of the tachyon $A = (A_1 - A_2 - \partial\theta)$. However,

$$F^V_{\mu u} = -F^V_{u\mu} = \partial_\mu V_u - R^3\partial_u V_\mu, \quad F^A_{\mu u} = -F^A_{u\mu} = \partial_\mu A_u - R^3\partial_u A_\mu. \tag{56}$$

The relative factor of R^3 simply reflects the change of variables (9).

The gauge field $V_\mu(x, u)$ gives rise to a tower of vector mesons while the fields $A_\mu(x, u)$ and $A_u(x, u)$, which are gauge-invariant, give rise to towers of axial and pseudoscalar mesons. Notice that the coefficients $a(u)$, $b(u)$ and $e(u)$ vanish if the tachyon is set to zero. In the absence of the tachyon, the vector and axial-vector mesons acquire masses because of a nonzero $d(u)$, but there is always a massless "pion."[23] The presence of the tachyon is thus essential to give a mass to the pion. Also note that with the tachyon present, the masses of the vector and axial-vector mesons are in principle different.

A. Vector mesons

We will be using the gauge $V_u = 0$. Expanding in modes, we have

$$V_\mu(x, u) = \sum_m V^{(m)}_\mu(x)W_m(u), \tag{57}$$

where $\{W_m(u)\}$ form a complete set of basis functions. These satisfy orthonormality conditions which will be determined presently. The fields $\{V^{(m)}_\mu(x)\}$ form a tower of vector mesons in the physical $(3 + 1)$-dimensional space-time. In terms of these fields, the vector part of the action (50) takes the form

$$\Delta S^V_{\rm gauge} = -\int d^4x\sum_{m,n}[Q^V_{mn}F^{V(m)}_{\mu\nu}F^{V(n)\mu\nu} + L^V_{mn}V^{(m)}_\mu V^{(n)\mu}], \tag{58}$$

where $F^{V(m)}_{\mu\nu}$ are the usual $(3 + 1)$-dimensional $U(1)$-invariant field strengths for the vector potentials $\{V^{(m)}_\mu\}$. Also, we have defined

$$Q^V_{mn} = \int du c(u)W_m(u)W_n(u), \quad L^V_{mn} = R^6\int du d(u)W'_m(u)W'_n(u). \tag{59}$$

In addition, we choose the basis functions $\{W_m(u)\}$ to satisfy the eigenvalue equations

$$-R^6(d(u)W'_m(u))' = 2\lambda^V_m c(u)W_m. \tag{60}$$

Using these we see that

$$L^V_{mn} = \tfrac{1}{2}[R^6(d(u)W'_m(u)W_n(u))_{\partial u} + 2\lambda^V_m Q^V_{mn}] + m \leftrightarrow n, \tag{61}$$

where, as in the previous section, ∂u refers to boundaries in the u direction.

Note that a potential zero mode in the vector sector[24] can be gauged away using the residual symmetry of making u-independent gauge transformations, which is still available after fixing the gauge $V_u = 0$. This is because a zero mode in this sector can have only a single scalar degree of freedom. This follows from the requirement of finiteness of the action (58), which cannot be satisfied since the coefficient of the field strength term blows up for a zero mode. Hence its field strength must vanish, leaving behind only a longitudinal degree of freedom.

For the nonzero modes we may, without loss of generality, choose

$$Q^V_{mn} = \tfrac{1}{4}\delta_{mn}, \tag{62}$$

which, on using (61), gives

$$L^V_{mn} = \tfrac{1}{2}\lambda^V_m\delta_{mn}. \tag{63}$$

Using (62) and (63) in (58), we get

$$\Delta S^V_{\rm gauge} = -\int d^4x\sum_m\left[\frac{1}{4}F^{V(m)}_{\mu\nu}F^{V(m)\mu\nu} + \frac{1}{2}\lambda^V_m V^{(m)}_\mu V^{(m)\mu}\right]. \tag{64}$$

B. Axial-vector and pseudoscalar mesons

As we have already noted, A_μ and A_u are gauge-invariant. Expanding in modes, we have

$$A_\mu(x, u) = \sum_m A^{(m)}_\mu(x)P_m(u), \quad A_u(x, u) = \sum_m \phi^{(m)}(x)S_m(u), \tag{65}$$

where $\{P_m(u)\}$ and $\{S_m(u)\}$ form complete sets of basis functions. These satisfy orthonormality conditions which will be determined presently. The fields $\{A^{(m)}_\mu(x)\}$ and $\{\phi^{(m)}(x)\}$ form towers of axial-vector and pseudoscalar

[23]Strictly speaking, for the $U(1)$ case under discussion, this pseudoscalar is the η'. It is massless here because of the $N_c \to \infty$ limit in which we are working.

[24]A zero mode is defined as a mode which has zero eigenvalue and goes to a constant at infinity.

mesons in the physical (3 + 1)-dimensional space-time. In terms of these fields, the axial-vector and pseudoscalar part of the action (50) takes the form

$$\Delta S^A_{\rm gauge} = -\int d^4x \sum_{m,n}\Big[\frac{1}{2}\delta_{mn}\lambda^\phi_m \phi^{(m)}\phi^{(n)} + Q^A_{mn}F^{A(m)}_{\mu\nu}F^{A(n)\mu\nu} + L^A_{mn}A^{(m)}_\mu A^{(n)\mu} + K_{mn}\partial_\mu\phi^{(m)}\partial^\mu\phi^{(n)} + J_{mn}A^{(m)\mu}\partial_\mu\phi^{(n)}\Big], \tag{66}$$

where $F^{A(m)}_{\mu\nu}$ are the usual (3 + 1)-dimensional $U(1)$-invariant field strengths for the axial-vector potentials $\{A^{(m)}_\mu\}$. Also, we have defined

$$\begin{aligned} Q^A_{mn} &= \int du c(u)P_m(u)P_n(u),\\ L^A_{mn} &= \int du\Big(R^6 d(u)P'_m(u)P'_n(u) + \Big(b(u)+\frac{1}{2}R^3e'(u)\Big)P_m(u)P_n(u)\Big), \\ J_{mn} &= \int du(e(u)P_m(u) - 2R^3d(u)P'_m(u))S_n(u),\\ K_{mn} &= \int du d(u)S_m(u)S_n(u) \end{aligned} \tag{67}$$

and used the orthonormality condition in the pseudoscalar sector

$$\int du a(u)S_m(u)S_n(u) = \frac{1}{2}\lambda^\phi_m\delta_{mn}. \tag{68}$$

In addition, we choose the basis functions $\{P_m(u)\}$ to satisfy the eigenvalue equations

$$-R^6(d(u)P'_m(u))' + \Big(b(u)+\frac{1}{2}R^3e'(u)\Big)P_m(u) = 2\lambda^A_m c(u)P_m(u). \tag{69}$$

Using these we see that

$$L^A_{mn} = \tfrac{1}{2}[R^6(d(u)P'_m(u)P_n(u))_{\partial u} + 2\lambda^A_m Q^A_{mn}] + m\leftrightarrow n, \tag{70}$$

where, as before, ∂u refers to boundaries in the u direction.

We note that because of the last term in (66) the longitudinal component of $A^{(m)}_\mu$ and $\phi^{(m)}$ mix. So we need to define new field variables in terms of which the action (66) is diagonal. Before we do that, let us first note that the axial-vector potential $A_\mu(x,u)$ has a possible zero mode provided the corresponding (3 + 1)-dimensional field strength vanishes, for reasons explained in the previous subsection. Hence the zero mode, which we shall denote by $A^{(0)}_\mu$, can have only a longitudinal component. The zero mode is gauge-invariant and, because of its mixing with the pseudoscalars, plays a special role. Let us see this in some detail.

The zero mode $A^{(0)}_\mu$ is conjugate to the eigenfunction $P_0(u)$ which satisfies the equation

$$-R^6(d(u)P'_0(u))' + \Big(b(u)+\frac{1}{2}R^3e'(u)\Big)P_0(u) = 0. \tag{71}$$

If there is no solution to this equation, then the zero mode does not exist, and we should proceed directly to diagonalize the action (66). If, however, a solution $P_0(u)$ to this equation exists and is such that it goes to a constant at infinity, then the zero mode $A^{(0)}_\mu$ exists. Since it is purely longitudinal, for a reason identical to that discussed in the vector case, we make this explicit by writing it in terms of a pseudoscalar field: $A^{(0)}_\mu = \partial_\mu\phi^{(0)}(x)$. The terms in the action (66) which contain $\phi^{(0)}(x)$ can be separated out. These terms are

$$L^A_{00}\partial_\mu\phi^{(0)}\partial^\mu\phi^{(0)} + \sum_m J_{0m}\partial_\mu\phi^{(m)}\partial^\mu\phi^{(0)}.$$

The sums over the indices m and n no longer include the zero mode. Also, we have used $L^A_{m0} = L^A_{0m} = 0$ for $m \neq 0$, which follows from (70) using the fact that $\lambda^A_0 = 0$ and the boundary terms vanish because $P_m(u)$ vanishes sufficiently fast at infinity. Without loss of generality, we may choose $L^A_{00} = 1/2$ (to get the normalization of the kinetic term of $\phi^{(0)}$ right). Then, we can rewrite the above as

$$\frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{1}{2}\sum_{m,n}J_{0m}J_{0n}\partial_\mu\phi^{(m)}\partial^\mu\phi^{(n)}, \tag{72}$$

where $\pi \equiv (\phi^{(0)} + \sum_m J_{0m}\phi^{(m)})$.

With the zero modes explicitly separated out in this way, for the nonzero modes we may, without loss of generality, choose

$$Q^A_{mn} = \tfrac{1}{4}\delta_{mn}, \tag{73}$$

which, on using (70), gives

$$L^A_{mn} = \tfrac{1}{2}\lambda^A_m\delta_{mn}. \tag{74}$$

Putting (72)–(74) in the action (66), we get

$$\Delta S^A_{\rm gauge} = -\int d^4x\Big[\sum_m\Big(\frac{1}{2}\lambda^\phi_m\phi^{(m)}\phi^{(m)} + \frac{1}{4}F^{A(m)}_{\mu\nu}F^{A(m)\mu\nu} + \frac{1}{2}\lambda^A_m A^{(m)}_\mu A^{(m)\mu}\Big) + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi + \sum_{m,n}(\tilde K_{mn}\partial_\mu\phi^{(m)}\partial^\mu\phi^{(n)} + J_{mn}A^{(m)\mu}\partial_\mu\phi^{(n)})\Big], \tag{75}$$

where $\tilde K_{mn} = (K_{mn} - \frac{1}{2}J_{0m}J_{0n})$. The above action describes a massless particle π, besides other massive particles. The existence of this massless particle depends on the existence of a solution to Eq. (71), satisfying the normalization condition

$$R^6(d(u)P_0(u)P_0'(u))_{\partial u} = \tfrac{1}{2}. \quad (76)$$

Later we will see that the existence of the desired solution $P_0(u)$ depends on the absence of a non-normalizable part in $T(u)$.

To diagonalize the action (75) for the massive modes, we define the new variables

$$A_\mu^{(m)} = \tilde{A}_\mu^{(m)} - \sum_n (\lambda_m^A)^{-1} J_{mn} \partial_\mu \phi^{(n)}. \quad (77)$$

Putting in (75), we get

$$\Delta S_{\rm gauge}^A = -\int d^4x \Big[\sum_m \Big(\frac{1}{2}\lambda_m^\phi \phi^{(m)}\phi^{(m)} + \frac{1}{4} F_{\mu\nu}^{A(m)} F^{A(m)\mu\nu} + \frac{1}{2}\lambda_m^A \tilde{A}_\mu^{(m)} \tilde{A}^{(m)\mu} \Big) + \frac{1}{2}\partial_\mu \pi \partial^\mu \pi + \sum_{m,n} K'_{mn} \partial_\mu \phi^{(m)} \partial^\mu \phi^{(n)} \Big], \quad (78)$$

where $K'_{mn} = (\tilde{K}_{mn} - \frac{1}{2}\sum_p (\lambda_p^A)^{-1} J_{pm} J_{pn})$. The modes have now been decoupled. To get the standard action for massive pseudoscalars we may, without loss of generality, set

$$K'_{mn} = \frac{1}{2}\delta_{mn} = K_{mn} - \frac{1}{2} J_{0m} J_{0n} - \frac{1}{2}\sum_p (\lambda_p^A)^{-1} J_{pm} J_{pn}. \quad (79)$$

This condition can be rewritten in a more conventional form as follows. We define

$$\psi_m(u) \equiv \sum_n (\lambda_n^A)^{-1} P_n(u) J_{nm} + P_0(u) J_{0m} \quad (80)$$

and using (69) note that it satisfies the equation

$$-R^6(d(u)\psi_m'(u))' + (b(u) + \tfrac{1}{2}R^3 e'(u))\psi_m(u) = \tfrac{1}{2}e(u)S_m(u) + R^3(d(u)S_m(u))'. \quad (81)$$

Using (80) in (79), we get

$$\delta_{mn} = \int du \Big(d(u)S_m(u)(S_n(u) + R^3\psi_n'(u)) - \frac{1}{2}e(u)S_m(u)\psi_n(u) \Big) + m \leftrightarrow n. \quad (82)$$

In terms of new variables defined by

$$S_m(u) \equiv R^3\eta_m'(u), \qquad \theta_m(u) \equiv \psi_m(u) + \eta_m(u), \quad (83)$$

(82) can be written as

$$\int du\, \eta_m'(u) \Big(R^6 d(u)\theta_n'(u) - \frac{1}{2}R^3 e(u)(\theta_n(u) - \eta_n(u)) \Big) + m \leftrightarrow n = \delta_{mn}. \quad (84)$$

Moreover, in terms of these variables the differential equation (81) reads

$$-R^6(d(u)\theta_m'(u))' + (b(u) + \tfrac{1}{2}R^3 e'(u))(\theta_m(u) - \eta_m(u)) - \tfrac{1}{2}R^3 e(u)\eta_m'(u) = 0. \quad (85)$$

From these two equations one can obtain the orthonormality condition

$$\int du \Big(R^6 d(u)\theta_m'(u)\theta_n'(u) + \Big(b(u) + \frac{1}{2}R^3 e'(u) \Big) \times (\theta_m(u) - \eta_m(u))(\theta_n(u) - \eta_n(u)) - \frac{1}{2}R^3 e(u)\eta_m'(u)(\theta_n(u) - \eta_n(u)) - \frac{1}{2}R^3 e(u)\eta_n'(u)(\theta_m(u) - \eta_m(u)) \Big) = \frac{1}{2}\delta_{mn}. \quad (86)$$

Also, rewriting (68) in terms of the new variables, we have

$$R^6 \int du\, a(u)\eta_m'(u)\eta_n'(u) = \frac{1}{2}\lambda_m^\phi \delta_{mn}. \quad (87)$$

Finally, (84) and (87) give

$$R^6 a(u)\eta_n'(u) = \lambda_n^\phi (R^6 d(u)\theta_n'(u) - \tfrac{1}{2}R^3 e(u)(\theta_n(u) - \eta_n(u))). \quad (88)$$

Equations (85) and (88) are the final form of the eigenvalue equations, and (86) and (87) are the orthonormality conditions in the pseudoscalar sector.

It is interesting to note from (85) that if η is constant, then the variable $(\theta - \eta)$ satisfies a differential equation that is identical to Eq. (71) satisfied by the zero mode P_0. Also, using (85) and (86) one can show that, for constant η, $(\theta - \eta)$ satisfies the normalization condition (76). From (88) it follows that, if η is constant, the eigenvalue λ^ϕ vanishes. Thus, the presence of a massless pseudoscalar can be naturally considered to be identical to the question of the existence of a solution to Eqs. (85)–(88) with zero eigenvalue, and so it becomes a part of the spectrum in the pseudoscalar tower of states. Hence, the action in this sector can be written in the form

$$\Delta S_{\rm gauge}^A = -\int d^4x \sum_m \Big[\frac{1}{4} F_{\mu\nu}^{A(m)} F^{A(m)\mu\nu} + \frac{1}{2}\lambda_m^A \tilde{A}_\mu^{(m)} \tilde{A}^{(m)\mu} + \frac{1}{2}\partial_\mu \phi^{(m)} \partial^\mu \phi^{(m)} + \frac{1}{2}\lambda_m^\phi \phi^{(m)}\phi^{(m)} \Big]. \quad (89)$$

Note that we have dropped the field $\pi(x)$ but extended the sum over m to cover a possible zero mode as well. If there is a solution to Eqs. (85)–(88) with constant η_0 and $\lambda_0^\phi = 0$, then a massless pion field will reappear as the zero mode $\phi^{(0)}$ in the pseudoscalar tower. Otherwise, the lowest mode in this sector will be massive, whose mass can be computed as in the following subsection.

C. Relation between pion mass and non-normalizable part of tachyon

In this subsection we will derive a relation between the pion mass and the non-normalizable part of the tachyon parametrized by T_-. This will give us further evidence for identifying the parameters T_+ and T_- with the chiral condensate and quark mass, respectively. We first note that, for $T(u) = 0$, $a(u)$ vanishes and hence λ_m^ϕ also vanishes by (88). However, as we will see from the following calculations, $T(u) = 0$ is a sufficient condition but not necessary to guarantee the presence of a massless pion. The necessary condition is that the non-normalizable piece in $T(u)$ should be absent, i.e. $T_- = 0$.

Let us assume that $T(u) \neq 0$ so that $a(u) \neq 0$. Then, (88) can be used to solve for $\eta_m(u)$ in terms of $\psi_m(u)$, which is related to $\theta_m(u)$ and $\eta_m(u)$ by (83). We get

$$\eta_m'(u) = \frac{\lambda_m^\phi}{a(u) - \lambda_m^\phi d(u)}\left(d(u)\psi_m'(u) - \frac{e(u)}{2R^3}\psi_m(u)\right). \tag{90}$$

Let us now denote by λ_0^ϕ the lowest mass eigenvalue. The corresponding eigenfunctions are $\psi_0(u)$ and $\eta_0(u)$. Assuming $\lambda_0^\phi \ll a(u)/d(u)$,[25] we can approximate the above equation for $\eta_0(u)$:

$$\eta_0'(u) \approx \frac{\lambda_0^\phi}{a(u)}\left(d(u)\psi_0'(u) - \frac{e(u)}{2R^3}\psi_0(u)\right). \tag{91}$$

If we know $\psi_0(u)$, then using the above in (87) we can compute the mass. Now, $\psi_0(u)$ satisfies the following differential equation, which can be obtained from (85) using (91) and the approximation $\lambda_0^\phi \ll a(u)/d(u)$:

$$-R^6(d(u)\psi_0'(u))' + (b(u) + \tfrac{1}{2}R^3 e'(u))\psi_0(u) \approx 0. \tag{92}$$

Also, using (92) and the approximation under which it was obtained, the normalization condition on $\psi_0(u)$ given by (86) can be approximated as

$$R^6 d(u)\psi_0'(u)\psi_0(u)|_{u=u_{\max}} \approx \tfrac{1}{2}. \tag{93}$$

These equations cannot be solved analytically in general. However, analytic solutions can be obtained in the IR and UV regimes. In the UV regime, for $u \lesssim u_{\max}$, we use (16) and (18) to approximate the coefficients in (92); we get

$$b(u) \approx \frac{V_4 V(0)}{R^3} u T^2(u), \qquad d(u) \approx \frac{V_4 V(0)}{4R^9} u^{5/2}, \qquad e(u) \approx \frac{9V_4 V(0)}{4R^6} h_0 h_1 u^{-3/2} T^2(u). \tag{94}$$

[25] This approximation can be justified *a posteriori* by the solution because the eigenvalue λ_0^ϕ turns out to be parametrically much smaller by a factor of $1/R^3$ [see (105)] compared to the ratio $a(u)/d(u)$.

In writing these, we have used $f(u) \approx 1$, which is a good approximation for large u. We see that we can clearly neglect $e(u)$ compared to $b(u)$ in (92), while $b(u)$ is itself negligible compared to $d(u)$. Using these approximations in (92) and (93) then gives

$$-(u^{5/2}\psi_0'(u))' \approx 0, \qquad \frac{V_4 V(0)}{4R^3} u^{5/2}\psi_0'(u)\psi_0(u)|_{u=u_{\max}} \approx \frac{1}{2}, \tag{95}$$

which are solved by

$$\psi_0(u) \approx c_0 - \frac{1}{3c_0}\frac{4R^3}{V_4 V(0)} u^{-3/2}. \tag{96}$$

Here c_0 is a parameter which is related to the pion decay constant. This can be argued by analyzing the 4D axial current correlator and using AdS/CFT along the lines of [46,47]. Using the AdS/CFT dictionary, one can compute the axial current correlator from the action (66), evaluated on shell, by differentiating twice with respect to the transverse part of the axial-vector field on the UV boundary. This is the source which couples to the axial current on the boundary. The source arises from the same zero mode solution $P_0(u)$ which we discussed in connection with a possible zero mode (the pion) in the longitudinal component of the axial gauge field. $P_0(u)$ satisfies Eq. (71), which is identical to that satisfied by $\psi_0(u)$, (92). However, the boundary condition now is different; it is the boundary condition for a source $P_0(u_{\max}) = 1$. In addition, one imposes the condition

$$R^6 d(u) P_0'(u) P_0(u)|_{u=u_{\max}} \approx \frac{f_\pi^2}{2}, \tag{97}$$

which is required to reproduce the correct zero momentum axial current correlator [46,47]. This follows from the action (66). Now, $P_0(u)$ satisfies (71) and the condition (97) if we set $P_0(u) = f_\pi \psi_0(u)$. Then, requiring $P_0(u_{\max}) = 1$ gives $c_0 = 1/f_\pi$.

In the IR regime $u \gtrsim u_0$, we use (21) and (22) to approximate the coefficients in (92); we get

$$b(u) \approx \frac{\pi^{3/2} V_4 u_0^{17/4}}{26R^3}\frac{V(T)}{(u-u_0)^4}, \qquad d(u) \approx \frac{13V_4 u_0^{9/4}}{32\sqrt{\pi}R^9} V(T), \qquad e(u) \approx \frac{13V_4 u_0^{9/4}}{16\sqrt{\pi}R^6}\frac{V(T)}{(u-u_0)}. \tag{98}$$

In writing these, we have used $f(u_0) \approx 1$, which is a good approximation for large u_0. Using $dV(T)/du = T'(u)V'(T)$, we see that $b(u)$ and $R^3 e'(u)$ both go as $(u-u_0)^{-4}$ in this regime. However, the coefficient of the latter is suppressed by a relative factor of $u_0^{-1/2}$, so for large u_0 we may neglect it compared to $b(u)$. But, unlike in the UV regime, $b(u)$ cannot be neglected compared to $d(u)$. In fact, this term is crucial for getting a nontrivial solution. In this regime, then, the leading terms in Eq. (92) give

$$\psi_0'(u) \approx \frac{32\pi R^6 u_0^{1/2}}{169} \frac{\psi_0(u)}{(u-u_0)}, \qquad (99)$$

which has the solution

$$\psi_0(u) \approx \tilde{c}_0 (u-u_0)^{(32\pi R^6 u_0^{1/2})/169}, \qquad (100)$$

where $\tilde{c}_0$ is an integration constant. Note that the normalization condition remains unchanged and cannot be used here because it receives a contribution only from the UV end due to the exponentially vanishing tachyon potential for large $T(u)$ at the IR end.

Let us now consider the formula (87), for the lowest mode, using which one can compute the eigenvalue λ_0^ϕ:

$$R^6 \int_{u_0}^{u_{\max}} du a(u)(\eta_0'(u))^2 = \frac{1}{2}\lambda_0^\phi. \qquad (101)$$

Using $a(u) \approx \frac{\sqrt{\pi}V_4 u_0^{19/4}}{8R^{15}} \frac{V(T)}{(u-u_0)}$ in the IR and (100) in (91), we see that $\eta_0'(u) \propto \psi_0(u)$ vanishes very rapidly as $u \to u_0$, with a power which grows as $u_0^{1/2}$ for large u_0. Moreover, since $V(T)$ vanishes exponentially for large T, the IR region makes a negligible contribution to the integral. Therefore, it is reasonable to calculate the integral by substituting the UV estimate of the integrand in it. In the UV region, $a(u) \approx \frac{V_4 V(0)}{R^{15}} u^4 T^2(u)$. Moreover, in this region the second term on the right-hand side of (91) can be neglected. So, we get

$$\begin{aligned}\frac{1}{2}\lambda_0^\phi &= R^6 \int_{u_0}^{u_{\max}} du a(u)(\eta_0'(u))^2 \\ &\approx R^6(\lambda_0^\phi)^2 \int_{\tilde{u}_0}^{u_{\max}} du \frac{d^2(u)}{a(u)}(\psi_0'(u))^2 \\ &\approx (\lambda_0^\phi)^2 \kappa \int_{\tilde{u}_0}^{u_{\max}} \frac{h_0 du}{(T_+ e^{-h_0 u} + T_- e^{h_0 u})^2},\end{aligned}$$

where $\tilde{u}_0 > u_0$ avoids the IR region in the integral and we have defined

$$\kappa \equiv \frac{f_\pi^2 R^9}{4h_0 V_4 V(0)}. \qquad (102)$$

The integral is easily done, giving

$$\lambda_0^\phi \approx \frac{1}{\kappa} \frac{(T_+ e^{-h_0\tilde{u}_0} + T_- e^{h_0\tilde{u}_0})(T_+ e^{-h_0 u_{\max}} + T_- e^{h_0 u_{\max}})}{e^{h_0(u_{\max}-\tilde{u}_0)} - e^{-h_0(u_{\max}-\tilde{u}_0)}}. \qquad (103)$$

From our numerical solutions we see that it is possible to choose $\tilde{u}_0$ to be relatively large and also satisfy the conditions $|T_+|e^{-h_0\tilde{u}_0} \gg |T_-|e^{h_0\tilde{u}_0}$ and $e^{h_0(u_{\max}-\tilde{u}_0)} \gg e^{-h_0(u_{\max}-\tilde{u}_0)}$. For such values of the parameters, then, to a good approximation (103) gives

$$\lambda_0^\phi \approx \frac{1}{\kappa}(T_+T_- + T_+^2 e^{-2h_0 u_{\max}}). \qquad (104)$$

Now, let us tune $u_{\max}$ to large values. We will do this in a manner consistent with the inequality (19). As explained in Sec. II D, one way of maintaining this inequality is to keep $|T_+|$ and $|T_-|e^{h_0 u_{\max}}$ fixed as $u_{\max}$ becomes large. In that case, the second term on the right-hand side of (104) becomes exponentially smaller than the first term as the cutoff is increased beyond some value. We may then neglect this term compared with the first term. This gives

$$\lambda_0^\phi \approx \frac{1}{\kappa} T_+ T_-. \qquad (105)$$

Finally, using $\lambda_0^\phi = m_\pi^2$ and (49) in this relation, we get

$$m_\pi^2 \approx \frac{m_q \chi}{f_\pi^2}. \qquad (106)$$

This is the well-known Gell-Mann-Oakes-Renner formula, up to a factor of 2.

VI. SUMMARY AND DISCUSSION

This paper further explores our proposal [21] of a modified SS model, which includes the degree of freedom associated with the open string tachyon between the flavor branes and antibranes. Here we have extended the analytic treatment of various aspects of the problem and supplemented it with extensive numerical calculations. We have argued that taking the tachyon into account is essential for the consistency of the setup and shown numerically that the solution which includes the tachyon is energetically favored. Our modification preserves the nice geometric picture of chiral symmetry breaking of the SS model and at the same time relates chiral symmetry breaking to tachyon condensation; the tachyon becomes infinitely large in the infrared region where the joining of the flavor branes signals chiral symmetry breaking.

We have identified a parameter in the non-normalizable part of the tachyon field profile with the quark mass. It is important to stress that this is the only tunable parameter in the modified SS model. It can be traded for the asymptotic brane-antibrane separation or the location of the point in the bulk where the brane and antibrane join. This provides a natural explanation for the latter parameter, which is also present in the SS model, but in that model it does not find any counterpart in the QCD-like theory at the boundary. In this paper we have presented numerical evidence to show that the point where the brane and antibrane meet is monotonically shifted towards ultraviolet as we tune the mass parameter to larger values. It would seem, therefore, that in our model a brane-antibrane pair disappears from the bulk consistently with a quark flavor becoming infinitely massive.

The presence of a non-normalizable part in the tachyon solution requires us to introduce an ultraviolet cutoff. The cutoff is needed not only because this part grows as one moves towards the ultraviolet region, as in any standard AdS/CFT example that includes a non-normalizable solution, but also because the asymptotic form of the solution is derived from an approximate equation which is valid only for small values of the tachyon. Therefore, the asymptotic solution itself is not valid beyond a certain maximum value of the holographic coordinate. We have presented sufficient numerical evidence of this phenomenon. Removing the ultraviolet cutoff, then, requires tuning the mass parameter to zero. We have explained one scheme by which this can be done. This scheme gives an exponential dependence on the cutoff to the mass parameter, similar to that discussed recently in [28]. The quark mass arises from an apparently very different mechanism in this work, and the cutoff is related to the location of a $D6$-brane that is present in this model. It would be interesting to see if there is any connection between this model and our model.

Once we have identified the quark mass as a parameter in the non-normalizable part of the tachyon, it is natural to expect, by the usual AdS/CFT rules, the normalizable part of the tachyon solution to give rise to the chiral condensate. To derive an expression for it, however, we need to go beyond the space-time-independent solutions of Sec. II. As we have seen, this requires an exact expression for the five-dimensional action for tachyon and brane-antibrane separation fields which are now taken to depend on space-time as well as the holographic coordinate. We have derived this action in this paper. Using the generalized solutions to the equations for this action, then, one can compute the chiral condensate. However, one also needs to add counterterms to the boundary brane action to remove from it contributions that diverge when the cutoff is removed.

We have studied in detail the fluctuations of flavor gauge fields on the brane-antibrane system. These give rise to vector, axial-vector and pseudoscalar towers of mesons, which become massive through a kind of Higgs mechanism, except for the pions. These arise from a gauge-invariant combination of the tachyon phase and the longitudinal zero mode of the axial-vector field. We have shown that the pions remain massless, unless a quark mass (non-normalizable part of the tachyon solution) is switched on. For a small quark mass, we have derived an expression for the mass of the lowest pseudoscalar meson in terms of the chiral condensate and shown that it satisfies the Gell-Mann-Oakes-Renner relation. The vector and axial-vector spectra are expected to be nondegenerate because they arise from eigenvalue equations with different tachyon contributions. We have not computed these spectra, but it would be interesting to see whether they have the Regge behavior for large masses.

A nonzero quark mass is essential to correctly reproduce phenomenology in the low-energy sector of QCD. Therefore, our modified SS model can be the starting point of a more quantitative version of the phenomenology initiated in [1]. For this purpose, our treatment needs to be extended to the non-Abelian case, which should be a straightforward exercise. The correct tachyon brane-antibrane action for curved directions transverse to the branes is not known. It is important to have such an action since this would extend the applicability of the present treatment to such interesting cases as e.g. the antipodal configuration of the flavor brane system and its connection with massless quarks. Another direction in which the present ideas can be extended is to discuss this model at finite temperature and describe the chiral symmetry restoration transition and study the phase diagram in some detail. The connection of chiral symmetry breaking with tachyon condensation seems fascinating, and a deeper understanding would be useful. Finally, baryons have been discussed in the SS model. It turns out that they have a very small size. This may change in the presence of the tachyon. This is because in the presence of the tachyon, the flavor energy momentum tensor is concentrated away from the infrared region where the branes meet. In other words, there is a new scale provided by the quark mass. It would be very interesting to investigate whether this effect makes any difference to the baryon size.

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APPENDIX A: OVERLAPPING $D8$-$\overline{D8}$-BRANE SYSTEM

In this case the appropriate Dirac-Born-Infeld action is

$$S = -\int d^9\sigma g_s V(T) e^{-\phi}(\sqrt{-\det A_L} + \sqrt{-\det A_R}),$$

$$(A_i)_{ab} = g_{MN}\partial_a x_i^M \partial_b x_i^N + F^i_{ab} + \tfrac{1}{2}((D_a\tau(D_b\tau)^* + (D_a\tau)^* D_b\tau)), \tag{A1}$$

where $D_a\tau = \partial_a\tau - i(A_{L,a} - A_{R,a})\tau$. The classical equation for the profile of the magnitude T of the tachyon τ can be obtained from (13) by substituting $h = 0$ in it everywhere. We get

$$\left(\frac{u^{13/4}}{\sqrt{d_T}} T'(u)\right)' = \frac{u^{7/4} f(u)^{-1}}{\sqrt{d_T}} \frac{V'(T)}{V(T)}, \tag{A2}$$

where now $d_T = f(u)^{-1}u^{-3/2} + T'(u)^2$. In the UV region, assuming T is small for large u, we can approximate this equation as

$$(u^4 T'(u))' = -\pi u^{5/2} T(u), \tag{A3}$$

where we have used the universal small T expansion $V(T) = \mathcal{T}_8(1 - \frac{\pi}{2}T^2 + \cdots)$. The general solution[26] to this equation is

[26]Equation (A3) can be solved exactly in terms of the Bessel functions $H^{(1)}$ and $H^{(2)}$. Here we give only the leading term.

$$T(u) = u^{-13/8}(c_1 \cos(4\sqrt{\pi}u^{1/4}) + c_2 \sin(4\sqrt{\pi}u^{1/4})) + \cdots, \tag{A4}$$

where c_1 and c_2 are arbitrary constants. Both of the independent solutions in this case are normalizable, so the interpretation of one of the parameters corresponding to a source for the quark mass term is not clear. In view of this, it is not clear how to apply the general treatment of [19] to this case.

In the IR region, a singular tachyon solution is obtained only for $u \sim u_k$. In this region $f(u)^{-1}$ blows up as $(u - u_k)^{-1}$, and this drives a singularity in the tachyon. Both of the potentials in (5) and (6) exhibit singular solutions, although the solutions and the nature of the singularity are different. For the potential (5) we find the solution

$$T(u) = \left(\pi + \frac{39}{2\sqrt{u_k}}\right)^{-1/2} \ln\frac{1}{(u - u_k)} + b_1 + \cdots, \tag{A5}$$

while for (6) we get

$$T(u) = b_2(u - u_k)^{-\alpha} + \cdots, \tag{A6}$$

where b_1 and b_2 are arbitrary constants and $\alpha = \frac{4\pi\sqrt{u_k}}{39}$. As in the case with nonzero brane-antibrane separation, the IR solution for which the tachyon blows up exhibits a smaller number of independent parameters than the UV solution, one in the IR as opposed to two in the UV in the present case. A solution with two independent parameters in the IR exists (for any potential), but this solution is finite:

$$T(u) = T_0 + T_1(u - u_k)^{1/2} + \left(\frac{2}{3\sqrt{u_k}} + \frac{T_1^2}{2}\right) \times \frac{V'(T_0)}{V(T_0)}(u - u_k) + \cdots. \tag{A7}$$

Here T_0 and T_1 are the two arbitrary parameters.

APPENDIX B: SCALAR FLUCTUATIONS

Here we will assume that $T(u, x)$ and $h(u, x)$ are weakly dependent on x^μ and expand $\det(1 + K)$ in (27) in powers of space-time derivatives. The action correct to quadratic terms in the derivatives is

$$S = -\frac{2V_4}{R^9}\int d^4x \int du u^{13/4} V(T)\sqrt{d_T}\Big[1 + \frac{u^{-3/2}}{2Q} \times \Big\{\Big(1 - \frac{T'^2}{d_T}\Big)(\partial T)^2 + \Big(1 - \frac{\frac{1}{4}fu^{3/2}h'^2}{d_T}\Big)\frac{1}{4}fu^{3/2}(\partial h)^2 - \frac{fu^{3/2}h'T'}{2d_T}(\partial h).(\partial T)\Big\}\Big], \tag{B1}$$

where d_T is given by (11), with $T(u)$ replaced by $T(u, x)$ and $h(u)$ by $h(u, x)$. Also, the notation $(\partial T)^2$ stands for $\eta^{\mu\nu}\partial_\mu T(u, x)\partial_\nu T(u, x)$; similar expressions hold for $(\partial h)^2$ and $(\partial h).(\partial T)$. For the expansion in derivatives to be valid, we must require the following conditions to be satisfied: (i) For large values of u, near the cutoff $u_{\rm max}$, we must have $|\partial T| \ll u_{\rm max}^{3/4}$ and $|\partial h| \ll 1$; (ii) for $u \sim u_0$, we must have $|\partial T| \ll |T'| \sim (u - u_0)^{-3}$ and $|\partial h| \ll |hT| \sim (u - u_0)^{-3/2}$.

Let us now consider small fluctuations around the x-independent solutions. We write $T(u, x) = T_c(u) + T_q(u, x)$ and $h(u, x) = h_c(u) + h_q(u, x)$, where $T_c(u)$ and $h_c(u)$ are the x-independent solutions of the classical equations (13) and (14). We now expand the above action and retain only terms up to second order in the fluctuations $T_q(u, x)$ and $h_q(u, x)$. We get

$$\begin{aligned} S = -\frac{2V_4}{R^9}\int d^4x \int_{u_0}^\infty du A\sqrt{d_c}\Big[1 &+ \Big\{\frac{V_c'}{V_c}T_q + \frac{1}{d_c}\Big(\frac{1}{4}fu^{3/2}h_c'h_q' + T_c'T_q' + h_cT_c^2h_q + h_c^2T_cT_q\Big)\Big\} \\ &+ \Big\{\frac{V_c''}{2V_c}T_q^2 + \frac{V_c'}{V_cd_c}\Big(\frac{1}{4}fu^{3/2}h_c'h_q'T_q + T_c'T_q'T_q + h_cT_c^2h_qT_q + h_c^2T_cT_q^2\Big) \\ &+ \frac{1}{2d_c}\Big(\frac{1}{4}fu^{3/2}h_q'^2 + T_q'^2 + T_c^2h_q^2 + h_c^2T_q^2 + 4h_cT_ch_qT_q\Big) - \frac{1}{2d_c^2}\Big(\frac{1}{4}fu^{3/2}h_c'h_q' + T_c'T_q' + h_cT_c^2h_q + h_c^2T_cT_q\Big)^2 \\ &+ \frac{u^{-3/2}}{2Q_0}\Big(\Big(1 - \frac{T_c'^2}{d_c}\Big)(\partial T_q)^2 - \frac{fu^{3/2}}{2d_c}h_c'T_c'(\partial h_q).(\partial T_q) + \Big(1 - \frac{\frac{1}{4}fu^{3/2}h_c'^2}{d_c}\Big)\frac{1}{4}fu^{3/2}(\partial h_q)^2\Big)\Big\} + \cdots\Big], \end{aligned} \tag{B2}$$

where we have used the notation $V_c = V(T_c)$, $d_c = d_{T_c}$, and $A = u^{13/4}V_c$. As before, a prime denotes derivative with respect to u, except on V_c, for which it denotes a derivative with respect to its argument. The part of this action linear in fluctuations S_1, which arises from the term in the first curly brackets above, is given by

$$S_1 = -\frac{2V_4}{R^9}\int d^4x \int_{u_0}^\infty du A\Big[\frac{V_c'}{V_c}\sqrt{d_c}T_q + \frac{1}{\sqrt{d_c}}\Big(\frac{1}{4}fu^{3/2}h_c'h_q' + T_c'T_q' + h_cT_c^2h_q + h_c^2T_cT_q\Big)\Big]. \tag{B3}$$

It is easy to verify that S_1 leads to the background equations (13) and (14). This part of the action, therefore, vanishes, except for a boundary term. It is this boundary term that gives rise to the chiral condensate.

The term in the second curly brackets becomes S_2, the action quadratic in fluctuations, after some manipulations. First, we open the square in the coefficient of $1/2d_c^2$ term and combine it with the term just before it. That is, we have

$$\frac{1}{2d_c}\left(\frac{1}{4}fu^{3/2}h_q'^2 + T_q'^2 + T_c^2h_q^2 + h_c^2T_q^2 + 4h_cT_ch_qT_q\right) - \frac{1}{2d_c^2}\left(\frac{1}{4}fu^{3/2}h_c'h_q' + T_c'T_q' + h_cT_c^2h_q + h_c^2T_cT_q\right)^2$$
$$= \frac{1}{2d_c}\left\{\left(1 - \frac{\frac{1}{4}fu^{3/2}h_c'^2}{d_c}\right)\frac{1}{4}fu^{3/2}h_q'^2 + \left(1 - \frac{T_c'^2}{d_c}\right)T_q'^2 + \left(1 - \frac{h_c^2T_c^2}{d_c}\right)(h_c^2T_q^2 + T_c^2h_q^2) + 2\left(2 - \frac{h_c^2T_c^2}{d_c}\right)h_cT_ch_qT_q\right\}$$
$$- \frac{1}{d_c^2}\left\{\frac{1}{4}fu^{3/2}h_c'(T_c'h_q'T_q' + T_c^2h_ch_qh_q' + h_c^2T_ch_q'T_q) + T_c'h_cT_c(T_cT_q'h_q + h_cT_q'T_q)\right\}. \tag{B4}$$

Furthermore, we can rewrite

$$A\frac{V_c'}{V_c\sqrt{d_c}}T_c'T_qT_q' \sim V_c'\left(\frac{u^{13/4}T_c'}{\sqrt{d_c}}\right)\left(\frac{T_q^2}{2}\right)' \to -A\sqrt{d_c}\left[\frac{V_c''}{V_c}\frac{T_c'^2}{d_c} + \frac{V_c'}{V_c}\left(\frac{h_c^2T_c}{d_c} + \frac{V_c'}{V_c}\left(1 - \frac{T_c'^2}{d_c}\right)\right)\right]\frac{T_q^2}{2}, \tag{B5}$$

where in the last step we have done an integration by parts over u, used the equation of motion (13) for T_c, h_c and ignored a possible boundary term since it is quadratic in fluctuations and so will not contribute to the calculation of the condensate. A similar manipulation gives

$$-\frac{A}{d_c\sqrt{d_c}}T_c'h_c^2T_cT_qT_q' \sim -V_c\left(\frac{u^{13/4}T_c'}{\sqrt{d_c}}\right)\left(\frac{h_c^2T_c}{d_c}\right)\left(\frac{T_q^2}{2}\right)' \to A\sqrt{d_c}\left[\left(\frac{V_c'}{V_c} + \frac{h_c^2T_c}{d_c}\right)\frac{h_c^2T_c}{d_c} + \frac{T_c'}{d_c}\left(\frac{h_c^2T_c}{d_c}\right)'\right]\frac{T_q^2}{2}. \tag{B6}$$

Combining the above with the other three $T_q^2/2$ terms, we find its net coefficient to be

$$A\left\{\left(\frac{V_c''}{V_c} - \left(\frac{V_c'}{V_c}\right)^2\right)\left(1 - \frac{T_c'^2}{d_c}\right)\sqrt{d_c} + 2\frac{V_c'}{V_c}\frac{h_c^2T_c}{\sqrt{d_c}} + \frac{h_c^2}{\sqrt{d_c}} + \frac{T_c'}{\sqrt{d_c}}\left(\frac{h_c^2T_c}{d_c}\right)'\right\}. \tag{B7}$$

Similarly, a partial integration using the equation of motion (14) allows us to combine the two $h_q^2/2$ terms, giving its net coefficient to be

$$A\left\{\left(\frac{h_cT_c^2}{d_c}\right)'\frac{\frac{1}{4}fu^{3/2}h_c'}{\sqrt{d_c}} + \frac{T_c^2}{\sqrt{d_c}}\right\}. \tag{B8}$$

Collecting all of this together, we get the action quadratic in fluctuations:

$$S_2 = -\frac{2V_4}{R^9}\int d^4x\int_{u_0}^{\infty}duA\left[\frac{1}{2}c_1T_q^2 + \frac{1}{2}c_2h_q^2 + \frac{1}{2}c_3h_q'^2 + \frac{1}{2}c_4T_q'^2 + c_5h_qT_q + c_6h_q'T_q' + c_7h_q'T_q + c_8h_qT_q'\right.$$
$$\left. + \frac{c_9}{8u^3Q_c}(\partial T_q)^2 + \frac{c_{10}}{4u^3Q_c}(\partial h_q).(\partial T_q) + \frac{c_{11}}{8u^3Q_c}(\partial h_q)^2\right], \tag{B9}$$

where the coefficients $\{c_i\}$ are given by

$$c_1 = \left(\frac{V_c'}{V_c}\right)'\left(1 - \frac{T_c'^2}{d_c}\right)\sqrt{d_c} + 2\frac{V_c'}{V_c}\frac{T_ch_c^2}{\sqrt{d_c}} + \frac{h_c^2}{\sqrt{d_c}} + \frac{T_c'}{\sqrt{d_c}}\left(\frac{h_c^2T_c}{d_c}\right)', \tag{B10}$$

$$c_2 = \left(\frac{h_cT_c^2}{d_c}\right)'\frac{\frac{1}{4}fu^{3/2}h_c'}{\sqrt{d_c}} + \frac{T_c^2}{\sqrt{d_c}}, \tag{B11}$$

$$c_3 = \frac{1}{\sqrt{d_c}}\left(1 - \frac{\frac{1}{4}fu^{3/2}h_c'^2}{d_c}\right)\frac{1}{4}fu^{3/2}, \tag{B12}$$

$$c_4 = \frac{1}{\sqrt{d_c}}\left(1 - \frac{T_c'^2}{d_c}\right), \tag{B13}$$

$$c_5 = \frac{V_c'}{V_c}\frac{h_cT_c^2}{\sqrt{d_c}} + \left(2 - \frac{h_c^2T_c^2}{d_c}\right)\frac{h_cT_c}{\sqrt{d_c}}, \tag{B14}$$

$$c_6 = -\frac{T_c'}{d_c\sqrt{d_c}}\frac{1}{4}fu^{3/2}h_c', \tag{B15}$$

$$c_7 = \frac{1}{\sqrt{d_c}}\left(\frac{V_c'}{V_c} - \frac{h_c^2T_c}{d_c}\right)\frac{1}{4}fu^{3/2}h_c', \tag{B16}$$

$$c_8 = -\frac{h_cT_c^2T_c'}{d_c\sqrt{d_c}}, \tag{B17}$$

$$c_9 = 4u^{3/2}\sqrt{d_c}\left(1 - \frac{T_c'^2}{d_c}\right), \tag{B18}$$

$$c_{10} = -u^3 \frac{f}{\sqrt{d_c}} h'_c T'_c, \tag{B19}$$

$$c_{11} = u^3 f\sqrt{d_c}\left(1 - \frac{\frac{1}{4} f u^{3/2} h_c'^2}{d_c}\right), \tag{B20}$$

with $Q_c = (1 + f u^{3/2} h_c^2 T_c^2)$. For later convenience, we have explicitly written out a factor of $1/4u^3 Q_c$ in the coefficients in the last three terms in (B9).

This action mixes T_q and h_q, and the equations of motion derived from it reflect this mixing. After some manipulations, the equations can be cast in the form

$$\partial^2 T_q = a_1 T_q + a_2 T'_q + a_3 T''_q + a_4 h_q + a_5 h'_q, \tag{B21}$$

$$\partial^2 h_q = b_1 h_q + b_2 h'_q + b_3 h''_q + b_4 T_q + b_5 T'_q, \tag{B22}$$

where the coefficients $\{a_i\}$ and $\{b_i\}$ are given by

$$\begin{aligned} a_1 &= c_{10}(\bar{c}_7 - c_5) + c_{11}c_1, \\ a_2 &= c_{10}(\bar{c}_6 + c_7 - c_8) - c_{11}\bar{c}_4, \\ a_3 &= c_{10}c_6 - c_{11}c_4, \\ a_4 &= -c_{10}c_2 + c_{11}(c_5 - \bar{c}_8), \\ a_5 &= c_{10}\bar{c}_3 - c_{11}(\bar{c}_6 - c_7 + c_8), \end{aligned} \tag{B23}$$

and

$$\begin{aligned} b_1 &= c_{10}(\bar{c}_8 - c_5) + c_9 c_2, \\ b_2 &= c_{10}(\bar{c}_6 - c_7 + c_8) - c_9\bar{c}_3, \\ b_3 &= c_{10}c_6 - c_9 c_3, \\ b_4 &= -c_{10}c_1 + c_9(c_5 - \bar{c}_7), \\ b_5 &= c_{10}\bar{c}_4 - c_9(\bar{c}_6 + c_7 - c_8). \end{aligned} \tag{B24}$$

Here we have used the notation $\bar{c}_i = (Ac_i)'/A$. As usual, a prime denotes a derivative with respect to u. Moreover, $\partial^2 = (-\partial_t^2 + \partial_x^2)$ is the flat space-time Laplacian. A possible term proportional to h''_q is not present in (B21) because its coefficient $(c_{10}c_3 - c_{11}c_6)$ vanishes. Similarly, in (B22) the term proportional to T''_q is absent because its coefficient $(c_{10}c_4 - c_9 c_6)$ vanishes.

The equations of motion derived from (B9) are quite complicated in general, but they simplify in the two asymptotic regimes of u.

$u \to u_{\max}$: In this limit, many of the c_i are small because they have at least one factor of T_c or its derivatives in them. The exceptions are $c_1 \sim h_0^2 u^{3/4}$, $c_3 \sim u^{9/4}/4$, $c_4 \sim u^{3/4}$, $c_9 \sim 4u^{3/4}$ and $c_{11} \sim u^{9/4}$. Retaining only the dominant terms in the equations, we get

$$-(u^4 T'_q(u,x))' + h_0^2 u^4 T_q(u,x) = 0, \tag{B25}$$

$$(u^{11/2} h'_q(u,x))' = 0. \tag{B26}$$

The term involving the space-time Laplacian on the fluctuations can be consistently neglected at the leading order since it is nonleading in powers of u, as can be verified *a posteriori*. These equations are identical to (15) and (17) and so have solutions similar to (16) and (18), but now with parameters that are functions of x:

$$\begin{aligned} T_q(u,x) &= \frac{1}{u^2}(T_{q+}(x)e^{-h_0 u} + T_{q-}(x)e^{h_0 u}), \\ h_q(u,x) &= h_{q0}(x) - h_{q1}(x)u^{-9/2}. \end{aligned} \tag{B27}$$

$u \to u_0$: This limit is more involved, requiring a more detailed analysis. One expands T_q and h_q in powers of $\epsilon \equiv (u - u_0)$ with arbitrary x-dependent coefficients:

$$\begin{aligned} T_q(u,x) &= \frac{\sqrt{\pi}}{4} u_0^{3/2} \epsilon^{\omega}(\varphi_0(x) + \epsilon\varphi_1(x) + \cdots), \\ h_q(u,x) &= \sqrt{\frac{26}{\pi u_0}} u_0^{-3/4} \epsilon^{\tau}(\vartheta_0(x) + \epsilon\vartheta_1(x) + \cdots). \end{aligned} \tag{B28}$$

Here, and in the following, we have set $f_0 = 1$. One also needs to expand the a_i's and b_i's in powers of ϵ. Retaining up to the first nonleading power in ϵ, we get

$$\begin{aligned} a_1 &= 8\xi\epsilon^{-1}\left(1 + \frac{23\epsilon}{12u_0}\right), \\ a_2 &= 2\xi\left(1 + \frac{2\epsilon}{u_0}\right), \\ a_3 &= \frac{4u_0^{-3/2}}{\pi}\xi\epsilon^3\left(1 + \frac{23\epsilon}{12u_0}\right), \\ a_4 &= \frac{2\pi u_0^{11/4}}{\sqrt{26}}\xi\epsilon^{-7/2}\left(1 + \frac{65\epsilon}{24u_0}\right), \\ a_5 &= \frac{4u_0^{5/4}}{\sqrt{26}}\xi\epsilon^{-1/2}\left(1 + \frac{21\epsilon}{8u_0}\right), \end{aligned} \tag{B29}$$

and

$$\begin{aligned} b_1 &= -3\xi\epsilon^{-1}\left(1 + \frac{3\epsilon}{4u_0}\right), \\ b_2 &= 2\xi\left(1 + \frac{2\epsilon}{u_0}\right), \\ b_3 &= \frac{4u_0^{-3/2}}{\pi}\xi\epsilon^3\left(1 + \frac{23\epsilon}{12u_0}\right), \\ b_4 &= \frac{16\sqrt{26}u_0^{-11/4}}{\pi}\xi\epsilon^{3/2}\left(-1 + \frac{\epsilon}{24u_0}\right), \\ b_5 &= -\frac{4\sqrt{26}u_0^{-11/4}}{\pi}\xi\epsilon^{5/2}\left(1 + \frac{\epsilon}{24u_0}\right), \end{aligned} \tag{B30}$$

where $\xi = -13u_0^2/8$. Substituting these expansions in Eqs. (B21) and (B22) and comparing different orders of ϵ, we see that a consistent solution exists only for $\omega = -3$ and $\tau = -1/2$, and then we get

$$\vartheta_0(x) = -\frac{1}{4}\varphi_0(x), \qquad \varphi_1(x) = \frac{5}{6u_0}\varphi_0(x),$$
$$\vartheta_1(x) = \frac{1}{8\xi}\left(\partial^2 + \frac{65u_0}{32}\right)\varphi_0(x). \tag{B31}$$

The first of these relations is precisely what is needed to think of the leading terms in (B28) as coming from expanding $(u - u_0(x))^{-1}$ around a constant u_0. The last relation shows that when x dependence is allowed, not all coefficients get uniquely determined. In fact, the part of $\varphi_0(x)$ annihilated by the operator on the right-hand side does not show up in $\vartheta_1(x)$.

The above analysis shows that perturbation expansion in "small" fluctuations around a constant u_0 is singular, although we have obtained a solution by a formal expansion.

APPENDIX C: CALCULATION OF THE EXACT (u, x)-DEPENDENT ACTION

This involves calculating the determinant of the matrix $(1 + K)$, whose elements are given in (28). We will simplify this calculation by making use of the following trick. Consider the family of determinants $D(\lambda) \equiv \det(1 + \lambda K)$, where λ is an arbitrary parameter. We actually need only to calculate $D(1)$, but this calculation can be reduced essentially to the calculation of the inverse of the matrix $(1 + \lambda K)$, which turns out to be much easier than a direct computation of the determinant. Consider the following:

$$\frac{d}{d\lambda}D(\lambda) = D(\lambda)\,\mathrm{tr}[(1 + \lambda K)^{-1}K]. \tag{C1}$$

We can obtain Δ by integrating this equation, using the boundary condition $D(0) = 1$:

$$\ln D(1) = \int_0^1 d\lambda D(\lambda)^{-1}\frac{d}{d\lambda}D(\lambda) = \int_0^1 d\lambda\,\mathrm{tr}[(1 + \lambda K)^{-1}K]. \tag{C2}$$

This reduces the required calculation to finding the inverse matrix $M(\lambda) = (1 + \lambda K)^{-1}$, which may be done as follows. Using the defining equation $(1 + \lambda K)M(\lambda) = 1$, one can express all components of M in terms of $M^\mu{}_\nu$:

$$M^u{}_\nu = -\lambda K^u{}_\mu M^\mu{}_\nu, \qquad M^u{}_u = 1 - \lambda^2 K^u{}_\mu K^\nu{}_u M^\mu{}_\nu,$$
$$M^\mu{}_u = -\lambda M^\mu{}_\nu K^\nu{}_u. \tag{C3}$$

Moreover, one can show that $M^\mu{}_\nu$ satisfies

$$P^\mu{}_\sigma M^\sigma{}_\nu = \delta^\mu{}_\nu,$$
$$P^\mu{}_\sigma \equiv (\delta^\mu{}_\sigma + \lambda K^\mu{}_\sigma - \lambda^2 K^\mu{}_u K^u{}_\sigma). \tag{C4}$$

Thus, to find $M(\lambda)$ we need to find the inverse of the $P^\mu{}_\sigma(\lambda)$ matrix. First note that using (28) we can write

$$P^\mu{}_\sigma(\lambda) = \delta^\mu{}_\nu + \beta_1(\lambda)\partial^\mu T\partial_\nu T + \beta_2(\lambda)\partial^\mu h\partial_\nu h + \beta_3(\lambda)(\partial^\mu T\partial_\nu h + \partial^\mu h\partial_\nu T), \tag{C5}$$

where

$$\beta_1(\lambda) = \frac{\lambda u^{-3/2}}{Q}\left(1 - \lambda\frac{T'^2}{d_T}\right),$$
$$\beta_2(\lambda) = \frac{\lambda f}{4Q}\left(1 - \lambda\frac{fu^{3/2}h'^2}{4d_T}\right),$$
$$\beta_3(\lambda) = -\frac{\lambda^2 fh'T'}{4Qd_T},$$
$$\beta_4(\lambda) = \beta_1(\lambda)\beta_2(\lambda) - (\beta_3(\lambda))^2. \tag{C6}$$

For $\lambda = 1$ these reduce to the β's in (31). Now, from the general structure of the $P^\mu{}_\nu$ matrix, we can parametrize the $M^\mu{}_\nu$ matrix as

$$M^\mu{}_\nu(\lambda) = \delta^\mu{}_\nu + \alpha_1(\lambda)\partial^\mu T\partial_\nu T + \alpha_2(\lambda)\partial^\mu h\partial_\nu h + \alpha_3(\lambda)(\partial^\mu T\partial_\nu h + \partial^\mu h\partial_\nu T). \tag{C7}$$

We have calculated the α's. They work out to be

$$\alpha_1(\lambda) = -\frac{1}{\Delta(\lambda)}[\beta_1(\lambda) + \beta_4(\lambda)(\partial h)^2],$$
$$\alpha_2(\lambda) = -\frac{1}{\Delta(\lambda)}[\beta_2(\lambda) + \beta_4(\lambda)(\partial T)^2],$$
$$\alpha_3(\lambda) = -\frac{1}{\Delta(\lambda)}[\beta_3(\lambda) - \beta_4(\lambda)\partial h.\partial T]. \tag{C8}$$

Here $\Delta(\lambda)$ is a generalization of Δ defined in (30). It has the same form but with the above λ-dependent β's replacing those in (30). By definition, $\Delta(1) = \Delta$.

Armed with the inverse matrix $M(\lambda)$, we can now compute the trace on the right-hand side of (C2). Using and (28) and (C3), we first note that

$$\mathrm{tr}[(1 + \lambda K)^{-1}K] = M^\mu{}_\sigma(\lambda)\frac{d}{d\lambda}P^\sigma_\mu(\lambda). \tag{C9}$$

Given Eqs. (C5)–(C8), it is straightforward, though tedious, to compute the right-hand side of the above equation. One gets the simple result

$$M^\mu{}_\sigma(\lambda)\frac{d}{d\lambda}P^\sigma{}_\mu(\lambda) = \Delta(\lambda)^{-1}\frac{d}{d\lambda}\Delta(\lambda). \tag{C10}$$

It follows from this and (C2) that $D(1) = \Delta(1) = \Delta$. Hence the complete five-dimensional action is that given in (29).

To compute the equations of motion for $T(u, x)$ and $h(u, x)$ that follow from this action, we will need the following, which can be easily calculated from the relation $\Delta_T = d_T\Delta$ and the definition of Δ given in (30):

$$\frac{1}{2}\frac{\partial\Delta_T}{\partial T'} = T' + \frac{fT'}{4Q}(\partial h)^2 - \frac{fh'}{4Q}\partial T.\partial h,$$

$$\frac{1}{2}\frac{\partial\Delta_T}{\partial(\partial_\mu T)} = d_T\beta_1\partial^\mu T + d_T\beta_3\partial^\mu h + \frac{u^{-3}}{4Q}(\partial^\mu T(\partial h)^2 - \partial^\mu h(\partial h.\partial T)),$$

$$\frac{1}{2}\frac{\partial\Delta_T}{\partial T} = Th^2\bigg[1 - \frac{f^2u^{3/2}}{4Q^2}(h'^2(\partial T)^2 + T'^2(\partial h)^2 - 2T'h'(\partial T.\partial h) + f^{-1}u^{-3}((\partial T)^2(\partial h)^2 - (\partial T.\partial h)^2))\bigg],$$

$$\frac{1}{2}\frac{\partial\Delta_T}{\partial h'} = \frac{fu^{3/2}}{4}h' + \frac{fh'}{4Q}(\partial T)^2 - \frac{fT'}{4Q}\partial T.\partial h,$$

$$\frac{1}{2}\frac{\partial\Delta_T}{\partial(\partial_\mu h)} = d_T\beta_2\partial^\mu h + d_T\beta_3\partial^\mu T + \frac{u^{-3}}{4Q}(\partial^\mu h(\partial T)^2 - \partial^\mu T(\partial h.\partial T)),$$

$$\frac{1}{2}\frac{\partial\Delta_T}{\partial h} = T^2h\bigg[1 - \frac{f^2u^{3/2}}{4Q^2}(h'^2(\partial T)^2 + T'^2(\partial h)^2 - 2T'h'(\partial T.\partial h) + f^{-1}u^{-3}((\partial T)^2(\partial h)^2 - (\partial T.\partial h)^2))\bigg]. \quad \text{(C11)}$$

Using these one can show that

$$\Delta_T - T'\frac{1}{2}\frac{\partial\Delta_T}{\partial T'} - \partial_\mu T\frac{1}{2}\frac{\partial\Delta_T}{\partial(\partial_\mu T)} = d_T - T'^2 + \frac{u^{-3/2}}{4}(\partial h)^2,$$

$$T'\frac{1}{2}\frac{\partial\Delta_T}{\partial h'} + \partial_\mu T\frac{1}{2}\frac{\partial\Delta_T}{\partial(\partial_\mu h)} = \frac{fu^{3/2}}{4}T'h' + \frac{u^{-3/2}}{4}(\partial T.\partial h). \quad \text{(C12)}$$

We can now give the equations of motion obtained from the action (29):

$$\frac{u^{13/4}}{\sqrt{\Delta_T}}\bigg[\frac{1}{2}\frac{\partial\Delta_T}{\partial T} + \frac{V'}{V}\bigg(d_T - T'^2 + \frac{u^{-3/2}}{4}(\partial h)^2\bigg)\bigg] = \bigg(\frac{u^{13/4}}{\sqrt{\Delta_T}}\frac{1}{2}\frac{\partial\Delta_T}{\partial T'}\bigg)' + \partial_\mu\bigg(\frac{u^{13/4}}{\sqrt{\Delta_T}}\frac{1}{2}\frac{\partial\Delta_T}{\partial(\partial_\mu T)}\bigg), \quad \text{(C13)}$$

$$\frac{u^{13/4}}{\sqrt{\Delta_T}}\bigg[\frac{1}{2}\frac{\partial\Delta_T}{\partial h} - \frac{V'}{V}\bigg(\frac{fu^{3/2}}{4}T'h' + \frac{u^{-3/2}}{4}(\partial T.\partial h)\bigg)\bigg] = \bigg(\frac{u^{13/4}}{\sqrt{\Delta_T}}\frac{1}{2}\frac{\partial\Delta_T}{\partial h'}\bigg)' + \partial_\mu\bigg(\frac{u^{13/4}}{\sqrt{\Delta_T}}\frac{1}{2}\frac{\partial\Delta_T}{\partial(\partial_\mu h)}\bigg). \quad \text{(C14)}$$

These can be further simplified using the expressions given in (C11), but we will not do so here since we will be interested only in a leading solution to these equations in the limit $u \sim u_0$. As a check, we note that these equations reduce to Eqs. (13) and (14) if T and h are x-independent.

[1] T. Sakai and S. Sugimoto, Prog. Theor. Phys. **113**, 843 (2005).
[2] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).
[3] A. Karch and A. Katz, J. High Energy Phys. 06 (2002) 043.
[4] J. Babington, J. Erdmenger, N. J. Evans, Z. Guralnik, and I. Kirsch, Phys. Rev. D **69**, 066007 (2004).
[5] M. Kruczenski, D. Mateos, R. C. Myers, and D. J. Winters, J. High Energy Phys. 05 (2004) 041.
[6] T. Sakai and J. Sonnenschein, J. High Energy Phys. 09 (2003) 047.
[7] J. L. F. Barbon, C. Hoyos, D. Mateos, and R. C. Myers, J. High Energy Phys. 10 (2004) 029.
[8] H. Nastase, arXiv:hep-th/0305069.
[9] T. Sakai and S. Sugimoto, Prog. Theor. Phys. **114**, 1083 (2005).
[10] H. Hata, T. Sakai, and S. Sugimoto, arXiv:hep-th/0701280.
[11] D. K. Hong, M. Rho, H. U. Yee, and P. Yi, Phys. Rev. D **76**, 061901 (2007).
[12] K. Nawa, H. Suganuma, and T. Kojo, Prog. Theor. Phys. Suppl. **168**, 231 (2007).
[13] O. Bergman, G. Lifschytz, and M. Lippert, J. High Energy Phys. 11 (2007) 056.
[14] D. Yamada, arXiv:0707.0101.

[15] O. Aharony, J. Sonnenschein, and S. Yankielowicz, Ann. Phys. (N.Y.) **322**, 1420 (2007).
[16] E. Antonyan, J. A. Harvey, S. Jensen, and D. Kutasov, arXiv:hep-th/0604017.
[17] A. Parnachev and D. A. Sahakyan, Phys. Rev. Lett. **97**, 111601 (2006).
[18] A. Sen, Int. J. Mod. Phys. A **20**, 5513 (2005).
[19] R. Casero, E. Kiritsis, and A. Paredes, Nucl. Phys. **B787**, 98 (2007).
[20] O. Bergmann, S. Seki, and J. Sonnenschein, J. High Energy Phys. 12 (2007) 037.
[21] A. Dhar and P. Nag, J. High Energy Phys. 01 (2008) 055.
[22] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[23] V. Balasubramanian, P. Kraus, and A. E. Lawrence, Phys. Rev. D **59**, 046003 (1999).
[24] V. Balasubramanian, P. Kraus, A. E. Lawrence, and S. P. Trivedi, Phys. Rev. D **59**, 104021 (1999).
[25] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).
[26] S. Sugimoto and K. Takahashi, J. High Energy Phys. 04 (2004) 051.
[27] O. Aharony and D. Kutasov, Phys. Rev. D **78**, 026005 (2008).
[28] K. Hashimoto, T. Hirayama, F. Lin, and H. Yee, J. High Energy Phys. 07 (2008) 089.
[29] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, Phys. Rev. D **58**, 046004 (1998).
[30] A. A. Tseytlin, Nucl. Phys. **B501**, 41 (1997).
[31] A. Sen, Phys. Rev. D **68**, 066008 (2003).
[32] M. R. Garousi, J. High Energy Phys. 01 (2005) 029.
[33] K. Bitaghsir-Fadafan and M. R. Garousi, Nucl. Phys. **B760**, 197 (2007); M. R. Garousi, J. High Energy Phys. 12 (2007) 089; M. R. Garousi and H. Golchin, Nucl. Phys. **B800**, 547 (2008); M. R. Garousi and E. Hatefi, J. High Energy Phys. 02 (2008) 109; M. R. Garousi, arXiv:0712.1954.
[34] C. j. Kim, H. B. Kim, Y. b. Kim, and O. K. Kwon, J. High Energy Phys. 03 (2003) 008.
[35] F. Leblond and A. W. Peet, J. High Energy Phys. 04 (2003) 048.
[36] N. Lambert, H. Liu, and J. M. Maldacena, J. High Energy Phys. 03 (2007) 014.
[37] J. A. Minahan and B. Zwiebach, J. High Energy Phys. 03 (2001) 038.
[38] D. Kutasov, M. Marino, and G. W. Moore, arXiv:hep-th/0010108.
[39] P. Kraus and F. Larsen, Phys. Rev. D **63**, 106004 (2001).
[40] T. Takayanagi, S. Terashima, and T. Uesugi, J. High Energy Phys. 03 (2001) 019.
[41] K. Hashimoto, J. High Energy Phys. 07 (2002) 035.
[42] M. Bianchi, D. Z. Freedman, and K. Skenderis, Nucl. Phys. **B631**, 159 (2002).
[43] K. Skenderis, Classical Quantum Gravity **19**, 5849 (2002).
[44] A. Karch, A. O'Bannon, and K. Skenderis, J. High Energy Phys. 04 (2006) 015.
[45] J. Erdmenger, N. Evans, I. Kirsch, and E. Threlfall, Eur. Phys. J. A **35**, 81 (2008).
[46] J. Erlich, E. Katz, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. **95**, 261602 (2005).
[47] L. Da Rold and A. Pomarol, Nucl. Phys. **B721**, 79 (2005).