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Quantum corrections to energy of short spinning string in AdS_5

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Motivated by a desire to shed light on the strong-coupling behavior of dimensions of short gauge-theory operators, we consider the famous example of folded spinning string in AdS_5 in the limit of small semiclassical spin parameter $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$. In this limit the string becomes short and is moving in a near-flat central region of AdS_5. Its energy scales with spin as $E = \sqrt{2S}[a_0 + a_1 \mathcal{S} + a_2 \mathcal{S}^2 + \ldots]$. We explicitly compute the leading 1-loop quantum $\mathrm{AdS}_5 \times S^5$ superstring correction to the short string energy and show that the coefficient a_0 is not renormalized from its classical value while a_1 receives a nontrivial contribution containing $\zeta(3)$.

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I. INTRODUCTION

The remarkable progress achieved recently in uncovering the integrable structure underlying the spectrum of planar $\mathcal{N} = 4$ SYM theory or the free $\mathrm{AdS}_5 \times S^5$ superstring theory was largely limited to a sector of gauge-theory operators with large number of fields/derivatives or strings with large values of quantum numbers like spins. It is important to try to learn more about dimensions/energies of short operators/strings and a step in that direction is to study quantum corrections to energies of strings carrying parametrically small values of spins.

With this motivation in mind here we revisit the computation of the 1-loop quantum correction to the energy of the prototypical example of rotating string-folded rotating string located at the center of AdS_5 [1,2].

The classical energy of this string is proportional to string tension, i.e. $E_0 = \sqrt{\lambda}\mathcal{E}(\mathcal{S})$, $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$ and in the limit of large $\mathcal{S}$ one finds [2]: $E_0 = S + \frac{\sqrt{\lambda}}{\pi}\ln S + \ldots$. In general, the radial coordinate ρ of the global AdS_5 space ($ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2$) is expressed in terms of an elliptic function of the spatial string coordinate σ and thus finding the explicit form of the 1-loop correction [3] to the energy E_1 of this soliton solution of 2d string sigma model appears to be technically challenging. The analytic form of the quantum correction can be found in the limit of large $\mathcal{S}$ when the ends of the string reach the boundary of the AdS_5. Then the solution drastically simplifies (ρ becomes linear in σ) [3,4] and one finds that $E_1 = c_1 \ln S + \ldots, c_1 = -3\ln 2$.

Since rotation of the string balances the contracting effect of its tension, smaller values of the spin correspond to smaller values of the length of the string whose center of mass is at $\rho = 0$: $\mathcal{S}$ essentially measures the length of the string. Since the AdS_5 space is nearly flat at the vicinity of $\rho = 0$, the slowly rotating (i.e. small) string with $\mathcal{S} \ll 1$ should have essentially the same classical energy as in flat space [2], i.e. $E_0 = \sqrt{2\sqrt{\lambda}S} + \ldots$.

Below we shall expand the general expression for the 1-loop correction to the energy of the spinning string in [3] (given by a sum of logarithms of determinants of the 2d second order differential operators depending on the string background) in the "short string" limit $\mathcal{S} \ll 1$ and find explicitly the coefficients of the first two leading terms in the small spin expansion of the 1-loop energy.

Our results can be summarized as follows. Given the energy $E(S, \lambda)$ of the corresponding state in the AdS/CFT spectrum we may expand it at large λ with $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$ fixed, i.e. in the semiclassical string limit. Expanding *then* in the limit $\mathcal{S} \ll 1$, i.e. $S \ll \sqrt{\lambda}$, and reexpressing E as a function of S and λ one is to find

$$E(S, \lambda) = \lambda^{1/4}\sqrt{2S}[h_0(\lambda) + h_1(\lambda)S + h_2(\lambda)S^2 + \ldots], \tag{1.1}$$

$$h_n = \frac{1}{(\sqrt{\lambda})^n}\left(a_{n0} + \frac{a_{n1}}{\sqrt{\lambda}} + \frac{a_{n2}}{(\sqrt{\lambda})^2} + \ldots\right), \qquad \lambda \gg 1, \qquad \frac{S}{\sqrt{\lambda}} = \text{fixed} \ll 1. \tag{1.2}$$

In the classical string theory limit

$$a_{00} = 1, \qquad a_{10} = \tfrac{3}{8}, \qquad a_{20} = -\tfrac{21}{128}, \ldots \tag{1.3}$$

while our 1-loop string computation gives

$$a_{01} = 1, \qquad a_{11} = \tfrac{13}{64} - \tfrac{1}{2}\zeta(3) \approx -0.398. \tag{1.4}$$

Since the leading $\sqrt{2S}$ term is essentially the same as in the

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flat-space string theory, is natural to conjecture that $h_0(\lambda) = 1$ to all orders in string coupling expansion.[1] This is indeed confirmed by our 1-loop computation. Then (1.1) can be written as

$$E(S,\lambda) = \lambda^{1/4}\sqrt{2S}\left[1 + \left(a_{10} + \frac{a_{11}}{\sqrt{\lambda}} + \ldots\right)\frac{S}{\sqrt{\lambda}} + O(S^2)\right]. \quad (1.5)$$

In contrast to the large spin or "long string" limit where the limits of large λ and large S appear to commute[2] (and thus one finds the same S dependence of the gauge-theory anomalous dimension and string theory energy at both weak and strong coupling, i.e. $E = S + f(\lambda)\ln S + \ldots$, with $f(\lambda \ll 1) = c_1\lambda + c_2\lambda^2 + \ldots$, $f(\lambda \gg 1) = \sqrt{\lambda}(b_0 + \frac{b_1}{\sqrt{\lambda}} + \ldots)$) here one cannot directly continue (1.1) to small λ and small S.

Indeed, the anomalous dimensions of low-twist gauge-theory operators like $\mathrm{tr}(\Phi D_+^S \Phi)$ computed for small λ and fixed S (see, e.g., [5]) and then formally expanded in small S limit scale as

$$E(\lambda, S) = q_1(\lambda)S + q_2(\lambda)S^2 + O(S^3), \qquad \lambda \ll 1, \quad S = \text{fixed}, \quad (1.6)$$

where

$$q_1(\lambda) = 1 + d_{01}\lambda + d_{02}\lambda^2 + \ldots, \quad q_2(\lambda) = d_{21}\lambda + d_{22}\lambda^2 + \ldots, \ldots. \quad (1.7)$$

To relate the "small spin" string theory (1.1) and gauge-theory (1.6) expansions one would need to resum the series in both arguments (λ, S), e.g., first sum up the weak-coupling expansion in (1.6) and then reexpand the result first in large λ for fixed $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$ and then in small $\mathcal{S}$.

In view of the need of this resummation which is, in fact, a generic situation in comparing the semiclassical string theory and the perturbative gauge-theory expansions[3] it is not clear at the moment how to directly interpret our result (1.5) as strong-coupling limit of a gauge-theory anomalous dimension. One interesting question is if the conjectured nonrenormalization (in strong-coupling expansion) of the leading $\sqrt{S}$ term in (1.5) has some counterpart in the properties of the corresponding gauge-theory anomalous dimensions.

We shall start in Sec. II with a review of the folded spinning string solution and its small spin expansion [2].

In Sec. III we shall first recall the general expression for the quadratic fluctuation Lagrangian $\tilde{L}$ [3] of the $AdS_5 \times S^5$ superstring [11] near the folded spinning string solution. We shall then expand the coefficients in $\tilde{L}$ in the small spin or short string parameter $\epsilon = \sqrt{2\mathcal{S}} + \ldots$. This expansion may be viewed as a particular case of a near flat space expansion of the quantum $AdS_5 \times S^5$ superstring. We will then show that the leading $O(\epsilon)$ term in the 1-loop string energy vanishes.

In Sec. IV we shall expand the 2d determinants that enter the expression for the 1-loop partition function to first two leading orders in ϵ and compute the value of the first nonzero correction to the string energy, i.e. the coefficient a_{11} in (1.5).

In the appendix we shall discuss a generalization to the short string expansion of the folded spinning string solution which also carries a momentum J in S^5 [3].

II. SHORT STRING LIMIT OF FOLDED SPINNING STRING SOLUTION

Let us start with a review of the classical solution for the folded string spinning in the AdS_3 part of AdS_5,

$$t = \kappa\tau, \qquad \phi = w\tau, \qquad \rho = \rho(\sigma), \quad ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2, \quad (2.1)$$

where

$$\rho'^2 = \kappa^2\cosh^2\rho - w^2\sinh^2\rho. \quad (2.2)$$

ρ varies from 0 to its maximal value ρ_*

$$\coth^2\rho_* = \frac{w^2}{\kappa^2} \equiv 1 + \frac{1}{\epsilon^2}. \quad (2.3)$$

Thus ϵ measures the length of the string. The solution of the differential equation (2.2), i.e.

$$\rho' = \pm\kappa\sqrt{1 - \epsilon^{-2}\sinh^2\rho}, \qquad \rho(0) = 0 \quad (2.4)$$

can be written in terms of the Jacobi function sn

$$\sinh\rho = \epsilon\,\mathrm{sn}(\kappa\epsilon^{-1}\sigma, -\epsilon^2). \quad (2.5)$$

The periodicity in σ implies the following condition on the parameters [2]

$$\kappa = \epsilon\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; -\epsilon^2). \quad (2.6)$$

The classical energy $E_0 = \sqrt{\lambda}\mathcal{E}_0$ and the spin $S = \sqrt{\lambda}\mathcal{S}$

[1] In flat space superstring (treated, e.g., in the in light-cone gauge) the fluctuation Lagrangian is quadratic and any possible quantum shifts are actually canceled due to supersymmetry. Supersymmetry should also be behind the cancellation of higher-loop corrections to h_0 in $AdS_5 \times S^5$.

[2] The perturbative string theory and perturbative gauge-theory limits are actually different as limits of functions on the two-parameter space (λ, S): in string theory one assumes $\lambda \gg 1$ with $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$ fixed and then takes $\mathcal{S}$ large; in gauge theory one assumes $\lambda \ll 1$ with $\mathcal{S}$ fixed and then takes S large. However, this appears not to matter for the leading $\ln S$ term which can be described by a single universal interpolating function of λ (cusp anomaly).

[3] Analogous resummation is needed to compare the weak-coupling gauge-theory expansion for anomalous dimensions of sl(2) sector operators in the limit $\lambda \ll 1$ with $J \gg 1$, $S \gg 1$, $j = \frac{J}{\ln S}$ = fixed and $j < 1$ with the strong-coupling string theory expansion in the limit $\lambda \gg 1$ with $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$, $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$, $\ell \frac{\mathcal{J}}{\ln \mathcal{S}}$ = fixed and $\ell < 1$ (see [3,4,6–10]).

are found to be

$$\mathcal{E}_0 = \epsilon_2 F_1\Big(-\frac{1}{2},\frac{1}{2};1;-\epsilon^2\Big),$$
$$\mathcal{S} = \frac{\epsilon^2}{2}\sqrt{1+\epsilon^2}{}_2F_1\Big(\frac{1}{2},\frac{3}{2};2;-\epsilon^2\Big) \tag{2.7}$$

Here we will be interested in the short string limit $0 < \epsilon \ll 1$ in which

$$\rho_* = \epsilon - \tfrac{1}{6}\epsilon^3 + O(\epsilon^5). \tag{2.8}$$

In the strict limit $\epsilon = 0$ or $\kappa = 0$ we get $\rho = \rho_* = 0$, so that the string shrinks to a point with $E = 0$.[4]

From (2.7) we obtain in the small ϵ or the small $\mathcal{S}$ limit

$$\epsilon = \sqrt{2\mathcal{S}} - \frac{1}{4\sqrt{2}}\mathcal{S}^{3/2} + \ldots,$$
$$\mathcal{E}_0 = \sqrt{2\mathcal{S}} + \frac{3}{4\sqrt{2}}\mathcal{S}^{3/2} + \ldots, \tag{2.9}$$

so the short string limit corresponds to $\mathcal{S} \ll 1$ and the expansion of the energy looks like

$$E_0(S,\lambda) = \lambda^{1/4}\sqrt{2S} + \frac{3}{4\sqrt{2}}\lambda^{-1/4}S^{3/2} + O(S^{5/2}). \tag{2.10}$$

For the purpose of computing the 1-loop correction to the energy to order $O(S^{3/2})$ we will need the expression for $\rho(\sigma)$ to order ϵ^4. Expanding the exact solution (2.5) in powers of ϵ we obtain

$$\sinh\rho = \epsilon\sin\sigma - \frac{\epsilon^3}{4}\sin\sigma\cos^2\sigma + O(\epsilon^5) \tag{2.11}$$

Other useful expansions are

$$\kappa = \epsilon\Big(1 - \frac{\epsilon^2}{4} + \ldots\Big), \qquad w = 1 + \frac{\epsilon^2}{4} + \ldots,$$
$$\rho' = \epsilon\cos\sigma - \frac{\epsilon^3}{4}\cos^3\sigma + \ldots, \tag{2.12}$$

$$\kappa\sinh\rho = \epsilon^2\sin\sigma - \frac{\epsilon^4}{8}(3+\cos2\sigma)\sin\sigma + \ldots, \tag{2.13}$$

$$w\cosh\rho = 1 + \frac{\epsilon^2}{4}(1+2\sin^2\sigma) - \frac{\epsilon^4}{64}(8-\cos4\sigma) + \ldots. \tag{2.14}$$

[4]Note that in this limit the string disappears instead of reducing to a massless point particle with nonzero momentum moving along null geodesic. This corresponds in flat space to considering a massive string state in the rest frame (which is possible in covariant quantization). In contrast to the flat space case where adding a nonzero center of mass momentum can be achieved by a Lorentz boost, adding a motion of the spinning string center of mass in curved $AdS_5 \times S^5$ space is a nontrivial operation (different parts of the string move along different geodesics) leads in general to a new nontrivial configuration.

The above small spin expansion is an example of a near flat space expansion: the leading-order in ϵ solution can be identified with the folded spinning string solution in the flat space

$$t = \epsilon\tau, \qquad \rho = \epsilon\sin\sigma, \qquad \phi = \tau,$$
$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2, \tag{2.15}$$

where ϵ is an arbitrary constant amplitude. The energy and the spin then satisfy the usual flat-space Regge relation (we use string tension $T = \frac{\sqrt{\lambda}}{2\pi}$)

$$E_0 = \epsilon\sqrt{\lambda}, \qquad S = \frac{\epsilon^2}{2}\sqrt{\lambda}, \qquad \text{i.e.} \qquad \mathcal{E}_0 = \lambda^{1/4}\sqrt{2S}. \tag{2.16}$$

In the flat space case this is the exact expression for any value of S [cf. (2.10)] which also does not receive quantum corrections.

III. VANISHING OF 1- LOOP CORRECTION TO SHORT STRING ENERGY AT ORDER $\sqrt{S}$

Following [3] and expanding the $AdS_5 \times S^5$ superstring action [11] in conformal gauge to quadratic order in fluctuations near the folded spinning string solution one finds $\tilde{S} = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau \int_0^{2\pi} d\sigma \tilde{L}$ with the bosonic part

$$\begin{aligned}\tilde{L}_B = &-\partial_a\tilde{t}\partial^a\tilde{t} - \mu_t^2\tilde{t}^2 + \partial_a\tilde{\phi}\partial^a\tilde{\phi} + \mu_\phi^2\tilde{\phi}^2 \\ &+ 4\tilde{\rho}(\kappa\sinh\rho\,\partial_0\tilde{t} - w\cosh\rho\,\partial_0\tilde{\phi}) + \partial_a\tilde{\rho}\partial^a\tilde{\rho} \\ &+ \mu_\rho^2\tilde{\rho}^2 + \partial_a\beta_u\partial^a\beta_u + \mu_\beta^2\beta_u^2 + \partial_a\varphi\partial^a\varphi \\ &+ \partial_a\chi_s\partial^a\chi_s,\end{aligned} \tag{3.1}$$

where

$$\mu_t^2 = 2\rho'^2 - \kappa^2, \qquad \mu_\phi^2 = 2\rho'^2 - w^2,$$
$$\mu_\rho^2 = 2\rho'^2 - w^2 - \kappa^2, \qquad \mu_\beta^2 = 2\rho'^2. \tag{3.2}$$

Here β_u $(u = 1, 2)$ are two AdS_5 fluctuations transverse to the AdS_3 subspace in which the string is moving, while φ, χ_s $(s = 1, 2, 3, 4)$ are fluctuations in S^5. The fermionic part of the quadratic fluctuation Lagrangian can be put into the form [3]

$$\tilde{L}_F = 2i(\bar{\Psi}\gamma^a\partial_a\Psi - \mu_F\bar{\Psi}\Gamma_{234}\Psi), \qquad \mu_F^2 = \rho'^2, \tag{3.3}$$

and can be interpreted as describing a system of 4 + 4 2d Majorana fermions with σ-dependent mass μ_F.

We now expand the coefficients in this fluctuation Lagrangian in ϵ as discussed in the previous section. To leading order in ϵ we get

$$\mu_t^2 = \epsilon^2\cos2\sigma + \ldots,$$
$$\mu_\phi^2 = -1 + (\cos2\sigma + \tfrac{1}{2})\epsilon^2 + \ldots, \tag{3.4}$$

$$\mu_\rho^2 = -1 + (\cos 2\sigma - \tfrac{1}{2})\epsilon^2 + \ldots, \quad \mu_\beta^2 = 2\mu_F^2 = 2\epsilon^2 \cos^2\sigma + \ldots, \tag{3.5}$$

$$4\tilde{\rho}(\kappa \sinh\rho \partial_0 \tilde{t} - w \cosh\rho \partial_0 \tilde{\phi}) = \tilde{\rho}\{4\epsilon^2 \sin\sigma \partial_0 \tilde{t} - [4 + \epsilon^2(1 + 2\sin^2\sigma)]\partial_0 \tilde{\phi}\}. \tag{3.6}$$

If we set ϵ to zero we are back to the flat space case: indeed, the only two coupled modes that are not massless are then described by

$$\tilde{L}_0 = \partial_a \tilde{\phi} \partial^a \tilde{\phi} - \tilde{\phi}^2 - 4\tilde{\rho}\partial_0 \tilde{\phi} + \partial_a \tilde{\rho} \partial^a \tilde{\rho} - \tilde{\rho}^2, \tag{3.7}$$

which becomes the Lagrangian for two massless modes after a τ-dependent rotation

$$\tilde{\rho} = \eta_1 \cos\tau + \eta_2 \sin\tau, \qquad \tilde{\phi} = -\eta_1 \sin\tau + \eta_2 \cos\tau. \tag{3.8}$$

If we perform this rotation also at order ϵ^2 we get $\tilde{L} = \tilde{L}_0 + \epsilon^2 \tilde{L}_1 + O(\epsilon^4)$ where $\tilde{L}_0$ is the same as in flat space and a nontrivial part of the subleading term is[5]

$$\begin{aligned}\tilde{L}_1 = &- \cos 2\sigma \tilde{t}^2 + (\sin^2\tau + \cos^2\sigma)\eta_1^2 \\ &+ (\cos^2\tau + \cos^2\sigma)\eta_2^2 + 2(\eta_1 \cos\tau + \eta_2 \sin\tau)\dot{\tilde{t}} \sin\sigma \\ &- 2(\dot{\eta}_1 \cos\tau + \dot{\eta}_2 \sin\tau)\tilde{t} \sin\sigma \\ &+ 2(\eta_1 \sin\tau - \eta_2 \cos\tau)\tilde{t} \sin\sigma - \eta_1 \eta_2 \sin 2\tau \\ &- \eta_1 \dot{\eta}_2 (1 + 2\sin^2\sigma).\end{aligned} \tag{3.9}$$

One can then argue on general grounds that the leading ϵ^2 part of 1-loop correction to string energy should vanish. Indeed, the 1-loop correction to string energy will look like (assuming all propagators were diagonalized)[6]

$$\Gamma_1 = \frac{1}{2}\sum_i (-1)^{n_i} \ln \frac{\det[\partial_0^2 - \partial_1^2 + \epsilon^2 M_i^2]}{\det[\partial_0^2 - \partial_1^2]} \sim \epsilon^2 \int d\tau d\sigma \, \mathrm{Tr} \sum_i (-1)^{n_i} M_i^2 + O(\epsilon^4). \tag{3.10}$$

Since $t = \kappa\tau$, $\kappa = \epsilon + \ldots$ the 1-loop correction to string energy is given by

$$E_1 = \frac{1}{\kappa \mathcal{T}}, \qquad \mathcal{T} \equiv \int d\tau \to \infty. \tag{3.11}$$

Here M_i^2 may be nontrivial matrices which depend on τ, σ. We are now to recall that the 1-loop logarithmic UV divergencies in the $AdS_5 \times S^5$ superstring action expanded near an arbitrary string solution manifestly cancel in the conformal gauge [3,12]. The nontrivial UV logarithmic divergencies have as their coefficient precisely the sum of the mass squared terms in the r.h.s. of (3.10); it vanishes for a generic on-shell string background, thus implying the absence of the ϵ^2 term in the 1-loop string partition function.

Let us now verify this by a direct computation. For the contribution of the β_u fields we get (rotating to euclidean space, $\tau \to i\tau$, and factorizing the infinite time interval $\mathcal{T}$)

$$\det[-\partial_1^2 - \partial_0^2 + 2\epsilon^2 \cos^2\sigma] = \mathcal{T} \int \frac{d\omega}{2\pi} \det[-\partial_1^2 + \omega^2 + 2\epsilon^2 \cos^2\sigma]. \tag{3.12}$$

We can now use perturbation theory in ϵ^2, i.e.

$$\ln \frac{\det[A + \epsilon^2 B]}{\det A} = \epsilon^2 \, \mathrm{Tr}[A^{-1}B] + O(\epsilon^4). \tag{3.13}$$

Then to order ϵ^2 [here $\sigma \in (0, 2\pi)$]

$$\ln \frac{\det[-\partial_1^2 + \omega^2 + 2\epsilon^2 \cos^2\sigma]}{\det[-\partial_1^2 + \omega^2]} \approx \epsilon^2 \sum_n \frac{2}{n^2 + \omega^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2\sigma = \epsilon^2 \sum_n \frac{1}{n^2 + \omega^2}. \tag{3.14}$$

Similarly, the ϵ^2 contribution of the fermionic modes is proportional to

$$\ln \frac{\det[-\partial_1^2 + \omega^2 + \epsilon^2 \cos^2\sigma]}{\det[-\partial_1^2 + \omega^2]} \approx \epsilon^2 \sum_n \frac{1}{n^2 + \omega^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2\sigma = \frac{\epsilon^2}{2} \sum_n \frac{1}{n^2 + \omega^2}. \tag{3.15}$$

The nontrivial part of the total Euclidean partition function[7]

$$Z = \frac{\det^{8/2}[-\partial_0^2 - \partial_1^2 + \epsilon^2 \cos^2\sigma] \det^{2/2}[-\partial_0^2 - \partial_1^2]}{\det^{2/2}[-\partial_0^2 - \partial_1^2 + 2\epsilon^2 \cos^2\sigma] \det^{5/2}[-\partial_0^2 - \partial_1^2] \det^{1/2} Q} \tag{3.16}$$

involves the operator Q on the space of the three mixed fluctuations $\tilde{\rho}$, $\tilde{\phi}$, $\tilde{t}$ in (3.1)

[5] We shall not use this τ-dependent form of the fluctuation Lagrangian for explicit computations below.

[6] To cancel the leading flat space term, i.e. to ensure that the total number of effective degrees of freedom is zero, one is of course to include also the conformal gauge ghost contribution.

[7] Here we choose not to rotate $\tilde{t} \to i\tilde{t}$ to make all fluctuations having physical norm but this can be easily done at any stage of what follows; we shall assume this rotation in the free (flat) part of the partition function.

$$Q = \begin{pmatrix} \partial_0^2 + \partial_1^2 - \epsilon^2 \cos 2\sigma & 0 & -2i\epsilon^2 \sin\sigma \partial_0 \\ 0 & -\partial_0^2 - \partial_1^2 - 1 + \epsilon^2(\frac{1}{2} + \cos 2\sigma) & 2i\partial_0 + i\epsilon^2(\frac{1}{2} + \sin^2\sigma)\partial_0 \\ 2i\epsilon^2 \sin\sigma\partial_0 & -2i\partial_0 - i\epsilon^2(\frac{1}{2} + \sin^2\sigma)\partial_0 & -\partial_0^2 - \partial_1^2 - 1 - \epsilon^2(\frac{1}{2} - \cos 2\sigma) \end{pmatrix}$$

Since there is no explicit τ dependence in the functional determinants we can write the 1-loop correction as

$$\Gamma_1 = -\ln Z_1 = -\frac{\mathcal{T}}{4\pi}\int_{-\infty}^{\infty} d\omega \ln\frac{\det^8[-\partial_1^2 + \omega^2 + \epsilon^2\cos^2\sigma]}{\det^2[-\partial_1^2 + \omega^2 + 2\epsilon^2\cos^2\sigma]\det^3[-\partial_1^2 + \omega^2]\det[Q_\omega]}, \tag{3.17}$$

where $Q_\omega = Q(\partial_0 \to i\omega)$.

Let us expand: $Q_\omega = Q_\omega^{(0)} + \epsilon^2 Q_\omega^{(2)} + \ldots$, where

$$Q_\omega^{(0)} = \begin{pmatrix} -(-\partial_1^2 + \omega^2) & 0 & 0 \\ 0 & -\partial_1^2 + \omega^2 - 1 & -2\omega \\ 0 & 2\omega & -\partial_1^2 + \omega^2 - 1 \end{pmatrix}, \tag{3.18}$$

$$Q_\omega^{(2)} = \begin{pmatrix} -\cos 2\sigma & 0 & 2\omega\sin\sigma \\ 0 & \cos 2\sigma + \frac{1}{2} & -\omega(\frac{1}{2} + \sin^2\sigma) \\ -2\omega\sin\sigma & \omega(\frac{1}{2} + \sin^2\sigma) & \cos 2\sigma - \frac{1}{2} \end{pmatrix}. \tag{3.19}$$

$$P_\omega = \begin{pmatrix} -(-\partial_1^2 + \omega^2) & 0 & 0 \\ 0 & -\partial_1^2 + \omega^2 & 0 \\ 0 & 0 & -\partial_1^2 + \omega^2 \end{pmatrix}, \tag{3.20}$$

the remaining nontrivial part of Γ_1 which was not already computed is given by

$$\frac{\mathcal{T}}{4\pi}\int d\omega\left(\ln\frac{\det[Q_\omega]}{\det[Q_\omega^{(0)}]} - \ln\frac{\det[P_\omega]}{\det[Q_\omega^{(0)}]}\right). \tag{3.21}$$

The second term here vanishes for the same reason why the rotation in (3.8) lead to the standard massless kinetic terms for the two originally coupled modes and thus to the trivial flat-space partition function. Indeed, the "mixed" 2 by 2 block contribution to $\ln\det[Q_\omega^{(0)}]$ can be written as $\ln\det[-\partial_1^2 + (\omega + i)^2] + \ln\det[-\partial_1^2 + (\omega - i)^2]$. Under the integral over ω one can then shift ω by $-i$ in one term and by $+i$ in another to get the cancellation against other massless determinants. These separate shifts are thus consistent with the trivial (supersymmetric) result for Γ_1 in flat space, and we shall perform similar shifts of the corresponding terms in what follows [in particular in $\det[Q_\omega^{(0)}]$ contribution of the first term in (3.21)].

To compute the first term in (3.21) we expand in ϵ as in (3.13)

$$\ln\frac{\det[Q_\omega]}{\det[Q_\omega^{(0)}]} = \epsilon^2\,\mathrm{Tr}[(Q_\omega^{(0)})^{-1}Q_\omega^{(2)}] + \ldots$$
$$= \epsilon^2\sum_n\int_0^{2\pi}\frac{d\sigma}{2\pi}(Q_\omega^{(0)})^{-1}_{ij}(Q_\omega^{(2)})_{ji} + \ldots. \tag{3.22}$$

The momentum-space propagator corresponding to $Q_\omega^{(0)}$ is

$$(Q_\omega^{(0)})^{-1} = \begin{pmatrix} -\frac{1}{n^2+\omega^2} & 0 & 0 \\ 0 & \frac{n^2+\omega^2-1}{n^4+2n^2(\omega^2-1)+(\omega^2+1)^2} & \frac{2\omega}{n^4+2n^2(\omega^2-1)+(\omega^2+1)^2} \\ 0 & -\frac{2\omega}{n^4+2n^2(\omega^2-1)+(\omega^2+1)^2} & \frac{n^2+\omega^2-1}{n^4+2n^2(\omega^2-1)+(\omega^2+1)^2} \end{pmatrix}. \tag{3.23}$$

It can be diagonalized by a rotation

$$M^{-1}(Q_\omega^{(0)})^{-1}M \equiv D_\omega^{(0)} = \begin{pmatrix} -\frac{1}{n^2+\omega^2} & 0 & 0 \\ 0 & \frac{1}{n^2+(\omega+i)^2} & 0 \\ 0 & 0 & \frac{1}{n^2+(\omega-i)^2} \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{i}{2} & -\frac{i}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{3.24}$$

$Q_\omega^{(2)}$ gets rotated into

$$M^{-1}Q_\omega^{(2)}M \equiv D_\omega^{(2)} = \begin{pmatrix} -\cos 2\sigma & \omega\sin\sigma & \omega\sin\sigma \\ -2\omega\sin\sigma & i\omega(1-\frac{1}{2}\cos 2\sigma)+\cos 2\sigma & -\frac{1}{2} \\ -2\omega\sin\sigma & -\frac{1}{2} & -i\omega(1-\frac{1}{2}\cos 2\sigma)+\cos 2\sigma \end{pmatrix}$$

and the ϵ^2 term in (3.22) becomes

$$\epsilon^2\sum_n\int_0^{2\pi}\frac{d\sigma}{2\pi}D_\omega^{(0)}D_\omega^{(2)} = \epsilon^2\sum_n\Big[\frac{i\omega}{n^2+(\omega+i)^2}-\frac{i\omega}{n^2+(\omega-i)^2}\Big]. \quad (3.25)$$

Thus finally

$$E_1 = \frac{\Gamma_1}{\kappa\mathcal{T}} = -\frac{\epsilon}{4\pi}\int_{-\infty}^{\infty}d\omega\sum_n\Big[\frac{2}{n^2+w^2}-\frac{i\omega}{n^2+(\omega+i)^2}+\frac{i\omega}{n^2+(\omega-i)^2}\Big]+O(\epsilon^3). \quad (3.26)$$

Doing the opposite shifts of ω in each of the last two terms we conclude that the order ϵ term in E_1 indeed vanishes, i.e.

$$E_1 = 0 + O(\epsilon^3). \quad (3.27)$$

The above formal argument overlooks an important subtlety of IR divergences that we have so far postponed to discuss but which will become crucial at the next order in ϵ studied in the following section. Indeed, if the sum over n in (3.25) runs over all values from $-\infty$ to $+\infty$ one may get different results by interchanging the order of integration over ω and summation over n: the integral over ω has an IR divergence at $n = 0$.

In fact, as in the usual perturbative expansion near a soliton, there is an issue of possible IR singularities due to a zero mode associated to the translational symmetry $\sigma \to \sigma + \sigma_0$. In the present case of expansion in ϵ the "free" propagator is essentially the massless one on $R\times S^1$ and thus the zero mode that is not damped in the path integral corresponds to $n = 0$. Its contribution can be either regularized by introducing a small mass or $i\epsilon$ in the propagator as in [13] or by isolating the modes constant in σ in the path integral and thus not including the $n = 0$ contributions in the propagators (as is done, e.g., in quantizing a sigma model on a compact 2d space). This is the prescription we shall adopt here, i.e. the sums over n in (3.14), (3.15), (3.25), and (3.26) will be understood not to include the $n = 0$ term.

IV. 1- LOOP CORRECTION TO THE $S^{3/2}$ TERM IN THE STRING ENERGY

After a warm-up in the previous section we are now ready to compute the first nontrivial 1-loop correction to the short string energy: the coefficient a_{11} of the $S^{3/2}$ term in (1.1) or (1.5). For that we shall consider the next order of the near flat space or $\epsilon \to 0$ expansion of the fluctuation Lagrangian (3.1) and (3.3). As in (3.13) we shall use that

$$\ln\frac{\det[A+\epsilon^2B+\epsilon^4C]}{\det A} = \epsilon^2\,\mathrm{Tr}[A^{-1}B]-\frac{\epsilon^4}{2}\mathrm{Tr}[A^{-1}BA^{-1}B]+\epsilon^4\,\mathrm{Tr}[A^{-1}C] + O(\epsilon^6) \quad (4.1)$$

Expanding the fluctuation Lagrangian in ϵ using (2.11), etc., we get

$$\tilde L = \tilde L_0 + \epsilon^2\tilde L_1 + \epsilon^4\tilde L_2 + \ldots, \quad (4.2)$$

where the ϵ^4 terms in the masses and the mixing terms are

$$\delta\mu_t^2 = \epsilon^4(\tfrac{1}{2}-\cos^4\sigma), \qquad \delta\mu_\phi^2 = \epsilon^4(\tfrac{5}{32}-\cos^4\sigma),$$
$$\delta\mu_\rho^2 = \epsilon^4(\tfrac{21}{32}-\cos^4\sigma), \qquad \delta\mu_\beta^2 = -\epsilon^4\cos^4\sigma,$$
$$\delta\mu_F^2 = -\tfrac{1}{2}\epsilon^4\cos^4\sigma,$$
$$\delta[4\tilde\rho(\kappa\sinh\rho\,\partial_0\tilde t - w\cosh\rho\,\partial_0\tilde\phi)] = -\tfrac{1}{2}\epsilon^4\tilde\rho[(3+\cos 2\sigma)\sin\sigma\,\partial_0\tilde t-(1-\tfrac{1}{8}\cos 4\sigma)\partial_0\tilde\phi\Big] \quad (4.3)$$

Let us first compute the ϵ^4 contribution to 1-loop effective action coming from the terms like $\epsilon^4\,\mathrm{Tr}[A^{-1}C]$ in (4.1). Going to momentum space in the (Euclidean) world-sheet time direction ($\partial_0 \to i\omega$) the operator Q acting on the $\tilde t$, $\tilde\rho$, $\tilde\phi$ subspace can be expanded as [cf. (3.18) and (3.19)]

$$Q_\omega = Q_\omega^{(0)} + \epsilon^2 Q_\omega^{(2)} + \epsilon^4 Q_\omega^{(4)} + \ldots, \quad (4.4)$$

$$Q_\omega^{(4)} = \begin{pmatrix} -(\frac{1}{2}-\cos^4\sigma) & 0 & -\frac{\omega}{4}(3+\cos 2\sigma)\sin\sigma \\ 0 & \frac{5}{32}-\cos^4\sigma & -\frac{\omega}{32}(\cos 4\sigma-8) \\ \frac{\omega}{4}(3+\cos 2\sigma)\sin\sigma & \frac{\omega}{32}(\cos 4\sigma - 8) & \frac{21}{32}-\cos^4\sigma \end{pmatrix}. \quad (4.5)$$

As in (3.24) we rotate this to $M^{-1}Q_\omega^{(4)}M = D_\omega^{(4)}$ whose diagonal elements are

$$\mathrm{diag}\,[D_\omega^{(4)}] = \{-\tfrac{1}{2} + \cos^4\sigma; \tfrac{1}{32}(13 - 8i\omega - 32\cos^4\sigma + i\omega\cos 4\sigma); \tfrac{1}{32}(13 + 8i\omega - 32\cos^4\sigma - i\omega\cos 4\sigma)\}$$

The computation of the ϵ^4 term in (4.1) coming from the coupled part gives

$$\begin{aligned}\mathrm{Tr}[(Q_\omega^{(0)})^{-1}Q_\omega^{(4)}] &= \sum_n \int_0^{2\pi}\frac{d\sigma}{2\pi}\mathrm{Tr}[(Q_\omega^{(0)})^{-1}Q_\omega^{(4)}] \\ &= \sum_n \int_0^{2\pi}\frac{d\sigma}{2\pi}\mathrm{Tr}[D_\omega^{(0)}D_\omega^{(4)}] \\ &= \frac{1}{32}\sum_n\Big[\frac{4}{n^2+w^2} + \frac{1-8i\omega}{n^2+(\omega+i)^2} \\ &\quad + \frac{1+8i\omega}{n^2+(\omega-i)^2}\Big]. \end{aligned} \tag{4.6}$$

The ϵ^4 contribution of the decoupled modes β_u coming from the single insertion of the ϵ^4 perturbation, i.e. an $\epsilon^4\,\mathrm{Tr}[A^{-1}C]$ type term is

$$\begin{aligned}&\frac{\det[-\partial_1^2 + \omega^2 + 2\epsilon^2\cos^2\sigma - \epsilon^4\cos^4\sigma]}{\det[-\partial_1^2+\omega^2]} \\ &\to -\epsilon^4\sum_n\frac{1}{n^2+\omega^2}\int_0^{2\pi}\frac{d\sigma}{2\pi}\cos^4\sigma \\ &= -\epsilon^4\frac{3}{8}\sum_n\frac{1}{n^2+\omega^2}. \end{aligned}\tag{4.7}$$

The single fermionic field gives just half of this contribution (up to the sign).

Putting together all of the contributions of the type $\epsilon^4\,\mathrm{Tr}[A^{-1}C]$ we get

$$\begin{aligned}\Gamma_1 \to -\frac{\mathcal{T}\epsilon^4}{4\pi}\int_{-\infty}^{\infty}d\omega\sum_n\Big[-\frac{7}{8}\frac{28}{n^2+w^2} - \frac{1}{32} \\ \times\frac{1-8i\omega}{n^2+(\omega+i)^2} - \frac{1}{32}\frac{1+8i\omega}{n^2+(\omega-i)^2}\Big]. \end{aligned}\tag{4.8}$$

Now let us compute the contributions of the type $\frac{1}{2}\epsilon^4\,\mathrm{Tr}[A^{-1}BA^{-1}B]$ in (4.1). Let us start with the decoupled fields β_u. Using the form of the $O(\epsilon^4)$ correction to the corresponding mass we get

$$\begin{aligned}\Big(\frac{\epsilon^4}{2}\mathrm{Tr}[A^{-1}BA^{-1}B]\Big)_\beta &= \frac{\epsilon^4}{2}\sum_{n_1,n_2}\frac{1}{n_1^2+\omega^2}\frac{1}{n_2^2+\omega^2}4\int_0^{2\pi}\frac{d\sigma_1}{2\pi}\frac{d\sigma_2}{2\pi}\cos^2\sigma_1 e^{i\sigma_1(n_1-n_2)}\cos^2\sigma_2 e^{-i\sigma_2(n_1-n_2)} \\ &= \frac{\epsilon^4}{2}\sum_n\frac{1}{n^2+\omega^2}\Big[\frac{1}{n^2+\omega^2} + \frac{1}{4[(n-2)^2+\omega^2]} + \frac{1}{4[(n+2)^2+\omega^2]}\Big] \end{aligned}\tag{4.9}$$

As discussed at the end of the previous section, to project out the zero mode contribution the sums over n in the massless propagators should not include the $n = 0$ point. Thus the sum in (4.8) should be over all $n \neq 0$. In computing the integrals over σ in (4.9) we have formally shifted n by ± 2, so the last line in the above equation should be understood as a combination of the three sums where in the first sum $n \neq 0$, in the second $n \neq 0, 2$ and in the third $n \neq 0, -2$.

The corresponding fermionic contribution is essentially $\frac{1}{4}$ of (4.8), as μ_F^2 is half of μ_β^2, but here there are two mass insertions. Putting together such contributions from the decoupled bosons and the fermions we observe that they cancel each other.

Next, let us find the $\epsilon^4\,\mathrm{Tr}[A^{-1}BA^{-1}B]$ type contribution of the coupled set of fluctuations. It can be written as [see (3.18) and (3.19)]

$$\begin{aligned}&\frac{\epsilon^4}{2}\mathrm{Tr}[(Q_\omega^{(0)})^{-1}Q_\omega^{(2)}(Q_\omega^{(0)})^{-1}Q_\omega^{(2)}] \\ &= \frac{\epsilon^4}{2}\sum_{n_1,n_2}\int_0^{2\pi}\frac{d\sigma_1}{2\pi}\frac{d\sigma_2}{2\pi}\mathrm{Tr}[(Q_\omega^{(0)})^{-1}(n_1)Q_\omega^{(2)}(\sigma_2) \\ &\quad\times(Q_\omega^{(0)})^{-1}(n_2)Q_\omega^{(2)}(\sigma_1)]e^{i(n_1-n_2)(\sigma_1-\sigma_2)} \end{aligned}\tag{4.10}$$

To compute this expression we again first diagonalize the propagator matrix and then integrate over σ. Putting together all the contributions from the two insertions of the ϵ^2 perturbations and adding the contribution with single ϵ^4 insertion (4.8) we get the following result for the 1-loop effective action to order ϵ^4

$$\Gamma_1(\epsilon^4) = -\frac{\mathcal{T}\epsilon^4}{4\pi}\int_{-\infty}^{\infty} d\omega\Big\{\sum_n\Big[-\frac{7}{8}\frac{1}{n^2+w^2} - \frac{1}{32}\frac{1-8i\omega}{n^2+(\omega+i)^2} - \frac{1}{32}\frac{1+8i\omega}{n^2+(\omega-i)^2}\Big]$$
$$+\frac{1}{2}\sum_n\Big[-\frac{\omega^2}{[n^2+(\omega+i)^2]^2} - \frac{\omega^2}{[n^2+(\omega-i)^2]^2} + \frac{1}{4}\frac{1}{n^2+w^2}\Big(\frac{1}{(n-2)^2+\omega^2}+\frac{1}{(n+2)^2+\omega^2}\Big)$$
$$+\frac{1}{2}\frac{1}{[n^2+(\omega+i)^2][n^2+(\omega-i)^2]} + \omega^2\Big(\frac{1}{(n+1)^2+\omega^2}+\frac{1}{(n-1)^2+\omega^2}\Big)\Big(\frac{1}{n^2+(\omega+i)^2}+\frac{1}{n^2+(\omega-i)^2}\Big)$$
$$+\frac{(1+\frac{i\omega}{2})^2}{4}\frac{1}{n^2+(\omega-i)^2}\Big(\frac{1}{(n-2)^2+(\omega-i)^2}+\frac{1}{(n+2)^2+(\omega-i)^2}\Big)$$
$$+\frac{(1-\frac{i\omega}{2})^2}{4}\frac{1}{n^2+(\omega+i)^2}\Big(\frac{1}{(n-2)^2+(\omega+i)^2}+\frac{1}{(n+2)^2+(\omega+i)^2}\Big)\Big]\Big\} \quad (4.11)$$

Again, this expression should be understood as a combination of sums over n where the values of n for which the effective (shifted) value of n vanishes should be projected out as it came from the original n_i in the propagator after doing the integral over σ and shifting the summation index. For example, we have

$$\sum_{n_1\neq 0, n_2\neq 0}\frac{1}{n_1^2+\omega^2}\frac{1}{n_2^2+\omega^2}\int_0^{2\pi}\frac{d\sigma_1}{2\pi}\frac{d\sigma_2}{2\pi}\cos 2\sigma_1\cos 2\sigma_2 e^{i(n_1-n_2)(\sigma_1-\sigma_2)}$$
$$=\frac{1}{4}\sum_{n\neq 0,2}\frac{1}{n^2+\omega^2}\frac{1}{(n-2)^2+\omega^2}+\frac{1}{4}\sum_{n\neq 0,-2}\frac{1}{n^2+\omega^2}\frac{1}{(n+2)^2+\omega^2}. \quad (4.12)$$

The first three terms in (4.11) can be simplified as in (3.26) by doing separate shifts of w by $\pm i$ in the last two terms; this gives

$$-\frac{1}{32}\sum_{n\neq 0}\int_{-\infty}^{\infty} d\omega\Big[\frac{28}{n^2+w^2}+\frac{1-8i\omega}{n^2+(\omega+i)^2}+\frac{1+8i\omega}{n^2+(\omega-i)^2}\Big] = -\frac{7}{16}\sum_{n\neq 0}\int_{-\infty}^{\infty} d\omega\frac{1}{n^2+\omega^2}. \quad (4.13)$$

Similar separate shifts of w under the integral $\int_{-\infty}^{\infty} d\omega$ can be used to transform some other terms in (4.11). For example, we get

$$\frac{\omega^2}{[n^2+(\omega+i)^2]^2}+\frac{\omega^2}{[n^2+(\omega-i)^2]^2}\to 2\frac{\omega^2-1}{(n^2+\omega^2)^2},$$
$$\frac{1}{[n^2+(\omega+i)^2][n^2+(\omega-i)^2]}=\frac{i}{2\omega}\Big[\frac{1}{n^2+(\omega+i)^2}-\frac{1}{n^2+(\omega-i)^2}\Big]\to -\frac{1}{2(\omega^2+1)}\frac{1}{n^2+\omega^2} \quad (4.14)$$

Using the identity $\frac{1}{ab}=(\frac{1}{a}-\frac{1}{b})\frac{1}{b-a}$ with a, b being $(n+k)^2+(\omega+v)^2$, $(k=0, \pm 2, v=0, \pm i)$ and shifting ω in terms containing only propagator factors with $(\omega\pm i)$ one finds that

$$\sum_{n\neq 0,-1}\frac{\omega^2}{[(n+1)^2+\omega^2][n^2+(\omega+i)^2]}+\sum_{n\neq 0,1}\frac{\omega^2}{[(n-1)^2+\omega^2][n^2+(\omega+i)^2]}+\text{c.c.}$$
$$\to\sum_{n\neq 0,1}\frac{\omega^2(n-1)}{[(n-1)^2+\omega^2][(n-1)^2+\omega^2]}-\sum_{n\neq 0,-1}\frac{\omega^2(n+1)}{[(n+1)^2+\omega^2][(n+1)^2+\omega^2]}-\sum_{n\neq 0,-1}\frac{n-\omega^2(n+2)}{(n^2+\omega^2)^2}$$
$$-\sum_{n\neq 0,1}\frac{-n+\omega^2(n-2)}{(n^2+\omega^2)^2}=-\frac{2}{(\omega^2+1)^2}+\sum_{n\neq 0}\frac{4\omega^2}{(n^2+\omega^2)^2}. \quad (4.15)$$

The second line above comes from the unshifted terms, while the third line from the ω-shifted terms. Performing similar shifts of ω and n in the last two lines in (4.11) we get

$$\frac{(1+\frac{i\omega}{2})^2}{4}\frac{1}{n^2+(\omega-i)^2}\Big[\frac{1}{(n-2)^2+(\omega-i)^2}+\frac{1}{(n+2)^2+(\omega-i)^2}\Big]+\text{c.c.}\to -\frac{\omega^2-1}{4(n^2+\omega^2)(n-2)^2+\omega^2}, \quad (4.16)$$

where the final term should be summed over $n\neq 0, 2$.

Collecting the above expressions we get for (4.11)

$$\Gamma_1(\epsilon^4) = -\frac{\mathcal{T}\epsilon^4}{4\pi}\int_{-\infty}^{\infty} d\omega\Big(C_0 + C_1 + C_2 + \sum_{n=3}^{\infty} S_n\Big), \tag{4.17}$$

where

$$C_0 = -\frac{1}{(\omega^2+1)^2}, \qquad C_1 = \frac{7\omega^4 + 84\omega^2 + 93}{8(\omega^2+1)^2(\omega^2+9)},$$
$$C_2 = \frac{8\omega^6 + 137\omega^4 - 89\omega^2 - 308}{8(\omega^2+4)^2(\omega^4+17\omega^2+16)} \tag{4.18}$$

$$S_n = \frac{1}{8(n^2+\omega^2)^2}\Big[9\omega^2 - 7n^2 + 16 - \frac{2(n^2+\omega^2)}{\omega^2+1} - (n^2+\omega^2)(\omega^2-3) \times\Big(\frac{1}{(n+2)^2+\omega^2} + \frac{1}{(n-2)^2+\omega^2}\Big)\Big]. \tag{4.19}$$

The result is UV finite as expected [3]. It is also IR finite (which would not be the case if the zero mode contributions were not properly projected out). The integrals over ω give

$$\int_{-\infty}^{\infty} d\omega C_0 = -\frac{\pi}{2}, \qquad \int_{-\infty}^{\infty} d\omega C_1 = \frac{9\pi}{8},$$
$$\int_{-\infty}^{\infty} d\omega C_2 = \frac{17\pi}{128}, \tag{4.20}$$

$$\int_{-\infty}^{\infty} d\omega S_n = -\frac{\pi(n^4 - 7n^3 + 3n^2 + 16n - 16)}{4n^3(n^2-4)(n-1)}. \tag{4.21}$$

Remarkably, the remaining sum over n can be also computed exactly

$$\sum_{n=3}^{\infty}\frac{n^4 - 7n^3 + 3n^2 + 16n - 16}{n^3(n^2-4)(n-1)} = \frac{149}{32} - 4\zeta(3), \tag{4.22}$$

giving

$$E_1 = \frac{\Gamma_1}{\mathcal{T}\kappa} = \frac{\Gamma_1(\epsilon^4)}{\mathcal{T}\epsilon} + O(\epsilon^5)$$
$$= \frac{1}{4}\Big[\frac{13}{32} - \zeta(3)\Big]\epsilon^3 + O(\epsilon^5). \tag{4.23}$$

Using (2.9) this can be written also as

$$E_1 = \frac{1}{\sqrt{2}}\Big[\frac{13}{32} - \zeta(3)\Big]\mathcal{S}^{3/2} + O(\mathcal{S}^{5/2}), \tag{4.24}$$

which corresponds to the value of a_{11} given in (1.4).

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APPENDIX: GENERALIZATION TO NONZERO S^5 ANGULAR MOMENTUM

The above discussion can be generalized to the case of the (S, J) string which is spinning with spin S in AdS_3 and also moving with momentum J around big circle in S^5 [3]. This generalization is potentially important as it allows one to relate the corresponding string states to operators like $\mathrm{tr}(D_+^S\Phi^J)$ in the closed $sl(2)$ sector of the SYM theory (with J having the interpretation of the length of the corresponding spin chain [14]).

The relations in Sec. II have straightforward generalization to the case when the string also moves along the S^1 in S^5:

$$\varphi = \nu\tau, \qquad J = \sqrt{\lambda}\nu,$$
$$\rho'^2 = \kappa^2\cosh^2\rho - w^2\sinh^2\rho - \nu^2, \tag{A1}$$

$$0 \le \rho \le \rho_*, \qquad \coth^2\rho_* = \frac{w^2-\nu^2}{\kappa^2-\nu^2} \equiv 1 + \frac{1}{\epsilon^2},$$
$$\rho_* = \epsilon - \frac{1}{6}\epsilon^3 + \ldots. \tag{A2}$$

Here $\nu \equiv \mathcal{J}$ plays the role of the semiclassical S^5 momentum parameter and ϵ again measures the length of the string. To include nonzero ν one is to shift $w \to \sqrt{w^2-\nu^2}$, $\kappa \to \sqrt{\kappa^2-\nu^2}$. We get [cf. (2.6) and (2.7)] [3]

$$\sqrt{\kappa^2-\nu^2} = \epsilon\, {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2\Big),$$
$$\mathcal{E}_0 = \frac{\kappa}{\sqrt{\kappa^2-\nu^2}}\epsilon\, {}_2F_1\Big(-\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2\Big), \tag{A3}$$

$$\mathcal{S} = \frac{w}{\sqrt{\kappa^2-\nu^2}}\frac{\epsilon^2}{2}\sqrt{1+\epsilon^2}\, {}_2F_1\Big(\frac{1}{2}, \frac{3}{2}; 2; -\epsilon^2\Big). \tag{A4}$$

To consider the short string limit we should expand in small ϵ while keeping ν arbitrary. Then we find

$$\mathcal{E}_0^2 = \nu^2 + \epsilon^2(1+\nu^2) + \frac{\epsilon^4}{2}\Big(1 + \frac{\nu^2}{4}\Big) + O(\epsilon^6),$$
$$\mathcal{S}^2 = \frac{\epsilon^4}{4}(1+\nu^2) + \frac{\epsilon^6}{16}(1-\nu^2) + O(\epsilon^8) \tag{A5}$$

i.e.

$$\epsilon^2 = \frac{2\mathcal{S}}{\sqrt{1+\nu^2}} + O(\mathcal{S}^2),$$
$$\mathcal{E}_0^2 = \nu^2 + 2\mathcal{S}\sqrt{1+\nu^2} + O(\mathcal{S}^2). \tag{A6}$$

The short string limit $\epsilon \ll 1$ [3] can thus be achieved by, e.g., considering a slowly spinning string $\mathcal{S} \ll 1$ or by assuming large momentum in S^5, i.e. $\nu \gg 1$. The latter is the fast string or BMN-like limit while the former may be called a near flat space limit in which ν may be kept arbitrarily small.

Below we shall concentrate on the short string limit $\epsilon \ll 1$. If we further assume that $\epsilon \ll \nu$ then the classical energy will be

$$\mathcal{E}_0 = \nu + \frac{\mathcal{S}}{\nu}\sqrt{\nu^2+1} + O(\mathcal{S}^2). \tag{A7}$$

If we then expand in large $\nu \gg 1$ that will correspond to the usual fast short string limit where one takes ν large at fixed $\frac{\mathcal{S}}{\nu} = \frac{S}{J}$ and then expands in $\frac{\mathcal{S}}{\nu} \ll 1$ [3]

$$\mathcal{E}_0 = \nu + \mathcal{S} + \frac{\mathcal{S}}{2\nu^2} + \ldots, \qquad \nu \gg 1, \qquad \frac{\mathcal{S}}{\nu} \ll 1. \tag{A8}$$

In the slow short string limit we have $\epsilon \ll 1, \mathcal{S} \ll 1$; if we assume in addition that the S^5 rotational energy is smaller than the spinning one, then $\nu \ll \sqrt{\mathcal{S}} \ll 1$. In this case $\nu \ll \epsilon$ which is opposite to the above assumption that led to (A7). Here we get $\epsilon = \sqrt{2\mathcal{S}} - \frac{1}{4\sqrt{2}}\mathcal{S}^{3/2}(1+\frac{2\nu^2}{\mathcal{S}}) + \ldots$ so that the classical energy has a "near flat space" expansion form

$$\mathcal{E}_0 = \sqrt{2\mathcal{S}}\Big(1 + \frac{\nu^2}{4\mathcal{S}} + \ldots\Big) + \frac{3}{4\sqrt{2}}\mathcal{S}^{3/2}\Big(1 + \frac{5\nu^2}{12\mathcal{S}} + \ldots\Big) + \ldots, \qquad \nu \ll \sqrt{\mathcal{S}} \ll 1. \tag{A9}$$

The fluctuation Lagrangian will now have 4 of S^5 fields having mass ν^2 and while the masses of the other fluctuation fields become [3] [cf. (3.1), (3.2), and (3.3)]:

$$\mu_t^2 = 2\rho'^2 - \kappa^2 + \nu^2, \qquad \mu_\phi^2 = 2\rho'^2 - w^2 + \nu^2,$$
$$\mu_\rho^2 = 2\rho'^2 - w^2 - \kappa^2 + 2\nu^2, \qquad \mu_\beta^2 = 2\rho'^2 + \nu^2,$$
$$\mu_F^2 = \rho'^2 + \nu^2. \tag{A10}$$

We can then compute the 1-loop correction to string energy by expanding in the short string limit, i.e. in $\epsilon \ll 1$ while keeping ν fixed.

Expanding the masses and the coefficients in the mixing term in the fluctuation Lagrangian we get the following expression for the 1-loop effective action [cf. (3.17), (3.18), (3.19), (3.20), and (3.21)]

$$\Gamma_1(\epsilon^2) = -\frac{\mathcal{T}}{4\pi}\int_{-\infty}^{\infty} d\omega\Big(8\ln\frac{\det[\Delta_0+\nu^2+\epsilon^2\cos^2\sigma]}{\det[\Delta_0+\nu^2]} - 2\ln\frac{\det[\Delta_0+\nu^2+2\epsilon^2\cos^2\sigma]}{\det[\Delta_0+\nu^2]} - \ln\frac{\det[Q_\omega]}{\det[Q_\omega^{(0)}]} + \ln\frac{\det[P_\omega]}{\det[Q_\omega^{(0)}]}\Big), \tag{A11}$$

where now

$$P_\omega = \begin{pmatrix} -\Delta_0 & 0 & 0 \\ 0 & \Delta_0+\nu^2 & 0 \\ 0 & 0 & \Delta_0+\nu^2 \end{pmatrix}, \qquad \Delta_0 \equiv -\partial_1^2 + \omega^2 \tag{A12}$$

and the mixing term operator Q_ω is given to order ϵ^2 by the following matrix ($i = 1, 2, 3$)[8]

$$\begin{aligned}(Q_\omega)_{1i} &= \{-(\Delta_0 + \epsilon^2\cos 2\sigma); 0; 2\epsilon w \sin\sigma\sqrt{\nu^2+\epsilon^2}\}\\ (Q_\omega)_{2i} &= \Big\{0; \Delta_0 - 1 + \epsilon^2\Big(\cos 2\sigma + \frac{1}{2}\Big); -2\omega\Big(1 + \frac{1}{2}\epsilon^2\sin^2\sigma\Big)\sqrt{\nu^2+1+\frac{1}{2}\epsilon^2}\Big\}\\ (Q_\omega)_{3i} &= \Big\{-2\epsilon w\sin\sigma\sqrt{\nu^2+\epsilon^2}; 2\omega\Big(1+\frac{1}{2}\epsilon^2\sin^2\sigma\Big)\sqrt{\nu^2+1+\frac{1}{2}\epsilon^2}; \Delta_0 - 1 + \epsilon^2\Big(\cos 2\sigma - \frac{1}{2}\Big)\Big\}\end{aligned} \tag{A13}$$

So far we considered $\epsilon \ll 1$ with ν arbitrary. Next, we may specify either to the fast short string case ($\nu \gg \epsilon$) or to the slow short string case ($\nu \ll \epsilon$). In the fast string case we get $Q_\omega = Q_\omega^{(0)} + \epsilon Q_\omega^{(1)} + \epsilon^2 Q_\omega^{(2)} + \ldots$ where

$$Q_\omega^{(0)} = \begin{pmatrix} -\Delta_0 & 0 & 0 \\ 0 & \Delta_0 - 1 & -2\omega\sqrt{1+\nu^2} \\ 0 & 2\omega\sqrt{1+\nu^2} & \Delta_0 - 1 \end{pmatrix}, \qquad Q_\omega^{(1)} = \begin{pmatrix} 0 & 0 & 2\omega\nu\sin\sigma \\ 0 & 0 & 0 \\ -2\omega\nu\sin\sigma & 0 & 0 \end{pmatrix}$$
$$Q_\omega^{(2)} = \begin{pmatrix} -\cos 2\sigma & 0 & 0 \\ 0 & \cos 2\sigma + \frac{1}{2} & -\frac{\omega}{\sqrt{1+\nu^2}}[\frac{1}{2} + (1+\nu^2)\sin^2\sigma] \\ 0 & \frac{\omega}{\sqrt{1+\nu^2}}[\frac{1}{2} + (1+\nu^2)\sin^2\sigma] & \cos 2\sigma - \frac{1}{2} \end{pmatrix}. \tag{A14}$$

[8]Here we expanded to order ϵ^2 in small ϵ at fixed ν but in some terms formally kept ϵ^2 contributions under the square roots to allow for a smooth $\nu \to 0$ limit.

We can again diagonalize the propagator matrix

$$D_\omega^{(0)} = M^{-1}(Q_\omega^{(0)})^{-1}M = \begin{pmatrix} -\frac{1}{n^2+\omega^2} & 0 & 0 \\ 0 & \frac{1}{n^2+(\omega+i\sqrt{1+\nu^2})^2+\nu^2} & 0 \\ 0 & 0 & \frac{1}{n^2+(\omega-i\sqrt{1+\nu^2})^2+\nu^2} \end{pmatrix}, \tag{A15}$$

where M is the same as in (3.24). Similarly,

$$D_\omega^{(1)} = M^{-1}Q_\omega^{(1)}M = \begin{pmatrix} 0 & \omega\nu\sin\sigma & \omega\nu\sin\sigma \\ -2\omega\nu\sin\sigma & 0 & 0 \\ -2\omega\nu\sin\sigma & 0 & 0 \end{pmatrix},$$
$$D_\omega^{(2)} = M^{-1}Q_\omega^{(2)}M = \begin{pmatrix} -\cos 2\sigma & 0 & 0 \\ 0 & \cos 2\sigma + \frac{i\omega[\frac{1}{2}+(1+\nu^2)\sin^2\sigma]}{\sqrt{1+\nu^2}} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \cos 2\sigma - \frac{i\omega[\frac{1}{2}+(1+\nu^2)\sin^2\sigma]}{\sqrt{1+\nu^2}} \end{pmatrix} \tag{A16}$$

One can show that the last term in (A11) vanishes. The leading term in the short-string limit of Γ_1 is of order ϵ^2. To compute it we note that

$$\ln\frac{\det[-\partial_1^2+\omega^2+\nu^2+2\epsilon^2\cos^2\sigma]}{\det[-\partial_1^2+\omega^2+\nu^2]} \approx \epsilon^2\sum_n \frac{2\int_0^{2\pi}\frac{d\sigma}{2\pi}\cos^2\sigma}{n^2+\omega^2+\nu^2} = \epsilon^2\sum_n\frac{1}{n^2+\omega^2+\nu^2}. \tag{A17}$$

and use the expansion

$$\ln\frac{\det[A+\epsilon B_1+\epsilon^2 B_2]}{\det A} = \epsilon\,\mathrm{Tr}[A^{-1}B_1] + \epsilon^2\,\mathrm{Tr}[A^{-1}B_2] - \frac{\epsilon^2}{2}\mathrm{Tr}[A^{-1}B_1A^{-1}B_1] + \ldots \tag{A18}$$

in the third nontrivial term in (A11). The order ϵ contribution vanishes. The ϵ^2 terms come from $\mathrm{Tr}[D_\omega^{(0)}D_\omega^{(2)}]$ and $\mathrm{Tr}[D_\omega^{(0)}D_\omega^{(1)}D_\omega^{(0)}D_\omega^{(1)}]$. Summing them up we get for the ϵ^2 term in the effective action

$$\Gamma_1(\epsilon^2) = \frac{\mathcal{T}\epsilon^2}{4\pi}\int_{-\infty}^{\infty}d\omega\sum_n\Big(-\frac{2}{n^2+\omega^2+\nu^2} + \frac{\nu^2+2}{2\sqrt{\nu^2+1}}\times\Big[\frac{i\omega}{n^2+\nu^2+(\omega+i\sqrt{\nu^2+1})^2}+\text{c.c.}\Big] - \frac{\nu^2\omega^2}{2(n^2+\omega^2)}\times\Big[\frac{1}{(n+1)^2+\nu^2+(\omega+i\sqrt{\nu^2+1})^2} + \frac{1}{(n-1)^2+\nu^2+(\omega+i\sqrt{\nu^2+1})^2}+\text{c.c.}\Big]\Big) \tag{A19}$$

Performing separate shifts of ω under the integrals in various terms as discussed in Secs. III and IV gives

$$\Gamma_1(\epsilon^2) = \frac{\mathcal{T}\epsilon^2}{4\pi}\int_{-\infty}^{\infty}d\omega\sum_{n=-\infty}^{\infty}\nu^2\Big[\frac{1}{n^2+\omega^2+\nu^2} + \frac{\omega^2(n-1-\nu^2)-(\nu^2+1)(n+1+\nu^2)}{[(n+1)^2+\nu^2+\omega^2][(n+1+\nu^2)^2+\omega^2(\nu^2+1)]}\Big]. \tag{A20}$$

Here we used the symmetry of the sum under shifts of n; it further allows one to simplify the square bracket under the sum into $\frac{(n-1)(\omega^2+\nu^2+n)}{(n^2+\omega^2+\nu^2)[(n+\nu^2)^2+\omega^2(\nu^2+1)]}$. Then the sum over n can be performed exactly and we get

$$\Gamma_1(\epsilon^2) = \frac{\mathcal{T}\epsilon^2}{4\pi}\times\int_{-\infty}^{\infty}d\omega\frac{\pi\sin(2\pi\nu^2)}{\cos(2\pi\nu^2)-\cosh(2\pi\omega\sqrt{\nu^2+1})}, \tag{A21}$$

or finally

$$\Gamma_1 = \frac{\mathcal{T}\epsilon^2}{4}\frac{2\nu^2-1}{\sqrt{\nu^2+1}} + O(\epsilon^4). \tag{A22}$$

Recalling that $E_1 = \frac{\Gamma_1}{\kappa\mathcal{T}}$ and that in the "short fast string" limit under the consideration (i.e. $\epsilon \ll 1$, $\epsilon \ll \nu$) one has $\kappa = \nu + \frac{\epsilon^2}{2\nu} + \ldots$, we finally obtain

$$E_1 = \frac{\mathcal{S}}{2\nu}\frac{2\nu^2-1}{\nu^2+1} + O(\mathcal{S}^2), \tag{A23}$$

where we have replaced ϵ by $\mathcal{S}$ using (A6). So far ν here is arbitrary apart from the condition $\nu \gg \epsilon$, i.e. $2\mathcal{S} \ll \nu^2\sqrt{1+\nu^2}$, so that (A23) is the 1-loop correction to the classical energy in (A7).

Assuming further that $\nu \gg 1$ we get

$$E_1 = \frac{\mathcal{S}}{\nu} - \frac{3}{2}\frac{\mathcal{S}}{\nu^3} + \ldots = \frac{S}{J}\left(1 - \frac{3}{2}\frac{\lambda}{J^2} + \ldots\right) + \ldots, \tag{A24}$$

which should be the correction to (A8).

This expression may be compared to the 1-loop correction to the folded spinning string energy found by quantizing the $sl(2)$ Landau-Lifshitz model in Appendix D of [15]

$$E_1 = -\frac{\mathcal{S}}{2\nu^3} + O(\mathcal{S}^2) = -\frac{\lambda}{2J^2}\frac{S}{J} + O(\mathcal{S}^2). \tag{A25}$$

There one first has taken the large ν limit with $\frac{\mathcal{S}}{\nu}$ kept fixed and then expanded in $\frac{\mathcal{S}}{\nu} \ll 1$. Here the order of limits was different (we first expanded in ϵ for fixed ν) and that could be a possible reason for a disagreement between (A24) and (A25).[9] To recover the standard fast string result one would need to start with the short string fluctuation operators in (A13), where no assumption on $\frac{\mathcal{S}}{\nu}$ was made, use them and (A17) without expanding in ϵ, compute the determinants needed in (A11), then expand in large ν with $\frac{\mathcal{S}}{\nu}$ kept fixed, and at the end take $\frac{\mathcal{S}}{\nu}$ to be small.

Let us now consider the 1-loop correction in the small ν region by taking ϵ to zero while keeping the parameter $x \equiv \frac{\nu}{\epsilon}$ fixed, i.e. scaling ν to zero together with ϵ so that $\frac{\nu}{\sqrt{2\mathcal{S}}} \approx x$ remains finite. We can then expand in small x and recover the case of $\nu \ll \epsilon \ll 1$. This will correspond to a correction to the near flat space expression (A9). In this limit $\kappa = \epsilon\sqrt{1+x^2} + O(\epsilon^3)$ and from (A13) we get

$$Q_\omega = Q_\omega^{(0)} + \epsilon^2 Q_\omega^{(2)} + \ldots, \qquad Q_\omega^{(0)} = \begin{pmatrix} -\Delta_0 & 0 & 0 \\ 0 & \Delta_0 - 1 & -2\omega \\ 0 & 2\omega & \Delta_0 - 1 \end{pmatrix},$$
$$Q_\omega^{(2)} = \begin{pmatrix} -\cos 2\sigma & 0 & 2\omega\sqrt{1+x^2}\sin\sigma \\ 0 & \cos 2\sigma + \frac{1}{2} & -\omega(\frac{1}{2} + x^2 + \sin^2\sigma) \\ -2\omega\sqrt{1+x^2}\sin\sigma & \omega(\frac{1}{2} + x^2 + \sin^2\sigma) & \cos 2\sigma - \frac{1}{2} \end{pmatrix} \tag{A26}$$

Diagonalizing $Q_\omega^{(0)}$ and using perturbation theory in ϵ we get

$$\mathrm{Tr}[(Q_\omega^0)^{-1}Q_\omega^1] = \sum_{n\neq 0}(1+x^2)\left[\frac{i\omega}{n^2 + (\omega+i)^2} - \frac{i\omega}{n^2 + (\omega - i)^2}\right] \to \sum_{n\neq 0}\frac{2(1+x^2)}{n^2+\omega^2}, \tag{A27}$$

where as in Sec. III we made shifts of ω under the integral and also excluded the zero mode terms. For the decoupled modes we find

$$\ln\frac{\det[-\partial_1^2 + \omega^2 + x^2\epsilon^2 + 2\epsilon^2\cos^2\sigma]}{\det[-\partial_1^2 + \omega^2 + x^2\epsilon^2]} = \epsilon^2\sum_{n\neq 0}\frac{1}{n^2+\omega^2} + O(\epsilon^4), \tag{A28}$$

and a similar expression is obtained for the fermionic contribution. In the last term in (A11) now P_ω includes ϵ^2 while Q_ω^0 does not, so that term is no longer zero. Computing it perturbatively in ϵ^2 we get

$$\ln\frac{\det[P_\omega]}{\det[Q_\omega^0]} = 2x^2\epsilon^2\sum_{n\neq 0}\frac{1}{n^2+\omega^2} + O(\epsilon^4). \tag{A29}$$

Finally, for the leading term in (A11) we obtain

$$\Gamma_1(\epsilon^2) = -\frac{\mathcal{T}\epsilon^2}{4\pi}\int d\omega\sum_{n\neq 0}\left[\frac{2}{n^2+\omega^2} - \frac{2(1+x^2)}{n^2+\omega^2} + \frac{2x^2}{n^2+\omega^2}\right] = 0. \tag{A30}$$

Thus the 1-loop correction to (A9) at order $\sqrt{\mathcal{S}}$ vanishes not only for $\nu = 0$ (as we have seen already in Sec. III) but also for any $\nu \sim \epsilon \ll 1$. As in Sec. III, this can be again related to the UV finiteness property of the $AdS_5 \times S^5$ superstring.

It would be interesting to extend the above computation to compute the first nonvanishing 1-loop correction to (A9).

[9]The presence of an unusual $\frac{S}{J}$ term in the 1-loop correction (A24) may be an artifact of the limit of the above expansion procedure.

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