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7D Randall-Sundrum cosmology, brane-bulk energy exchange, and holography

Liuba Mazzanti*

Dip. di Fisica G. Occhialini, Milano-Bicocca University, 20126 Milano, Italy, INFN, Milano-Bicocca, and CPhT, École Polytechnique, 91128 Palaiseau, France

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We discuss the cosmological implications and the holographic dual theory of the 7D Randall-Sundrum gravitational setup. Adding generic matter in the bulk on the 7D gravity side, we study the cosmological evolution inferred by the nonvanishing value of the brane-bulk energy exchange parameter. This analysis is achieved in detail for specific assumptions on the internal space evolution, including analytical considerations and numerical results. The dual theory is then constructed, making use of the holographic renormalization procedure. The resulting renormalized 6D conformal field theory is anomalous and coupled to 6D gravity plus higher order corrections. The critical point analysis on the brane is performed. Finally, we sketch a comparison between the two dual descriptions. We moreover generalize the Ads/CFT dual theory to the nonconformal and interacting case, relating the energy exchange parameter of the bulk gravity description to the new interactions between hidden and visible sectors.

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I. INTRODUCTION

Many aspects of brane models have been recently developed both from the point of view of standard model building (see for example [1–3] and references therein) and of cosmology ([4–9] and references therein). In particular, Randall-Sundrum (RS) [10] cosmology [11] is among the most interesting insights related to brane-world cosmology (others are mirage cosmology [12], brane inflation [13], brane induced gravity [14–18], brane/antibrane inflation [19–21], cosmologies from higher derivative corrections [22–24], particular examples with varying speed of light [25] and cosmological evolution induced by the rolling tachion [26], inflation in flux compactification scenarios [27–29], and recent brane-world models [30–36]). Branes can be used to give origin to four dimensional gauge theories (living on a 3-brane) and hence localizing matter in four dimensions embedded in the ten dimensional space-time in which string theory lives (or 11 dimensional for M theory). On the gravity side, since gravitons propagate in all dimensions admitted by string theory, the validity of Newton's law seems to constrain the number of non compact dimensions to be four.

Randall and Sundrum [10] proposed an alternative way of localizing gravity in four dimensions without compactifying the extra dimensions. This was achieved in RSII model by assuming a warped extra direction, instead of a compact one (compact extra dimensions have been used [37] as an attempt of giving an explanation to the hierarchy problem, to which RSI [10] represents an alternative way out). The setup of RSII is five dimensional gravity in a bulk space-time cut by a 3-brane. There is, in addition, a $\mathbb{Z}_2$ reflection of the extra dimension transverse to the brane, having as fixed point the location of the brane. The result is a bound state graviton mode localized on the brane and a tower of Kaluza-Klein (KK) modes, without mass gap, that give negligible corrections to the effective 4D gravity description. The metric solving the equations of motion for the five dimensional action is a slice of AdS_5 copied and reflected with respect to (w.r.t.) the $\mathbb{Z}_2$ symmetry. It can be viewed as coming from the Type IIB string theory background for a stack of N D3-branes, in the low energy effective field theory approximation, which is indeed dual to the $AdS_5 \times S^5$ near horizon geometry for a 3-brane supergravity solution. In RS analysis only gravity on AdS_5 is considered, since the S^5 is factored out from the anti de Sitter space, giving KK modes. The truncation of AdS_5 with the 3-brane cuts out its boundary.

In a recent work [38], a different string background related to RS setup has been considered. The analogous analysis to the 5D RS model has been made by Bao and Lykken [38] in a seven dimensional anti de Sitter background rather than in the five dimensional original model. The background may come from the near horizon geometry of M5-branes in the 11 dimensional M theory, which gives $AdS_7 \times S^4$. As for the five dimensional model, the sphere is factored out and only the physics of gravity in AdS_7 plus KK modes is considered. A further step performed in [38] is to reduce $AdS_7 \to AdS_5 \times \Sigma^2$, where Σ^2 is a two dimensional internal space (namely a two-sphere or a torus). In [38] the RS spectrum of KK modes gets modified and supplemental KK and winding modes appear, because of the Σ^2 compactification. In [39] the 7D supergravity orbifold compactification on $S^1/\mathbb{Z}_2$ is considered in the context of anomaly cancellation on the boundary of the background, showing that the matter contents of the theory cannot be completely generic.

On the cosmological side, RS models can give new descriptions of the cosmological evolution of our universe. A realistic model should be able to explain the existence of

*liuba.mazzanti@cpht.polytechnique.fr

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dark energy and the nature of dark matter, early time inflation and eventually the exit from this phase, as well as the late time acceleration coming from the present observations—additional issues are related to the cosmological constant, temperature anisotropies, etc. (see [40–42] for recent reviews on the observed cosmology). RS cosmologies have been studied and found to exhibit some of these features. Generally, brane-world cosmological models should take account of the energy exchange between brane and bulk that naturally arises because of the non factorized extra direction. The implications of this energy exchange has been analyzed in [43]. The authors propose some scenarios describing the cosmological evolution of a universe with two accelerating phases, as we expect from experimental data. Moreover, most of the fixed points were shown to be stable. Earlier attempts include [44]. Subsequent papers [45–57] have been written on the subject, also finding new solutions, some of which are exact.

The aim of this paper is to investigate the 7D RS cosmology with brane-bulk energy exchange and to explore the model from the holographic point of view, making an explicit comparison between the two descriptions. Our starting point is gravity in the 7D bulk cut by a five-brane and with the usual RS $\mathbb{Z}_2$ reflection plus a generic matter term on the brane. In order to study the cosmological evolution of the brane-world, the ansatz for the metric is time dependent. Besides, the direction transverse to the 5-brane is the warped direction characterizing RS models. Unlike in [38], we have different warp factors for the 3D extended space and for the two dimensional compact internal space. It is worth noticing that the gravitational coupling constant of the 4D space-time is dynamically related to the 7D Newton constant, since the compact space volume is generally time dependent. Indeed, the 4D energy density and pressures are also dynamical functions of the density and the pressures defined on the brane. We analyze the generic Friedmann and (non) conservation equations, also including the energy exchange terms, in order to get the expressions for the Hubble parameter of the 4D space-time as a function of the density and to describe realistic cosmologies. It is interesting to study both analytically and numerically the system of Einstein equations. The analogous critical point analysis in the 5D bulk was performed in [43]. Some explicit solutions, derived with simplifying assumptions on the parameter of the internal space and on its geometry are also illustrated in our work.

Further investigations on the 5D RS cosmology with brane-bulk energy exchange have been made from the holographic point of view [9], studying the dual theory in one lower dimension. The gauge/gravity duality [58,59] (see [60] for a complete review) has undergone great improvements over the last ten years and provides a new approach to the analysis of brane-world models. As it is explained in [61–63], the truncation of the AdS_{d+1} space is equivalent to introducing in the dual picture a UV cutoff for the d-dimensional gauge theory (conformal field theory) coupled to d-dimensional gravity. Earlier suggestions about this idea are present in [64]. The presence of brane-bulk exchange corresponds to interactions between the gauge theory and the matter fields while the bulk "self-interaction" is shown to be related to the perturbation of the conformal field theory (CFT) (that becomes a strongly coupled gauge theory). In [9], explicit examples of cosmological evolutions in the holographic 5D/4D picture as well as comparison between the two dual theories have been discussed. Other cosmological models have been analyzed in the context of the holographic correspondence [65–71].

Exploiting the AdS/CFT results, we build the holographic theory corresponding to the 7D RS background. The 7D RS dual theory (see [9] for the analogous 5D/4D derivation) is then a renormalized 6D CFT (the theory corresponding to the M5 system is an anomalous [72–77] (0, 2) SCFT, but any other six dimensional large-N CFT can be chosen) coupled to 6D gravity. See also [78] for other examples of holographic Weyl anomaly derivation. There are in addition higher order corrections to gravity and the six dimensional matter action. Higher derivative terms driven by conformal four dimensional anomaly [79] were proved to lead to an inflationary critical point in the 4D Starobinsky model [80] and to a successive graceful exit from the long primordial inflation. As illustrated in [81], higher derivative contributions to the Einstein equations cause the universe to enter a matter dominated era where the scale factor oscillates after inflation and to proceed through thermalization to a radiation dominated era. In our 6D holographic cosmological model we specially look for de Sitter fixed point solutions to the equations of motion describing late time acceleration or critical points suitable for early time inflation studying the associated stability matrix. A comparison with the 7D bulk analysis results shows some peculiar features of the 7D/6D setup.

Interesting results for cosmologies with compactification emerge in the context of dynamical compactification [82,83]. The compact space is treated with a different scale factor (as in the approach that will be used in this paper). In particular, in the context of dynamical compactification, the scale factor for the internal space has an inverse power dependence on the scale factor for the visible directions. The extra dimensions thus contract as the extended space expands. We also include some remarks on dynamical compactification applications in our setup. Other attempts to reduce to conventional cosmology and investigate issues such as the cosmological constant from models with arbitrary number of extra dimensions are given in [84,85]. In particular, cosmologies in 6 dimensions are analyzed in [86].

The structure of the paper is as follows. The next section will describe the setup of the seven dimensional RS model.

Section III will be focused on the cosmological evolution from the 7D point of view, admitting brane-bulk energy exchange and particularly investigating the form of Einstein equations with some specific ansatz, while in section IV the critical point analysis will be illustrated, including numerical phase space portrait and explicit solutions. In section V and the following we will derive the 6D holographic dual to the 7D RS and the associated equations of motion. Section VII summarizes the fixed points in the holographic description and their stability. Some examples of the correspondence between the brane and bulk points of view will be given in section VIII. The generalization to nonconformal and interacting theory, corresponding to nonvanishing brane-bulk energy exchange and bulk self-interaction in the 7D approach, will be exposed in section IX. Finally, the last section will summarize the results, give some conclusions and further considerations. The first appendix gives the form of the general anomaly for a 6D CFT in a curved space and of the other trace terms in the 6D theory and appendix B is devoted to the critical point analysis in the six dimensional cosmology.

II. 7D RS SETUP

As we announced in the introduction, we will work in a seven dimensional bulk with a 5-brane located at the origin of the direction z transverse to the brane itself and with a $z \rightarrow -z$ $\mathbb{Z}_2$ identification. In analogy to the 5D RS model, the action in this seven dimensional setup is given by the sum of the Einstein-Hilbert action with 7D cosmological constant plus a contribution localized on the brane that represents the brane tension. Besides, we also put a matter term both in the bulk and on the brane. In formula we thus have

$$\begin{aligned} S &= S_{\rm EH} + S_{\rm GH} + S_{m,B} + S_{\rm tens} + S_m \\ &= \int \mathrm{d}^7 x \sqrt{-G}(M^5 R - \Lambda_7 + \mathcal{L}_B^{\rm mat}) \\ &\quad + \int \mathrm{d}^6 x \sqrt{-\gamma}(-V + \mathcal{L}_b^{\rm mat}) + S_{\rm GH} \end{aligned} \tag{1}$$

where V is the brane tension and we will call the associated contribution to the action $S_{\rm tens}$. $S_{\rm EH}$ is the usual Einstein-Hilbert action with the seven dimensional bulk cosmological constant Λ_7 and $\mathcal{L}_B^{\rm mat}$, $\mathcal{L}_b^{\rm mat}$ are, respectively, the bulk and brane matter Lagrangians. $S_{\rm GH} = \int \mathrm{d}^6 x \sqrt{-\gamma} K$, where K denotes the trace of the extrinsic curvature on the boundary, is the Gibbons-Hawking action added in order to cancel the boundary term arising computing the variation on the Einstein-Hilbert action and to get the usual Einstein equations. The action for the matter in the bulk $S_{m,B}$ is an additional term with respect to the usual RS setup, whereas the matter on the brane contribution will be referred to as S_m. The metric $\gamma_{\mu\nu}$ is the induced metric on the brane. The brane tension is necessary in the RS models in order to compensate for the presence of the cosmological constant in the bulk.

The classical solution of the equations of motion for the theory above, neglecting all the matter terms and with a warped geometry of the kind $\mathrm{d}s^2 = \mathrm{e}^{-W}\mathrm{d}x^2 + \mathrm{d}z^2$ ($W = W(z)$ is the warp factor and the 6D x-directions are flat), is the analogue of the solution described by Randall and Sundrum [10] for the 5D RSII model. The 7D solution gives as a result $W(z) = 2|z|\sqrt{-\frac{\Lambda_7}{30M^5}}$, so that the space-time is a slice of AdS_7 with the $\mathbb{Z}_2$ typical reflection, where there should exist a relation between the brane tension and the bulk cosmological constant $3V^2 = -40M^5\Lambda_7$.

The aim of this section and of the next one is to generalize the RS ansatz to a time dependent background and to wrap the 5-brane over a two dimensional internal space, ending up with an effective 4D cosmology. Taking account of the seventh warped extra dimension and of the compactification over the other two extra dimensions, giving two different warp factors to the 3D space and the internal 2D space, the time dependent ansatz for the metric is of the form

$$\begin{aligned} \mathrm{d}s^2 = &-n^2(t,z)\mathrm{d}t^2 + a^2(t,z)\zeta_{ij}\mathrm{d}x^i\mathrm{d}x^j \\ &+ b^2(t,z)\xi_{ab}\mathrm{d}y^a\mathrm{d}y^b + f^2(t,z)\mathrm{d}z^2, \end{aligned} \tag{2}$$

with the maximally symmetric ζ_{ij} background in three spatial dimensions (with spatial curvature k) and ξ_{ab} for the 2D internal space (with spatial curvature κ). We use capital indices $A, B, \ldots$ to run over the seven dimensions, i, $j, \ldots$ for the three spatial dimensions of the 4D space-time, and a, b, $\ldots$ for the two internal dimensions. In our notations z represents the seventh warped extra direction, the y coordinates belong to the 2D internal space, while the $\{x_\mu\} = \{t, x_i, y_a\}$ run over the 6D space-time on the brane. Summarizing, the structure of the bulk is thus made of a time coordinate, three extended maximally symmetric spatial dimensions (that gives, together with the time, the visible 4D space-time), two compact dimensions and a warped direction. The 3D and 2D spaces have two different scale factors $a(t,z)$ and $b(t,z)$ respectively, while a gauge choice can be made for the values of the $n(t,z)$ and $f(t,z)$ factors on the brane, i.e. when $z = 0$.

A less physically meaningful background, but better understood, would be to have a five dimensional maximally symmetric space with some 5D metric $\tilde{\zeta}_{ij}$ and just one scale factor $\tilde{a}(t,z)$, without compactifying on any two dimensional internal space. The solution to the equations of motion in this case is much simpler. The results related to this background will be briefly mentioned along with the more realistic analysis with brane wrapping over the two dimensional internal space.

III. COSMOLOGICAL EVOLUTION IN THE BULK

In this section we will analyze some aspects of the cosmological evolution from the 7D bulk point of view. We will write the equations of motion for the bulk action and solve them by making assumptions to simplify their form and get explicit results evaluated on the brane.

Given the setup described in the previous section, we parametrize all the contributions to the stress-energy tensor as

$$\begin{aligned} T^A_C|_{v,b} &= \frac{\delta(z)}{f}\mathrm{diag}(-V,-V,-V,-V,-V,-V,0)\\ T^A_C|_{v,B} &= \mathrm{diag}(-\Lambda_7,-\Lambda_7,-\Lambda_7,-\Lambda_7,-\Lambda_7,-\Lambda_7,-\Lambda_7)\\ T^A_C|_{m,b} &= \frac{\delta(z)}{f}\mathrm{diag}(-\rho,p,p,p,\pi,\pi,0)\\ T^A_C|_{m,B} &= T^A_C, \end{aligned} \tag{3}$$

with the subindices v and m indicating the vacuum and matter stress-energy tensors, while b and B stand for the brane and bulk contributions, respectively. A difference between this $(4+2+1)$D background and the simpler $(6+1)$D analysis without the 2D compactification cited at the end of the previous section, is having in (3) two different pressures in the 3D space and in the 2D compact dimensions for the matter on the brane, while for the $(6+1)$D background we would put $\pi = p$. This generalization is due to the fact that we do not assume homogeneity for the matter fluid in the whole $(3+2)$-dimensional space, but only in the 3D and 2D spaces separately.

Having calculated the Einstein tensor, we can put the explicit expression in the equation

$$G_{\mathrm{AC}} = \frac{1}{2M^5}T_{\mathrm{AC}} \tag{4}$$

evaluated on the brane (from now on all the functions are evaluated on the brane, i.e. at $z\to 0$), in the specific background (2). As a consequence, for the 00, ij and ab components we obtain the jump equations

$$\begin{aligned} a'_+ &= -a'_- = -\frac{fa}{20M^5}(V+\rho+2p-2\pi)\\ b'_+ &= -b'_- = -\frac{fb}{20M^5}(V+\rho-3p+3\pi)\\ n'_+ &= -n'_- = \frac{fn}{20M^5}(-V+4\rho+3p+2\pi). \end{aligned} \tag{5}$$

These are the values of the warp factors in the limit $z\to 0$, where the subscripts + and − distinguish the limit taken from below from the limit taken from above. The prime denotes the partial derivative with respect to the z coordinate, while the dot indicates the time derivative. For the 07 and 77 components, substituting the expressions (5) and choosing a gauge with $f(t,0)=1$ and $n(t,0)=1$, we get the (non) conservation equation

$$\dot\rho + 3\frac{\dot a}{a}(\rho+p) + 2\frac{\dot b}{b}(\rho+\pi) = 2T_{07} \tag{6}$$

and the Friedmann equation

$$\begin{aligned} &3\frac{\ddot a}{a} + 2\frac{\ddot b}{b} + 3\frac{\dot a^2}{a^2} + \frac{\dot b^2}{b^2} + 6\frac{\dot a}{a}\frac{\dot b}{b} + 3\frac{k}{a^2} + \frac{\kappa}{b^2}\\ &= -\frac{5}{(20M^5)^2}\Big[V(6p-\pi-\rho) + \rho(6p-\pi+\rho)\\ &\quad + \rho^2 + \frac{1}{5}(p-\pi)(V-19\rho-3p-7\pi)\Big]\\ &\quad + 15\lambda_{\mathrm{RS}} - \frac{1}{2M^5}T^7_7. \end{aligned} \tag{7}$$

We have defined the constant

$$\lambda_{\mathrm{RS}} = \frac{1}{30M^5}\Big(\Lambda_7 + \frac{3}{40M^5}V^2\Big), \tag{8}$$

which plays the role of an effective cosmological constant on the brane. These (6) and (7) are two equations in five variables H, F, ρ, p, π. We will thus have to make an ansatz for some of those variables.

The pure RS system correspond to setting $T^0_7 = T^7_7 = 0$, that means putting to zero the brane-bulk energy exchange and no cosmological term on the brane, i.e. λ_{RS}, to restore RS fine-tuning.

We can now write a simplified version of the differential Eqs. (6) and (7), using the usual ansatz for the equation of state of the matter fluid on the brane, i.e.

$$p = w\rho, \qquad \pi = w_\pi\rho. \tag{9}$$

The set of equations, in terms of the Hubble parameters $H\equiv\frac{\dot a}{a}$, $F\equiv\frac{\dot b}{b}$, is

$$\begin{aligned} &3\dot H + 2\dot F + 6H^2 + 6HF + 3F^2 + 3\frac{k}{a^2} + \frac{\kappa}{b^2}\\ &\quad = -\frac{1}{M^{10}}(c_V V + c_\rho\rho)\rho + 15\lambda_{\mathrm{RS}} - \frac{T^7_7}{2M^5} \end{aligned} \tag{10}$$

$$\dot\rho + [3(1+w)H + 2(1+w_\pi)F]\rho = 2T_{07} \tag{11}$$

with

$$\begin{aligned} c_V &= \frac{31w - 6w_\pi - 5}{400},\\ c_\rho &= \frac{11w + 14w_\pi + 10 - (w-w_\pi)(3w-7w_\pi)}{400}. \end{aligned} \tag{12}$$

Looking at the definition of the two coefficients c_V, c_ρ, we can note that Eq. (10) gets further simplified when the two pressures p and π are equal. This can be seen also from the previous Eq. (7), where the "nonstandard" term on the right-hand side (standard with respect to the homogeneous background analysis) is proportional to $(p-\pi)$. We will first examine some cosmological solutions assuming $p=\pi$ and then we will drop this equal pressure condition to find an expression for H, in terms of the energy density, in the particular limits of static compact extra dimensions and equal scale factors.

A. Equal pressures in 3D and 2D compact space

We can first try to find an interesting solution by simplifying the computation assuming $\pi = p$. In this case, the Eq. (7) written in terms of the Hubble parameters of the 3D space and 2D extra dimensions, defined as $H = \dot{a}/a$, $F = \dot{b}/b$ respectively, together with the (non) conservation Eq. (6), becomes

$$3\dot{H} + 2\dot{F} + 6H^2 + 6HF + 3F^2 + 3\frac{k}{a^2} + \frac{\kappa}{b^2} = -\frac{1}{80M^{10}}[V(5p - \rho) + \rho(5p + 2\rho)] + 15\lambda_{\mathrm{RS}} - \frac{T_7^7}{2M^5} \tag{13}$$

$$\dot{\rho} + (3H + 2F)(\rho + p) = 2T_{07} \tag{14}$$

We note that the system of equations written above still contains three variables $H(t)$, $F(t)$, $\rho(t)$ but only two equations. So we are able to just determine the value of the 3D Hubble parameter $H(t)$ as a function of the 2D one $F(t)$. Moreover, given the complicated form of this set of equations, we will make some assumptions on the internal space, such as flat compact extra dimensions ($\kappa = 0$) or static extra dimensions ($F(t) \equiv 0$) in the following subsections.

Manipulating (13) the system takes the form

$$5\frac{\mathrm{d}}{\mathrm{d}t}(3H+2F)^2 + 6(3H+2F)^3 + 6(3H+2F)(H-F)^2 = \frac{1}{8M^{10}}[5V\dot{\rho} + 6(3H+2F)V\rho + 5\rho\dot{\rho} + 3(3H+2F)\rho^2] + 150(3H+2F)\lambda_{\mathrm{RS}} + \frac{5}{8M^{10}}(V+\rho)2T_7^0 - \frac{5}{M^5}(3H+2F)T_7^7 - 10(3H+2F)\left(3\frac{k}{a^2} + \frac{\kappa}{b^2}\right). \tag{15}$$

It is interesting to derive the first order ordinary differential equation (ODE)s from the second order one (15), in order to find the expression for H^2 as a function of the localized matter energy density ρ and to perform the critical point analysis.

1. *Flat compact extra dimensions with equal pressures*

In flat compact extra dimensions ($\kappa = 0$) and flat 3D space ($k = 0$), Eq. (15) shows that in the limit in which the two Hubble parameters are almost equal we can deduce the solution for $(3H + 2F)$ in terms of the localized energy density ρ and of a mirage density χ that we will define below through a differential equation. In fact, in this case the third term on the left-hand side of (15) is negligible, leaving an analogous differential expression on both sides of the equality.

The solution for $(H - F) \ll (3H + 2F)$ and $k = \kappa = 0$ is given by

$$(3H + 2F)^2 = \frac{1}{16M^{10}}\rho^2 + \frac{V}{8M^{10}}(\rho + \chi) + 25\lambda_{\mathrm{RS}}. \tag{16}$$

The solution is written in terms of the mirage density χ and the localized energy density ρ. The mirage density must satisfy

$$\dot{\chi} + \frac{6}{5}(3H + 2F)\chi = \left(\frac{\rho}{V} + 1\right)2T_7^0 - \frac{8M^5}{V}(3H + 2F)T_7^7, \tag{17}$$

and the equation for ρ is the (non) conservation equation

$$\dot{\rho} + (3H + 2F)(\rho + p) = 2T_{07}. \tag{18}$$

Here we get a linear and quadratic ρ dependence for the Hubble parameter H^2, as well as a dependence on the mirage density χ and on the hidden sector Hubble parameter F. The quadratic and linear terms in ρ are analogous to those in the 5D analysis [9], implying that for $\rho \ll V$ the cosmological evolution looks four dimensional, while it moves away from the 4D behavior for $\rho \gg V$. The term in χ also already appears in the 5D model, as well as the λ_{RS} constant term. Besides, the mirage energy density dynamics are controlled by the bulk parameters T_7^0 and T_7^7, that represent the brane-bulk energy exchange and bulk pressure, as in [9]. However, a new variable F, the internal dimension Hubble parameter, arises and remains undetermined unless we make an ansatz for it. We can argue that the solution (16) and (17) is written in terms of a "total" Hubble parameter $\frac{1}{5}(3H + 2F)$, that carries the same characteristics as the H Hubble parameter in the 5D model, but also includes the dynamics of the evolution of the extra dimensions. For equal scale factors, $F = H$, this total Hubble parameter reduces to H alone, giving the exact analogue to the 5D RS cosmology.

2. *Equal scale factors with equal pressures*

A special case in which the $(H - F) \ll (3H + 2F)$ limit is valid is the equal scale factor case $F = H$. The results can directly be obtained from the previous subsection, yielding

$$H^2 = \frac{1}{400M^{10}}\rho^2 + \frac{V}{200M^{10}}(\rho + \chi) + \lambda_{\mathrm{RS}} - \frac{1}{10}\left(3\frac{k}{a^2} + \frac{\kappa}{a^2}\right), \tag{19}$$

$$\dot{\chi} + 6H\chi = \left(\frac{\rho}{V} + 1\right)2T_7^0 - \frac{40M^5}{V}HT_7^7, \tag{20}$$

$$\dot{\rho} + 5H(\rho + p) = 2T_{07}. \tag{21}$$

We added the curvature contributions that can be computed exactly in this limit. This solution is particularly simple thanks to the simultaneous vanishing of the $(H - F)$ and $(p - \pi)$ terms. It shows the quadratic dependence of H^2 on ρ and the linear term in $(\rho + \chi)$. The mirage density reduces to free radiation in 6D space-time when we restrict to pure RS configuration, with no energy exchange. In this same limit, the localized matter energy density obeys the standard conservation equation in 6D.

We will now drop the equal pressure ansatz and derive the expression for the Hubble parameter of the 3D space making particular assumptions on the internal space scale factor. We suppose from now on to live in a spatially flat universe ($k = 0$), where nevertheless the extra dimensions may be curved ($\kappa \neq 0$ generally).

B. Equal scale factors (generic pressures)

For generic pressures, we make use of the parametrization by means of w, w_π for the pressures of both the non compact and internal dimensions p, π that we introduced in (9), coming from the generalization of the equation of state for a fluid with energy density ρ.

We evaluate the Friedmann and (non) conservation differential Eqs. (10) and (11) assuming the scale factors of the 3D space and of the 2D internal space to be equal and consequently assuming the Hubble parameters governing the cosmological evolution of the two spaces to be equal, $F = H$. The system (10) and (11) takes the form

$$\frac{5}{2}(\dot{H}^2 + 6H^3) = -\frac{1}{M^{10}}(c_V V + c_\rho \rho)H\rho - \frac{T_7^7}{2M^5} + 15H\lambda_{\rm RS} - \frac{\kappa}{a^2}H \tag{22}$$

$$\dot{\rho} + (3(1 + w) + 2(1 + w_\pi))H\rho = 2T_{07}, \tag{23}$$

with the coefficients c_V, c_ρ still given by (12). With the help of (22) and (23) can be brought in a form from which we can explicitly deduce H as a function of ρ and χ

$$\frac{5}{2}(\dot{H}^2 + 6H^3) = \frac{1}{M^{10}}[\tilde{c}_{V(\rm eq)}V(\dot{\rho} + 6H\rho) + \tilde{c}_{\rho(\rm eq)}((\dot{\rho^2}) + 6H\rho^2)] + \frac{2T_7^0}{M^{10}}(\tilde{c}_{V(\rm eq)}V + \tilde{c}_{\rho(\rm eq)}\rho) - H\frac{T_7^7}{2M^5} + 15H\lambda_{\rm RS} - \frac{\kappa}{a^2}H, \tag{24}$$

yielding

$$H^2 = \frac{\tilde{c}_{\rho(\rm eq)}}{5M^{10}}\rho^2 + \frac{2\tilde{c}_{V(\rm eq)}V}{5M^{10}}(\rho + \chi) + \lambda_{\rm RS} - \frac{1}{10}\frac{\kappa}{a^2}, \tag{25}$$

$$\dot{\chi} + 6H\chi = 2T_7^0\left(1 + \frac{\tilde{c}_{\rho(\rm eq)}}{\tilde{c}_{V(\rm eq)}}\frac{\rho}{V}\right) - \frac{M^5}{2\tilde{c}_{V(\rm eq)}V}HT_7^7, \tag{26}$$

$$\dot{\rho} + (3(1 + w) + 2(1 + w_\pi))H\rho = 2T_{07}. \tag{27}$$

The two constants $\tilde{c}_{V(\rm eq)}$, $\tilde{c}_{\rho(\rm eq)}$ must satisfy

$$\tilde{c}_{V(\rm eq)} = \frac{c_V}{3(1 + w) + 2(1 + w_\pi) - 6}, \qquad \tilde{c}_{\rho(\rm eq)} = \frac{c_\rho}{3(1 + w) + 2(1 + w_\pi) - 3} \tag{28}$$

in order to have the right coefficients in Eq. (24). For some values of w, w_π the denominator of $\tilde{c}_{V(\rm eq)}$ or $\tilde{c}_{\rho(\rm eq)}$ may vanish. However we can fix w_π such that c_V (or c_ρ) becomes proportional to $3(1 + w) + 2(1 + w_\pi) - 6$ (or $3(1 + w) + 2(1 + w_\pi) - 3$ for c_ρ) and $\tilde{c}_{V(\rm eq)}$ (or $\tilde{c}_{\rho(\rm eq)}$) is finite. As an example consider $w_\pi = w$ (equal pressure in the internal space and 3D space, $\pi = p$) and check that both $\tilde{c}_{V(\rm eq)}$ and $\tilde{c}_{\rho(\rm eq)}$ remains finite and equal to $1/80$. Clearly, when $\tilde{c}_{V(\rm eq)}$ (or $\tilde{c}_{\rho(\rm eq)}$) diverges we cannot write the Friedmann Eq. (22) in the form (25).

C. Static compact extra dimensions (generic pressures)

We can follow the same procedure as in the equal scale factor limit for the case of static compact extra dimensions $F = 0$. While in the previous subsection the two internal and observed spaces were evolving according to the same dynamics, in this limit the extra dimensions do not evolve and remain static.

The two differential equations of motion for the gravity action in this case become

$$\frac{3}{2}(\dot{H}^2 + 4H^3) = -\frac{1}{M^{10}}(c_V V + c_\rho \rho)H\rho - \frac{T_7^7}{2M^5} + 15H\lambda_{\rm RS} - \frac{\kappa}{b_0^2}H \tag{29}$$

$$\dot{\rho} + 3(w + 1)H\rho = 2T_{07}, \tag{30}$$

where c_V, c_ρ are as before (12). We introduce the new coefficients $\tilde{c}_{V(\rm st)}$, $\tilde{c}_{\rho(\rm st)}$ defined by

$$\tilde{c}_{V(\rm st)} = \frac{c_V}{3w - 1}, \qquad \tilde{c}_{\rho(\rm st)} = \frac{c_\rho}{3w + 1}. \tag{31}$$

After plugging (30) into the Friedmann Eq. (29) we come to the expressions for H and χ

$$H^2 = \frac{\tilde{c}_{\rho(\rm st)}}{3M^{10}}\rho^2 + \frac{2\tilde{c}_{V(\rm st)}V}{3M^{10}}(\rho + \chi) + \frac{5\lambda_{\rm RS}}{2} - \frac{1}{6}\frac{\kappa}{b_0^2} \tag{32}$$

$$\dot{\chi} + 4H\chi = \left(1 + \frac{\tilde{c}_{\rho(\rm st)}}{\tilde{c}_{V(\rm st)}}\frac{\rho}{V}\right)2T_7^0 - \frac{M^5}{2\tilde{c}_{V(\rm st)}V}HT_7^7 \tag{33}$$

$$\dot{\rho} + 3(w + 1)H\rho = 2T_{07}. \tag{34}$$

In analogy to the equal scale factor case, these expressions are valid as long as we do not have $w = 1/3$ ($w = -1/3$) with $c_V \neq 0$ ($c_\rho \neq 0$).

D. Proportional Hubble parameters

We are going to combine in the same description the two limits of $a(t) = b(t)$ and $F = 0$, implying, in the first case, an equal cosmological evolution for the internal space and the 3D visible spatial dimensions and, in the second case, the absence of evolution for the compact space.

Both the two systems of differential Eqs. (25)–(27) and (32)–(34) obtained in the two different limits can be written in a unified formulation that encloses the two just cited sets of equations, defining some appropriate constant parameters. We introduce an "effective" number of dimensions d that takes the values $d = 6$ in the equal scale factor limit and $d = 4$ in the static compact extra dimensions. If we look at the Eqs. (26) and (33), we see that d appears as the number of dimensions for which the energy density χ satisfies the free radiation conservation equation in the limit of pure RS ($T^7_7 = T^0_7 = 0$). In fact, the Friedmann equation plus the two (non) conservation equations can be rewritten as

$$H^2 = \frac{\tilde{c}_{\rho,d}}{(d-1)M^{10}}\rho^2 + \frac{2\tilde{c}_{V,d}V}{(d-1)M^{10}}(\rho+\chi) - \frac{1}{2(d-1)}\frac{\kappa}{b^2} + \frac{30}{d(d-1)}\lambda_{\rm RS} \tag{35}$$

$$\dot{\chi} + dH\chi = 2T^0_7\left(1 + \frac{\tilde{c}_{\rho,d}}{\tilde{c}_{V,d}}\frac{\rho}{V}\right) - \frac{M^5}{2\tilde{c}_{V,d}V}HT^7_7 \tag{36}$$

$$\dot{\rho} + w_d H\rho = 2T_{07}. \tag{37}$$

We have in addition defined $w_d = 3(1+w) + (d-4)\times(1+w_\pi)$ and $\tilde{c}_{V,d} = c_V/(w_d - d)$, $\tilde{c}_{\rho,d} = c_\rho/(w_d - d/2)$ where c_V and c_ρ are given in (12). We remind that in order to get an algebraic equation for H^2 as a function of the energy densities ρ and χ (35), we have to keep $\tilde{c}_{V,d}$, $\tilde{c}_{\rho,d}$ finite, i.e. respectively $w_d \neq d$, $w_d \neq d/2$ unless $c_V = 0$, $c_\rho = 0$. For example we cannot write H in the form (35) if $w = 1/3$, $w_\pi = 0$ in both the equal scale factor and the static compact extra dimension limit, since the linear term in ρ has a diverging coefficient.

Moreover, we can further generalize this analysis introducing a parameter ξ such that $F = \xi H$. This description contains all the above studied limits. The analogous of previous relations (35)–(37) can be written as

$$H^2 = \frac{\tilde{c}_{\rho,\xi}}{(3+2\xi)M^{10}}\rho^2 + \frac{2\tilde{c}_{V,\xi}V}{(3+2\xi)M^{10}}(\rho+\chi) - \frac{1}{\xi^2+3\xi+6}\frac{\kappa}{a^{2\xi}} + \frac{5}{\xi^2+2\xi+2}\lambda_{\rm RS} \tag{38}$$

$$\dot{\chi} + d_\xi H\chi = 2T^0_7\left(1 + \frac{\tilde{c}_{\rho,\xi}}{\tilde{c}_{V,\xi}}\frac{\rho}{V}\right) - \frac{M^5}{2\tilde{c}_{V,\xi}V}HT^7_7 \tag{39}$$

$$\dot{\rho} + w_\xi H\rho = 2T_{07}, \tag{40}$$

where now d_ξ is a more complicated function of the proportionality constant ξ between the two Hubble parameters $d_\xi \equiv 6\frac{\xi^2+2\xi+2}{3+2\xi}$ and it reduces to $d = 6$, $d = 4$ in the two previously examined limits of equal scale factors and static compact extra dimensions ($\xi = 1$, $\xi = 0$). The constant w_ξ reduces to w_d for $\xi = 0$, $\xi = 1$ and is defined by $w_\xi \equiv 3(1+w) + 2\xi(1+w_\pi)$. The two coefficients $\tilde{c}_{V,\xi}$, $\tilde{c}_{\rho,\xi}$ are defined as $\tilde{c}_{V,d}$, $\tilde{c}_{\rho,d}$, with $w_d \to w_\xi$, $d \to d_\xi$. The result (38) is valid unless $\xi = -3/2$. In that case the equation for H becomes algebraic—though we still have the curvature term explicitly depending on the scale factor—, thus

$$H^2 = -\frac{\tilde{c}_{\rho,\xi}}{33M^{10}}\rho^2 - \frac{\tilde{c}_{V,\xi}V}{33M^{10}}\rho - \frac{\kappa}{33a^{2\xi}} + 4\lambda_{\rm RS}$$

$$\dot{\rho} + 3(w - w_\pi)H\rho = 2T_{07}. \tag{41}$$

No mirage density appears and the Hubble parameter is a quadratic polynomial in the localized energy density ρ alone. Besides, if the pressures are equal $w_\pi = w$ in the pure RS setup $T^0_7 = 0$, the energy density is constant in time and so is $H^2 + \frac{\kappa a^3}{33}$, for $\xi = -3/2$ (41). The set of Eqs. (41) also does not depend on T^7_7 at all.

Again we have to restrict to $w_\xi \neq d_\xi, d_\xi/2$ to keep $\tilde{c}_{V,\xi}$, $\tilde{c}_{\rho,\xi}$ finite (unless $c_V \propto w_\xi - d_\xi$, $c_\rho \propto w_\xi - d_\xi/2$).

We remark that for the scale factors satisfying $b(t) = 1/a(t)$, i.e. dynamical compactification with $\xi = -1$, the equation for the mirage energy density χ in the pure RS setup is still an effective 6D free radiation conservation equation, as for $b(t) = a(t)$. In fact, the only solutions to $d_\xi = 6$ are $\xi = \pm 1$. To obtain a 4D free radiation equation for χ we have to require $\xi = 0$, since the second solution to $d_\xi = 4$ is $\xi = -3/2$, for which we do not define a mirage density (41).

However, this is not the end of the story. Introducing the effective 4D densities $\varrho = V_{(2)}\rho$ and $\mathcal{X} = V_{(2)}\chi$ (where $V_{(2)} = v b^2(t) = v a^{2\xi}(t)$ is the volume of the 2D internal space), we have to replace the left-hand side of Eqs. (39) and (40) respectively by $\dot{\mathcal{X}} + (d_\xi - 2\xi)H\mathcal{X}$ and $\dot{\varrho} + (w_\xi - 2\xi)H\varrho$ (the right-hand side equations are also modified and we will explicitly write them at the end of subsection IV B). This tells us that the 4D mirage density is a free radiation energy density for pure RS in four dimensions for $d_\xi - 2\xi = 4$, which has solutions $\xi = 0$, $\xi = 1$—i.e. static internal space or equal scale factors, justifying the study of these two limits.

E. Comments

We here summarize some considerations about the bulk evolution equations derived in the previous subsections and, in particular, about the explicit expressions we have found for the Hubble parameters in the discussed limits.

(i) With the assumption of having the same pressure for the matter fluid in the two dimensional internal space and in the 3D visible space (i.e. $\pi = p$), we found a form of the Friedmann equation that has the advantage of keeping both the Hubble parameters not constrained by any particular ansatz. The Friedmann Eq. (15) provides an expression for $(3H + 2F)$ in terms of ρ and χ (for spatially flat spaces). This solution, though, is satisfactory only in the limit of small $(H - F)$. When $(H - F)$ is not negligible w.r.t. $(3H + 2F)$, the mirage density equation may be written introducing an extra term independent of the bulk parameters T_7^0, T_7^7. This prevents χ to obey to a free radiation equation in the pure RS setup ($T_7^0 = T_7^7 = 0$), as it should instead be in the context of the AdS/CFT correspondence (we will discuss the comparison between the bulk and the dual brane analysis in section VIII).

(ii) In the simple limits of equal scale factors (25) and static compact extra dimensions (32) we recovered an expression for H^2 containing a quadratic term in ρ and a linear term in $(\rho + \chi)$, where ρ is the localized energy density and χ is an artificially introduced mirage density that accounts for the bulk dynamics. In fact it depends on the bulk parameters T_7^0, T_7^7. When $T_7^0 = T_7^7 = 0$, the mirage density obeys to 4D free radiation equation for the static compact extra dimension case and to 6D free radiation equation for the equal scale factor case. This is in complete analogy to the 5D analysis [43], where the same dependence on ρ and χ occurs and the mirage energy satisfies 4D free radiation for pure RS (i.e. $T_5^0 = T_5^5 = 0$).

(iii) When $w_\pi = w$ (or equivalently $\pi = p$) in the equal scale factor limit (25), we find the results given by the equal pressures subsection in the case $F = H$ (19). The two limits of equal pressures and equal scale factors then commute and the results are consistent.

(iv) The description of section III D encloses in a unifying way the results in the limits of static compact extra dimensions and equal scale factors. It moreover generalizes these results to the case of evolutions governed by proportional Hubble parameters $F(t) = \xi H(t)$. We will use the set of equations written in terms of the effective number of dimensions d (that describes the two limits of static internal space, with $d = 4$, and equal scaling for the compactification space and the 3D space, with $d = 6$) to study the corresponding cosmological evolution in the next section.

We are now going to proceed to the analysis of the critical points for this seven dimensional universe in a 7D Randall-Sundrum setup, including the energy exchange term.

IV. BULK CRITICAL POINT ANALYSIS WITH ENERGY EXCHANGE

We have until now transformed the second order differential Eq. (7) plus the (non) conservation Eq. (6) in a set of three linear differential Eqs. (35)–(37) for combined equal scale factor and static compact extra dimension limits, or more generally (38)–(40) for proportional Hubble parameters. We have introduced the mirage density χ defined by its differential equation. In this section we will use the first system of Eqs. (35)–(37), obtained to describe both the limit of equal scale factors and static compact extra dimensions, to find its fixed points and the corresponding stability. The critical point analysis will allow us to study the cosmological evolution in the bulk description for $F = 0$ or $F = H$.

We make an assumption on the bulk components of the stress-energy tensor that appears in the differential equations for the energy densities ρ and χ. As in [43], we will take the diagonal elements of the stress-energy tensor to satisfy the relation

$$\left| \frac{T_{m,B}^{(\mathrm{diag})}}{T_{v,B}^{(\mathrm{diag})}} \right| \ll \left| \frac{T_{m,b}^{(\mathrm{diag})}}{T_{v,b}^{(\mathrm{diag})}} \right|. \tag{42}$$

This enforces the solution to the Friedmann equation to be reasonably independent of the bulk dynamics, since the T_7^7 term in (7) becomes negligible with respect to the first term on the right-hand side of the same equation. Imposing such a relation, T_7^7 disappears from the sets of linear differential equations, while we remain left with the T_7^0 component. For the future bulk calculations we will define $T \equiv 2T_7^0$ to simplify the notation.

Before starting the critical point analysis we note that when $T = 0$ the system of Eqs. (35)–(37) have only trivial critical points characterized by the zero visible Hubble parameter when the internal space is flat. There are two of these critical points. One is given by $H_\star^2 = -\kappa/2(d-1)b^2$, $\rho_\star = \chi_\star = 0$ (which is valid only if we are compactifying on hyperbolic or flat spaces) and the other is $H_\star = 0$, $\tilde{c}_{\rho,d}\rho_\star^2 + 2\tilde{c}_{V,d}V(\rho_\star + \chi_\star) = M^{10}\kappa/b$.

We will first restrict to small density [87] $\rho \ll V$ and flat internal space $\kappa = 0$ (remember that the 3D space is already supposed to be flat, having put $k = 0$) and then go through the generic density analysis. Eventual de Sitter stable solutions (for the 4D visible space-time) could represent the present accelerated era, while inflationary phases at early times may be associated to primordial inflation.

A. Small energy density and flat compact extra dimensions

When the localized energy density is small and the internal space curvature vanishes, $\rho \ll V$, $\kappa = 0$, the bulk Einstein Eqs. (35)–(37) in terms of H, χ and ρ become

$$H^2 = \frac{2\tilde{c}_{V,d}V}{(d-1)M^{10}}(\rho + \chi) \tag{43}$$

$$\dot{\chi} + dH\chi = T \tag{44}$$

$$\dot{\rho} + w_d H\rho = -T. \tag{45}$$

We note that in this approximation expression (43) is not valid for $w_d = d$, unless $c_V \propto (w_d - d)$ (this would, for example, determine a specific value for w_π as a function of w). Nevertheless we will find in the rest of the section that $\tilde{c}_{V,d}$ always appears with the coefficient $(w_d - d)$ in the critical point analysis. We thus have to keep in mind that $(w_d - d)\tilde{c}_{V,d} = c_V = (31w - 6w_\pi - 5)/400$ always remains finite.

Fixed point solutions In the small density approximation, the fixed points in terms of the critical energy density can immediately be found (to have a full solution we have to make an ansatz on the form of the energy exchange parameter).

The first solution is given by

$$H_\star = -\frac{B_\star}{w_d}\rho_\star^{1/2} \qquad \chi_\star = -\frac{w_d}{d}\rho_\star \qquad T_\star = B_\star\rho_\star^{3/2}, \tag{46}$$

where $B_\star \equiv -w_d(\frac{(d-w_d)2\tilde{c}_{V,d}V}{d(d-1)M^{10}})^{1/2}$. This represents an inflationary critical point for the cosmological evolution, for $(w_d - d)\tilde{c}_{V,d} = c_V > 0$, i.e. both $\tilde{c}_{V,d} > 0$ and $d > w_d$ or $\tilde{c}_{V,d} < 0$ and $d < w_d$ [88]. The acceleration factor at the fixed point is simply given by $q_\star = H_\star^2$. Since we assume w_d to be positive, $B_\star$ is negative and $H_\star$ in (46) is positive. At this fixed point the universe is thus expanding.

Another fixed point leaves $\chi_\star$ unchanged, while $T_\star$ and $H_\star$ have switched signs with respect to (46), meaning that $H_\star$ is negative and the universe in contracting (there is a symmetry $T \to -T$, $H \to -H$)

$$H_\star = \frac{B_\star}{w_d}\rho_\star^{1/2} \qquad \chi_\star = -\frac{w_d}{d}\rho_\star \qquad T_\star = -B_\star\rho_\star^{3/2}. \tag{47}$$

The trivial critical point is characterized by mirage density equal and opposite to $\rho_\star$, but zero Hubble parameter and energy exchange (in the case we admit for the energy exchange the form $T = A\rho^\nu$ all the variables are zero at the trivial fixed point).

For positive critical energy densities $\rho_\star$, we obtain a negative brane-bulk energy exchange parameter at the critical point if $H_\star > 0$ and, vice versa, we have positive critical energy exchange for a contracting universe at the critical point. During the evolution, we can expect a change of regime going from negative to positive T as the energy density localized on the brane grows, as for the 5D RS critical point analysis with energy exchange carried in [43]. Even though most of the analysis in this section will be performed supposing that the energy exchange parameter has fixed sign determined by the sign of A (since we will mainly assume $T = A\rho^\nu$), we can argue that for small energy density ρ the generic energy exchange is presumably negative, meaning that energy would be transferred from the bulk onto the brane. In this hypothesis, an equilibrium can be reached, such that the energy density would have a large limiting value for which energy starts to flow back into the bulk (with positive energy exchange).

Stability analysis The real parts of the eigenvalues of the stability matrix for the $(\delta\chi, \delta\rho)$ linear perturbations corresponding to the critical point (46) can have opposite signs or be both negative. This depends on the value of T as a function of the energy density ρ at the fixed point (46) describing an expanding universe with energy influx. The explicit form of the eigenvalues is

$$\lambda_\pm = \frac{B_\star}{2w_d}\rho_\star^{1/2}[d + (1-\tilde{\nu})w_d \pm \sqrt{(d + (1-\tilde{\nu})w_d)^2 - 2(3-2\tilde{\nu})dw_d}], \tag{48}$$

where we have defined

$$\tilde{\nu} \equiv \frac{\partial \log|T|}{\partial \log\rho}\bigg|_\star. \tag{49}$$

Expression (48) then shows that the two eigenvalues have negative real part when $\tilde{\nu} < 3/2$. There is a second upper bound on $\tilde{\nu}$ derived from requiring negative real part for the eigenvalues. Nonetheless, for the range of values $-1 \leq w, w_\pi \leq 1$ and $d = 4, 6$ in which we are interested, this bound is always equal or greater than $3/2$. If $|T|$ is a decreasing or constant function of ρ near $T_\star$, the nontrivial inflationary critical point always is an attractor. Also for growing $|T|$ we can have stable inflationary fixed points, as long as $\tilde{\nu} < 3/2$. In particular, the linear $\tilde{\nu} = 1$ case is included in the stable inflationary fixed point window and will be analyzed both solving the Einstein equations numerically, in the next subsection, and deriving an explicit solution, in subsection IV C.

Besides, when

$$1 - \sqrt{\frac{2d}{w_d}} - \frac{d}{w_d} < \tilde{\nu} < 1 + \sqrt{\frac{2d}{w_d}} - \frac{d}{w_d}, \tag{50}$$

the eigenvalues have non zero imaginary part and the critical point will hence be a stable spiral for $\tilde{\nu} < 3/2$. For values of $\tilde{\nu}$ out of the range (50), we get a node. As an example, let us assume the value $\tilde{\nu} = 1$ in the equal scale factor background. This gives a stable spiral for $w_\pi > 1/2$ or $w_\pi < 1/2$ and $w > -2(1 + w_\pi)$ (considering $w, w_\pi > -1$). This means that in the case $w \simeq 1/3$ and $w_\pi = 0$ the critical point is a stable spiral, while for both w and w_π null we instead have a stable node.

For energy outflow $T_\star > 0$ (which goes along with contraction $H_\star < 0$), we get a minus sign overall modifying the eigenvalues (48) referring to the linearized system around the (47) critical point (characterized indeed by energy outflow). The eigenvalues cannot be both negative in this case. In fact we should demand $w_d > d/(\tilde{\nu} - 1)$ with $\tilde{\nu} > 1$ but also $\tilde{\nu} < 3/2$ to get a stable fixed point. Only the trivial point, as we will discuss later, can be attractive for energy outflow dynamics.

Assumption $T = A\rho^\nu$ *and numerical solutions* Assuming the brane-bulk energy exchange parameter to take the form $T = A\rho^\nu$ (so that $\tilde{\nu} = \nu$ referring to (49)), we can rewrite the system of differential equations in term of dimensionless quantities $\check{\rho} = \gamma^6\rho$, $\check{\chi} = \gamma^6\sigma$, $\check{H} = \gamma H$, $\check{T} = \gamma^7 T$, where we called $\gamma^4 \equiv \frac{2V}{(d-1)M^{10}}$. The dimensionless variable ρ/V used to perform the small energy density expansion at the beginning of this section is related to the dimensionless variable $\check{\rho}$ by $\check{\rho} = (2\tilde{c}_{V,d}/(d-1))^{3/2} \times (V/M^6)^{5/2}(\rho/V)$. So, considering the small ρ/V approximation is equivalent to considering small $\check{\rho}$ approximation if the brane tension V satisfies $V \lesssim M^6$ with respect to the 7D Planck mass and $\tilde{c}_{V,d}$ is reasonably of the order $\tilde{c}_{V,d} \lesssim 1$. The complete set of fixed point solutions (discarding the trivial ones) can be calculated in terms of the parameters A, ν characterizing the energy exchange, w_d, d denoting the background (static extra dimensions or equal scale factors) and the equations of state for both the 3D and internal spaces.

The Einstein equations become

$$\check{H}^2 = \tilde{c}_{V,d}(\check{\rho} + \check{\chi}) \qquad \dot{\check{\chi}} + d\check{H}\check{\chi} = \check{A}\check{\rho}^\nu$$
$$\dot{\check{\rho}} + w_d\check{H}\check{\rho} = -\check{A}\check{\rho}^\nu, \tag{51}$$

where $\check{A} = \gamma^{1+6(1-\nu)}A$.

The acceleration $\check{q}$ can be evaluated independently of ν and $\check{A}$

$$\check{q} = \left(1 - \frac{w_d}{2}\right)\tilde{c}_{V,d}\check{\rho} + \left(1 - \frac{d}{2}\right)\tilde{c}_{V,d}\check{\chi} \tag{52}$$

as a function of the localized matter density and of the mirage density. Because of the positiveness constraint on $\tilde{c}_{V,d}(\check{\rho} + \check{\chi})$ coming from the first equation in (51), the trajectories in the phase space must satisfy

$$\check{q} \le (d - w_d)\tilde{c}_{V,d}\rho \tag{53}$$

as it is indeed showed in the numerical plots of Fig. 1. For $w_d > 2$ and $\tilde{c}_{V,d} > 0$ we have positive acceleration only if the mirage density $\check{\chi}$ is negative and smaller than $-\check{\rho}(2 - w_d)/(2 - d)$. On the other hand, $\check{\chi}$ gets positive (suppose $\tilde{c}_{V,d} > 0$) only if $\check{q} < (2 - w_d)\check{\rho}/2$.

The fixed points are given by

$$\check{H}_\star^{3-2\nu} = (-)^{3-2\nu}\left(\frac{\tilde{c}_{V,d}(d - w_d)}{d}\right)^{1-\nu}\frac{\check{A}}{w_d} \tag{54}$$

$$\check{\chi}_\star^{3-2\nu} = (-)^{3-2\nu}\frac{d^{2(\nu-1)}\check{A}^2}{\tilde{c}_{V,d}(d - w_d)w_d^{2\nu-1}} \tag{55}$$

$$\check{\rho}_\star^{3-2\nu} = \frac{d\check{A}^2}{w_d^2\tilde{c}_{V,d}(d - w_d)}. \tag{56}$$

For $\nu < 3/2$, when the nontrivial fixed point is stable, we have two roots of (56) with opposite signs if $\nu = 1/2 + m$, $m \in \mathbb{Z}$. Only one real root exists for integer ν and it carries the sign of the right-hand side in (56). Finally, for $\nu = (2m + 1)/4$, we have a positive root if the right-hand side in (56) is negative. The two eigenvalues corresponding to a negative $\check{\rho}_\star$ always have opposite real part, implying that this fixed point is not be stable at linear order, it is a saddle. We further note that for $w_d > d$ and positive $\tilde{c}_{V,d}$ (or alternatively $d > w_d$ and negative $\tilde{c}_{V,d}$) we can only have real and positive $\rho_\star$ fixed point if $\nu = (2m + 1)/4$, $m \in \mathbb{Z}$. This implies that for $\nu = 1$ there is no nontrivial fixed point with positive $\rho_\star$, when $(d - w_d)$ and $\tilde{c}_{V,d}$ have

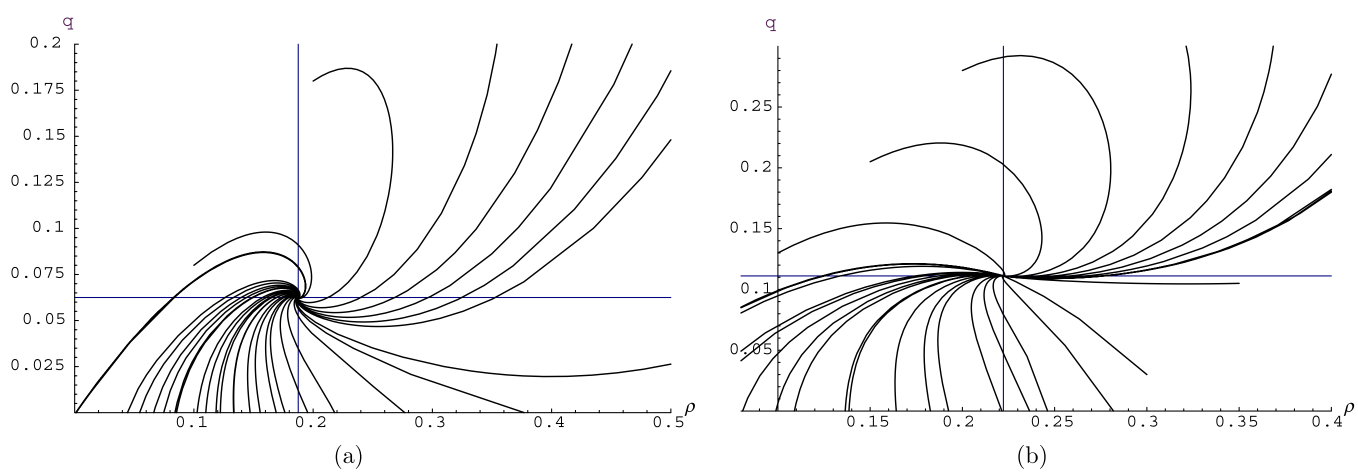

FIG. 1 (color online). Phase spaces $\check{q}/\check{\rho}$ for different initial conditions $\check{\rho}_0$ and $\check{q}_0$, with $\check{A} = -1$, $\nu = 1$, $d = 6$ (equal scale factors—analogous pictures come from the static extra dimension case) and: (a) $w_d = 4$ (for instance $w = 0$, $w_\pi = -1/2$) or $w = -1/5$, $w_\pi = 0$) leading to a spiral like stable critical point, (b) $w_d = 3$ (for instance $w = 0$, $w_\pi = 1/2$) determining a stable node. The extra blue grid lines intersection represents the fixed point.

opposite signs. Moreover, these negative $\rho_\star$ points are always characterized by negative $H_\star$, so that they would not be inflationary.

We can check the stability of the critical points by means of a numerical analysis of the differential system of Eqs. (51). In the case of energy influx $\check{A} < 0$, putting $\nu = 1$ and different values for the d, w_d parameters, we get the phase spaces in Fig. 1, plotting the acceleration factor $\check{q}(t) \equiv \ddot{\check{a}}(t)/\check{a}(t)$ as a function of the energy density $\check{\rho}(t)$. We thus check that, solving the system of differential equations for variable initial conditions for $\check{\chi}$ and $\check{\rho}$, including both positive and negative initial $\check{q}$, all the different trajectories converge to the nontrivial fixed point, designated by the intersection of the two perpendicular lines in the picture. Besides, as we expect, in part 1(a) they have a spiral behavior, while in the 1(b) case they denote a node.

In the limit $w_d \to d$ we numerically recover the analytical solution discussed in subsection IV C, neglecting the very large density behavior.

For $\nu > 3/2$, i.e. when the nontrivial fixed point is no more an attractor, we find that some of the trajectories go to the trivial critical point, while another branch of solutions to the Eintein Eqs. (51) are characterized by diverging energy density $\check{\rho}$ (they become unreliable when $\check{\rho}^2 \gtrsim (2\tilde{c}_{V,d}/(d-1))^3(V/M^6)^5$). This happens because, as it is suggested by the integration of the third equation in (51) with energy influx hypothesis, for $\check{\rho}$ big enough—precisely for $\check{\rho}^{\nu-(3/2)} \gtrsim (w_d/A)\tilde{c}_{V,d}^{(1/2)}(1+\check{\chi}/\check{\rho})^{1/2}$—the function $\check{\rho}(t)$ starts growing, while for small $\check{\rho}$, $\check{\rho}(t)$ eventually goes to zero. Depending on the initial conditions we will have solutions ending in the trivial fixed point or diverging.

The behavior of the system with energy outflow can be analytically deduced. Given the hypothesis $T > 0$ it is clear that the trajectories cannot be attracted by the fixed point solution $T_\star = B_\star\rho_\star^{3/2}$. They may go to the critical point characterized by negative Hubble parameter and $T_\star = -B_\star\rho_\star^{3/2}$. However, we already determined the non attractive nature of this fixed point. Another way to describe outflow dynamics is to conclude from the form of the Einstein Eqs. (45) that all the trajectories in the phase space $\check{q}/\check{\rho}$ go toward the trivial point, since for positive T the density ρ is suppressed at late time. The way in which the trajectories go to the critical point depends as before on the positiveness of the function under the square root in (48). We numerically checked that for $\nu = 1$ and $d = 6$, $w_d = 4$ the null fixed point is an attractor and, in particular, a stable node.

B. Critical points with general energy density

Allowing κ to be different from zero, not restricting the localized energy density to be small and with the assumptions (42) on the bulk matter stress-energy tensor, we have to solve the following set of equations

$$H^2 = \frac{\tilde{c}_{\rho,d}}{(d-1)M^{10}}\rho^2 + \frac{2\tilde{c}_{V,d}V}{(d-1)M^{10}}(\rho+\chi) - \frac{1}{2(d-1)}\frac{\kappa}{b^2} \tag{57}$$

$$\dot{\chi} + dH\chi = T\left(1 + \frac{\tilde{c}_{\rho,d}}{\tilde{c}_{V,d}}\frac{\rho}{V}\right) \tag{58}$$

$$\dot{\rho} + w_d H\rho = -T. \tag{59}$$

As in the small energy density regime, we can immediately notice that the analytical behavior for energy outflow is characterized by decreasing ρ in time. The trajectories will thus be attracted to the trivial fixed point characterized by vanishing $\rho_\star$.

We will now carry a general critical point analysis. Equation (57) exhibits divergences if $w_d = d$ or $w_d = d/2$, unless $c_V \propto (w_d - d)$ or $c_\rho \propto (w_d - d/2)$. When the divergence arises, the right system of equations is (10). Terms like $(w_d - d)\tilde{c}_{V,d} = c_V$ and $(w_d - d/2)\tilde{c}_{\rho,d} = c_\rho$ are always finite (where we recall $c_V = 31w - 6w_\pi - 5$ and $c_\rho = 11w + 14w_\pi + 10 - (w - w\pi)(3w - 7w_\pi)$).

Fixed point solutions If we demand H to be positive, i.e. expanding universe, the solution for $H_\star$ and $T_\star$ is given by

$$H_\star = -\frac{B_\star}{w_d}\rho_\star^{1/2} \qquad \chi_\star = -\frac{w_d}{d}\left(1 + \frac{\tilde{c}_{\rho,d}\rho_\star}{\tilde{c}_{V,d}V}\right)\rho_\star \qquad T_\star = B_\star\rho_\star^{3/2}, \tag{60}$$

where $B_\star = B_\star(\rho_\star)$ depends also on κ and $\rho_\star$ and is defined by

$$B_\star(\rho_\star) = -w_d\left[\frac{(d-2w)\tilde{c}_{\rho,d}\rho_\star + 2(d-w)\tilde{c}_{V,d}V}{d(d-1)M^{10}} - \frac{\kappa}{2(d-1)b_\star^2\rho_\star}\right]^{1/2}. \tag{61}$$

We have a negative energy exchange parameter, as in the small density limit, and a positive Hubble parameter.

The second fixed point solution is equal to the first except for the $H_\star$ and $T_\star$ signs reversed (keep in mind the $H \to -H$, $T \to -T$ symmetry), such that $H_\star$ would be negative and we would have energy outflow at the critical point

$$H_\star = \frac{B_\star}{w_d}\rho_\star^{1/2} \qquad \chi_\star = -\frac{w_d}{d}\left(1 + \frac{\tilde{c}_{\rho,d}\rho_\star}{\tilde{c}_{V,d}V}\right)\rho_\star \qquad T_\star = B_\star\rho_\star^{3/2}. \tag{62}$$

The trivial critical point is characterized by vanishing $H_\star$ and $T_\star$, while the mirage density becomes

$$\chi_\star = \frac{M^{10}\kappa}{4b_\star^2\tilde{c}_{V,d}V} - \left(1 + \frac{\tilde{c}_{\rho,d}\rho_\star}{2\tilde{c}_{V,d}V}\right)\rho_\star. \tag{63}$$

If the energy exchange is supposed to be of the form $T = A\rho^\nu$, the trivial fixed point is characterized by zero value for all the variables except for the mirage density χ which becomes $\chi_\star = M^{10}\kappa/4b_\star^2\tilde{c}_{V,d}V$ and is zero for flat compact spaces.

As in the limit of small energy density considered in the previous section, the constant $B_\star$ is negative whenever the argument of the square root in (61) is positive, i.e. when

$$\frac{\kappa}{2(d-1)b_\star^2} < \frac{(d-2w_d)\tilde{c}_{\rho,d}\rho_\star + 2(d-w)\tilde{c}_{V,d}V}{d(d-1)M^{10}}\rho_\star. \tag{64}$$

If the square root gives an imaginary number, we do not have any real valued fixed point except for the trivial one.

Stability analysis The positiveness of the eigenvalues of the stability matrix depends now on all the parameters and constants of the theory and not only on $\tilde{\nu}$, as for the small density, flat compact extra dimension simple case. We consider the situation in which the variation of κ/b^2 vanishes (this happens in the static compact extra dimension limit or for $\kappa = 0$ in the equal scale factor background, as the ratio κ/b^2 remains constant). Otherwise we would have a linearized system of two differential equations plus one algebraic equation in the four variables $\delta\kappa$, $\delta\rho$, $\delta\chi$, δH.

There are two conditions that must be satisfied, in order to get two negative eigenvalues and hence a stable fixed point. These conditions give two upper bounds for $\tilde{\nu}$ in terms of the constants w_d, d and V, $\rho_\star$, $\kappa/b_\star^2$, M^{10}

$$\begin{aligned}(\tilde{\nu}-1) &< \frac{w_d}{d}\frac{w_d}{(d-1)B_\star^2M^{10}}[(d-2w_d)\tilde{c}_{\rho,d}\rho_\star \\ &\quad + (d-w_d)\tilde{c}_{V,d}V] \\ (\tilde{\nu}-1) &< \frac{d}{w_d}.\end{aligned} \tag{65}$$

The second bound in (65) satisfies $d/w_d + 1 > 3/2$ in the range $-1 \le w, w_\pi \le 1$. Besides, the first bound reduces to $\nu < 3/2$ when we take the limit $\rho/V \ll 1$ and put $\kappa = 0$. So the results are in agreement with the previous small density analysis.

The bounds (65) depend on the fixed point value of ρ, which cannot be determined without making any assumption on the form of T. However, we can make some remarks on the nature of the fixed points. For values of $\tilde{\nu}$ in the range

$$1 - \frac{d}{w_d} - R_\star < \tilde{\nu} < 1 - \frac{d}{w_d} + R_\star, \tag{66}$$

where we defined

$$R_\star \equiv 2\sqrt{\frac{w_d}{(d-1)B_\star^2M^{10}}[(d-2w_d)\tilde{c}_{\rho,d}\rho_\star + (d-w_d)\tilde{c}_{V,d}V]}.$$

the stability matrix eigenvalues have non null imaginary part and the trajectories near to the critical point have a spiral like behavior. When $R_\star^2 < 0$ we always have node like fixed points. In agreement with the small density case (50), when $\rho/V \ll 1$ and $\kappa = 0$ we get $R_\star \to \sqrt{2d/w_d}$.

As an example, we assign the value $\tilde{\nu} = 1$. Since the first bound (65) can be rewritten as

$$(\tilde{\nu}-1) < \frac{w_d}{d}\frac{R_\star^2}{4}. \tag{67}$$

This means that we should have $R_\star^2 > 0$ to get stability. We also find that the fixed points have spiral shape when $R_\star > d/w_d$ or $R_\star < -d/w_d$, they will be nodes otherwise.

Assumption $T = A\rho^\nu$ and numerical solutions To do a more quantitative analysis we have to make an ansatz on the form of the energy exchange parameter T. As in the previous section, we suppose a power dependence on the energy density ρ such that $T = A\rho^\nu$. The equations for generic energy densities and internal space curvature can be rewritten introducing dimensionless variables as in (51)

$$\begin{aligned}\check{H}^2 &= \tilde{c}_{\rho,d}\alpha\check{\rho}^2 + \tilde{c}_{V,d}(\check{\rho}+\check{\chi}) - \check{\kappa}\,\dot{\check{\chi}} + d\check{H}\,\check{\chi} \\ &= \check{A}\rho^\nu\left(1 + 2\frac{\tilde{c}_{\rho,d}}{\tilde{c}_{V,d}}\alpha\check{\rho}\right) \\ \dot{\check{\rho}} + w_d\check{H}\,\check{\rho} &= -\check{A}\check{\rho}^\nu,\end{aligned} \tag{68}$$

where α is a dimensionless constant defined by $\alpha^2 \equiv \frac{(d-1)^3}{64}(\frac{M^6}{V})^5$, $\check{\kappa}$ is the dimensionless variable $\check{\kappa} = \frac{\gamma^2\kappa}{2(d-1)b^2}$ we remind that we restrict to constant $\check{\kappa}$ approximation.

To obtain real observables, we have to restrict the possible values for $\check{\rho}$ and $\check{\chi}$ such that $\tilde{c}_{\rho,d}\alpha\check{\rho}^2 + \tilde{c}_{V,d}(\check{\rho} + \check{\chi}) - \check{\kappa} \ge 0$. In fact, the plots show the presence of a prohibited zone in the phase space—in particular, in Fig. 2(a) it is clear that the region of the possible trajectories is delimited by a parabola. The relation that must be satisfied, in terms of the acceleration parameter and the energy density, is

$$\check{q} \le (d-2w_d)\tilde{c}_{\rho,d}\alpha\check{\rho}^2 + (d-w_d)\tilde{c}_{V,d}\check{\rho} - d\kappa. \tag{69}$$

In fact, the analytical expression for the acceleration $\check{q} = \dot{\check{H}} + \check{H}^2$ can be written using (68) in terms of the visible energy density and the mirage density. For any ν

$$\begin{aligned}\check{q} &= (1-w_d)\tilde{c}_{\rho,d}\alpha\check{\rho}^2 + \left(1-\frac{w_d}{2}\right)\tilde{c}_{V,d}\check{\rho} \\ &\quad + \left(1-\frac{d}{2}\right)\tilde{c}_{V,d}\check{\chi} - \check{\kappa}.\end{aligned} \tag{70}$$

So, taking a specific value for $\check{\rho}$, we can have positive acceleration for our universe only if

$$-(\tilde{c}_{\rho,d}\alpha\check{\rho}^2 + \tilde{c}_{V,d}\check{\rho} - \check{\kappa}) \le \tilde{c}_{V,d}\check{\chi} < -\frac{2(w_d-1)\tilde{c}_{\rho,d}\alpha\check{\rho}^2 + (w_d-2)\tilde{c}_{V,d}\check{\rho} + 2\check{\kappa}}{d-2} \tag{71}$$

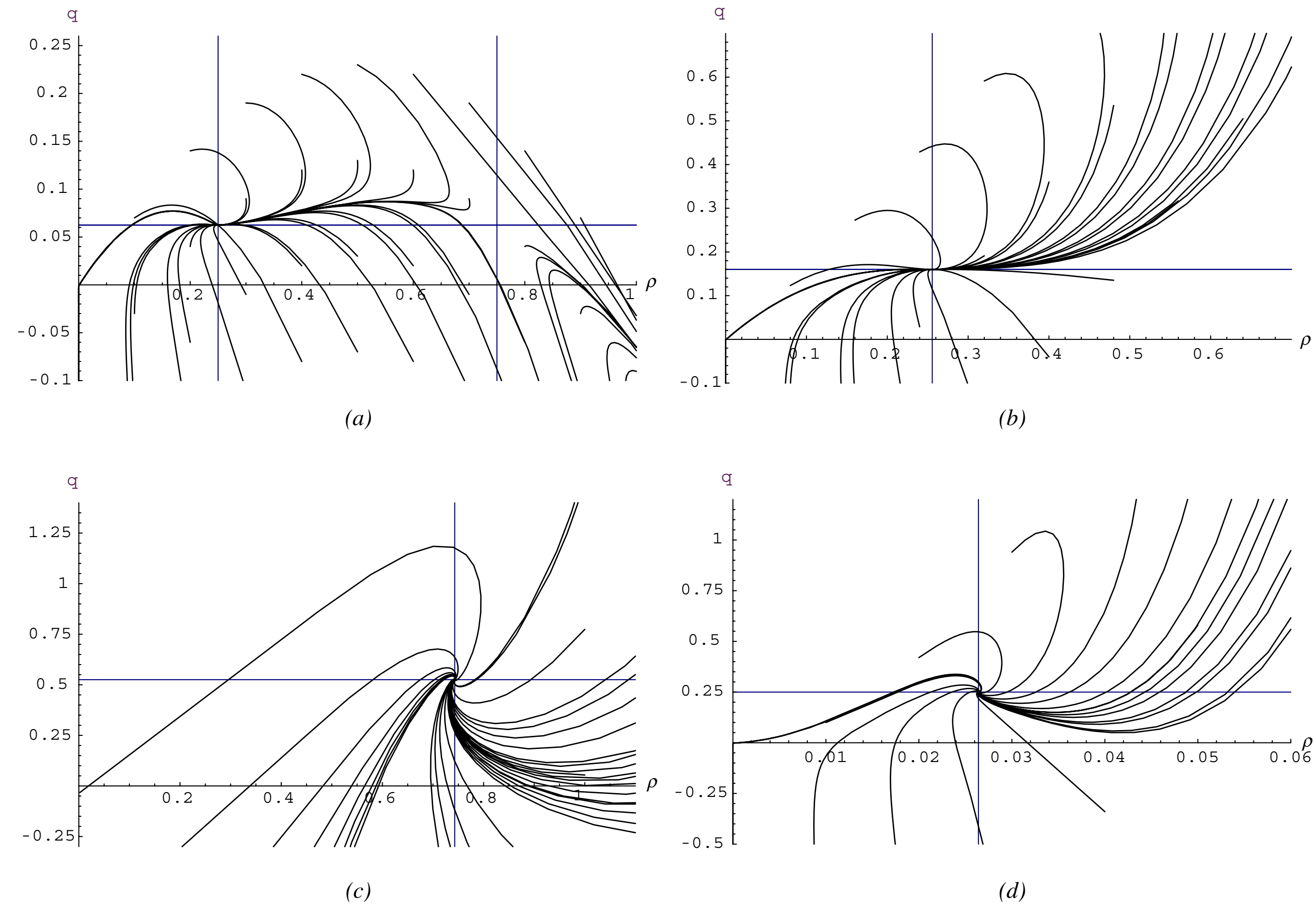

FIG. 2 (color online). Trajectories in the phase spaces $\check{q}/\check{\rho}$ with varying $\check{\rho}_0$ and $\check{\chi}_0$, $d = 6$, $\check{A} = -1$ (analogous pictures come from the static extra dimension case) and: (a) $w_d = 4 > d/2$, $\alpha = 1$, $\nu = 1$ leading to stable node in $\check{\rho}_\star = 1/4$ plus a second repulsive node in $\check{\rho}_\star = 3/4$, (b) $w_d = 2.5 < d/2$, $\alpha = 1$, $\nu = 1$ determining a stable node, (c) $w_d = 2.5 < d/2$, $\alpha = 1$, $\nu = -1$ an example of a stable spiral, (d) $w_d = 2 < d/2$, $\alpha = 10^3$, $\nu = 1$ a spiral behavior for large α (large M^6/V).

and a necessary condition for this to be possible is a bound on the energy density $(d - 2w_d)\tilde{c}_{V,d}\alpha\check{\rho}^2 + (d - w_d)\tilde{c}_{V,d}\check{\rho} > d\check{\kappa}$, as we can deduce from (69). The mirage density $\check{\chi}$ has to be negative to get positive acceleration for $w_d \geq 2$, $\kappa \geq 0$. If instead $w_d \leq 1$, $\kappa \leq 0$, the mirage density is positive for negative $\check{q}$.

Manipulating the set of Eq. (68), we write the following differential equations in terms of the generic energy exchange parameter T

$$a\frac{\mathrm{d}\check{\chi}}{\mathrm{d}a} = -d\check{\chi} + \eta\check{T}\left(1 + 2\frac{\tilde{c}_{\rho,d}}{\tilde{c}_{V,d}}\alpha\check{\rho}\right) \times [\tilde{c}_{\rho,d}\alpha\check{\rho}^2 + \tilde{c}_{V,d}(\check{\rho} + \check{\chi}) - \check{\kappa}]^{-(1/2)} \quad (72)$$

$$a\frac{\mathrm{d}\check{\rho}}{\mathrm{d}a} = -w_d\check{\rho} - \eta\check{T}[\tilde{c}_{\rho,d}\alpha\check{\rho}^2 + \tilde{c}_{V,d}(\check{\rho} + \check{\chi}) - \check{\kappa}]^{-(1/2)}. \quad (73)$$

We thus come to the differential equation for the acceleration factor

$$\left(w_d\check{\rho} + \frac{\eta\check{T}}{\check{H}}\right)\frac{\mathrm{d}\check{q}}{\mathrm{d}\check{\rho}} = -\frac{\eta\check{T}}{2\check{H}}(2\alpha c_\rho\check{\rho} + c_V) + \quad (74)$$

$$+\left[2\alpha(1 - w_d)c_\rho\check{\rho} + \frac{1}{2}(2 - w_d)c_V\right]\check{\rho} + d\check{q}, \quad (75)$$

where $\check{H} = \sqrt{(2\alpha c_\rho\check{\rho}^2 + c_V\check{\rho} + d\check{\kappa})/2 + \check{q}}$ and $\eta = \pm 1$ denotes the two possible roots for $\check{H}$. Again we note the presence of the symmetry $\check{H} \to -\check{H}$, $\check{T} \to -\check{T}$. We have used the definitions $(w_d - d)\tilde{c}_{V,d} = c_V$ and $(w_d - d/2)\tilde{c}_{\rho,d} = c_\rho$. From this equation we can infer that positive $\check{q}$ implies growing $\check{q}$ in an expanding universe ($\eta = +1$) with energy outflow ($\check{T} > 0$) if $c_V < 0$, $c_\rho > 0$ and $\check{\rho} < \frac{w_d - 2}{w_d - 1}\frac{|c_V|}{4\alpha c_\rho} \equiv \check{\rho}_{\mathrm{lim}}$. For $\tilde{c}_{V,d}$, $\tilde{c}_{\rho,d} > 0$, $c_V < 0$, $c_\rho > 0$ this is realized if $d/2 < w_d < d$. In the case of energy influx ($\check{T} < 0$), we get increasing positive acceleration if c_V, $c_\rho > 0$ and $w_d < 1$ for all positive energy densities ($\tilde{c}_{V,d}$, $\tilde{c}_{\rho,d}$ has to be negative). Or else, $\check{q}$ grows as $\check{\rho}$ grows if $c_V > 0$, $c_\rho < 0$ and $w_d < 1$, until the energy density reaches the bound $\check{\rho}_{\rm lim}$.

The nontrivial fixed points for energy influx are determined by the roots of the equation for $\check{\rho}_\star$

$$\frac{1}{d}(d-2w_d)\tilde{c}_{\rho,d}\alpha\check{\rho}_\star^2 + \frac{1}{d}\tilde{c}_{V,d}(d-w_d)\check{\rho}_\star - \frac{\check{A}^2}{w_d^2}\check{\rho}_\star^{2(\nu-1)} - \check{\kappa} = 0 \tag{76}$$

while for $\check{\chi}_\star$ and $\check{H}_\star$ we get the two functions of $\check{\rho}_\star$

$$\check{\chi}_\star = -\frac{w_d}{d}\left(1 + 2\frac{\tilde{c}_{\rho,d}}{\tilde{c}_{V,d}}\alpha\check{\rho}_\star\right)\check{\rho}_\star, \qquad \check{H}_\star = -\frac{\check{A}}{w_d}\check{\rho}_\star^{\nu-1} \tag{77}$$

We thus have to fix a particular value for ν in order to establish the precise number of roots and the explicit solution for the critical points. For integer and seminteger ν the number of roots we can obtain, keeping $d \neq 2w_d$, is

$$\nu \geq 2 \Rightarrow \#\text{roots} = 2(\nu - 1) \geq 2$$
$$1 \leq \nu < 2 \Rightarrow \#\text{roots} = 2$$
$$\nu < 1 \Rightarrow \#\text{roots} = 2(2 - \nu) > 2.$$

These are all the roots of (76), including trivial and complex roots. When $d = 2w_d$ the critical value for the mirage energy density diverges due to the divergence of $\tilde{c}_{\rho,d}$, unless we fix w_π to keep it finite. In this case, the number of roots changes to

$$\nu > 1 \Rightarrow \#\text{roots} = 2(\nu - 1) \geq 1$$
$$\nu = 1 \Rightarrow \#\text{roots} = 1$$
$$\nu < 1 \Rightarrow \#\text{roots} = 3 - 2\nu > 1.$$

For $\nu > 1$, one of the roots of Eq. (76) is null if $\kappa = 0$.

There are moreover two trivial fixed point solutions given by $\check{\rho}_\star = \check{H}_\star = 0$, $\check{\chi}_\star = \tilde{c}_{V,d}\check{\kappa}$ and $\check{\rho}_\star = \check{\chi}_\star = 0$, $\check{H}_\star = \sqrt{-\check{\kappa}}$, that reduce to a unique point with all vanishing variables when the internal space is flat.

Let us study in more detail the case $\nu = 1$, since solutions can be written explicitly being the case with the minimum number of roots for (76), together with the $\nu = 2$ case. As a result we get a trivial fixed point solution with $\check{H}_\star = \check{\rho}_\star = 0$, $\check{\chi}_\star = \tilde{c}_{V,d}\check{\kappa}$ and the trivial solution, acceptable only for negative and zero curvature, $\check{H}_\star = \sqrt{-\check{\kappa}}$, $\check{\rho}_\star = \check{\chi}_\star = 0$. Finally, the two nontrivial solutions (for $d \neq 2w_d$) are given by

$$\check{H}_\star = -\frac{\check{A}}{w_d} \qquad \check{\chi}_\star = \frac{-\tilde{c}_{V,d}(d-w_d) - 4(d-2w_d)\tilde{c}_{\rho,d}\alpha K^2 \pm \sqrt{(d-w_d)^2\tilde{c}_{V,d}^2 - 4d(d-2w_d)\tilde{c}_{\rho,d}\alpha K^2}}{2(d-2w_d)\tilde{c}_{\rho,d}\alpha}$$
$$\check{\rho}_\star = \frac{-\tilde{c}_{V,d}(d-w_d) \pm \sqrt{(d-w_d)^2\tilde{c}_{V,d}^2 - 4d(d-2w_d)\tilde{c}_{\rho,d}\alpha K^2}}{2(d-2w_d)\tilde{c}_{\rho,d}\alpha}, \tag{78}$$

where K corresponds to a shift and rescaling of $\check{A}^2$ due to the nonvanishing value of κ and is defined by $K^2 \equiv \check{A}^2/w_d^2 + \check{\kappa}$. The two roots are both real only if the argument of the square root in (78) is positive, i.e. when w_d lies outside the two roots $-\tilde{\alpha}d(1 \pm \sqrt{(\tilde{\alpha}+1)/\tilde{\alpha}})$, with $\tilde{\alpha} \equiv 4\tilde{c}_{\rho,d}\alpha K - 1$. If $\tilde{\alpha}$ is in the range bounded by -1 and 0 the square root is always real, whatever d, w_d we choose. If two or all among $\tilde{c}_{V,d}$, $\tilde{c}_{\rho,d}$, $(d-w_d)$, $(d-2w_d)$ have equal sign, one of the two solutions (78) always is characterized by a negative $\check{\rho}_\star$. We note that we can have at list a nontrivial fixed point with positive energy density if $(d-2w_d)\tilde{c}_{\rho,d}\alpha > 0$, $(d-w_d)\tilde{c}_{V,d} < 0$—exactly two positive $\check{\rho}_\star$ fixed points—, or for $(d-2w_d)\tilde{c}_{\rho,d}\alpha < 0$—only one critical point with positive energy density.

The nontrivial solution for $d = 2w_d$ is

$$\check{H}_\star = -\frac{\check{A}}{w_d}, \qquad \check{\rho}_\star = 2\tilde{c}_{V,d}K^2,$$
$$\check{\chi}_\star = -\tilde{c}_{V,d}K^2(4\tilde{c}_{\rho,d}K^2\alpha + 1). \tag{79}$$

For this unique fixed point solution to be characterized by positive $\check{\rho}_\star$ we have to demand a positive $\tilde{c}_{V,d}K^2$. Moreover, we can derive from (67) that for $\check{\kappa} > \tilde{c}_{V,d}w_d\check{\rho}_\star/4d$ the fixed point is a spiral, so that for instance, in a flat internal space, we always obtain a node since $\tilde{c}_{V,d}$ must be positive in order to have a positive $\check{\rho}_\star$ fixed point ($K = A$ in this case).

The numerical analysis can now show some of the features that we commented for the cosmological evolution with generic density. The differential system of Eqs. (68) (substituting some precise values for ν) can be solved numerically in order to check the existence of stable inflationary critical points. In Fig. 2 we plot the dimensionless acceleration factor $\check{q}(t)$ as a function of the dimensionless energy density $\check{\rho}(t)$, as we did in the previous section for small densities. In the plots, we consider for simplicity positive $\tilde{c}_{V,d}$, $\tilde{c}_{\rho,d}$ and flat compact extra dimensions $\kappa = 0$ ($\kappa \neq 0$ results in a shift for $\check{A}^2/w_d^2$ in the critical point evaluation and in some scaling of the suitable value of ν in order to have stability).

All the critical points we get are characterized by positive $\check{q}$, i.e. they represent an inflationary point. In the phase space portrait 2(a) trajectories starting with positive acceleration factor and energy density lower than the critical one pass through an era of larger acceleration and then slow down to the fixed point where we have inflation. There are then solutions starting with negative acceleration and going to the positive $\check{q}$ critical point, eventually passing through a larger acceleration phase or through a smaller density phase. The families of solutions that distinguish the diagram 2(a) from the others are characterized by both initial and final very high energy density, since they are repelled by the second non attractive fixed point. Some start with high energy density and negative acceleration at late time, go through an era of larger acceleration (eventually positive) and then, while $\check{\rho}$ becomes very large, they go to a region of large and negative $\check{q}$. Other go from positive acceleration to large negative $\check{q}$ and large $\check{\rho}$. The mirage density $\check{\chi}$ can start from an initial condition smaller or bigger than the critical value, has to be positive for $\check{q} < -(3\check{\rho} + 1)\check{\rho}$—as we deduce from (70) plugging in the values for the parameters—and approaches a negative constant value.

In diagram 2(b), trajectories starting with negative acceleration go to the positive $\check{q}$ fixed point, eventually reaching a maximum $\check{q}$ before ending into the critical point. Positive acceleration initial condition lead to growing acceleration at very early times, when usually the energy density grows as well, then both $\check{q}$ and $\check{\rho}$ decrease to reach the fixed point, passing through a minimum for acceleration. The mirage density goes to the negative critical value being initially positive for the trajectories that come from negative acceleration conditions, with $\check{q} < -(4\check{\rho} + 1)\check{\rho}/2$. With the choice of parameters we used, we get $w_d < d/2$. We could also have used the fixed point solution (79) if $w_d = d/2 = 3$ and the diagram for the phase space would have been analogous to that in plot 2(c), keeping the same values of plot 2(b) for the other parameters.

In the phase space 2(c) all solutions converge to the fixed point with a spiral behavior. Energy density and acceleration parameter thus oscillate around the critical values. Here $\nu = -1$ and $d < 2w_d$. The number of roots corresponding to the critical point solutions are six, but four of them are complex roots and one is characterized by negative energy density. Only one real critical point with positive energy density exists in this case and it has Hubble parameter. Since $\check{q}$ can be negative at some time of the evolution, even if starting with a positive value, $\check{\chi}$ can pass through a positive phase, crossing zero before reaching the negative critical value, if $\check{q} < -(4\check{\rho} + 1)\check{\rho}/2$. A similar plot can be drawn also if $w_d = d/2$. There would be only five roots for $\check{\rho}_\star$, four of which would be complex conjugated and the last would have positive energy density, representing the stable spiral.

Another stable spiral is represented in Fig. 2(d). Here, the dimensionless parameter α, which is proportional to M^6/V, is large and $\nu = 1$. We find only one nontrivial fixed point with positive energy density and spiral behavior, so that trajectories has a shape analogous to the ones in 2(c).

For values of ν different from $\nu = 1$ the number of fixed point roots may vary according to the previous discussion. Nonetheless, (as it is shown as an example in Fig. 2(c) for $\nu = -1$) some of the roots may be complex conjugated and thus not acceptable. Another simple case is $\nu = 2$, where we get two solutions to (76), as with the $\nu = 1$ assumption. We will not discuss this situation in detail since the phase spaces we can find are analogous to the $\nu = 1$ ones.

The case of energy outflow is analogous to the small energy density analysis. In fact, the fixed point solution (62) cannot be characterized by positive energy exchange parameter T. We can nevertheless have a critical point with energy outflowing from the brane into the bulk and negative Hubble parameter (as we can deduce from the expansion → contraction, influx → outflow symmetry). As the differential equation for ρ (59) shows, the energy density decreases and go to the trivial fixed point, meaning that the negative H critical point is not an attractor.

The 4D energy density in the static compact extra dimension case is just given by a constant rescaling of the 6D density. Thus, the phase portraits are given by the plots in Fig. 1 for small energy density, and Fig. 2 for generic density (up to constant rescaling). However, for equal scale factors $d = 6$, the 4D effective energy density ϱ is dynamically determined by the energy density localized on the 5-brane ρ and the volume of the compact space $V_{(2)}$: $\varrho = V_{(2)}\rho$. The 4D mirage density can be similarly defined as $\mathcal{X} = V_{(2)}\chi$. We can single out the volume time dependence defining the dimensionful constant υ such that $V_{(2)} \equiv \upsilon b^2(t)$, where $b(t)$ is as usual the compact space scale factor—in this particular case $b(t) = a(t)$. The generic set of equations for the dimensionless variables $\check{H}$, $\check{\varrho}$, $\check{\mathcal{X}}$ is

$$\begin{aligned}\check{H}^2 &= \frac{\tilde{c}_{\rho,d=6}\alpha}{\upsilon^2}\frac{\check{\varrho}^2}{a^4} + \frac{\tilde{c}_{V,d=6}}{\upsilon}\frac{(\check{\varrho} + \check{\mathcal{X}})}{a^2} - \check{\kappa} \\ \dot{\check{\mathcal{X}}} + 4\check{H}\check{\mathcal{X}} &= \check{\mathcal{A}}\frac{\varrho^{\nu}}{a^{2(\nu-1)}}\left(1 + 2\frac{\tilde{c}_{\rho,d=6}}{\tilde{c}_{V,d=6}}\frac{\alpha}{\upsilon}\check{\varrho}\right)\end{aligned} \tag{80}$$

$$\dot{\check{\varrho}} + (3(1 + w) + 2w_\pi)\check{H}\check{\varrho} = -\check{\mathcal{A}}\frac{\check{\varrho}^{\nu}}{a^{2(\nu-1)}}, \tag{81}$$

where $\mathcal{A} \equiv A/\upsilon^{\nu-1}$. We first note that in the case of zero energy exchange $T = 0$ ($\check{\mathcal{A}} = 0$) the 4D mirage density satisfies the 4D free radiation equation, as in the static internal space hypothesis. The 4D energy density ϱ does not have a definite behavior in the case of energy outflow. While in the static 2D compact space background it is clear that ϱ, just as ρ, is suppressed in time since $w_{d=4} > 0$, here we may have a negative coefficient for the linear term in $\check{\varrho}$ in (81). If $w_\pi < -3(1 + w)/2$—which is possible only for $w < -1/3$ if $w_\pi > -1$—we could have nontrivial stable critical points, as in the energy influx context. This scenario would need further investigations.

We can make some considerations regarding the more generic assumption on the relation between the Hubble parameters H and F, $F = \xi H$, of subsection III D. For positive ξ the qualitative behavior is analogous to what we deduced in the case of static compact extra dimensions and equal scale factors. When ξ is negative, meaning that we are using a dynamical compactification approach [82], we could instead have some differences. In particular, it is worth noticing that w_ξ (appearing in the conservation equation for ρ) can become negative, always implying a diverging behavior for the localized energy density at late time in the case of energy influx. Thus, there will not be stable critical point in the dynamical compactification scenario with energy flowing from the bulk onto the brane.

C. Small density and free radiation equation of state: an explicit solution

We can write an explicit solution to the set of Eqs. (35)–(37) in the special limit of small localized energy density $\rho \ll V$ and when $w_d = d$. This last condition is realized if $w = 1/3$ with static compact extra dimensions and if $w_\pi = (1 - 3w)/2$ with equal scale factors. We must be careful though, because in this limit $\tilde{c}_{V,d}$ generally diverges. It is thus important to keep it finite by imposing *a priori* a specific value for w and w_π. With these assumptions, H^2 only depends on the sum $(\rho + \chi)$ and the equation for this sum can be easily integrated independently of the explicit form of T. To be more specific, we get the following solution

$$H^2 = \frac{2\tilde{c}_{V,d} V}{(d-1)M^{10}}(\rho_0 + \chi_0)\frac{a_0^d}{a^d} - \frac{1}{2(d-1)}\frac{\kappa}{b^2} \quad (82)$$

$$\rho + \chi = (\rho_0 + \chi_0)\frac{a_0^d}{a^d}. \quad (83)$$

As in the four dimensional RS model with energy exchange analyzed in [43], the evolution is determined by the initial value of the energy density (if we put, for example, $\chi_0 = 0$) weighted by the expansion in a (effective six or four dimensional) radiation dominated era.

We deduce from (82) that $|a(t)|$ must have an upper limiting value for $\kappa > 0$. For small positive $a(t)$ the rate $\dot{a}(t)$ is a positive function and the universe expands until it reaches the limiting value. If the compactification is over an hyperbolic space, the scale factor grows without bound. In particular, in a universe with extra dimensions evolving according to the same Hubble parameter as for the observed space-time, the expansion rate goes to a constant positive value as the scale factor grows. In a static extra dimension setup instead, $a(t)$ exponentially grows at infinity. When $\kappa = 0$ the explicit solutions for $a(t)$ reduce to $a(t) \sim t^{1/2}$ for static internal space, and $a(t) \sim t^{1/3}$ for equal scale factors. These represent exactly a radiation dominated flat universe in four or six effective dimensions, respectively. The evolution $a(t) \sim t^{1/3}$ can also be traced in 4D to the Friedmann equation for scalar field subject to a null potential.

If we further assume the energy exchange to be linear in the localized energy density $T = A\rho$, also imposing the initial condition $\chi_0 = 0$ and $w_d = d$, the integration of the χ and ρ equations yields

$$\chi = \rho_0\frac{a_0^d}{a^d}(1 - e^{-At}), \qquad \rho = \rho_0\frac{a_0^d}{a^d}e^{-At} \quad (84)$$

This solution shows that, for energy outflow $A > 0$, the initial amount of radiation energy density decays in favor of the mirage energy density. The late time evolution is thus governed by the mirage density.

For equal scale factors it is interesting to write the explicit solution (remember $w_{d=6} = 6$ in this case) in terms of the 4D densities ϱ, $\mathcal{X}$ ($\varrho = \upsilon b^2(t)\rho$, $\mathcal{X} = \upsilon b^2(t)\chi$). The ansatz implying static internal directions only results in a constant rescaling of the 6D quantities. The equal scale factor solution is given by

$$H^2 = \frac{2\tilde{c}_{V,d=6}V}{5M^{10}\upsilon}(\varrho_0 + \mathcal{X}_0)\frac{a_0^4}{a^6} - \frac{1}{10}\frac{\kappa}{a^2} \quad (85)$$

$$\varrho + \mathcal{X} = (\varrho_0 + x_0)\frac{a_0^4}{a^4} \quad (86)$$

The evolution is still weighted by the characteristic effective 6D radiation dominated era $1/a^6$. But we can see, exploring the solutions for ϱ and $\mathcal{X}$ in the case of energy exchange parameter determined by $T = A\rho$ (with positive A), that the 4D localized energy density evolves as 4D radiation, exponentially suppressed in time (as for the static internal space background). In fact, assuming for example $\mathcal{X}_0 = 0$, we get

$$\mathcal{X} = \varrho_0\frac{a_0^4}{a^4}(1 - e^{-At}), \qquad \varrho = \varrho_0\frac{a_0^4}{a^4}e^{-At} \quad (87)$$

In this case $\mathcal{A} = A$.

The case of energy influx is also analogous to the analysis in [43]. For both static compact extra dimensions and equal scale factors we rewrite the conservation equation for ρ for flat internal space as

$$\dot{\rho} + \frac{2}{t}\rho = -T \quad (88)$$

where we have used the equation determining H (82). This shows that for negative T, ρ should increase without bounds at late time. Since we are in the low density approximation, we can only rely on the generic ρ analysis of the previous section for large ρ. However, assuming $T = A\rho^\nu$, we can deduce that for $\nu > 3/2$ the energy density can flow to zero (for certain values of the parameters). Indeed, $\nu > 3/2$ corresponds to non stable critical point in the small density analysis.

We remark that introducing the 4D density ϱ for a universe with equal scale factors brings to

$$\dot{\varrho} + \frac{4}{3t}\varrho = -T. \tag{89}$$

Equation (89) tells us that the 4D localized energy density still grows unlimited at late time, if T is linear in ϱ—more general considerations are analogous to the 6D density case. Again, we would need the full treatment for generic density.

The acceleration parameter $q \equiv \ddot{a}/a$ in this context is equal to

$$q = -\frac{(d-2)\tilde{c}_{V,d}V}{(d-1)M^{10}}(\rho_0 + \chi_0)\frac{a_0^d}{a^d}. \tag{90}$$

For non zero κ, the value of the acceleration can be either positive or negative. It has to be negative when $\kappa \geq 0$, but may be positive for compactification on hyperbolic spaces ($\kappa < 0$), giving as a result a loitering universe.

V. CONSTRUCTION OF THE HOLOGRAPHIC DUAL

The theory dual to the 7D RS model, via the AdS/CFT correspondence [58], will be derived in complete analogy to the 5D setup considered in [9]. The RS model, with a time independent warped geometry, gives AdS_7 metric as a solution to the equations of motion for the gravity action in the bulk. It will be useful to parametrize it according to Fefferman and Graham [89]

$$G_{AB}\mathrm{d}x^M\mathrm{d}x^N = \frac{\ell^2}{4}\rho^{-2}\mathrm{d}\rho^2 + \ell^2\rho^{-1}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu, \tag{91}$$

where the indices $M, N\ldots$ run over the 7D bulk space-time, $\mu, \nu, \ldots$ span the 6D space-time on the 5-brane and ρ is a reparametrization of the z coordinate transverse to the brane. The location of the brane, translated to this new set of coordinates, is $\rho = 0$ which represents the boundary of the background (91). Generally, for all seven dimensional asymptotically AdS space-times the 6D metric $g_{\mu\nu}$ can be expanded as [89]

$$g = g_{(0)} + \rho g_{(2)} + \rho^2 g_{(4)} + \rho^3 g_{(6)} + \rho^3 \log\rho h_{(6)} + \mathcal{O}(\rho^4). \tag{92}$$

The logarithmic piece appears only for space-times with an odd number of dimensions and is responsible for the cutoff dependent counterterm in the renormalized action. As a consequence, it is also responsible for the conformal anomaly of the holographic dual CFT [72,76]—in fact, we do not have conformal anomaly in even dimensional backgrounds. The subindices in the coefficients of the metric expansion stand for the number of derivatives contained in each term.

More precisely, RS background is a slice of AdS_7, where the boundary gets replaced by the 5-brane and the IR part is reflected, eliminating the UV slice. To describe seven dimensional gravity we will take the usual Einstein-Hilbert action in the bulk, adding as usual a Gibbons-Hawking term [90] to take account of the boundary extrinsic curvature. Since the gravitational theory exhibits divergences in a space-time with boundaries, we also have to regularize the Einstein-Hilbert plus Gibbons-Hawking action, cutting off the boundary of the space-time. We are now going to illustrate the regularization procedure.

A. Regularization on the gravitational side

The renormalization for a gravitational theory in a background with boundaries has been explained in [72,73,76,91,92] for a generic number of dimensions. We will apply those computations to the case of a seven dimensional bulk space-time.

In general, the bulk action for gravity gets modified with the Gibbons-Hawking boundary term [90] and with some counterterms also localized on the boundary

$$S_{\mathrm{gr}} = S_{\mathrm{EH}} + S_{\mathrm{GH}} - S_{\mathrm{count}}. \tag{93}$$

Using the Fefferman and Graham parametrization of the metric (91) and cutting off the boundary at $\rho = \epsilon$, we have

$$\begin{aligned} S_{\mathrm{EH}} &= M^5\int_{\rho\geq\epsilon}\mathrm{d}^7x\sqrt{-G}\left(R[G] + \frac{30}{\ell^2}\right), \\ S_{\mathrm{GH}} &= 2M^5\int_{\rho=\epsilon}\mathrm{d}^6x\sqrt{-\gamma}K, \end{aligned} \tag{94}$$

where $R[G]$ is the bulk Ricci scalar, K is the trace of the extrinsic curvature and $\gamma_{\mu\nu}$ is the induced metric on the boundary. Putting the brane at $\rho = \epsilon$ corresponds to regularize the gravity action. The counterterm contributions necessary to make it finite in the limit $\epsilon \longrightarrow 0$ are given by

$$S_{\mathrm{count}} = S_0 + S_1 + S_2 + S_3 \tag{95}$$

S_i are terms of order i in the brane curvature R (the curvature of the induced metric $\gamma_{\mu\nu}$ on the boundary). In fact they can be written in terms of the induced metric $\gamma_{\mu\nu}$ and its Riemann tensor $R_{\mu\nu\rho\sigma}$, using the perturbative expansion relating $\gamma_{\mu\nu}$ to $g_{(0)\mu\nu}$ (see for instance [76])

$$S_0 = 10\frac{M^5}{\ell}\int_{\rho=\epsilon}\mathrm{d}^6x\sqrt{-\gamma} \tag{96}$$

$$S_1 = -\frac{1}{4}M^5\ell\int_{\rho=\epsilon}\mathrm{d}^6x\sqrt{-\gamma}R \tag{97}$$

$$S_2 = \frac{1}{32}M^5\ell^3\int_{\rho=\epsilon}\mathrm{d}^6x\sqrt{-\gamma}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{10}R^2\right) \tag{98}$$

$$S_3 = \frac{\log\epsilon}{64} M^5 \ell^5 \int_{\rho=\epsilon} d^6x \sqrt{-\gamma}\Big(\frac{1}{2} R R_{\mu\nu} R^{\mu\nu} + \frac{3}{50} R^3 + R^{\mu\nu} R^{\rho\sigma} R_{\mu\rho\nu\sigma} + \frac{1}{5} R^{\mu\nu} \nabla_\mu \nabla_\nu R - \frac{1}{2} R^{\mu\nu} \Box R_{\mu\nu}\Big). \tag{99}$$

The third order term S_3 depends on the cutoff ϵ and is thus responsible for the breaking of the scale invariance, i.e. it gives rise to the conformal anomaly for the dual 6D CFT in the context of the AdS/CFT correspondence. We also note that the zeroth order term is related to the brane tension term of the RS model (1) $S_{\rm tens}$ by $S_{\rm tens} = -2S_0$, if we fine-tune $\lambda_{\rm RS} = 0$. In fact, in the pure RS setup, where the effective cosmological constant is null $\lambda_{\rm RS} = 0$, the brane tension is $V = 20M^5/\ell$, since the bulk cosmological constant is given by $\Lambda_7 = -30M^5/\ell^2$ as a function of the background length scale ℓ and the bulk Planck mass M, in our background metric parametrization (91). We will now use the AdS/CFT correspondence to compute the dual theory.

B. Gauge/gravity duality

The AdS/CFT duality [58] is realized between gravity (string theory or M theory in the decoupling limit) in a background with one or more stacks of some kind of branes and the gauge theory that lives on the boundary of the near horizon geometry inferred by the branes [58,59]. In our particular case, $AdS_7 \times S^4$ is the near horizon geometry of a system of N parallel M5-branes in 11 dimensional M theory. The radius of the AdS space is given in terms of the 11 dimensional Planck length $\ell_{\rm Pl}$ and of the number N of M5-branes

$$\ell = 2(\pi N)^{1/3} \ell_{\rm Pl}. \tag{100}$$

The radius of the four sphere is half the radius of AdS_7. The supergravity approximation for M theory is valid if $N \gg 1$ and $\ell_{\rm Pl} \sim N^{-1/3} \to 0$, keeping the radius of the AdS large and finite in units of $\ell_{\rm Pl}$. The six dimensional theory that Maldacena [58] conjectured to be dual to M theory in the background described above is a (0, 2) SCFT. This theory is realized as the open string theory in the world-volume of the M5-branes, in the low energy decoupling limit, and it does not contain dimensionless nor dimensionful parameters. The $AdS_7 \times S^4$ supergravity background is characterized by a 4-form flux quantized in terms of the number of M5-branes and is not conformally flat, since the radii of the four-sphere and the AdS space are not coincident.

The AdS/CFT correspondence relates the gravity (M theory) partition function for the bulk fields Φ_i (which is a function of the value of the fields on the boundary of AdS_7, ϕ_i) to the generating functional of correlation functions of the dual CFT operators with sources ϕ_i

$$Z_{\rm gr}[\phi_i] \equiv \int \mathcal{D}\Phi_i e^{-S_{\rm gr}} = e^{-W_{\rm CFT}(\phi_i)}. \tag{101}$$

Knowing that gravity on AdS_7 (the S^4 geometry can be factored out) corresponds to the specific CFT suggested by Maldacena [58], we can now obtain as a consequence the theory dual to the 7D RS model, in analogy to [9]. In fact, the action of the gravitational theory that we want to analyze via holography is

$$S_{\rm RS} = S_{\rm EH} + S_{\rm GH} + S_{\rm tens} + S_m. \tag{102}$$

We just add the Gibbons-Hawking term to (1). We expect that the hidden sector of the holographic theory reflects the bulk nontrivial contents encoded in the bulk components of the 7D stress-energy tensor (T^7_0, T^7_7) when we go to the nonconformal interacting generalization.

We now have to keep in mind that the duality for gravity on AdS_7 can be stated as

$$Z_{\rm gr}[\phi_i] \equiv \int_{\rho>\epsilon} \mathcal{D}\Phi_i e^{-S_{\rm EH} - S_{\rm GH} + S_0 + S_1 + S_2 + S_3} = e^{-W_{\rm CFT}(\phi_i)}. \tag{103}$$

Second, we have to remember that $S_{\rm tens} = -2S_0$. Furthermore, we note that the integration in (103) is over one half of the space-time appearing in the RS model, because of the $\mathbb{Z}_2$ reflection along the z direction. Since the integrals over the two specular regions are independent and equal we can write

$$Z_{\rm RS}[\phi_i, \chi_i] \equiv \int_{{\rm all}\,\rho} \mathcal{D}\Phi_i \mathcal{D}\chi_i e^{-S_{\rm EH} - S_{\rm GH} + 2S_0 - S_m} = \int_{\rho>\epsilon} \mathcal{D}\Phi_i \mathcal{D}\chi_i e^{-2S_{\rm EH} - 2S_{\rm GH} + 2S_0 - S_m} \tag{104}$$

where χ_i are the matter fields on the brane. Finally, putting all together, using Eq. (103), we obtain

$$Z_{\rm RS}[\phi_i, \chi_i] = \int_{\rho>\epsilon} \mathcal{D}\Phi_i \mathcal{D}\chi_i e^{-2W_{\rm CFT} - 2S_1 - 2S_2 - 2S_3 - S_m} \tag{105}$$

The RS dual theory is

$$S_{\tilde{\rm RS}} = S_{\rm CFT} + S_R + S_{R^2} + S_{R^3} + S_m \tag{106}$$

having defined

$$S_{\rm CFT} = 2W_{\rm CFT}, \quad S_R = 2S_1, \quad S_{R^2} = 2S_2, \quad S_{R^3} = 2S_3 \tag{107}$$

The 6D Planck mass is thus given by $M^4_{\rm Pl} = \frac{M^5 \ell}{2}$.

We are now ready to calculate the equations of motion for the holographic 6D RS cosmology.

VI. HOLOGRAPHIC EVOLUTION EQUATIONS

As we know, the RS classical solution in a 7D bulk with a warped geometry is AdS_7. Since our purpose is to study the cosmology associated to the 7D RS setup, we have generalized the ansatz for the metric to be time dependent in section II and III. We have successively reviewed the notion of holographic dual theory in the previous section. What we want to do now is to describe the cosmology of the seven dimensional RS model from the six dimensional holographic point of view, using the correspondence relation obtained in the previous section and generalizing the ansatz for the 6D induced metric on the 5-brane to a time dependent geometry, as we had done for the 7D bulk analysis.

We consider a 6D space-time, compactified on a 2D internal space, with a Friedmann-Robertson-Walker (FRW) metric for the four large dimensions. The induced metric tensor can be expressed as

$$\gamma_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -\mathrm{d}t^2 + \frac{a^2(t)}{1-kr^2}\mathrm{d}r^2 + a^2(t)r^2\mathrm{d}\theta^2 + a^2(t)r^2\sin^2\theta\mathrm{d}\phi^2 + \frac{b^2(t)}{1-\kappa\rho^2}\mathrm{d}\rho^2 + b^2(t)\rho^2\mathrm{d}\psi^2, \tag{108}$$

where k and κ, $a(t)$ and $b(t)$, $H(t)$ and $F(t)$ are, respectively, the curvatures, the scale factors, the Hubble parameters for the 3D and 2D spaces.

The action we are considering is

$$S_{\tilde{\mathrm{RS}}} = S_{\mathrm{CFT}} + S_m + S_\lambda + S_R + S_{R^2} + S_{R^3}, \tag{109}$$

S_R, S_{R^2}, S_{R^3} being, respectively, twice the first, second and third order terms in the curvature contributing to the counterterm action, as defined in (96)–(99). S_λ is an effective cosmological term on the brane—that represents a generalization to the case of a non exact RS fine-tuning with respect to the action (106). S_m and S_{CFT} are the matter and (twice the) CFT actions of the 6D description.

We want to solve the Friedmann equations, imposing the conservation and anomaly equations, defining

$$T_{\mu\nu} = \frac{1}{\sqrt{-\gamma}}\frac{\delta S_m}{\delta\gamma^{\mu\nu}} \qquad W_{\mu\nu} = \frac{1}{\sqrt{-\gamma}}\frac{\delta S_{\mathrm{CFT}}}{\delta\gamma^{\mu\nu}}, \tag{110}$$

$$Y_{\mu\nu} = \frac{1}{\sqrt{-\gamma}}\frac{\delta S_{R^2}}{\delta\gamma^{\mu\nu}} \qquad Z_{\mu\nu} = \frac{1}{\sqrt{-\gamma}}\frac{\delta S_{R^3}}{\delta\gamma^{\mu\nu}}, \tag{111}$$

and $V_{\mu\nu} = W_{\mu\nu} + Y_{\mu\nu} + Z_{\mu\nu}$. The equations of motion take the form

$$M_{\mathrm{Pl}}^4 G_{\mu\nu} + \lambda\gamma_{\mu\nu} = T_{\mu\nu} + V_{\mu\nu} \qquad \nabla^\nu T_{\mu\nu} = 0$$
$$\nabla^\nu V_{\mu\nu} = 0 \qquad V^\mu_\mu = \mathcal{A}_{(6)} + Y. \tag{112}$$

Here $\mathcal{A}_{(6)}$ is the general anomaly for a 6D conformal theory [77,79] that comes uniquely from the S_3 contribution to the renormalized action [72–77,93], while Y is the trace Y^μ_μ of the variation of (twice) the second order counterterm action S_{R^2}. The trace of $Z_{\mu\nu}$ is null [94]. The trace of $Y_{\mu\nu}$ is quadratic in the curvature of the metric (108)

$$Y = \frac{1}{32}M^5\ell\left(R^{\mu\nu}R_{\mu\nu} - \frac{3}{10}R^2\right). \tag{113}$$

The explicit form for the anomaly is a complicated expression of dimensions 6, cubic in the curvature, and is discussed in appendix A. The effective cosmological constant on the brane is λ. The stress-energy tensors are parametrized as

$$T_{00} = \rho(t), \qquad T_{ij} = p(t)\gamma_{ij}, \qquad T_{ab} = \pi(t)\gamma_{ab}$$
$$V_{00} = \sigma(t), \qquad V_{ij} = \sigma_p(t)\gamma_{ij}, \qquad V_{ab} = \sigma_\pi(t)\gamma_{ab} \tag{114}$$

where the indices $ij\ldots$ parametrize the space part of the 4D FRW space-time and run from 1 to 3, while $ab\ldots$ belong to the 2D internal space and takes the values (4, 5) [95].

Eqs. (112) take the following form when we choose the metric (108) and the stress-energy tensors written in (114). The Friedmann equations become

$$M_{\mathrm{Pl}}^4\left(3H^2 + 6HF + F^2 + 3\frac{k}{a^2} + \frac{\kappa}{b^2}\right) = \rho + \sigma + \lambda$$
$$M_{\mathrm{Pl}}^4\left(2\dot{H} + 3H^2 + 4HF + 2\dot{F} + 3F^2 + \frac{k}{a^2} + \frac{\kappa}{b^2}\right) = -p - \sigma_p + \lambda \tag{115}$$
$$M_{\mathrm{Pl}}^4\left(3\dot{H} + 6H^2 + 3HF + \dot{F} + F^2 + \frac{k}{a^2}\right) = -\pi - \sigma_\pi + \lambda,$$

then the conservation equations

$$\dot{\sigma} + 3(\sigma + \sigma_p)H + 2(\sigma + \sigma_\pi)F = 0$$
$$\dot{\rho} + 3(\rho + p)H + 2(\rho + \pi)F = 0, \tag{116}$$

and finally the anomaly equation

$$\sigma - 3\sigma_p - 2\sigma_\pi = \mathcal{A}_{(6)} + Y. \tag{117}$$

As we said, the anomaly comes from the cubic counterterm, so that it is cubic in the curvature. We make some more precise statement about its form in appendix A, where we also explicitly give Y.

Simplifications and ansatz

The set of Eqs. (115)–(117) does not contains six independent equations. Plugging the conservation in the first Friedmann equation differentiated w.r.t. time, we get a linear combination of the other two Friedmann equations.

So we will discard the last of (115) from now on. We further note that the system contains only one algebraic equation: the Friedmann equation. We will start by solving the anomaly equation in terms of one of the pressures coming from the hidden theory.

Plugging the expression for σ_p obtained evaluating (117) into the first of the conservation Eqs. (116), we get a differential equation for σ depending on σ_π

$$\dot{\sigma} + 2(2H + F)\sigma + 2(H - F)\sigma_\pi = \mathcal{A}_{(6)} + Y. \quad (118)$$

To obtain a solvable decoupled equation for σ we can consider the limit in which the internal space has the same CFT pressure $\sigma_\pi = \sigma_p$ of the three large dimensions [96]. Else, we can also consider the limit of zero pressure—for the CFT—in the internal space. Putting these two limits together, we can try to solve the Friedmann equations imposing a more general ansatz

$$\sigma_\pi = \Omega\sigma_p \quad (119)$$

So that

$$\sigma - \frac{1}{\omega}\sigma_p = \mathcal{A}_{(6)} + Y, \quad (120)$$

where $\omega \equiv 1/(3 + 2\Omega)$ (ω is equal to $1/5$, $1/3$ in the two limits considered above) and the differential equation for σ becomes

$$\dot{\sigma} + 3[(1 + \omega)H + (1 - \omega)F]\sigma - [3\omega H + (1 - 3\omega)F](\mathcal{A}_{(6)} + Y) = 0. \quad (121)$$

We could now evaluate σ solving the following integral

$$\sigma = \chi + \frac{1}{a^{3(1+\omega)}b^{3(1-\omega)}}\int dt a^{3(1+\omega)}b^{3(1-\omega)} \times \left[3\omega\frac{\dot{a}}{a} + (1 - 3\omega)\frac{\dot{b}}{b}\right][c_A E_{(6)} + c_B I_{(6)} + Y], \quad (122)$$

where χ is a solution for the homogeneous equation

$$\dot{\chi} + 3[(1 + \omega)H + (1 - \omega)F]\chi = 0 \Rightarrow \chi = \frac{\chi_0}{a^{3(1+\omega)}b^{3(1-\omega)}}. \quad (123)$$

We observe that generally (122) is not explicitly integrable. In (122) we have written the anomaly in terms of its contributions that are the Euler density in six dimensions $E_{(6)}$ and the local covariants included in $I_{(6)}$ (see appendix A for further details).

The set of independent equations we finally have to solve, once we use (120) to eliminate σ_p by means of

$$\sigma_p = \omega\sigma - \omega(\mathcal{A}_{(6)} + Y) \quad (124)$$

is then

$$M_{\rm Pl}^4\left(3H^2 + 6HF + F^2 + 3\frac{k}{a^2} + \frac{\kappa}{b^2}\right) = \rho + \sigma + \lambda \quad (125)$$

$$\begin{aligned} M_{\rm Pl}^4\left(2\dot{H} + 2\dot{F} + 3H^2 + 4HF + 3F^2 + \frac{k}{a^2} + \frac{\kappa}{b^2}\right) &= -w\rho - \omega\sigma + \omega(\mathcal{A}_{(6)} + Y) + \lambda \\ \dot{\sigma} + 3[(1 + \omega)H + (1 - \omega)F]\sigma &= [3\omega H + (1 - 3\omega)F](\mathcal{A}_{(6)} + Y) \\ \dot{\rho} + [3(1 + w)H + 2(1 + w_\pi)F]\rho &= 0, \end{aligned} \quad (126)$$

(the anomaly $\mathcal{A}_{(6)}$ will be written explicitly—in terms of H and F—in the particular cases that we will take under examination in the following). We also use the three following ansatz relating the pressures and the energy densities

$$\begin{aligned} p &= w\rho \qquad \pi = W \qquad p = w_\pi\rho \\ \sigma_\pi &= \Omega\sigma_p \qquad (3 + 2\Omega = 1/\omega). \end{aligned} \quad (127)$$

Now we are left with a system of four Eqs. (125) and (126) in four variables (H, F, ρ, σ). The other variables $(\sigma_p, \sigma_\pi, p, \pi)$ are determined by the ansatz (127) and by the Eq. (124). In the next section, this system of differential equations will be studied restricting to some special limits, such as flat or static internal space, or equal scale factors. We will find the critical point solutions and analyze the associated stability matrix.

VII. HOLOGRAPHIC CRITICAL POINT ANALYSIS

The fixed points of the cosmological evolution of the universe we are considering may represent its inflationary eras—for instance the early time or the late time acceleration—, since the Hubble parameters, just as the energy densities, are constant. If the constant value for the Hubble parameter is positive we have inflation. In this section, we are going to look for the existence of such inflationary points for our specific holographic model and to find what kind of dependence they have on the parameters of the theory.

We will describe the fixed point solutions in the special limits of flat extra dimensions, curved static extra dimensions and equal scale factors for the internal and extended spaces. We will then study the stability matrix associated with the critical points. Since the fixed points represents inflationary eras in the universe evolution, they could offer an explanation to the early inflation or to the late time acceleration. In the first case they will have to be unstable or saddle points to allow the trajectory of the cosmological evolution to flow away from inflation and exit from this phase. In the second case, on the other hand, the fixed points must be stable and act as attractors for the nearby trajectories.

In what follows we will always suppose that the effective cosmological constant on the brane λ will be zero, unless we specify it differently.

A. Flat compact extra dimensions

In the limit of zero spatial curvature for the extra dimensions and for the extended space, the Friedmann plus conservation Eqs. (125) take the form

$$M_{\rm Pl}^4(3H^2 + 6HF + F^2) = \rho + \sigma + \lambda \tag{128}$$

$$M_{\rm Pl}^4(2\dot{H} + 2\dot{F} + 3H^2 + 4HF + 3F^2) = -w\rho - \omega\sigma + \omega(\mathcal{A}_{(6)} + Y) + \lambda \tag{129}$$

$$\dot{\sigma} + 3[(1+\omega)H + (1-\omega)F]\sigma = [3\omega H + (1-3\omega)F](\mathcal{A}_{(6)} + Y) \tag{130}$$

$$\dot{\rho} + [3(1+w)H + 2(1+w_\pi)F]\rho = 0. \tag{131}$$

We find the fixed points of this system of differential equations and study stability with some further restrictions (see appendix B in the flat extra dimension subsections for the explicit calculations). As we point out in appendix A, the anomaly $\mathcal{A}_{(6)}$ generally depends on the Hubble parameters of the model, on their time derivatives up to the third order and on the spatial curvatures. This remains true also for the flat extra dimension limit that we are examining, ignoring the curvatures.

Fixed point solutions With the assumption of flat internal space and zero curvature for the 3D space as well, we can find different fixed points depending on the value of the extra dimensions Hubble parameter. They can be summarized as follows.

(1) As we illustrate in appendix B, there are two nontrivial time independent solution with F non vanishing at the fixed point, $F_\star \neq 0$, one for $\omega \neq 1/5$ and one for $\omega = 1/5$. The values of the 3D Hubble parameter (the measurable Hubble parameter) are given in terms of the constants ω, c_A, c_B, c_Y and the mass scale $M_{\rm Pl}$. We note that the anomaly parameters c_A, c_B are given by the CFT, while ω relates the hidden sector pressure of the internal space to the hidden pressure of the 3D space (127). In particular, the two Hubble parameters are related by the equality $H_\star = (\mathcal{C}_\epsilon + 1)F_\star$ (where $\mathcal{C}_\epsilon$ is a rather complicated function of ω, c_A, c_B, $c_Y M_{\rm Pl}^2$) when $\omega \neq 1/5$, while for $\omega = 1/5$ we have $H_\star = F_\star$. For $\omega = 1/5$, $F_\star \neq 0$ and we can choose it to be positive or negative, implying that the extra dimensions scale exponentially at those points, with, respectively, either positive or negative velocity. Consequently H would also describe either a contracting or expanding universe.

(2) We also found a fixed point solution for which the extra dimensions are static, i.e. $F_\star = 0$ ($\omega \neq 0, -1$ and $c_B \neq 0$), meaning that, while our visible universe is exponentially growing (or, in principle, decreasing) the internal space is not expanding nor collapsing. The corresponding solution is given by

$$\begin{aligned} H_\star^2 &= -\frac{20}{3c_B}\frac{\omega}{\omega+1}M_{\rm Pl}^2\Big[48c_Y \\ &\quad \pm\sqrt{6\Big(384c_Y^2 - c_B\frac{\omega}{\omega+1}\Big)}\Big] \\ \sigma_\star &= -\sigma_{\pi\star} = \frac{2\omega}{3\omega-1} \\ \sigma_{\pi\star} &= -\frac{20}{c_B}\frac{\omega}{\omega+1}M_{\rm Pl}^6\Big[48c_Y \\ &\quad \pm\sqrt{6\Big(384c_Y^2 - c_B\frac{\omega}{\omega+1}\Big)}\Big] \\ \rho_\star &= 0. \end{aligned} \tag{132}$$

The roots are real if $384c_Y^2 - c_B\omega/(\omega+1) > 0$ and cannot be both positive. We never have a couple of fixed points in the phase space diagram.

(3) A third fixed point is characterized by zero extra dimension Hubble parameter and $\omega = -1$. In this case the critical point exists only if the conformal field theory is characterized by a positive coefficient for the type B anomaly. The solution is

$$\begin{aligned} H_\star^2 &= \frac{640c_Y}{c_B}M_{\rm Pl}^2, \\ \sigma_\star &= -\sigma_{p\star} = \frac{1}{2}\sigma_{\pi\star} = \frac{640c_Y}{3c_B}M_{\rm Pl}^6, \\ \rho_\star &= 0. \end{aligned} \tag{133}$$

If c_B is zero (i.e. the anomaly vanishes at the fixed point) we are left only with the trivial fixed point.

(4) For vanishing ω, λ should be non zero to get the inflationary fixed point $H_\star^2 = \frac{\lambda}{3M_{\rm Pl}^4}$.

(5) There also exists a trivial fixed point with $H_\star = F_\star = 0$, where also the anomaly and the trace Y_μ^μ become zero, and $\rho_\star = \sigma_\star = 0$ if $\omega \neq w$ or $\rho_\star = -\sigma_\star$ if $\omega = w$.

In any case—i.e. for every ω and λ—the solution does not depend on w, w_π (in fact, the system of Eqs. (B1)–(B5) does not contain w, w_π). So, if a solution exists for some ω and c_A, c_B, c_Y, that solution always exists whatever values the two parameters relating matter pressures to energy density take. This marks a difference with the bulk analysis of section IV, since here we do not have any bulk dynamics perturbing the conservation equations, being the hidden sector theory conformal and non interacting.

All the critical points have zero localized energy density $\rho_\star$ (except for the trivial point with $\omega = w$). Also, when the Hubble parameter is non vanishing, we do not get a positive value for $\rho_\star$.

Stability analysis In appendix B we analyze the stability of the $F_\star = 0$ critical points linearizing the system of differential equations around the fixed point. We conclude that the studied fixed point (132) and (133), characterized by vanishing $F_\star$ and $\omega = -1$ or $\omega \neq -1$ can both be saddles or attractors, depending on the value of the anomaly parameter c_B, of c_Y and of ω (relating the two CFT pressures). The trivial fixed point cannot be analyzed at linear order, since its stability matrix is null. It is obvious from (131) that for positive Hubble parameters the energy density goes to zero at late time.

We can thus observe that, starting the cosmological evolution with a hidden energy density σ different from zero and with a suitable value of the anomaly coefficient c_B, choosing ω to be such that the $F_\star = 0$ stability matrix has negative eigenvalues, the corresponding $F_\star = 0$ fixed point could be a global attractor for the flat extra dimension universe. This critical point could eventually represent the present accelerated era. However, it is a zero energy density critical point.

B. Static compact extra dimensions

The Einstein equations of motion (125) get simplified when we take the $b =$ const ansatz for the internal space scale factor

$$M_{\rm Pl}^4\left(3H^2 + \frac{\kappa}{b_0^2}\right) = \rho + \sigma + \lambda \tag{134}$$

$$M_{\rm Pl}^4\left(2\dot{H} + \frac{\kappa}{b_0^2}\right) = -w\rho - \omega\sigma + \omega(\mathcal{A}_{(6)} + Y) + \lambda \tag{135}$$

$$\dot{\sigma} + 3(1+\omega)H\sigma = 3\omega H(\mathcal{A}_{(6)} + Y) \tag{136}$$

$$\dot{\rho} + 3(1+w)H\rho = 0. \tag{137}$$

From these equations we can deduce the corresponding fixed points and their criticality, following the calculations in appendix B.

Fixed point solutions We have found the inflationary fixed points for a universe with non evolving internal dimensions, i.e. with constant scale factor $b(t) \equiv b_0$. Besides the trivial fixed point $H_\star = 0$ there are other solutions.

(1) The existence of nontrivial fixed points is determined by the values of the parameter of the specific conformal theory c_A, c_B, c_Y and ω, but also by the mass scales $M_{\rm Pl}$ and κ/b_0. The form of the fixed point solutions are derived in appendix B. An easy critical point solution can be derived in the case of zero type B contribution to the anomaly and $\omega = -1$

$$\begin{aligned} H_\star^2 &= \frac{\kappa}{b_0^2}\left[9 \pm \sqrt{90 + \frac{5c_A}{192c_Y}\frac{1}{M_{\rm Pl}^2}\frac{\kappa}{b_0^2}}\right]^{-1} \\ \sigma_\star &= -\sigma_{p\star} = 2\sigma_{\pi\star} \\ &= M_{\rm Pl}^4\left(3\left[9 \pm \sqrt{90 + \frac{5c_A}{192c_Y}\frac{1}{M_{\rm Pl}^2}\frac{\kappa}{b_0^2}}\right]^{-1} + 1\right)\frac{\kappa}{b_0^2} \\ \rho_\star &= 0. \end{aligned} \tag{138}$$

For $\kappa = 0$ it reduces to the trivial fixed point. We have real roots for $H_\star$ if $192c_Y^2M_{\rm Pl}^2 > -c_A\kappa/45b_0^2$ and they will be both positive when $\kappa > 0$ and $-192c_Y^2M_{\rm Pl}^2 < c_A\kappa/45b_0^2 < -192c_Y^2M_{\rm Pl}^2/252$, so that c_A must be negative.

(2) We also have a trivial critical point with $H_\star = 0$. In particular both $\rho_\star$ and $\sigma_\star$ can be different from zero, they are functions of κ/b_0^2 as we can see from the equation $\rho_\star + \sigma_\star = M_{\rm Pl}^4\kappa/b_0^2$, and are also related by

$$\begin{aligned} &(1+w)\rho_\star + (1+\omega)\sigma_\star \\ &\quad = \omega\left(-\frac{c_B}{800}\frac{\kappa}{b_0^2} + \frac{4c_Y}{5}M_{\rm Pl}^2\right)\frac{\kappa^2}{b_0^4}. \end{aligned} \tag{139}$$

(3) Other fixed points can be found, for example, for $\omega = 0$. The internal space curvature κ would have to be negative in those cases, since $H_\star^2 = -\kappa/3b_0^2$ and we would have to compactify on an hyperbolic space.

Stability analysis For the static extra dimension fixed points we found that, depending on the values of c_A, c_B, c_Y, $M_{\rm Pl}^2$, and ω, we can get an attractor or a saddle.

We can thus choose the hidden sector parameters such that we can get a stable fixed point. There exists however another critical point, i.e. the trivial one characterized by $H_\star = 0$ but generally non zero $\rho_\star$ and $\sigma_\star$, which always is a saddle. So, trajectories may either be attracted by the nontrivial critical point or flow away from the saddle point.

C. Equal scale factors

Another limit that simplifies some of the calculations is the equal scale factor assumption. In this case the Hubble parameters of the internal space and the 3D space are equal, $F = H$, and we remain with the following set of equations for the variables H, ρ, σ

$$M_{\rm Pl}^4\left(10H^2 + \frac{\kappa}{b^2}\right) = \rho + \sigma + \lambda \tag{140}$$

$$M_{\rm Pl}^4\left(4\dot{H} + 10H^2 + \frac{\kappa}{b^2}\right) = -w\rho - \omega\sigma + \omega(\mathcal{A}_{(6)} + Y) + \lambda \quad (141)$$

$$\dot{\sigma} + 6H\sigma = H(\mathcal{A}_{(6)} + Y) \quad (142)$$

$$\dot{\rho} + (5 + 3w + 2w_\pi)H\rho = 0. \quad (143)$$

Fixed point solutions We find an inflationary fixed point where the internal space is also staying in an inflationary era, since the two Hubble parameters are equal (see appendix B for calculations). The existence of such fixed point depends on the values of c_A, c_B, c_Y, κ and ω.

It is necessary to impose $\omega = 1/5$, implying that the two pressures characterizing the CFT stress-energy tensor must be equal $\sigma_\pi = \sigma_p$, in order to obtain the following fixed point solutions.

(1) We write the explicit solution for a flat internal space $\kappa = 0$, where the critical value for the Hubble parameter is given in terms of the anomaly coefficients and of the only dimensionful parameter which is the 6D Planck mass. This solution restricts the possible values of c_A and c_B, coming from the conformal field theory anomaly

$$\begin{aligned} H_\star^2 &= -\frac{24}{c_A + 2c_B} M_{\rm Pl}^2 [24c_Y \pm \sqrt{576c_Y^2 + (c_A + 2c_B)}] \\ \sigma_\star &= -\sigma_{p\star} = -\sigma_{\pi\star} \\ &= -\frac{240}{c_A + 2c_B} M_{\rm Pl}^6 [24c_Y \pm \sqrt{576c_Y^2 + (c_A + 2c_B)}], \\ \rho_\star &= 0. \end{aligned} \quad (144)$$

For $(24c_Y)^2 > -(c_A + 2c_B)$ the two roots are real. We must satisfy the condition $(c_A + 2c_B) < 0$ on the anomaly parameters in order to have two positive $H_\star$ critical points (both denoted by zero energy density $\rho_\star$).

(2) For $\kappa \neq 0$ there exists no fixed points, since $(\mathcal{A}_{(6)} + Y)$ and $(10H_\star^2 + \frac{\kappa}{a^2})$ cannot be both constant unless we enforce the staticity condition on the Hubble parameter $H_\star = 0$ and $\kappa = 0$.

Stability analysis In this last case of equal scale factors we have again carried the stability analysis in appendix B. As a result, we found that the equal scale factor critical point with flat internal space is an attractor. It could thus represent the eternal acceleration of the universe.

D. Comments

We want to summarize the interesting features of the critical analysis of sections VII A, VII B, and VII C, in the perspective of the comparison with the 7D bulk gravitational dual description that we illustrated basically in sections III and IV.

(i) All the critical points we can find in the brane description are characterized by exactly zero value for the localized matter energy density ρ (except for some $H_\star = 0$ trivial points). To have a non vanishing energy density it is necessary to introduce an interaction term between the matter fields and the hidden sector fields. The reason for this is that it modifies the conservation equation for ρ, allowing for a non zero time independent solution. This intuitively corresponds to turning on the brane-bulk energy exchange on the bulk gravity side.

(ii) In most of the simple critical point solutions, the hidden sector pressure $\sigma_{p\star}$ is related to the energy density $\sigma_\star$ by $\sigma_{p\star} = -\sigma_\star$. This indicates a vacuum behavior for the equation of state of the hidden sector of the holographic dual theory at the inflationary fixed points.

(iii) There also appears to exist more than one critical point solution in some of the explicitly examined limits. Since the stability matrix analysis reveals that we can have either stable or saddle points (depending on the CFT and counterterm parameters c_A, c_B, c_Y and on the Plack mass), we can expect two kinds of behavior (if the number of critical point solution is two). Whenever one of the fixed points is attractive and the other one is a saddle, we can generally depict a phase portrait such that some of the trajectories are attracted by the stable point, while others can be repulsed by the saddle and go toward the large density region. If, on the other hand, we get two saddle points, trajectories bend near to the saddles and flow away. We note that the trivial critical point has undefined stability at linear order in the perturbations, so that it may either attract or repel trajectories in its neighborhood. However, if $H > 0$ and w, $w_\pi > -1$, late time evolution always is described by $\rho \to 0$.

(iv) Comparing these results with the bulk cosmology we note that we get both trivial and nontrivial accelerating critical points in the CFT description. The nontrivial fixed points are associated to the so called Starobinski branch of the solution to the Einstein, conservation and anomaly equations. Acceleration is produced due to conformal anomaly and higher derivative terms. These accelerating points can be interpreted to be in correspondence with the accelerating fixed points found in the gravity description with inflow. The trivial fixed points come from the smooth branch, instead, and they are smoothly connected to the gravity description. An analogous map can be found in the 5D-4D model [9].

(v) In the bulk description we only found positive acceleration critical points since $q = H^2$. With the holographic approach, we could in principle also get non constant H critical point solutions. In fact we could solve the system of Einstein equations on the brane asking time independence for $q = \dot{H} + H^2$, ρ and σ. We would get a new system of first order (non linear) differential equations in H, which in principle could have nontrivial solutions.

VIII. BRANE/BULK CORRESPONDENCE

We derive some solutions that can illustrate interesting aspects of the cosmological model we considered and, in particular, of the duality that relates the two descriptions. We will be able to find explicit expressions by making special simplifying assumptions on the parameters of the holographic theory and on the space-time background. These examples allow us to make a comparison between the results we will find in the holographic setup and the expressions we derived in the bulk gravity theory.

Since we are interested in comparing the two dual approaches, in the sense of AdS/CFT correspondence, we have to derive some expressions for H in terms of the localized matter energy density ρ and of a mirage density χ. In section III, we performed the cosmological analysis in the 7D bulk description, expressing the 3D Hubble parameter H in terms of the localized energy density ρ, reducing the first order ODE in H to an algebraic equation for H^2 plus a first order ODE for the mirage density χ. In the holographic dual theory, the mirage density is identified with the solution to the homogeneous equation associated to the conservation equation for the hidden sector density σ. It will thus have the property of obeying to the free radiation conservation equation in d effective dimensions (where d is the effective number of dimensions equal to 4 in a static compact space background and to 6 when the $a(t)$ and $b(t)$ scale factors are equal).

To obtain the explicit result for H^2 in the brane dual description, however, it is necessary to integrate the differential Eq. (122) for the energy density σ. It is established that the anomaly and the trace of the quadratic contribution to the variation of the dual theory action are highly nontrivial functions of the Hubble parameters and the spatial curvatures and contain derivatives of H, F up to order three. So, an analytical integration of the σ conservation equation is in general apparently unachievable. However, it is possible to neglect the $\mathcal{A}_{(6)}$ and Y contributions if we are in the slowly scaling approximation, which corresponds to a small curvature approximation. This is what we are going to discuss in the following.

A. Slowly scaling approximation

We give a rough idea on how the correspondence between the brane and the bulk dual theories works. In fact, we will neglect all the higher order terms in the holographic description, which is equivalent to ask that the Hubble parameter is nnegligiblewith respect to the Planck mass $H^2 \ll M_{\rm Pl}^2$ (i.e. $\dot{a} \ll M_{\rm Pl} a$). In this approximation, all the higher order curvature terms—including the anomaly and the trace Y^μ_μ—can be neglected in favor of contributions proportional to the Einstein tensor. The integration of the σ equation would give terms of the order of $M_{\rm Pl}^2 H^4$ (from Y) and H^6 (from $\mathcal{A}_{(6)}$). Once we plug the result for σ in the Einstein Eq. (125) these terms are suppressed, since the left-hand side of (125) is of order $M_{\rm Pl}^4 H^2$.

We anticipate that, as a consequence of the small Hubble parameter approximation on the brane, we get a linear dependence of H^2 on the mirage plus visible matter energy densities (the hidden sector density σ is identified with the mirage density χ). Higher order contributions due to the anomaly and to Y would give rise to higher power dependences in a small density expansion for H^2. Since in the holographic description we truncate to linear order in density, we also keep only linear terms in ρ for the bulk gravity results. The bulk equations for H and χ (35)–(37), can be formulated independently of w if we ignore higher (quadratic) order terms in ρ and assign a specific value to w_π. Neglecting the second order term in ρ we will only have one condition to determine M and V in terms of the brane parameter $M_{\rm Pl}$, so that only the ratio M^{10}/V will be identified.

Since in this approximation the quadratic and higher order dependence of H^2 on ρ are absent, we do not capture the eventual Starobinsky [80] behavior of the solutions to the Einstein equations [81]. The higher derivative terms are necessary in that case to calculate the exit from inflation to a matter dominated universe and the subsequent thermalization to radiation dominated era. In Starobinsky model the higher derivative terms are represented by the type D conformal anomaly contribution to the trace of the stress-energy tensor. Nonetheless, in the 5D RS holographic dual analysis of [9], where these terms are canceled, stringy corrections like Gauss-Bonnet terms can play the same role. We note that in our setup, the 6D conformal anomaly contains suitable higher derivative terms, not only in the total derivative contributions but also in type B anomaly.

We now explain the results to which the slowly scaling approximation leads in some particular limits.

1. Equal scale factors

We start looking at the Einstein, conservation and anomaly equations in the equal scale factor limit. The system of Eqs. (125) and (126) for the theory on the brane takes the form

$$M_{\rm Pl}^4\left(10H^2 + 3\frac{k}{a^2} + \frac{\kappa}{a^2}\right) = \rho + \sigma + \lambda \tag{145}$$

$$M_{\rm Pl}^4\left(4\dot{H} + 10H^2 + \frac{k}{a^2} + \frac{\kappa}{a^2}\right) = -w\rho - \omega\sigma + \omega(\mathcal{A}_{(6)} + Y) + \lambda \tag{146}$$

$$\dot{\sigma} + 6H\sigma = H(\mathcal{A}_{(6)} + Y) \qquad (147)$$

$$\dot{\rho} + (3(1 + w) + 2(1 + w_\pi))H\rho = 0. \qquad (148)$$

We note that the set of equations is independent of the hidden sector parameter ω and it is furthermore interesting that the homogeneous equation associated to (147) is indeed precisely the 6D free radiation equation, independently of the value for ω. In particular, the two conservation equations can be written in the integral form

$$\sigma = \chi + a^{-6}\int \mathrm{d}t a^6 H(\mathcal{A}_{(6)} + Y), \qquad \chi = \chi_0\left(\frac{a_0}{a}\right)^6 \qquad \rho = \rho_0\left(\frac{a_0}{a}\right)^{3(1+w)+2(1+w_\pi)}. \qquad (149)$$

Plugging the result for σ in the first equation of the system (145) and neglecting the curvature higher order terms that come from the integration of $(\mathcal{A}_{(6)} + Y)$, we obtain the following expression for H^2, together with the ρ and χ equations in their differential form

$$H^2 + \frac{1}{10}\left(3\frac{k}{a^2} + \frac{\kappa}{a^2}\right) = \frac{1}{10M_{\mathrm{Pl}}^4}(\rho + \chi) + \frac{1}{10M_{\mathrm{Pl}}^4}\lambda \qquad (150)$$

$$\dot{\chi} + 6H\chi = 0 \qquad (151)$$

$$\dot{\rho} + (3(1 + w) + 2(1 + w_\pi))H\rho = 0. \qquad (152)$$

It is now easy to compare (150)–(152) with the corresponding system of equations in the bulk description of the equal scale factor universe, with zero brane-bulk energy exchange T_7^0 and bulk "self-interaction" T_7^7. The expression for H^2 can also be written in a w-independent way fixing w_π (in this particular case we could also include the quadratic term in ρ in the w-independent formulation, i.e. when $w_\pi = w$). Neglecting the second order term in the energy densities we obtain

$$H^2 + \frac{1}{10}\left(3\frac{k}{a^2} + \frac{\kappa}{a^2}\right) = \frac{2\tilde{c}_{V(\mathrm{eq})}V}{5M^{10}}(\rho + \chi) + \lambda_{\mathrm{RS}} \qquad (153)$$

$$\dot{\chi} + 6H\chi = 0 \qquad (154)$$

$$\dot{\rho} + (3(1 + w) + 2(1 + w_\pi))H\rho = 0. \qquad (155)$$

The two systems of Eqs. (150)–(155) perfectly agree at this order in the approximation. The matching between the scales on the two sides of the duality is then

$$\frac{M^{10}}{V} = 4\tilde{c}_{V(\mathrm{eq})}M_{\mathrm{Pl}}^4 \overset{w_\pi = w}{\to} \frac{M^{10}}{V} = \frac{M_{\mathrm{Pl}}^4}{20}. \qquad (156)$$

As we announced, only the ratio M^{10}/V can be determined, since we only have one condition to match the two descriptions. When higher order corrections are included in the brane description we would generally find a matching for both M and V, which in principle would depend on the particular CFT parameters (c_A, c_B, c_Y). We can guess that the ratio M^{10}/V would not depend on them (indeed, for the pure RS setup we get $M_{\mathrm{Pl}}^4 V \propto M^{10}$). When $w_\pi = w$ (i.e. when the pressures of the matter perfect fluid relative to the 2D internal space and the 3D space are equal $\pi = p$) the coefficient $\tilde{c}_V$ becomes $\tilde{c}_{V(\mathrm{eq})} = 1/80$. It is interesting to note that for $w = w_\pi$ the matching exactly reduces to the RS condition for zero effective cosmological constant on the brane $\lambda_{\mathrm{RS}} = 0$. Since we are in the limit $F = H$, it seems natural to have $\pi = p$ too.

2. *Static compact extra dimensions*

We consider the static extra dimension limit $F = 0$. The Einstein equations plus conservation and anomaly equations in this limit read

$$M_{\mathrm{Pl}}^4\left(3H^2 + 3\frac{k}{a^2} + \frac{\kappa}{b_0^2}\right) = \rho + \sigma + \lambda \qquad (157)$$

$$M_{\mathrm{Pl}}^4\left(2\dot{H} + 3H^2 + \frac{k}{a^2} + \frac{\kappa}{b_0^2}\right) = -w\rho - \omega\sigma + \omega(\mathcal{A}_{(6)} + Y) + \lambda \qquad (158)$$

$$\sigma + 3(1 + \omega)H\sigma = 3\omega H(\mathcal{A}_{(6)} + Y) \qquad (159)$$

$$\dot{\rho} + 3(1 + w)H\rho = 0. \qquad (160)$$

The homogeneous equation associated to (159) is the 4D free radiation equation only if $\omega = 1/3$, implying that the hidden sector pressure of the internal space σ_π must be zero. With this assumption, the two conservation equations become

$$\sigma = \chi + a^{-4}\int \mathrm{d}t a^4 H(\mathcal{A}_{(6)} + Y), \qquad \chi = \chi_0\left(\frac{a_0}{a}\right)^4 \qquad \rho = \rho_0\left(\frac{a_0}{a}\right)^{3(1+w)} \qquad (161)$$

The results for χ and ρ agree with the bulk formulation for zero energy exchange. Plugging (161) into (157) and neglecting the curvature higher order term, as we are in the slowly scaling approximation, we find

$$H^2 + \frac{1}{3}\frac{\kappa}{b_0^2} = \frac{1}{3M_{\mathrm{Pl}}^4}(\rho + \chi) \qquad (162)$$

$$\dot{\chi} + 4H\chi = 0 \qquad (163)$$

$$\dot{\rho} + 3(1 + w)H\rho = 0 \qquad (164)$$

Although conservation equations agree, the Friedmann like equation does not give the expected $1/6$ coefficient in front of the κ/b_0^2 term in (162) that instead follows from the equations on the bulk gravity side

$$H^2 + \frac{1}{6}\frac{\kappa}{b_0^2} = \frac{2\tilde{c}_{V(\mathrm{st})}V}{3M^{10}}(\rho + \chi) \tag{165}$$

$$\dot{\chi} + 4H\chi = 0 \tag{166}$$

$$\dot{\rho} + 3(w+1)H\rho = 0. \tag{167}$$

The bulk equations are derived in the density linear approximation and for vanishing energy exchange T_7^0 and T_7^7. The coefficient $\tilde{c}_{V(\mathrm{st})}$ can be written in a w-independent way if we fix w_π.

As a consequence, the ratio of the two bulk parameters can be identified with

$$\frac{M^{10}}{V} = 2\tilde{c}_{V(\mathrm{st})}M_{\mathrm{Pl}}^4 \overset{w_\pi=(w+5)/6}{\longrightarrow} \frac{M^{10}}{V} = \frac{M_{\mathrm{Pl}}^4}{20} \tag{168}$$

The κ/b_0^2 terms differ in the two dual descriptions (in the static extra dimension background). The matching (168) gives a result that depends on the values of w, w_π in a different way if compared to the equal scale factor limit (156). It is thus interesting to further examine how the matching varies according to the value of the internal space Hubble parameter. We are indeed going to consider the proportionality ansatz $F = \xi H$ to better understand this behavior. We note that in the limit $w_\pi = (w+5)/6$ we recover in (168) the RS fine-tuning determining zero effective cosmological constant on the brane $\lambda_{\mathrm{RS}} = 0$, since $\tilde{c}_{V(\mathrm{st})} = 1/40$.

3. Proportional Hubble parameters

Following the computations in the last two sections and generalizing them, we derive the set of equations for H^2, χ and ρ for proportional and small Hubble parameters. Since, as before, $\sigma = \chi$ if we neglect higher order terms in the small curvature approximation, Eqs. (125) and (126) lead to the following equations

$$H^2 + \frac{1}{(\xi_b^2 + 6\xi_b + 3)}\frac{\kappa_b}{b^2} = \frac{1}{(\xi_b^2 + 6\xi_b + 3)M_{\mathrm{Pl}}^4}(\rho + \chi) \tag{169}$$

$$\dot{\chi} + d_{\xi_b}H\chi = 0 \tag{170}$$

$$\dot{\rho} + w_{\xi_b}H\rho = 0, \tag{171}$$

where $d_{\xi_b} \equiv 3(1+\omega) + 3\xi_b(1-\omega)$, $w_{\xi_b} \equiv 3(1+w) + 2\xi_b(1+w_\pi)$ and ξ_b is the proportionality factor $F = \xi_b H$ (or $b = a^{\xi_b}$). The bulk equations where derived in (38)–(40). For the moment we will keep two different proportionality factors: in the bulk $F = \xi_B H$ as in section III D (putting zero T_7^7 and T_7^0)

$$H^2 + \frac{1}{(\xi_B^2 + 3\xi_B + 6)}\frac{\kappa_B}{b^2} = \frac{2\tilde{c}_{V,\xi}V}{(2\xi_B + 3)M^{10}}(\rho + \chi) \tag{172}$$

$$\dot{\chi} + d_{\xi_B}H\chi = 0 \tag{173}$$

$$\dot{\rho} + w_{\xi_B}H\rho = 0. \tag{174}$$

We have used the following definitions: $d_{\xi_B} \equiv 6(\xi_B^2 + 2\xi_B + 2)/(2\xi_B + 3)$, $w_{\xi_B} \equiv 3(1+w) + 2\xi_B(1+w_\pi)$ and we recall that $\tilde{c}_{V,\xi}$ is $\tilde{c}_{V,\xi} \equiv c_V/(w_{\xi B} - d_{\xi B})$, where $c_V = (31w - 6w_\pi - 5)/400$.

In order for the two descriptions to be equivalent w.r.t. the χ and ρ differential equations (which do not get any correction from higher order contributions), we have to put $\xi_b = \xi_B = \xi$, as it was expected. Besides, the parameter relating σ_π to σ_p must be $\omega = 1/(2\xi + 3)$. However, assuming an equal proportionality relation on the two sides of the duality, the coefficient of the κ term in (169) and (172) differ if the two curvatures in the bulk and brane descriptions are equal, unless $\xi = 1$. So, the only setup that predicts the same effective spatial curvature for the internal space in the brane and bulk descriptions is the equal scale factor background (neglecting higher order corrections). However, we can determine an effective spatial curvature for the internal space in the brane description, given by $\kappa_b = (\xi^2 + 6\xi + 3)\kappa_B/(\xi^2 + 3\xi + 6)$.

The matching for the scales of the two dual theories is given by

$$\frac{M^{10}}{V} = \frac{\xi^2 + 6\xi + 3}{2\xi + 3}2\tilde{c}_{V,\xi}M_{\mathrm{Pl}}^4. \tag{175}$$

It is always possible to choose a w_π such that the matching relation (175) gives the RS fine-tuning condition $M_{\mathrm{Pl}}^4 = 20M^{10}/V$. Otherwise, missing the fine-tuning would amount to introducing a non vanishing effective cosmological constant on the RS brane.

IX. NON-CONFORMAL AND INTERACTING GENERALIZATION

To examine the general cosmological evolution that reflects the presence of energy exchange between the brane and the bulk in the seven dimensional bulk picture and the non zero value of the bulk component of the stress tensor T_7^7, we will drop the assumption of having a conformal non interacting field theory living on the brane. Intuitively, a non vanishing T_7^0 in the bulk description corresponds to interactions between the gauge theory and the visible matter. The diagonal T_7^7 component appears in the brane description as dual to a new trace term spoiling the conformal invariance. The generalization of the 6D RS dual action will be modified adding the new interaction term and substituting the hidden sector CFT with a strongly coupled gauge theory (SCGT). The following analysis will be done in analogy with the 4D holographic dual generalization of the 5D RS cosmology exposed in [9].

Using the notations of section V we write the generalized action as

$$S_{\rm gen} = S_{\rm SCGT} + S_R + S_{R^2} + S_{R^3} + S_m + S_{\rm int} \qquad (176)$$

where the new entry is the interaction term $S_{\rm int}$ and $S_{\rm CFT}$ has been changed into $S_{\rm SCGT}$. The strongly coupled fields can be integrated out, transforming the sum of the strongly coupled theory action plus the interaction term into an effective functional of the visible fields (and of the metric) $W_{\rm SCGT}$. As a result, the action (176) becomes

$$S_{\rm gen} = W_{\rm SCGT} + S_R + S_{R^2} + S_{R^3} + S_m. \qquad (177)$$

As in the conformal non interacting case, we are now ready to calculate the general 6D equations of motion for the holographic generalized RS cosmology.

A. Generalized evolution equations

The stress-energy tensors are defined in an analogous way

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{\sqrt{-\gamma}}\frac{\delta S_m}{\delta\gamma^{\mu\nu}}, \qquad W_{\mu\nu} = \frac{1}{\sqrt{-\gamma}}\frac{\delta W_{\rm SCGT}}{\delta\gamma^{\mu\nu}} \\ Y_{\mu\nu} &= \frac{1}{\sqrt{-\gamma}}\frac{\delta S_{R^2}}{\delta\gamma^{\mu\nu}}, \qquad Z_{\mu\nu} = \frac{1}{\sqrt{-\gamma}}\frac{\delta S_{R^3}}{\delta\gamma^{\mu\nu}} \\ V_{\mu\nu} &= W_{\mu\nu} + Y_{\mu\nu} + Z_{\mu\nu}, \qquad Y^\mu_\mu = Y, \end{aligned} \qquad (178)$$

and the Einstein equation, the (non) conservation conditions, the anomaly equation read

$$\begin{aligned} M^4_{\rm Pl} G_{\mu\nu} &= T_{\mu\nu} + W_{\mu\nu} + Y_{\mu\nu} + V_{\mu\nu} \qquad \nabla^\nu T_{\mu\nu} = T \\ \nabla^\nu V_{\mu\nu} &= -T \qquad V^\mu_\mu = \mathcal{A}_{(6)} + X + Y. \end{aligned} \qquad (179)$$

The total stress-energy tensor is still conserved. Taking account of the interactions between the hidden theory and the matter generally amounts to have non separately conserved $V_{\mu\nu}$ and $T_{\mu\nu}$. This is reflected by the introduction of a non homogenous term in the conservation equations. The anomaly equation contains the general expression for the conformal anomaly in six dimensions $\mathcal{A}_{(6)}$ and the trace term Y. Furthermore, it gets modified including an extra term X that accounts for classical and quantum breaking of the conformal symmetry in a FRW plus compact space background. The stress-energy tensors are parametrized as before

$$\begin{aligned} T_{00} &= \rho(t), \qquad T_{ij} = p(t)\gamma_{ij}, \qquad T_{ab} = \pi(t)\gamma_{ab} \\ V_{00} &= \sigma(t), \qquad V_{ij} = \sigma_p(t)\gamma_{ij}, \qquad V_{ab} = \sigma_\pi(t)\gamma_{ab} \end{aligned} \qquad (180)$$

The consequent changes in the equations written in terms of the Hubble parameters, of the energy densities and pressures are the following. The Friedmann equations remain the same (as for the non interacting conformal case)

$$\begin{aligned} M^4_{\rm Pl}\left(3H^2 + 6HF + F^2 + 3\frac{k}{a^2} + \frac{\kappa}{b^2}\right) &= \rho + \sigma + \lambda \\ M^4_{\rm Pl}\left(2\dot H + 3H^2 + 4HF + 2\dot F + 3F^2 + \frac{k}{a^2} + \frac{\kappa}{b^2}\right) &= -p - \sigma_p + \lambda \end{aligned} \qquad (181)$$

the conservation equations now involve the quantity T

$$\begin{aligned} \dot\sigma + 3(\sigma + \sigma_p)H + 2(\sigma + \sigma_\pi)F &= T \\ \dot\rho + 3(\rho + p)H + 2(\rho + \pi)F &= -T \end{aligned} \qquad (182)$$

and the anomaly equation includes the conformal breaking term, as a consequence of the masses and β-functions of the strongly coupled gauge theory

$$\sigma - 3\sigma_p - 2\sigma_\pi = \mathcal{A}_{(6)} + Y + X \qquad (183)$$

X has to be written in terms of the β-functions and operators of the strongly coupled gauge theory (SCGT) and matter theory. Taking the same ansatz for the pressures as for the non interacting conformal theory

$$\sigma_\pi = \Omega\sigma_p \quad (\omega^{-1} \equiv 3 + 2\Omega) \quad p = w\rho, \quad \pi = w_\pi\rho \qquad (184)$$

and using the anomaly equation to eliminate $\sigma_p = \omega(\sigma - \mathcal{A}_{(6)} - Y - X)$ from the set of remainig equations, we get

$$M^4_{\rm Pl}\left(3H^2 + 6HF + F^2 + 3\frac{k}{a^2} + \frac{\kappa}{b^2}\right) = \rho + \sigma + \lambda \qquad (185)$$

$$\begin{aligned} M^4_{\rm Pl}&\left(2\dot H + 3H^2 + 4HF + 2\dot F + 3F^2 + \frac{k}{a^2} + \frac{\kappa}{b^2}\right) \\ &= -w\rho - \omega\sigma + \lambda + \omega(\mathcal{A}_{(6)} + Y + X) \end{aligned} \qquad (186)$$

$$\begin{aligned} \dot\sigma &+ [3(1+\omega)H + 3(1-\omega)F]\sigma \\ &= [3\omega H + (1 - 3\omega)F](\mathcal{A}_{(6)} + Y + X) + T \end{aligned} \qquad (187)$$

$$\dot\rho + [3(1+w)H + 2(1 - w_\pi)F]\rho = -T. \qquad (188)$$

The cosmological evolution described by these differential equations which include non conformality (represented by the X term) and matter/hidden sector interactions (related to the T term) could be now investigated. In the spirit of AdS/CFT correspondence, the CFT generalization amounts to introducing nontrivial dynamics in the bulk and brane-bulk energy exchange (and bulk self-interaction) in the 7D picture. The bulk cosmology with T_7^0 parameter turned on has been analyzed in section IV.

B. Critical points and stability

The fixed points can be derived as we have done in appendix B for the conformal non interacting theory and their stability can then be studied for specific theories. We will not discuss this topic here. Since the new deformation parameters X, T depend on the 6D space-time curvature, they contain functions of the Hubble parameters and spatial curvatures and of the intrinsic energy scale of the background, the AdS_7 radius (or M_{Pl}). They will thus in general modify the equations for the fixed points and their stability in a sensible way, depending on the specific generalization one wants to consider.

We will instead try to understand how the comparison with the bulk description gets changed when we go to the generalized scenario. This will be the subject of the next subsection.

C. Comparison to 7D cosmology with energy exchange in slowly scaling regime

As we have done for $T = X = 0$, we want to illustrate some explicit examples with the aim of understanding the peculiar features of this cosmological model and its two dual descriptions. We will therefore make some assumptions simplifying the set of equations including Einstein, conservation and anomaly equations. First of all, we are going to neglect terms containing higher orders in the background curvature—namely the anomaly coming from S_{R^3} and the trace contribution Y coming from the second order action S_{R^2}.

Equal scale factors The correspondence works as in the conformalnoninteracting analysis of subsection VIII A 1. On the brane side (referring to Eqs. (185)–(188)), the slowly scaling approximation leads to the equations

$$H^2 + \frac{1}{10}\left(3\frac{k}{a^2} + \frac{\kappa}{a^2}\right) = \frac{1}{10M_{\rm Pl}^4}(\rho + \chi) + \frac{1}{10M_{\rm Pl}^4}\lambda \quad (189)$$

$$\dot{\chi} + 6H\chi = HX + T \quad (190)$$

$$\dot{\rho} + (3(1 + w) + 2(1 + w_\pi))H\rho = -T. \quad (191)$$

To get the bulk description expressions for H, ρ, χ we truncate Eqs. (35)–(37) to first order in the density, neglecting ρ/V w.r.t. order 1 terms

$$H^2 + \frac{1}{10}\left(3\frac{k}{a^2} + \frac{\kappa}{a^2}\right) = \frac{2\tilde{c}_{V({\rm eq})}V}{5M^{10}}(\rho + \chi) + \lambda \quad (192)$$

$$\dot{\chi} + 6H\chi = 2T_7^0 - \frac{40M^5}{V}HT_7^7 \quad (193)$$

$$\dot{\rho} + (3(1 + w) + 2(1 + w_\pi))H\rho = -2T_7^0. \quad (194)$$

The matching with the system of equations on the brane (189)–(191) is exact if we have the following relations among the brane and bulk parameters

$$M_{\rm Pl}^4 = \frac{M^{10}}{2\tilde{c}_{V({\rm eq})}V} \overset{w_\pi = w}{\longrightarrow} M_{\rm Pl}^4 = 20\frac{M^{10}}{V} \quad (195)$$

$$T = 2T_7^0 \quad (196)$$

$$X = -\frac{M^5}{2\tilde{c}_{V({\rm eq})}V}T_7^7 \Rightarrow \frac{X}{M_{\rm Pl}^4} = -2\frac{T_7^7}{M^{10}}. \quad (197)$$

In the previous equations we have also explicitly evaluated the matching for equal matter pressures $w_\pi = w$, that gives the RS fine-tuning $\lambda_{\rm RS} = 0$, as in the noninteracting conformal theory.

Static compact extra dimensions The condition of static internal space $F = 0$, toghether with the small Hubble parameter approximation, brings Eqs. (185)–(188), relative to the brane description, and Eqs. (35)–(37), relative to the bulk description, in the form

$$H^2 + \frac{1}{3}\frac{\kappa}{b_0^2} = \frac{1}{3M_{\rm Pl}^4}(\rho + \chi) \quad (198)$$

$$\dot{\chi} + 4H\chi = HX + T \quad (199)$$

$$\dot{\rho} + 3(1 + w)H\rho = -T, \quad (200)$$

(where we assumed $\omega = 1/3$) and

$$H^2 + \frac{1}{6}\frac{\kappa}{b_0^2} = \frac{2\tilde{c}_{V({\rm st})}V}{3M^{10}}(\rho + \chi) \quad (201)$$

$$\dot{\chi} + 4H\chi = 2T_7^0 - \frac{40M^5}{V}HT_7^7 \quad (202)$$

$$\dot{\rho} + 3(w + 1)H\rho = -2T_7^0. \quad (203)$$

The parameters in the gauge and gravity descriptions are thus related by the following expressions

$$M_{\rm Pl}^4 = \frac{M^{10}}{4\tilde{c}_{V({\rm st})}V} \overset{w_\pi = (w+5)/6}{\longrightarrow} M_{\rm Pl}^4 = 20\frac{M^{10}}{V} \quad (204)$$

$$T = 2T_7^0 \quad (205)$$

$$X = -\frac{M^5}{2\tilde{c}_{V({\rm st})}V}T_7^7 \Rightarrow \frac{X}{M_{\rm Pl}^4} = -\frac{T_7^7}{M^{10}}. \quad (206)$$

For $w_\pi = (w + 5)/6$ we get the zero effective cosmological constant on the RS brane, as before.

Proportional Hubble parameters In the limit of proportional Hubble parameters or equivalently scale factors related by $b = a^{\xi_b}$, we use the set of Eqs. (185)–(188), substituting $F = \xi_b H$, $\kappa \to \kappa_b$, and expanding in the slowly scaling approximation

$$H^2 + \frac{1}{(\xi_b^2 + 6\xi_b + 3)} \frac{\kappa_b}{b^2} = \frac{1}{(\xi_b^2 + 6\xi_b + 3)M_{\rm Pl}^4}(\rho + \chi) \tag{207}$$

$$\dot{\chi} + d_{\xi_b} H\chi = (3\omega + \xi_b(1 - 3\omega))HX + T \tag{208}$$

$$\dot{\rho} + w_{\xi_b} H\rho = -T. \tag{209}$$

As before $d_{\xi_b} \equiv 3(1 + \omega) + 3\xi_b(1 - \omega)$, $w_{\xi_b} \equiv 3(1 + w) + 2\xi_b(1 + w_\pi)$. The bulk dynamics is described by (38)–(40) with $F = \xi_B H$, $\kappa \to \kappa_B$

$$H^2 + \frac{1}{(\xi_B^2 + 3\xi_B + 6)} \frac{\kappa_B}{b^2} = \frac{2\tilde{c}_{V,\xi} V}{(2\xi_B + 3)M^{10}}(\rho + \chi) \tag{210}$$

$$\dot{\chi} + d_{\xi_B} H\chi = 2T_7^0 - \frac{40M^5}{V} HT_7^7 \tag{211}$$

$$\dot{\rho} + w_{\xi_B} H\rho = -2T_7^0, \tag{212}$$

with $d_{\xi_B} \equiv 6(\xi_B^2 + 2\xi_B + 2)/(2\xi_B + 3)$, $w_{\xi_B} \equiv 3(1 + w) + 2\xi_B(1 + w_\pi)$.

If w and w_π are the same on both sides of the duality, then we must make the identification $\xi_b = \xi_B = \xi$ to match ρ equations. As a consequence, $\omega = 1/(2\xi + 3)$ in order to have agreement for the mirage density (non) conservation equations and $\kappa_b = (\xi^2 + 6\xi + 3)\kappa_B/(\xi^2 + 3\xi + 6)$. With these conditions, the comparison between the two sets of equations thus gives the following matching relations

$$M_{\rm Pl}^4 = \frac{2\xi + 3}{\xi^2 + 6\xi + 3} \frac{M^{10}}{2\tilde{c}_{V,\xi} V} \tag{213}$$

$$T = 2T_7^0 \tag{214}$$

$$X = -\frac{2\xi + 3}{2\xi^2 + 3} \frac{M^5}{2\tilde{c}_{V,\xi} V} T_7^7 \Rightarrow \frac{X}{M_{\rm Pl}^4} = -\frac{\xi^2 + 6\xi + 3}{2\xi^2 + 3} \frac{T_7^7}{M^{10}}. \tag{215}$$

X. SUMMARY AND CONCLUSIONS

In the context of holographic cosmology, we have investigated the specific background of 7D RS gravity, including an energy exchange interaction between brane and bulk. Some novel features arise both on the bulk side of the duality and in the conformal holographic theory on the brane. In particular, we found distinctive results in the comparison between the two descriptions that need a better understanding. The originality with respect to the 5D/4D holographic cosmology [9,43] is due to the compactification over a 2D internal space, around which we wrap the 5-brane. The 6D space-time filled by the brane acquires a dishomogeneity that distinguishes the 3D visible space from the 2D internal directions. Evolution can generally be different in the two spaces and pressures are individually related to the energy density by the usual ansatz $p = w\rho$ and $\pi = w_\pi \rho$ (p and π are, respectively, the 3D and 2D pressures), with $w_\pi \neq w$ in general.

Concerning the gravity theory in the bulk, we studied the Friedmann like equation that comes along with the introduction of a mirage energy density satisfying to the non-homogeneous radiation equation in some effective number of dimensions (which is six for equal scale factors in both 3D and 2D spaces and is four when the internal space is static). The bulk cosmological evolution is then determined by the Friedmann equation and by the (non) conservation equations for the mirage density and the localized matter density on the 5-brane. Making use of some simple ansatz for the evolution of the 2D compactification space (such as putting the corresponding Hubble parameter F equal to the Hubble parameter of the visible space H or to make it vanish) we found a wide spectrum of possible cosmologies that reduce to the RS vacuum in the absence of matter (i.e. we imposed the RS fine-tuning $\lambda_{\rm RS} = 0$).

Assuming small density approximation, we have described the explicit analytical solution in case of radiation dominated universe. The Hubble parameter evolves as in an effective 6D (4D) radiation dominated era for equal scale factors (static compact extra dimensions), independently of the form of the brane-bulk energy exchange. The effective 4D mirage and matter energy densities obey to the 4D free radiation equation in absence of energy exchange. If energy flows from the brane into the bulk, the 4D localized energy density is suppressed in time, in favor of the mirage density, even with zero mirage initial condition. For influx to the bulk, the 4D matter energy density apparently grows unbounded (if T is linear in ρ, otherwise energy density may go to zero for suitable values of the theory parameters), eventually diverging at a finite time. The small density approximation must break down and the full analysis is needed. On the other hand, still for small densities but generic perfect fluid equation of state (non necessarily pure vacuum energy) and energy influx, we found inflationary fixed point solutions that are stable for a large class of energy exchange parametrizations $T = A\rho^\nu$. These thus represent stable de Sitter solutions for (the four visible space-time directions of) our universe. We moreover argued that, differently than in the 5D RS approach [43], we may have a stable de Sitter critical point solution even for energy outflowing to the bulk, in the case of equal scale factors with $w < -1/3$ (and $\nu = 1$). For dynamical compactification (i.e. $F = \xi H$, $\xi < 0$) [82] we could in principle also get an outflow stable inflationary fixed point. We note that the 4D mirage energy density evolution without energy exchange is governed by the effective 4D free radiation equation only in the two limits of equal scale factors $F = H$ and static compact extra dimensions $F = 0$.

Having dropped the small density approximation, more elaborate models of cosmologies were developed. The number of possible inflationary critical point solutions can be larger than 1, depending on the parametrization for the brane-bulk energy exchange. For energy influx we showed the 6D picture of a scenario with two fixed points, where trajectories in the phase space can either always be characterized by positive acceleration, either remain at all time with negative acceleration, or alternate acceleration and deceleration phases. The portraits are rigorously valid for the effective 4D energy density in the case of static internal space (up to a constant rescaling). If we have equal scale factors, the evolution equations become much more complicated functions of the 4D densities and the computation is beyond the scope of the paper. For $\nu = 1$ there seems to exist only the trivial critical point characterized by vanishing Hubble parameter, so that the energy density should grow without bounds as predicted by the small density approximation, until the full analysis is needed. For energy outflow, the 6D energy density localized on the 5-brane decreases and the trajectories in the phase space go toward the trivial fixed point, eventually passing through an accelerated era. The effective 4D picture may differ from this description in the equal scale factor case, since it would be possible in principle not to have decreasing density at all times.

The rather detailed study of the various cosmologies emerging from the 7D RS model with energy exchange has been embodied in the context of the AdS/CFT correspondence. We have examined the role played in the holographic critical point analysis by the 6D anomalous CFT coupled to 6D gravity—with the addition of the higher order counterterms to the dual action. The dual theory on the brane is conformal (classically) and non interacting (with the matter theory on the brane). This CFT would then correspond to the RS setup with no energy exchange. Despite this fact, we may find inflationary critical point solutions, depending on the anomaly parameters and on the coefficient of the second order counterterm (in the curvature). All this fixed points are characterized by zero matter energy density. Clearly, neglecting all the higher order contributions (including the anomaly $\mathcal{A}_{(6)}$ and the trace term Y) we recover the trivial fixed points of the pure RS gravity background.

The comparison between the two dual descriptions has been achieved in the approximation where all the higher order terms can be neglected, i.e. for small Hubble parameters. Since higher order terms are truncated, we cannot access to the typical non conventional ρ^2 dependence in the expression for H^2—only linear terms are present in this approximation. Comparing the bulk Friedmann equation with the corresponding equation derived in the holographic description, we have to match the ratio M^{10}/V (M is the 7D Planck Mass and V is the tension of the RS brane) in terms of the 6D Planck mass $M_{\rm Pl}^4$ (the 4D Planck mass $M_{(4)}$ is related to $M_{\rm Pl}$ by $M_{(4)}^2 = V_{(2)} M_{\rm Pl}^4$). The RS fine-tuning $M_{\rm Pl}^4 = 20M^{10}/V$ is restored when we recover homogeneity in the background, imposing $F = H$ and $w_\pi = w$. With these assumptions indeed, the matching is exact also with respect to the spatial curvature terms. As we move away from homogeneity, we have to define an effective spatial curvature for the compact extra dimensions in the holographic description. The matching between the scales of the theories reflects the RS fine-tuning for a specific value of the matter pressure in the internal space (determined by w_π), depending on the proportionality factor ξ relating the two Hubble parameters $F(t) = \xi H(t)$, or $b(t) = a^\xi(t)$. We finally matched the evolution equations in the generalized holographic dual theory with the general bulk description. The interactions between hidden and visible sectors encode the dynamics of the brane-bulk energy exchange, $T_7^0 \neq 0$, on the bulk gravity side, while the breaking of conformal invariance (via non zero β-functions or masses) amounts to turning on the bulk "self-interaction", $T_7^7 \neq 0$.

The 7D RS background has been quite accurately studied on the bulk side, though many profound cosmological aspects have not been explored. There could be space, however, to fit the cosmological evolution of the universe in this model, since one of the stable de Sitter critical point solutions we found could represent the actual accelerated era. Besides, trajectories can end into the stable point first passing through a decelerated phase representing the matter or radiation dominated universe. Another accelerated era may be present at early times, eventually corresponding primordial inflation. Still, there is no rigorous construction of such a precise evolution. The holographic dual theory could also give an interesting cosmological description of the brane-world. It would be interesting to exploit Starobinky argument of graceful exit from primordial inflation via higher derivatives term in this context. In conclusion, we showed the basics of the brane-bulk duality in 7D RS background with energy exchange and of its cosmological features, but many further questions can be addressed within the framework of the 7D RS holographic cosmology.

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APPENDIX A: CONFORMAL ANOMALY AND TRACES IN SIX DIMENSIONS

Conformal anomaly The conformal anomaly for 6D theories has been studied in [79]. It can be derived using AdS/CFT and the gravitational renormalization procedure as in [72,73,76].

In our notations, the general expression for the trace anomaly in a six dimensional CFT is

$$\mathcal{A}_{(6)} = -(c_A E_{(6)} + c_B I_{(6)} + \nabla_\mu J^\mu_{(5)}), \qquad \text{(A1)}$$

$E_{(6)}$ is the Euler density in six dimensions (type A anomaly), $I_{(6)}$ is a fixed linear combination of three independent Weyl invariants of dimension six (type B anomaly) and $\nabla_i J^\mu_{(5)}$ is an linear combinations of the Weyl variation of six independent local functionals (type D anomaly), so that at the end we have eight free coefficients in the general form of the anomaly, depending on the specific CFT. The type D anomaly is a trivial (it is a total derivative, indeed) scheme dependent term that can be canceled by adding local covariant counterterms to the action [77].

For our metric we obtain as a result that $E_{(6)}$ depends on the Hubble parameters and on their time derivatives up to order one (that is, up to the second time derivative of the scale factors). $I_{(6)}$ instead depends on H and F time derivatives up to order three, and so does the divergence term. To be more specific, $I_{(6)}$ is made up by three contributions I_1, I_2, I_3, with fixed coefficients; the first two are two different contractions of three Weyl tensors (and contain only derivatives of the Hubble parameters up to order one), while in I_3 there are second order derivatives of the Weyl tensor (i.e. third order time derivatives of the Hubble parameters).

For a 6D FRW background (i.e. requiring homogeneity in all six dimensions) the sum of type A plus type B anomalies depends on the Hubble parameters only up to the first time derivative, while the type D anomaly contains time derivatives up to order three.

In terms of the Riemann tensor, Ricci tensor and scalar curvature, the A, B, and D contributions to the anomaly read [72]

$$E_{(6)} = \frac{1}{6912}E_0 \qquad I_{(6)} = \frac{1}{1152}\left(-\frac{10}{3}I_1 - \frac{1}{6}I_2 + \frac{1}{10}I_3\right)$$
$$J^\mu_{(5)} = -\frac{1}{1152}[-R^{\mu\nu\rho\sigma}\nabla^\tau R_{\tau\nu\rho\sigma} + 2(R_{\nu\rho}\nabla^\mu R^{\nu\rho} - R_{\nu\rho}\nabla^\nu R^{\mu\rho})] - \frac{1}{2880}R^{\mu\nu}\nabla_\nu R + \frac{1}{5760}R\nabla^\mu R, \qquad \text{(A2)}$$

where

$$E_0 = K_1 - 12K_2 + 3K_3 + 16K_4 - 24K_5 - 24K_6 + 4K_7 + 8K_8$$
$$I_1 = \frac{19}{800}K_1 - \frac{57}{160}K_2 + \frac{3}{40}K_3 + \frac{7}{16}K_4 - \frac{9}{8}K_5 - \frac{3}{4}K_6 + K_8$$
$$I_2 = \frac{9}{200}K_1 - \frac{27}{40}K_2 + \frac{3}{10}K_3 + \frac{5}{4}K_4 - \frac{3}{2}K_5 - 3K_6 + K_7$$
$$I_3 = K_1 - 8K_2 - 2K_3 + 10K_4 - 10K_5 - \frac{1}{2}K_9 + 5K_{10} - 5K_{11},$$

and

$$(K_1, \ldots, K_{11}) = (R^3, RR_{\mu\nu}R^{\mu\nu}, RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R_\mu{}^\nu R_\nu{}^\rho R_\rho{}^\mu, R^{\mu\nu}R^{\rho\sigma}R_{\mu\rho\sigma\nu}, R_{\mu\nu}R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau}, R_{\mu\nu\rho\sigma}R^{\mu\nu\tau\lambda}R^{\rho\sigma}{}_{\tau n}, R_{\mu\nu\rho\sigma}R^{\mu\tau\lambda\sigma}R^\nu{}_{\tau\lambda}{}^\rho, R\Box R, R_{\mu\nu}\Box R^{\mu\nu}, R_{\mu\nu\rho\sigma}\Box R^{\mu\nu\rho\sigma}).$$

In the analysis of the solutions to the Friedmann equations we plug in the specific expression for the Riemann tensor obtained considering our ansatz (108) for the metric. But before doing this, we use the anomaly equation and some standard assumptions on the pressures that parametrize the stress-energy tensors to manipulate our system of differential equations.

To give an explicit result for the conformal anomaly in the specific case of 6D CFT on curved space-time, with the ansatz (108) for a 4D FRW plus a 2D compact internal space background, we write the type A contribution, in terms of the 3D and 2D spaces Hubble parameters H, F and spatial curvatures k, κ:

$$E_{(6)} = -\frac{1}{48}\left\{\frac{\kappa}{b^2}(\dot H + H^2)\left(H^2 + \frac{k}{a^2}\right) + F^2(\dot H + H^2)\left(3H^2 + \frac{k}{a^2}\right) + 2(\dot F + F^2)\left(H^2 + \frac{k}{a^2}\right)\right\}. \qquad \text{(A3)}$$

The type B and D contributions have a more complicated form and we write them when it is necessary, in the specific limits we consider throughout the paper.

For the (0, 2) SCFT dual to the N M5 background, the anomaly coefficients are given by $c_A = c_B = 4N^3/\pi^3$ [72].

Counterterm traces The dual RS theory action contains the three counterterms S_1, S_2, S_3 written in (96). Varying these contributions w.r.t. the six dimensional induced metric $\gamma_{\mu\nu}$ on the brane yields [76,97]

$$T^{\rm ct}_{\mu\nu} = -2M^5\bigg(5\gamma_{\mu\nu} + \frac{1}{4}\bigg(R_{\mu\nu} - \frac{1}{2}R\gamma_{\mu\nu}\bigg) - \frac{1}{32}\bigg[-\Box R_{\mu\nu} + 2R_{\mu\sigma\nu\rho}R^{\rho\sigma} + \frac{2}{5}\nabla_\mu\nabla_\nu R - \frac{3}{5}RR_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\bigg(R_{\rho\sigma}R^{\rho\sigma} - \frac{3}{10}R^2 - \frac{1}{5}\Box R\bigg)\bigg] - T^a_{\mu\nu}\log\epsilon\bigg), \tag{A4}$$

where $T^a_{\mu\nu}$ is a traceless tensor of cubic order in the curvature. The trace of the conformal variation of S_1 (corresponding to the linear part of (A4) in the curvature) gives a term proportional to the Einstein tensor. The variation of the $S_3 \propto \int\sqrt{-\hat{g}_{(0)}}a_{(6)}$ action (related to $T^a_{\mu\nu}$, where we introduced $a_{(3)}$ as in the standard notation of [73]) is traceless because it is proportional [76] to the traceless tensor $h_{(6)\mu\nu}$ that enters into the parametrization of the metric (91) due to Fefferman and Graham. However, the variation under conformal transformations of the cutoff dependent counterterm is nontrivial and is the only contribution to the conformal anomaly, so that $\mathcal{A}_{(6)} \propto a_{(6)}$ (see for instance [76,98] for a more detailed derivation). Finally, the trace of S_2 (equal to the trace of the quadratic contributions in (A4)) is

$$Y = \frac{1}{32}M^5\ell\bigg(R^{\mu\nu}R_{\mu\nu} - \frac{3}{10}R^2\bigg), \tag{A5}$$

which can be expressed in terms of the Hubble parameters H, F and of the spatial curvatures k, κ of the (4D FRW + 2D compact space) background (108) as

$$Y = -\frac{2M^5\ell}{160}\bigg\{-3\frac{k^2}{a^4} + 6\frac{k}{a^2}\bigg(3\frac{\kappa}{b^2} + F^2 + 8FH + 3H^2 + 6\dot{F} + 4\dot{H}\bigg) + 2\frac{\kappa}{b^4} + 2\frac{\kappa}{b^2}[-(F-6H)(F+3H) + \dot{F} + 9\dot{H}] - 3F^4 + 48F^3H + 6FH(21H^2 + 7\dot{F} + 13\dot{H}) + F^2(111H^2 - 4\dot{F} + 24\dot{H}) + 3[6H^2 - (\dot{F} - \dot{H})^2 + 2H^2(7\dot{F} + 3\dot{H})]\bigg\}. \tag{A6}$$

We define $c_Y \equiv M^5\ell/32M^2_{\rm Pl}$ wich is given as a function of the number N of M5-branes in the gravity background by $c_Y = \sqrt{2N^3/\pi^3}$.

APPENDIX B: FIXED POINTS IN THE HOLOGRAPHIC DESCRIPTION

In this appendix, we are going to look for the existence of inflationary points for our specific holographic model and to find what kind of dependence they have on the parameters of the theory. We will also study the stability matrix determining—in some special limits—whether the critical points are stable or saddles.

In the calculations, we suppose that the effective cosmological constant on the brane λ is null.

1. Flat compact extra dimensions

We start by considering the case of (locally) flat internal space, which could be, for example, a two-torus. The three spatial dimensions of the 4D FRW are already supposed to be flat, so that the system of equations of motion simplify, having dropped the terms proportional to both spatial curvatures.

The general flat extra dimension fixed points ($F_\star \neq 0$) are not easy to characterize. We choose to analyze the case in which the extra dimensions Hubble parameter is zero at the fixed point, meaning that the fixed point represents a universe with static flat extra dimensions.

a. $F_\star \neq 0$, $\omega \neq \frac{1}{5}$

Fixed point solutions Looking for the solution to the Friedmann plus conservation set of equations with constant Hubble parameters ($H \equiv H_\star$, $F \equiv F_\star$) and zero curvatures ($k = \kappa = 0$), we have to consider the simplified system of equations (where we have already solved the equation for σ)

$$M^4_{\rm Pl}(3H^2_\star + 6H_\star F_\star + F^2_\star) - [3\omega H_\star + (1-3\omega)F_\star](\tilde{\mathcal{A}}_{(6)\star} + \tilde{Y}_\star) = \lambda \tag{B1}$$

$$M^4_{\rm Pl}(3H^2_\star + 4H_\star F_\star + 3F^2_\star) - \omega(3H_\star + 2F_\star)(\tilde{\mathcal{A}}_{(6)\star} + \tilde{Y}_\star) = \lambda \tag{B2}$$

$$\sigma_\star = [3\omega H_\star + (1-3\omega)F_\star]\tilde{\mathcal{A}}_{(6)\star} = M^4_{\rm Pl}(3H^2_\star + 6H_\star F_\star + F^2_\star) - \lambda \tag{B3}$$

$$\sigma_{p\star} = -\omega(3H_\star + 2F_\star)(\tilde{\mathcal{A}}_{(6)\star} + \tilde{Y}_\star) = -M^4_{\rm Pl}(3H^2_\star + 4H_\star F_\star + 3F^2_\star) + \lambda \tag{B4}$$

$$\rho_\star = 0, \qquad \chi_\star = 0, \tag{B5}$$

where the relation between Ω and ω was defined to be $1/\omega = 2\Omega + 3$ and the trace contributions $\mathcal{A}_{(6)\star}$, $Y_\star$ take the form

$$\mathcal{A}_{(6)\star} = \frac{c_A}{48}[2F^3_\star H^3_\star + 3F^2_\star H^4_\star] + \frac{c_B}{4800}[12F^6_\star - 128F^5_\star H_\star + 291F^4_\star H^2_\star + 184F^3_\star H^3_\star + 557F^2_\star H^4_\star + 138FH^5_\star - 54H^6_\star] \tag{B6}$$

$$Y_\star = \frac{6}{5}c_Y M^2_{\rm Pl}(2H^2_\star + 2H_\star F_\star + F^2_\star) \times (3H^2_\star + 18H_\star F_\star - F^2_\star), \tag{B7}$$

and $\mathcal{A}_{(6)\star}$, $Y_\star$ have been defined as

$$\tilde{\mathcal{A}}_{(6)\star} \equiv \mathcal{A}_{(6)\star}/[3(1+\omega)H_\star + 3(1-\omega)F_\star]$$
$$\tilde{Y}_\star \equiv Y_\star/[3(1+\omega)H_\star + 3(1-\omega)F_\star],$$

for $(1+\omega)H_\star + (1-\omega)F_\star \neq 0$.

For $F_\star \neq 0$ and $\omega \neq 1/5$, we can reformulate Eqs. (B1) and (B2) in order to get

$$(3-21\omega)H_\star^2 + (4-30)F_\star H_\star - (3-11\omega)F_\star^2 = (1-5\omega)\frac{\lambda}{M_{\rm Pl}^4} \tag{B8}$$

$$(\tilde{\mathcal{A}}_{(6)\star} + \tilde{Y}_\star) = 2M_{\rm Pl}^4 \frac{H_\star - F_\star}{1-5\omega}. \tag{B9}$$

Imposing $\lambda = 0$ (no effective constant on the brane) greatly simplifies the solution since $H_\star \propto F_\star$. Under that assumption, defining $\mathcal{C}_\epsilon$ and $\mathcal{D}_\epsilon$—as functions of ω and $\epsilon = \pm 1$ ($\mathcal{D}_\epsilon$ is a function of the anomaly parameters c_A and c_B, of c_Y and the Planck mass as well)—such that $H_\star - F_\star = \mathcal{C}_\epsilon F$ and $(\tilde{\mathcal{A}}_{(6)\star} + \tilde{Y}_\star) = \mathcal{D}_\epsilon F_\star^5$, the solution takes the form

$$H_\star^2 = M_{\rm Pl}^2(\mathcal{C}_\epsilon + 1)^2\left[\frac{2\mathcal{C}_\epsilon}{(1-5\omega)\mathcal{D}_\epsilon}\right]^{1/2}, \quad F_\star^2 = M_{\rm Pl}^2\left[\frac{2\mathcal{C}_\epsilon}{(1-5\omega)\mathcal{D}_\epsilon}\right]^{1/2}. \tag{B10}$$

This solution exists for the values of ω such that $\mathcal{C}_\epsilon/\mathcal{D}_\epsilon > 0$ (for $\omega < 1/5$) or $\mathcal{C}_\epsilon/\mathcal{D}_\epsilon < 0$ (for $\omega > 1/5$).

The CFT energy density and pressures are then given by

$$\sigma_\star = M_{\rm Pl}^6(1 + 3\omega\mathcal{C}_\epsilon)\mathcal{D}_\epsilon\left[\frac{2\mathcal{C}_\epsilon}{(1-5\omega)\mathcal{D}_\epsilon}\right]^{3/2} \tag{B11}$$

$$\sigma_{p\star} = \frac{2\omega}{1-3\omega}\sigma_\pi = -M_{\rm Pl}^6\omega(5 + 3\mathcal{C}_\epsilon)\mathcal{D}_\epsilon\left[\frac{2\mathcal{C}_\epsilon}{(1-5\omega)\mathcal{D}_\epsilon}\right]^{3/2}, \quad \rho_\star = 0, \tag{B12}$$

(for $\omega = 1/3$ we have $\sigma_\pi = 0$).

b. $F_\star \neq 0$, $\omega = \frac{1}{5}$

Fixed point solutions To analyze the case $\omega = 1/5$, it is better to reformulate Eqs. (B1) and (B2) in the following way:

$$2M_{\rm Pl}^4(H_\star - F_\star)F_\star - (1-5\omega)F_\star(\tilde{\mathcal{A}}_{(6)\star} + \tilde{Y}_\star) = 0 \tag{B13}$$

$$M_{\rm Pl}^4(3H_\star^2 + 6H_\star F_\star + F_\star^2) - [3\omega H_\star + (1-3\omega)F_\star](\tilde{\mathcal{A}}_{(6)\star} + \tilde{Y}_\star) = \lambda. \tag{B14}$$

From the first equation we get $H_\star = F_\star$ and substituting in the second we find the equation for $H_\star$

$$-\frac{5}{288}(c_A + 2c_B)H_\star^6 - 120c_Y M_{\rm Pl}^2 H_\star^4 + 10M_{\rm Pl}^4 H_\star^2 = \lambda. \tag{B15}$$

For $\lambda = 0$ it has a nontrivial solution only if $(24c_Y)^2 > (c_A + 2c_B)$

$$H_\star^2 = F_\star^2 = -\frac{24}{c_A + 2c_B}M_{\rm Pl}^2\left[24c_Y \pm \sqrt{576c_Y^2 + (c_A + 2c_B)}\right]. \tag{B16}$$

The energy density and pressures are equal to

$$\sigma_\star = -\sigma_{p\star} = -\sigma_{\pi\star} = -\tfrac{240}{c_A+2c_B}M_{\rm Pl}^6\left[24c_Y \pm \sqrt{576c_Y^2 + (c_A + 2c_B)}\right], \quad \rho_\star = 0. \tag{B17}$$

c. $F_\star = 0$

Fixed point solutions Supposing instead $F_\star = 0$, the fixed point solution is

$$H_\star^2 = -\frac{20}{3c_B}\frac{\omega}{\omega+1}M_{\rm Pl}^2\left[48c_Y \pm \sqrt{6\left(384c_Y^2 - c_B\frac{\omega}{\omega+1}\right)}\right] \tag{B18}$$

$$\sigma_\star = -\sigma_{\pi\star} = \frac{2\omega}{3\omega - 1}\sigma_{\pi\star} = -\frac{20}{c_B}\frac{\omega}{\omega+1}M_{\rm Pl}^6\left[48c_Y \pm \sqrt{6\left(384c_Y^2 - c_B\frac{\omega}{\omega+1}\right)}\right], \quad \rho_\star = 0, \tag{B19}$$

for $\omega \neq -1$. This gives real Hubble parameter for $384c_Y^2 - c_B\omega/(\omega+1) > 0$. If $\omega = -1$, we find

$$H_\star^2 = \frac{640c_Y}{c_B}M_{\rm Pl}^2, \quad \sigma_\star = -\sigma_{p\star} = \frac{1}{2}\sigma_{\pi\star} = \frac{640c_Y}{3c_B}M_{\rm Pl}^6, \quad \rho_\star = 0. \tag{B20}$$

If the CFT is characterized by a positive c_B, there is no nontrivial critical point. For vanishing c_B the only fixed point with $\lambda \neq 0$ is the trivial one.

When $F_\star = -H(1+\omega)/(1-\omega)$, there is only one possible solution, for which the parameters must have the values: $\omega = -1$ (i.e. $\sigma_\pi = -2\sigma_p$), $c_B = 0$, $F_\star = 0$ and the fixed point is thus the one in (B20).

Stability analysis For both fixed points characterized by the zero value of the extra dimension Hubble parameter, i.e. both for $\omega \neq -1$ or $\omega = -1$, we must find the eigenvalues of a 4×4 matrix. In fact we have a third order linearized differential equation for the perturbation $\delta H(t)$ and a first order ODE for the energy density $\delta\rho$, while $\delta F(t)$ is found to be proportional to $\delta H(t)$ solving an algebraic equation

$$\delta H^{(3)} = -a_2\delta\ddot{H} - a_1\delta\dot{H} - a_0\delta H + c_0\delta\rho \qquad \text{(B21)}$$

$$\delta\dot{\rho} = -3(1+w)H_\star\delta\rho \qquad \text{(B22)}$$

$$\delta F = \alpha\delta H \qquad \text{(B23)}$$

The coefficients in the differential equations are functions of the anomaly parameters c_A, c_B, of the trace parameter c_Y, of ω and of $M_{\rm Pl}$.

The eigenvalues λ_1, λ_2, λ_3 are then given by the roots of the third degree polynomial

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \qquad \text{(B24)}$$

while $\lambda_4 = -3(1+w)H_\star < 0$. The coefficients a_0, a_1, a_2 are given by

$$a_0 = \frac{12}{25}\frac{8000M_{\rm Pl}^4(3+2\alpha+3(1+\alpha)\omega)}{c_BH_\star(1-\alpha)\omega} - (c_BH_\star^4(23\alpha-54) + 144c_YM_{\rm Pl}^2H_\star^2)\omega$$
$$a_1 = \frac{1}{25}\frac{960000M_{\rm Pl}^4(1+\alpha)}{c_BH_\star^2(1-\alpha)\omega} - (c_BH_\star^4(137\alpha-222) + 36c_YM_{\rm Pl}^2H_\star^2)\omega$$
$$a_2 = \frac{7-6\alpha}{1-\alpha}H_\star, \qquad \text{(B25)}$$

where

$$\alpha = -3\frac{800M_{\rm Pl}^4(1+\omega) - (-3c_BH_\star^4 + 480c_YM_{\rm Pl}^2H_\star^2)\omega}{800M_{\rm Pl}^4(5-3\omega) - (-3c_BH_\star^4 + 480c_YM_{\rm Pl}^2H_\star^2)(1-3\omega)}. \qquad \text{(B26)}$$

The sign of the eigenvalues λ_1, λ_2, λ_3 ($\lambda_3 = \bar{\lambda}_2$ iff $27a_0^2 + 4a_1^3 - 18a_0a_1a_2 - a_1^2a_2^2 + 4a_0a_2^3 > 0$, otherwise we get three real roots) determines the nature of the fixed point. Since $\lambda_4 < 0$, we find that we can only have a completely stable fixed point or a saddle. In the case of one real and two complex conjugated roots the critical point can be attractive only if $a_2 > 0$ and

$$-\frac{2a_2}{3} < A + B < \frac{a_2}{3}, \qquad \text{(B27)}$$

where

$$A = \text{sgn}(R)(|R| + \sqrt{R^2 - Q^3})^{1/3}, \qquad B = \frac{Q}{A} \qquad \text{(B28)}$$

$$R \equiv \frac{1}{54}(2a_2^3 - 9a_1a_2 + 27a_0), \qquad Q \equiv \frac{1}{9}(a_2^2 - 3a_1). \qquad \text{(B29)}$$

When the three roots are real, they are negative (corresponding to an attractive fixed point) iff $a_0, a_1, a_2 > 0$. For the other values of a_0, a_1, a_2 the critical point is a saddle.

The coefficients of the linearized differential equation do not depend on the anomaly parameter c_A corresponding to the type A anomaly, so that only type B anomaly influences the characteristics of this fixed point.

2. Static compact extra dimensions

We analyze the set of differential equations when the extra dimensions are (locally) compactified on a sphere, i.e. $\kappa > 0$, supposing that the corresponding acceleration factor $b(t)$ remains constant, so that $F(t) \equiv 0$.

Beside the $H_\star = 0$ fixed points, we only have two acceptable time independent solutions to the Friedmann equations. The $H_\star = 0$, $\kappa \neq 0$ fixed points are always saddle points as we can conclude from the linear order analysis, since the eigenvalues of the stability matrix (or their real parts) are one opposite to the other.

$\omega \neq 0$

Fixed point solutions The energy density $\rho_\star$ is zero, $H_\star$ and $\sigma_\star$ are then determined by

$$(\mathcal{A}_{(6)\star} + Y_\star) \equiv \frac{c_A}{48}\frac{\kappa}{b_0^2}H_\star^4 - \frac{c_B}{4800}\left(54H_\star^6 - 98\frac{\kappa}{b_0^2}H_\star^4 + 42\frac{\kappa^2}{b_0^4}H_\star^2 - 6\frac{\kappa^3}{b_0^6}\right) + \frac{4c_Y}{5}M_{\rm Pl}^2\left(9H_\star^4 + 18\frac{\kappa}{b_0^2}\frac{\kappa^2}{b_0^4}\right) = \frac{1+\omega}{\omega}M_{\rm Pl}^4\left(3H_\star^2 + \frac{\kappa}{b_0^2}\right) \qquad \text{(B30)}$$

$$\sigma_\star = M_{\rm Pl}^4\left(3H_\star^2 + \frac{\kappa}{b_0^2}\right) \qquad \text{(B31)}$$

$$\rho_\star = \mathbf{0}. \qquad \text{(B32)}$$

Restricting the possible values of ω, we can obtain at least one positive root $H_\star^2$ of Eq. (B30), without having limitations on the anomaly coefficients c_A, c_B.

We can illustrate an example, choosing the simple case $c_B = 0$ (i.e. there is no contribution from the conformal invariants in the anomaly) and also $\omega = -1$, which simplifies the Eq. (B30). The fixed point is thus determined by

$$H_\star^2 = \frac{\kappa}{b_0^2}\left[9 \pm \sqrt{90 + \frac{5c_A}{192c_Y}\frac{1}{M_{\rm Pl}^2}\frac{\kappa}{b_0^2}}\right]^{-1} \quad \text{(B33)}$$

$$\begin{aligned}\sigma_\star &= -\sigma_{p\star} = 2\sigma_{\pi\star}\\ &= M_{\rm Pl}^4\left(3\left[9 \pm \sqrt{90 - \frac{5c_A}{192c_Y}\frac{1}{M_{\rm Pl}^2}\frac{\kappa}{b_0^2}}\right]^{-1} + 1\right)\frac{\kappa}{b_0^2}\\ \rho_\star &= 0,\end{aligned} \quad \text{(B34)}$$

which is real for $192c_Y^2M_{\rm Pl}^2 > -c_A\kappa/45b_0^2$. We can moreover have two distinct positive $H_\star$ fixed points if $5c_A\kappa/9b_0^2 < -192c_Y^2M_{\rm Pl}^2$.

Stability analysis We now analyze the $H_\star \neq 0$ fixed points behavior.

Regarding the fixed point determined by (B30), we get a negative eigenvalue $\lambda_4 = -3(1+w)H_\star$ (given that $w > -1$, $H_\star > 0$) and the other three are the roots of the third degree polynomial

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \quad \text{(B35)}$$

where $a_i = \tilde{a}_i/\tilde{a}_3$, $i = 0, 1, 2$ and

$$\begin{aligned}\tilde{a}_0 &= -\tfrac{c_A\omega}{12}\tfrac{\kappa}{b_0^2}H_\star^3 + \tfrac{c_B\omega}{1200}H_\star\left(81H_\star^4 - 98\tfrac{\kappa}{b_0^2}H_\star^2 + 21\tfrac{\kappa^2}{b_0^4}\right)\\ &\quad + \frac{144c_Y\omega}{5}M_{\rm Pl}^2\left(H_\star^2 + \frac{\kappa}{b_0^2}\right) + 6M_{\rm Pl}^4(1+\omega)H_\star,\\ \tilde{a}_1 &= -\frac{c_A\omega}{48}\frac{\kappa}{b_0^2}H_\star^2 + \frac{c_B\omega}{4800}\left(111H_\star^4 - 68\frac{\kappa}{b_0^2}H_\star^2 + 21\frac{\kappa^2}{b_0^4}\right)\\ &\quad + \frac{36c_Y\omega}{5}M_{\rm Pl}^2\left(H_\star^2 + \frac{\kappa}{b_0^2}\right) + 2M_{\rm Pl}^4,\\ \tilde{a}_2 &= \frac{7c_B\omega}{1920}H_\star\left(H_\star^2 + \frac{\kappa}{b_0^2}\right),\\ \tilde{a}_3 &= \frac{c_B\omega}{1920}\left(H_\star^2 + \frac{\kappa}{b_0^2}\right).\end{aligned} \quad \text{(B36)}$$

We get a 4×4 stability matrix—despite the fact that we should have only 2 variables (H, ρ)—because the differential equations are of third order: the ρ eigenvalue is λ_4, but H is a superposition of the four modes corresponding to the four eigevalues of the matrix.

As in the previous analysis for the flat extra dimensions, the solutions $\lambda_{1,2,3}$ of the Eq. (B35) are such that $\lambda_1 \in \mathbb{R}$, $\lambda_3 = \bar{\lambda}_2$ or $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Besides, when we have the complex conjugated pair, there are only three possibilities:

(1) $\lambda_1, \Re(\lambda_2) = \Re(\lambda_3), \lambda_4 \leq 0 \Rightarrow$ the solution is stable (even if one of the eigenvalues are null, because that mode will not then contribute to the expression for H)

(2) $\lambda_1, \lambda_4 < 0, \Re(\lambda_2) = \Re(\lambda_3) > 0 \Rightarrow$ we get a saddle point

(3) $\lambda_1 > 0, \Re(\lambda_2) = \Re(\lambda_3), \lambda_4 < 0 \Rightarrow$ in this case too, the fixed point is a saddle

The equalities $\lambda_1 = 0$ and $\Re(\lambda_2) = \Re(\lambda_3)$ in the case (i) are possible, but not simultaneously. When the roots are all real we can get a stable point iff a_0, a_1, and $a_2 > 0$, which implies λ_1, λ_2, and $\lambda_3 < 0$, and a saddle otherwise, with one negative two positive, or two negative one positive roots.

3. Equal scale factors

Another limit that simplifies some of the calculations is the equal scale factor assumption. In this case the Hubble parameters of the internal space and the 3D space are equal, $F = H$, and we remain with a set of equations for the variables H, ρ, σ, as in the static extra dimension limit.

$\omega \neq 0$

Fixed point solutions We observe that when $\omega = 0$ there are no acceptable solutions to the time independent Einstein equations. So, we want to find the fixed points in the $H = F$, $\omega \neq 0$ limit. The time independent Friedmann plus conservation equations lead to

$$\begin{aligned}(\mathcal{A}_{(6)\star} + Y_\star) &\equiv \frac{5}{48}(c_A + 2c_B)H_\star^4\left(H_\star^2 + \frac{\kappa}{a^2}\right)\\ &\quad - \frac{1}{192}c_B\frac{\kappa^2}{a^4}H_\star^2 + \frac{1}{800}c_B\frac{\kappa^3}{a^6}\\ &\quad + \frac{4c_Y}{5}M_{\rm Pl}^2\left(150H_\star^4 + 20\frac{\kappa}{b_0^2}H_\star^2 - \frac{\kappa^2}{b_0^4}\right)\\ &= 6M_{\rm Pl}^4\left(10H_\star^2 + \frac{\kappa}{a^2}\right)\end{aligned} \quad \text{(B37)}$$

$$\sigma_\star = M_{\rm Pl}^4\left(10H_\star^2 + \frac{\kappa}{a^2}\right) \quad \text{(B38)}$$

$$\rho_\star = \mathbf{0}, \quad \text{(B39)}$$

provided that $5 + 3w + 2w_\pi \neq 0$ [99]. As a consequence of solving the system of equations we also obtain a constraint on the CFT pressure of the hidden sector σ_π, as we must impose $\omega = 1/5$, i.e. $\sigma_\pi = \sigma_p$. The Eq. (B38) yields the value of the Hubble parameter at the fixed point as a function of c_A, c_B, c_Y, $M_{\rm Pl}$, κ.

In the case of flat extra dimensions $\kappa = 0$ we immediately solve the system of equations finding (discarding the trivial $H_\star = 0$ solution)

$$\begin{aligned}H_\star^2 &= -\frac{24}{c_A + 2c_B}M_{\rm Pl}^2\left[24c_Y \pm \sqrt{576c_Y^2 + (c_A + 2c_B)}\right]\\ \sigma_\star &= -\sigma_{p\star} = -\sigma_{\pi\star}\\ &= -\frac{240}{c_A + 2c_B}M_{\rm Pl}^6\left[24c_Y \pm \sqrt{576c_Y^2 + (c_A + 2c_B)}\right],\\ \rho_\star &= 0.\end{aligned} \quad \text{(B40)}$$

The solution is acceptable if $-(c_A - 2c_B) < (24c_Y)^2$ and the two roots are both positive when $(c_A - c_B) < 0$.

Stability analysis The last situation that we are considering is the case of equal scale factors, as in the fixed point analysis. We only found one fixed point with $F = H$, that entails a relation between the two CFT pressures $\sigma_\pi = \sigma_p$ ($\omega = 1/5$). We could thus calculate the stability matrix eigenvalues corresponding to this particular limit.

When the extra dimensions spatial curvature is zero $\kappa = 0$, in addition to the vanishing 3D curvature ($k = 0$), the stability matrix can be studied straightforward. All the eigenvalues are coincident since $\delta H \propto \delta\sigma \propto \delta\rho$. They are given by

$$\lambda = -(5 + 3w + 2w_\pi)H_\star < 0. \tag{B41}$$

For $H_\star > 0$ the fixed point is hence stable.

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approach [82,83].
[88] As examples of negative $\tilde{c}_{V,d}$ and $(d - w_d)$, both negativeness conditions can be satisfied if $w = 1/3$ for $w_\pi > 8/9$ when $H = F$, but never when $F = 0$. If $w = 0$ we must have $w_\pi > 1/2$ in the equal scale factors limit, while no solution can be found with static compact extra dimensions. We instead get positive $\tilde{c}_V$ and $(d - w_d)$ if $w = 0$ for $-5/6 < w_\pi < 1/2$ with $H = F$ and for $w_\pi > 1/2$ with $F = 0$. No value of w_π satisfies the positiveness conditions if $w = 1/3$.
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[94] It is indeed proportional to the traceless tensor $h_{(6)}$ that appears in the Fefferman and Graham metric parametrization (108) [76]. We thank K. Skenderis for helpful discussion on this subject.
[95] The energy density $\rho(t)$ should not be confused with the 2D coordinate ρ, since the radius of the extra dimensions does not appear in the calculations.
[96] This limit comes from classical evaluation of stress-energy tensor derived from the action $S \propto \int d^6x\sqrt{-\gamma}H_{\mu\nu\rho}H^{\mu\nu\rho}$.
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[99] If the reasonable range of values for the pressures is given by w, $w_\pi \geq -1$, this condition is satisfied whenever w (or w_π) is strictly greater than -1 and, since we are interested in fixed points with a general $w \neq -1$, we will assume that this is the case.