

Applicability of the linear δ expansion for the $\lambda\phi^4$ field theory at finite temperature in the symmetric and broken phases

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The thermodynamics of a scalar field with a quartic interaction is studied within the linear δ expansion (LDE) method. Using the imaginary-time formalism the free energy is evaluated up to second order in the LDE. The method generates nonperturbative results that are then used to obtain thermodynamic quantities like the pressure. The phase transition pattern of the model is fully studied, from the broken to the symmetry restored phase. The results are compared with those obtained with other nonperturbative methods and also with ordinary perturbation theory. The results coming from the two main optimization procedures used in conjunction with the LDE method, the principle of minimal sensitivity (PMS) and the fastest apparent convergence (FAC) are also compared with each other and studied in which cases they are applicable or not. The optimization procedures are applied directly to the free energy.

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I. INTRODUCTION

Phase transition phenomena in quantum field theories are typically of nonperturbative nature and thus naive perturbation theory based on an expansion in the coupling constant cannot be employed. This is clearly the case of phase changes at high temperatures, where perturbation theory becomes unreliable because powers of the coupling constant become surmounted by powers of the temperature [1]. Problems with perturbation theory also happen in phenomena occurring close to critical points, because large fluctuations can emerge in the system due to infrared divergences, thus requiring nonperturbative methods as well in their studies. This is the case of studies involving second order phase transitions and also in weak first order phase transitions [2]. Typical examples where these problems can manifest are in studies of symmetry changing phenomena in a hot and dense medium, a subject of interest in quantum chromodynamics (QCD) in the context of heavy-ion collision experiments, and also in studies of the early universe. Consequently, there is a great deal of interest in investigating thermal field theories describing matter under extreme conditions [3–6].

Familiar nonperturbative methods that have been used to study symmetry changing phenomena at finite temperatures are resummationlike techniques, such as the daisy and superdaisy schemes [7,8], composite operator methods [9], and field propagator dressing methods [10,11]. Other methods used include expansions in parameters not related to a coupling constant, like the $1/N$ expansion and the ϵ -expansion [12]. In addition, there are numerical methods, the most notably ones are those based on lattice

Monte Carlo simulations [13]. Each method has its own advantages and disadvantages. For instance, in numerical methods there may be issues related to numerical precision, lattice spacing, and lattice sizes. In addition, there is the notorious problem of simulating fermions on the lattice at finite chemical potentials [13]. In any nonperturbative method based on an expansion in some parameter one has to face the problem of higher order terms becoming increasingly cumbersome, so stalling further analysis. This is usually the case when carrying out calculations beyond leading order in the $1/N$ expansion. Careless use of a nonperturbative method can also lead to problems like the lack of self-consistency or overcounting of effects. Known examples of such problems are the earlier resummation works dealing with daisy and superdaisy schemes, that at some point were giving wrong results, e.g. predicting a first order transition [14] for the $\lambda\phi^4$ theory, an unexpected result since the model belongs to the universality class of the Ising model, which is second order. These methods also predicted a strong first order phase transition in the electroweak standard model, a result proved to be misleading [15].

Let us recall that the breakdown of perturbation theory at high temperatures and its poor convergence properties have been dealt with many different methods. Examples are the use of self-consistent approximations [16], hard-thermal-loop (HTL) resummation [17,18], perturbative expansions in the coupling constant with resummation implemented with the use of a variational mass parameter, also known as Screened Perturbation Theory (SPT) [19,20], and the use of two-particle irreducible (2PI) effective actions [21]. The 2PI method, in particular, leads to a much better convergence of thermodynamic quantities (like the pressure) as compared to some of the other methods [22]. Related to the 2PI method is the Φ -derivable technique, which has been used to study the

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thermodynamics of scalar and gauge theories [23–26]. One difficulty with the 2PI effective actions is that the renormalization procedure is nontrivial [27]. In addition, there seems that the Φ -derivable technique breaks down for a coupling beyond some value [28].

In general, it is desirable that any analytical nonperturbative method obey two basic requirements. First, it should be self-consistent, and second, it should produce useful results already at lowest orders without the need for going to higher orders. That is, it should produce results that quickly converge at some order where calculations are still feasible analytically or semianalytically. Though some of the cited methods may satisfy one, or to some extent both of these requirements, in the present paper we are particularly interested in the one known as the linear δ expansion (LDE) [29], a nonperturbative method that has been used successfully in different contexts related to thermal field theories [30–32] and in many other theories—for a long, but far from complete list of references see Refs. [33,34]. In the LDE, a linear interpolation on the original model Lagrangian density is performed in terms of a fictitious expansion parameter δ , which is used only for bookkeeping purposes and set at the end equal to one. The standard application of the LDE to a theory described by a Lagrangian density \mathcal{L} starts with an interpolation defined by

$$\begin{aligned}\mathcal{L} \rightarrow \mathcal{L}^\delta &= (1 - \delta)\mathcal{L}_0(\eta) + \delta\mathcal{L} \\ &= \mathcal{L}_0(\eta) + \delta[\mathcal{L} - \mathcal{L}_0(\eta)],\end{aligned}\quad (1.1)$$

where \mathcal{L}_0 is the Lagrangian density of a solvable theory, which is modified by the introduction of an arbitrary mass parameter (or parameters) η . The Lagrangian density \mathcal{L}^δ interpolates between the solvable $\mathcal{L}_0(\eta)$ (when $\delta = 0$) and the original \mathcal{L} (when $\delta = 1$). The procedure defined by Eq. (1.1) leads to modified Feynman vertices that become multiplied by δ , and modified propagators that now depend on η . All quantities evaluated at any finite order in the LDE will then depend explicitly on η , unless one could perform a calculation to all orders. Up to this stage the results remain strictly perturbative and very similar to the ones obtained via an ordinary perturbative calculation. It is through the freedom in fixing η that nonperturbative results can be generated in this method. Since η does not belong to the original theory, one may fix it requiring that a physical quantity $\Phi^{(k)}$, calculated perturbatively to order δ^k , be evaluated at the value where it is less sensitive to this parameter. This criterion, known as the principle of minimal sensitivity (PMS), translates into the variational relation [35]

$$\left. \frac{d\Phi^{(k)}}{d\eta} \right|_{\bar{\eta}, \delta=1} = 0. \quad (1.2)$$

The optimum value $\bar{\eta}$ which satisfies Eq. (1.2) is a function of the original parameters of the theory. In particular, $\bar{\eta}$ is a

nontrivial function of the couplings and because of this nonperturbative results are generated. Another optimization procedure used is known as the fastest apparent convergence (FAC) criterion [35]. It requires from the k th coefficient of the perturbative expansion

$$\Phi^{(k)} = \sum_{i=0}^k c_i \delta^i, \quad (1.3)$$

that

$$[\Phi^{(k)} - \Phi^{(k-1)}]_{\delta=1} = 0, \quad (1.4)$$

which is just equivalent to taking the k th coefficient (at $\delta = 1$) in Eq. (1.3) equal to zero.

One should note that it is not at all guaranteed that the condition in Eq. (1.2) has a nontrivial solution. In cases where this may happen, the second criterion, Eq. (1.4), may be more appropriate. One example where the condition given by Eq. (1.2) fails to produce a nontrivial solution was in the problem studied by the authors in Ref. [36], who applied the LDE to compute the effective potential in superspace. There, the authors found that while the PMS condition was unable to give a nonperturbative solution to the effective potential, the FAC criterion worked perfectly well. Of course, in many situations both optimization criteria may work and in this case one may ask whether they lead to equivalent results. Previous studies indicated that this is indeed so, but a full comparison of results obtained with both optimization criteria is still lacking. Another issue associated with the LDE is its convergence. Rigorous LDE convergence proofs have been obtained for the problem of the quantum anharmonic oscillator, at zero temperature, considered in Ref. [37], while its partition function at finite temperatures was considered in [38]. For quantum field theories, Ref. [39] has proved convergence for a particular perturbative series in an asymptotically-free, renormalizable model at zero temperature. For a critical $\lambda\phi^4$ $O(N)$ theory in three dimensions the issue of convergence was studied in [40] employing both PMS and FAC optimization criteria. Finally, regarding the possible solutions that can emerge from the optimization criteria (PMS or FAC), we must use a definite approach in selecting the optimum root $\bar{\eta}$ from either Eq. (1.2) or (1.4). The problem of dealing with the many possible solutions for $\bar{\eta}$ was treated in detail in the first two papers cited in Ref. [40], where the convergence of the LDE was also studied in detail. Typically, the higher the order in δ , the more solutions can appear. As shown in those references, all solutions at each given order in δ can be classified into families. The optimum value for η is chosen as follows: The trivial solutions for $\bar{\eta}$, e.g. $\bar{\eta} = 0$ and those that are not dependent on the coupling constant (and thus cannot lead to nonperturbative results) are not considered. In addition to these, at first order there is only one nontrivial solution (first family), consistent with all our approximations (like the high-temperature approximation, used later in our cal-

culations). This family is then followed in the next orders and used in all our calculations. As proved in earlier references with the LDE method, this is a consistent and unambiguous way for choosing the optimum value for η .

It is important to stress that in the method of the LDE the selection and evaluation of Feynman diagrams proceed in the same fashion as in ordinary perturbation theory, including the renormalization procedure [33,39,40]. The results obtained are free from infrared divergences, even at the critical point and in its neighborhood, thus making it a particularly suitable method to study phase transition phenomena in quantum field theories. It is important to recall here that there are similarities between the LDE and the SPT methods. In particular, the implementation of the latter can be put in a form similar to the LDE by means of a modified loop expansion [41], named optimized perturbation theory (OPT) in this reference. But there are also some major differences between these methods. For instance, in the LDE no assumption is made *a priori* for the parameter η , while in the SPT/OPT it is assumed that such a mass term is already of some order in the coupling constant. The implication of this is that the order counting of loop expansion has to be readjusted accordingly.

In the present paper we study the application of the LDE to the $\lambda\phi^4$ theory. We will study the applicability of the PMS and FAC optimization criteria for the symmetric and broken phases of the theory and compare results obtained with both methods. In addition, in the present work we choose to optimize the free energy, instead of the self-energy like in many other works employing the LDE, particularly Refs. [31,32]. There are several reasons for doing so [30,34], but an important one is that in some situations it might happen that the optimization of the self-energy does not lead to nontrivial solutions, while optimization of the free energy with PMS or FAC are seen to lead to nontrivial solutions already at first order in δ . The critical temperature T_c , the pressure P , and the background dependent free energy F are obtained here in an explicit calculation up to order δ^2 . Calculations at this order require a calculation of vacuum terms up to three loops. Since the thermodynamics of this model has been extensively studied before in the literature with a number of methods, our calculation here will be useful to benchmark the application of the LDE and its two main optimization procedures against those previous applications. In addition, we compare our results with those obtained with standard perturbation theory. Besides correctly reproducing the expected second order phase transition pattern for the model, our results at order δ^2 are shown to be sufficient to obtain the thermodynamics of the model, in the sense that the results at $\mathcal{O}(\delta^2)$ are not much different from the ones at $\mathcal{O}(\delta)$. The results point towards a quickly convergent LDE, as already indicated in previous studies with different models under different conditions [39,40].

This work is organized as follows. In the next section we introduce the interpolation procedure for the model. In

Sec. III we compute the free energy in the symmetric and broken phases to $\mathcal{O}(\delta^2)$. In Sec. IV we present the results obtained from the optimization procedures. The pressure is evaluated and contrasted order by order with the one obtained within perturbation theory. The critical temperature, the temperature dependent vacuum expectation value of the scalar field, and the free energy are determined to $\mathcal{O}(\delta^2)$. Our conclusions are presented in Sec. V.

II. THE MODEL LAGRANGIAN DENSITY

The interpolation defined by Eq. (1.1) when applied for the standard $\lambda\phi^4$ model gives

$$\mathcal{L}^\delta = \mathcal{L}_0(\eta) - \delta \frac{\lambda}{4!} \phi^4 + \delta \frac{\eta^2}{2} \phi^2 + \mathcal{L}_{ct}^\delta, \quad (2.1)$$

where

$$\mathcal{L}_0(\eta) = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m_0^2}{2} \phi^2 - \frac{\eta^2}{2} \phi^2, \quad (2.2)$$

and \mathcal{L}_{ct}^δ is the part of the Lagrangian density carrying the renormalization terms needed to render the model finite. Details about this renormalization procedure in the LDE and the explicit form for \mathcal{L}_{ct}^δ are given e.g. in Ref. [31] for the case of background field dependent contributions (broken symmetry phase), while the field independent contributions (symmetric phase) were given in Ref. [20] within the context of the SPT, so we will not repeat those same renormalization details here. One should also note that the only “new” terms introduced by the δ -expansion interpolation are quadratic terms and so the renormalizability of the original theory is not changed. This means that the renormalization of the theory can be carried out in an analogous way as in ordinary perturbation theory [31]. Specifically, the interpolation procedure given by Eq. (2.1) introduces a new (quadratic) interaction term, with Feynman rule $i\delta\eta^2$. In addition to this modification, the original bare propagator,

$$S(k) = i(k^2 - m_0^2 + i\varepsilon)^{-1}, \quad (2.3)$$

now becomes

$$S_\delta(k) = i(k^2 - m_0^2 - \eta^2 + i\varepsilon)^{-1}, \quad (2.4)$$

while the original quartic vertex is changed from $-i\lambda$ to $-i\delta\lambda$.

In the next section we will show the results for the finite temperature free energy density using the interpolated model with the LDE at $\mathcal{O}(\delta^2)$. We will consider the cases of $m_0^2 = |m_0|^2$ and $m_0^2 = -|m_0|^2$ in Eq. (2.2), corresponding to the symmetric and broken phases, respectively.

III. THE FINITE TEMPERATURE FREE ENERGY IN THE LDE TO $\mathcal{O}(\delta^2)$

We perform the standard derivation of the free energy [42] up to $\mathcal{O}(\delta^2)$. With the constant field introduced

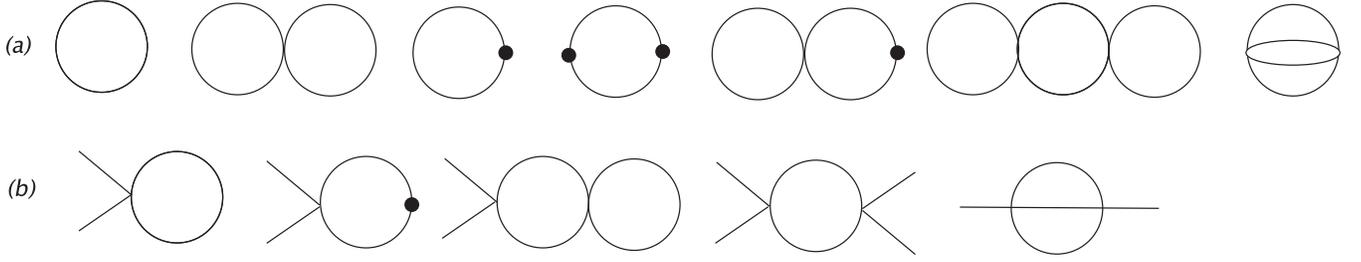


FIG. 1. Diagrams contributing to the free energy up to $\mathcal{O}(\delta^2)$, given by (a) vacuum diagrams and (b) background field (external legs). The black dots indicate a $\delta\eta^2$ insertion.

through the usual shift of the scalar field, $\phi \rightarrow \phi + \varphi$, the Lagrangian density is rewritten as

$$\mathcal{L}[\phi(x), \varphi] = \mathcal{L}_2[\phi(x), \varphi] + \mathcal{L}_I[\phi(x), \varphi], \quad (3.1)$$

where \mathcal{L}_2 is the part of the Lagrangian quadratic in the fields,

$$\mathcal{L}_2[\phi(x), \varphi] = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\Omega^2 \phi^2, \quad (3.2)$$

while \mathcal{L}_I is

$$\mathcal{L}_I[\phi(x), \varphi] = -\frac{\delta\lambda}{6} \varphi \phi^3 - \frac{\delta\lambda}{4!} \varphi^4, \quad (3.3)$$

where in Eq. (3.2) Ω^2 is given by

$$\Omega^2 = \pm m_0^2 + \frac{\delta\lambda}{2} \varphi^2 + (1 - \delta)\eta^2. \quad (3.4)$$

Note that in all loop contributions the propagators will carry a mass term as given by Eq. (3.4). These terms are then expanded in δ to the desired order, thus generating the insertions of η^2 that appear as a consequence of the quadratic vertex introduced in Eq. (2.1).

The free energy is

$$F[\varphi] = F_0(\varphi) + F_{1\text{-loop}}(\varphi) + \frac{1}{\mathcal{V}} i \ln \left\langle \exp \left\{ i \int d^4x \mathcal{L}_I[\phi(x), \varphi] \right\} \right\rangle, \quad (3.5)$$

where $F_0(\varphi)$ is the tree-level classical potential and $F_{1\text{-loop}}(\varphi)$ is the one-loop contribution to the free energy (\mathcal{V} is the space volume) given by

$$F_{1\text{-loop}}(\varphi) = \frac{1}{\mathcal{V}} i \ln \int d\phi e^{i \int d^4x \mathcal{L}_2[\varphi, \phi(x)]}. \quad (3.6)$$

Higher loops are given by the last term in Eq. (3.5), with the average $\langle \dots \rangle$ meaning

$$\langle \dots \rangle = \frac{\int \mathcal{D}\phi (\dots) e^{i \int d^4x \mathcal{L}_2[\phi(x), \varphi]}}{\int \mathcal{D}\phi e^{i \int d^4x \mathcal{L}_2[\phi(x), \varphi]}}. \quad (3.7)$$

As said above, the scalar field propagators in the diagrams are obtained from $\mathcal{L}_2[\phi(x), \varphi]$, and the vertices are determined from $\mathcal{L}_I[\phi(x), \varphi]$, with both as given at the end of Sec. II.

Our calculations are performed, as usual, in the imaginary-time formalism [42]. Thus, the scalar boson field has Euclidean four-momentum $P = (\omega_n, \mathbf{p})$, with $P^2 = \omega_n^2 + \mathbf{p}^2$, where ω_n are the discrete Matsubara bosonic frequencies $\omega_n = 2\pi n/\beta$, with $n = 0, \pm 1, \pm 2, \dots$, and $\beta = 1/T$. Loop diagrams involve sums over the Matsubara frequencies and integrals over the space momentum \mathbf{p} . All space momentum integrals are performed in arbitrary dimension $d = 3 - 2\epsilon$ and renormalization is performed in the modified minimal subtraction scheme ($\overline{\text{MS}}$). The measure used in the sum-integrals is then defined as

$$\int \! \! \! \int = \left(\frac{e^\gamma \mu^2}{4\pi} \right)^\epsilon \beta^{-1} \sum_n \int \frac{d^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}}, \quad (3.8)$$

where μ is an arbitrary momentum scale in dimensional regularization. The factor $(\frac{e^\gamma \mu^2}{4\pi})^\epsilon$ is introduced so that, after minimal subtraction of the poles in ϵ due to ultraviolet divergences, μ coincides with the renormalization scale in the $\overline{\text{MS}}$ scheme.

From Eq. (3.5), the free energy is expressed up to $\mathcal{O}(\delta^2)$ by expanding all appropriate terms in δ . Considering the vacuum contributions to the free energy, this means that terms up to three loops must be included. All bare (unrenormalized) contributions are shown in Fig. 1.

The renormalization procedure for the symmetric phase was performed in detail in Refs. [20,31]. The counterterms for the vacuum diagrams are given in Ref. [20], while those

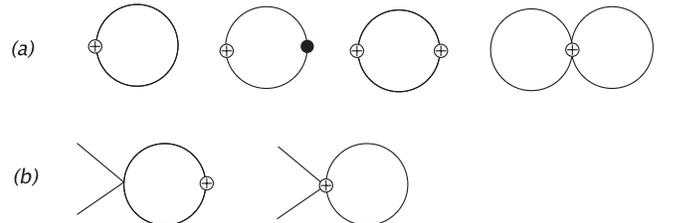


FIG. 2. Diagrams representing the counterterms for the free energy up to $\mathcal{O}(\lambda^2)$: (a) vacuum contribution, (b) background field contribution. As in Fig. 1, the black dot indicates a $\delta\eta^2$ insertion. The circle-cross denotes either insertion of a mass counterterm or of a vertex counterterm.

for the field dependent diagrams are given in Ref. [31]. We also note that the divergences in the broken phase can be removed by the same counterterms determined for the symmetric phase [41,43,44], so the renormalization for the broken phase does not require extra effort. The renormalization proceeds just as in standard perturbation theory and as shown in detail in Ref. [31], only temperature independent counterterms are required and the temperature dependent divergent terms cancel out exactly. All diagrams of counterterms contributing to $F[\varphi]$ up to $\mathcal{O}(\delta^2)$ are shown in Fig. 2.

The circle-cross in Fig. 2 denotes either a mass counterterm vertex Δm^2 , or a vertex counterterm $\Delta\lambda$, given, respectively, by [31]

$$\Delta m^2 = \delta \frac{\lambda}{32\pi^2\epsilon} [(m^2 + (1 - \delta)\eta^2)] - \delta^2 \frac{\lambda^2}{(32\pi^2)^2} \left(\frac{-2}{\epsilon^2} + \frac{1}{\epsilon} \right) (m^2 + \eta^2), \quad (3.9)$$

$$\Delta\lambda = -\delta^2 \frac{3\lambda^2}{32\pi^2\epsilon}. \quad (3.10)$$

The final expression for the renormalized free energy $F[\varphi]$, including all terms shown in Figs. 1 and 2, becomes

$$F[\varphi] = F_{\text{vacuum}} + F_\varphi, \quad (3.11)$$

where F_{vacuum} denotes the vacuum contributions,

$$\begin{aligned} F_{\text{vacuum}} = & -\frac{1}{8(4\pi)^2} \left[2\ln\left(\frac{\mu^2}{\mathcal{M}^2}\right) + 3 \right] \mathcal{M}^4 - \frac{1}{2(4\pi)^2} J_0(\beta\mathcal{M}) T^4 + \delta \frac{\lambda}{8(4\pi)^4} \left[\left(\ln\left(\frac{\mu^2}{\mathcal{M}^2}\right) + 1 \right) \mathcal{M}^2 - J_1(\beta\mathcal{M}) T^2 \right]^2 \\ & + \delta \frac{\eta^2}{2(4\pi)^2} \left[\left(\ln\left(\frac{\mu^2}{\mathcal{M}^2}\right) + 1 \right) \mathcal{M}^2 - J_1(\beta\mathcal{M}) T^2 \right] - \delta^2 \frac{\eta^4}{4(4\pi)^2} \left[\ln\left(\frac{\mu^2}{\mathcal{M}^2}\right) + J_2(\beta\mathcal{M}) \right] \\ & - \delta^2 \frac{\lambda}{4(4\pi)^4} \eta^2 \left[\ln\left(\frac{\mu^2}{\mathcal{M}^2}\right) + J_2(\beta\mathcal{M}) \right] \left[\left(\ln\left(\frac{\mu^2}{\mathcal{M}^2}\right) + 1 \right) \mathcal{M}^2 - J_1(\beta\mathcal{M}) T^2 \right] \\ & - \delta^2 \frac{\lambda^2}{48(4\pi)^6} \left\{ \left[5\ln^3\left(\frac{\mathcal{M}^2}{\mu^2}\right) + 17\ln^2\left(\frac{\mathcal{M}^2}{\mu^2}\right) + \frac{41}{2} \ln\left(\frac{\mathcal{M}^2}{\mu^2}\right) - 23 - \frac{23}{12\pi^2} - \psi''(1) + C_0 \right] \right. \\ & + 3 \left(\ln\left(\frac{\mathcal{M}^2}{\mu^2}\right) + 1 \right)^2 J_2(\beta\mathcal{M}) \left. \right] \mathcal{M}^4 - \left[12\ln^2\left(\frac{\mathcal{M}^2}{\mu^2}\right) + 28\ln\left(\frac{\mathcal{M}^2}{\mu^2}\right) - 12 - \pi^2 - 4C_1 \right. \\ & + 6 \left(\ln\left(\frac{\mathcal{M}^2}{\mu^2}\right) + 1 \right) J_2(\beta\mathcal{M}) \left. \right] J_1(\beta\mathcal{M}) \Omega^2 T^2 \\ & + \left. \left[3 \left(3\ln\left(\frac{\mathcal{M}^2}{\mu^2}\right) + 4 \right) J_1^2(\beta\mathcal{M}) + 3J_1^2(\beta\mathcal{M}) J_2(\beta\mathcal{M}) + 6K_2 + 4K_3 \right] T^4 \right\}, \quad (3.12) \end{aligned}$$

and F_φ denotes the background field dependent contributions,

$$\begin{aligned} F_\varphi = & F_0 + \left\{ \frac{\delta\lambda\mathcal{M}^2}{32\pi^2} \left[\log\left(\frac{\mathcal{M}^2}{\mu^2}\right) - 1 + \frac{T^2}{\mathcal{M}^2} J_1(\beta\mathcal{M}) \right] - \delta^2 \frac{\lambda\eta^2}{32\pi^2} \left[\ln\left(\frac{\mathcal{M}^2}{\mu^2}\right) - J_2(\beta\mathcal{M}) \right] \right. \\ & - \delta^2 \frac{\lambda^2\mathcal{M}^2}{2(32\pi^2)^2} \left[\left(\ln\left(\frac{\mathcal{M}^2}{\mu^2}\right) \right)^2 + \frac{\pi^2}{6} \right] - \delta^2 \frac{3\lambda^2\mathcal{M}^2}{2(32\pi^2)^2} \left[\left(\ln\left(\frac{\mathcal{M}^2}{\mu^2}\right) - 1 \right)^2 + 1 + \frac{\pi^2}{6} \right] \\ & + \delta^2 \frac{\lambda^2}{1024\pi^4} \left[\mathcal{M}^2 \left(1 + \frac{\pi^2}{6} \right) + 4\mathcal{M}^2 \ln\left(\frac{\mu}{\mathcal{M}}\right) \left[1 + J_2(\beta\mathcal{M}) \right] + J_2(\beta\mathcal{M}) \mathcal{M}^2 + 8\mathcal{M}^2 \ln^2\left(\frac{\mu}{\mathcal{M}}\right) \right. \\ & - 4\ln\left(\frac{\mu}{\mathcal{M}}\right) J_1(\beta\mathcal{M}) T^2 - J_2(\beta\mathcal{M}) J_1(\beta\mathcal{M}) T^2 \left. \right] + \delta^2 \frac{\lambda^2 T^2}{24(4\pi)^2} \left[\ln\left(\frac{\mathcal{M}^2}{T^2}\right) + 5.3025 \right] \\ & + \delta^2 \frac{\lambda^2\mathcal{M}^2}{256\pi^4} \left[\frac{\pi^2}{24} - 3\ln\left(\frac{\mathcal{M}}{\mu}\right) + 2\ln^2\left(\frac{\mathcal{M}}{\mu}\right) + 1.164032 \right] \left. \right\} \frac{\varphi^2}{2} - \delta^2 \frac{3\lambda^2}{32\pi^2} \left[\log\left(\frac{\mathcal{M}^2}{\mu^2}\right) - J_2(\beta\mathcal{M}) \right] \frac{\varphi^4}{4!} \quad (3.13) \end{aligned}$$

with F_0 given by

$$F_0 = \frac{1}{2} [\pm m_0^2 + (1 - \delta)\eta^2] \varphi^2 + \delta \frac{\lambda}{24} \varphi^4. \quad (3.14)$$

In Eqs. (3.12) and (3.13), $\mathcal{M}^2 = \pm m_0^2 + \eta^2$, and the constant terms appearing in Eq. (3.12) are defined as follows: $\psi''(1) = -2\zeta(3)$, where $\zeta(x)$ is the zeta function, $C_2 \simeq 39.429$, and $C_3 \simeq -9.8424$, while K_2 and K_3 are three-

dimensional integrals that can be evaluated numerically [45]. In the high-temperature limit, $\mathcal{M}/T \ll 1$, they are given by [20]

$$\begin{aligned} K_2 \simeq & \frac{32\pi^4}{9} [\ln(\beta\mathcal{M}) - 0.04597] \\ & - 372.65\beta\mathcal{M} [\ln(\beta\mathcal{M}) + 1.4658] \quad (3.15) \end{aligned}$$

and

$$K_3 \simeq 453.51 + 1600\beta\mathcal{M}[\ln(\beta\mathcal{M}) + 1.3045]. \quad (3.16)$$

In Eqs. (3.12) and (3.13), we have also defined the temperature dependent integrals J_n ($n = 0, 1, 2$) as follows:

$$J_n(a) = \frac{4\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2}-n)} \int_0^\infty dx \frac{x^{4-2n}}{\sqrt{x^2+a^2}} \frac{1}{e^{\sqrt{x^2+a^2}} - 1}, \quad (3.17)$$

which can be expressed as a series expansion as follows [1,42,46,47]:

$$\begin{aligned} J_0(a) &= \frac{8\pi}{3}a^3 + a^4 \left(\ln\left(\frac{a}{4\pi}\right) + \gamma - \frac{3}{4} \right) \\ &+ 128 \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!! \zeta(2n+1) a^{(2n+4)}}{32(n+2)! 2^{n+1} (2\pi)^{2n}} \\ &- \frac{4\pi^2}{3}a^2 + \frac{16}{45}\pi^4, \end{aligned} \quad (3.18)$$

$$\begin{aligned} J_1(a) &= -4\pi a - 2a^2 \left[\ln\left(\frac{a}{4\pi}\right) + \gamma - \frac{1}{2} \right] + \frac{4\pi^2}{3} \\ &- 16 \sum_{n=1}^{\infty} \left(\frac{(-1)^n (2n-1)!! \zeta(2n+1) a^{(2n+2)}}{4n! 2^{n+1} (n+1) (2\pi)^{2n}} \right), \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} J_2(a) &= \frac{2\pi}{a} + 2 \ln\left(\frac{a}{4\pi}\right) + 2\gamma \\ &+ 4 \left[\sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!! \zeta(2n+1) a^{2n}}{n! 2^{n+1} (2\pi)^{2n}} \right]. \end{aligned} \quad (3.20)$$

Equations (3.18), (3.19), and (3.20) are all convergent in the high-temperature limit as can be easily checked by considering a few terms in the sums in these equations.

We should note that when optimizing the free energy, since J_0 , J_1 , and J_2 are dependent on η , it is important to check the stability of the results when truncating the sums in Eqs. (3.18), (3.19), and (3.20). This is particularly critical for parameter values such that \mathcal{M}/T is not much smaller than 1, a situation that requires a fairly large number of terms in the sums. In all results shown in the next section we have used enough terms in Eqs. (3.18), (3.19), and (3.20) so to obtain stable results for all parameter and temperature values used.

IV. OPTIMIZATION AND NUMERICAL RESULTS

We now turn to the application of the optimization procedures in the LDE and show the results obtained by implementing the PMS, Eq. (1.2), and FAC, Eq. (1.4). As we explained in the introduction, the optimization criteria are applied directly to the free energy. The results obtained with each optimization criterion are contrasted with each other and with those available from other methods. This

will then allow us to gauge the performance of each optimization procedure regarding both reliability and convergence.

A. Symmetric phase

We initially restrict our calculations to the symmetric phase (with positive mass term in the classical potential) and evaluate the pressure, $P = -F$. In Fig. 3 we show our results for the pressure using the usual perturbation theory in λ up to $\mathcal{O}(\lambda^2)$ and where we have restricted to the case of high temperatures ($\mathcal{M}/T \ll 1$). In this figure the behavior of the pressure is shown as a function of the renormalized coupling constant, λ_R , and T_0 is a reference energy scale chosen as $T_0 = m_R(T_0)$, where m_R is the renormalized mass. This is similar as done in Ref. [22] using the 2PI method. Note that in Ref. [22] the authors define the quartic coupling differently from us. In their case, $g^2 = \lambda/24$, and their results are plotted as a function of the renormalized coupling g_R . We choose here the same scale as in Ref. [22] so to facilitate the comparison with their results for the pressure. It is clear in Fig. 3 the typical alternating behavior of the perturbative calculation, which indicates its very poor convergence.

Next, we use the result for the free energy evaluated up to $\mathcal{O}(\delta^2)$, given by Eq. (3.12). Note that in the symmetric phase the pressure depends only on vacuum terms, since the free energy is evaluated at the vacuum expectation value for the field, $\varphi = 0$. By optimizing the free energy using the PMS criterion, Eq. (1.2), we determine the root $\bar{\eta}$, which is then substituted back into the expression for the free energy, with the criterion used for choosing the optimum root as discussed below Eq. (1.4). This naturally brings nonlinear λ contributions and generates nonperturbative results. The pressure obtained in this case is shown in Fig. 4, where we show the results obtained up to orders δ and δ^2 . In the same figure we also show the perturbative

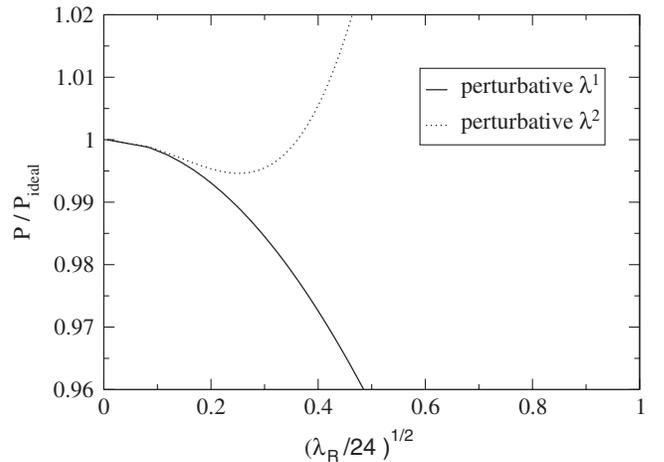


FIG. 3. The pressure in the symmetric phase using perturbation theory in λ up to $\mathcal{O}(\lambda^2)$. The parameters used are $T = 20T_0$ for the temperature and $\mu = T_0$ for the renormalization scale.

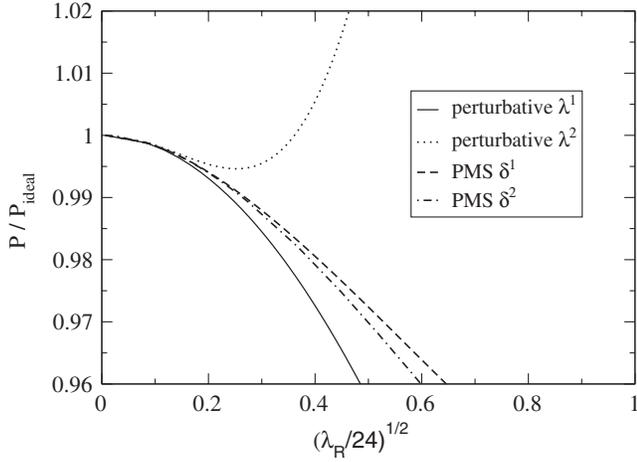


FIG. 4. The pressure in the symmetric phase to orders δ and δ^2 using the PMS optimization criterion—the perturbative results of Fig. 3 are also shown. The parameters used are the same as those in Fig. 3.

results of Fig. 3 for comparison. It becomes evident here the convergence of the results with the LDE-PMS, with both $\mathcal{O}(\delta)$ and $\mathcal{O}(\delta^2)$ results not differing too much, in contrast to the perturbative (in λ) results.

In Fig. 5 we show again the results for the pressure, but now using the FAC optimization criterion, Eq. (1.4). We once again see the excellent convergence for the pressure when contrasted to the perturbative results.

In Fig. 6 we plot side by side our results for the pressure at order δ^2 using the PMS and FAC optimization criteria. It is seen as an excellent agreement between the two optimization criteria and it shows the equivalence of these two optimization procedures.

A side by side comparison of the order δ^2 result for the pressure (from either the PMS or FAC) with the 2PI two-loop result of Ref. [22] (second panel of their Fig. 1) shows an excellent agreement between the two results. Since

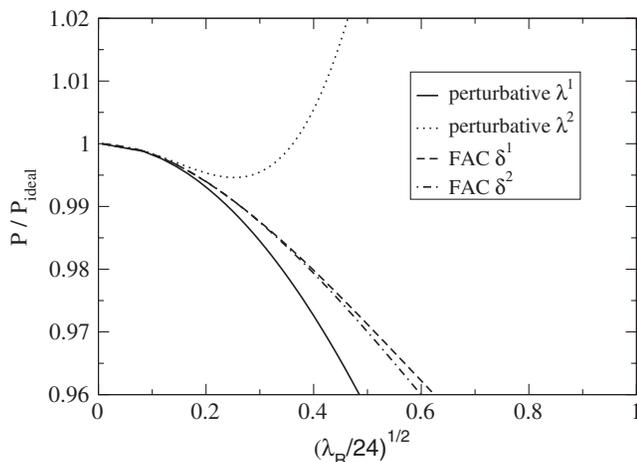


FIG. 5. The pressure in the symmetric phase to orders δ and δ^2 using the FAC optimization criterion.

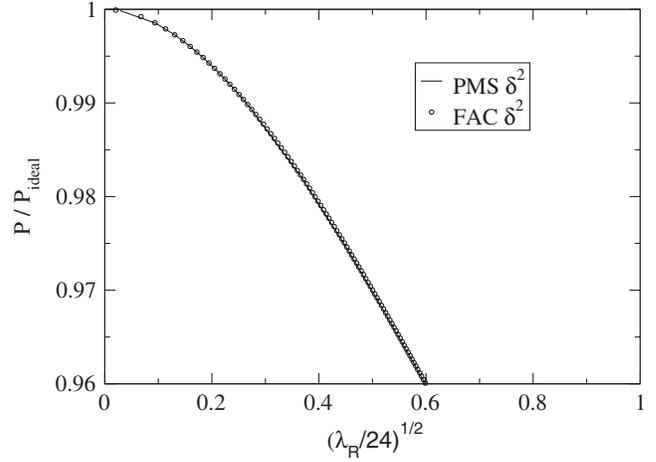


FIG. 6. The pressure for the symmetric phase at order δ^2 using the PMS and FAC optimization criteria.

operationally the LDE is much simpler to be implemented than the 2PI calculation and also when compared with other methods, like those based on the renormalization group and Schwinger-Dyson equations, this may be a great advantage of the LDE. Many previous applications of the LDE to a large variety of problems (cited previously) also confirm the strength of the method. Its strength comes basically from the fact that its implementation is similar to that of standard perturbation theory. The important and fundamental difference with standard perturbation theory resides in the optimization procedure that fixes an initial, *a priori*, arbitrary parameter, η . It is then interesting to investigate what kind of role the optimum η represents in the LDE after optimization. This is partially clarified in the plot shown in Fig. 7, where we show the optimum η as a function of the renormalized coupling constant. It shows

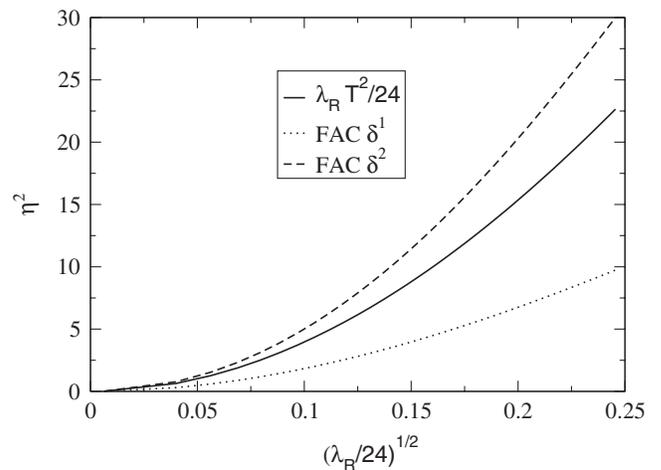


FIG. 7. The behavior of the optimum parameter η^2 with respect to the renormalized coupling constant and evaluated at order δ and δ^2 using the FAC optimization criterion. The parameters used are the same as in the previous figures.

that by increasing the order in δ , $\bar{\eta}$ becomes closer and closer to the thermal mass m_T , here computed at one-loop order for simplicity. In general, we can extrapolate this expectation and say that the expected optimum value for η should be the (quantum and) thermal mass (quadratic in the field) corrections, as would be derived from a true gap equation. This is in fact confirmed by the many applications of the LDE to the Gross-Neveu model [30], in which case exact results are known (in the large- N approximation) and can then be readily compared with the results obtained from the LDE method applied to that model.

B. Broken phase

Let us now turn to the symmetry broken case (with negative square mass term in the classical potential). For this case we found that only the FAC optimization criterion leads to nontrivial solutions for η . The FAC criterion is applied to the free energy and the resulting nonlinear equation is solved simultaneously with the equation giving the minimum condition for the field (thermal) expectation value, $\nu(T)$, given by the condition

$$\left. \frac{dF[\varphi]}{d\varphi} \right|_{\varphi=\nu(T)} = 0. \quad (4.1)$$

As is well known, the phase transition in the pure scalar theory is second order [15], as required by universality reasons. Our results for the free energy using the FAC criterion indicate a second order phase transition. This is shown in Fig. 8, where the free energy for $\lambda = 0.1$ is plotted for different temperature values. The critical temperature obtained here is $T_c/\mu \simeq 15.49$, consistent with the perturbative prediction [42] and other nonperturbative calculations [10]. Another quantity that indicates that the transition is a continuous one is the temperature evolution

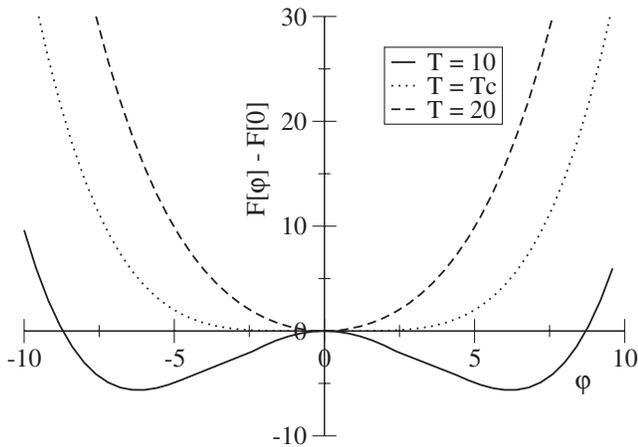


FIG. 8. The nonperturbative free energy for $\lambda = 0.1$ evaluated at order δ^2 and using the FAC optimization criterion, for three different temperatures: $T < T_c$, $T = T_c$, and $T > T_c$, where $T_c = 15.49$ (in units of the renormalization scale μ). Here we have set $\mu = m_0$.

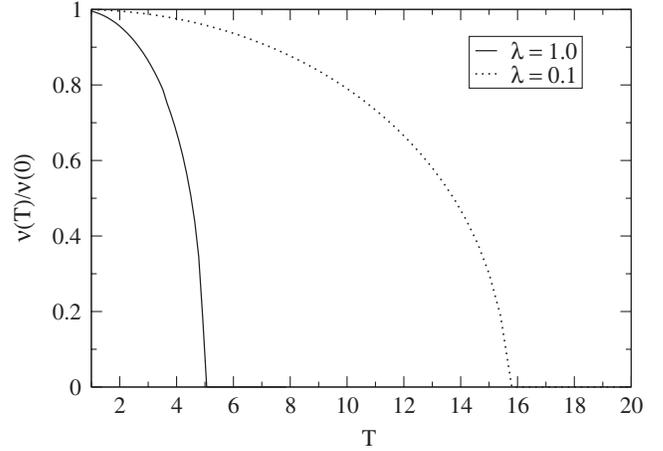


FIG. 9. Temperature dependence of the minimum of the free energy $\nu(T)$ at order δ^2 obtained with the FAC optimization criterion. Here $\nu(0)$ is the tree-level minimum of the bare free energy.

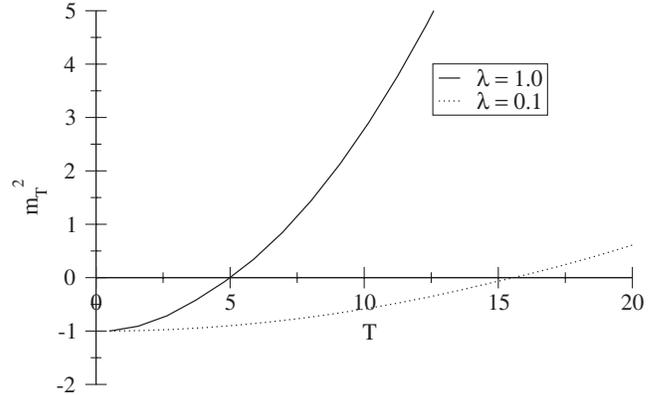


FIG. 10. Temperature dependence of the nonperturbative thermal mass at order δ^2 evaluated with the FAC criterion for two values of the coupling constant λ . We use in this plot $\mu = m_0$.

of the minimum of free energy, $\nu(T)$. This is shown in Fig. 9 for $\lambda = 0.1$ and $\lambda = 1.0$.

Finally, in Fig. 10 we show the temperature dependence of the thermal mass, m_T , as derived from the free energy,

$$m_T^2 = \left. \frac{d^2 F[\varphi]}{d^2 \varphi} \right|_{\varphi=0}. \quad (4.2)$$

We once again can notice a continuous and smooth transition. We note that one can determine the critical temperature by looking at which value of T m_T^2 changes sign and check whether this gives the same result for T_c as obtained from $\nu(T_c) = 0$ (as in Fig. 9).

V. CONCLUSIONS

One of the motivations for using the LDE to study the thermodynamics of the scalar field theory at high temperatures, as done in this work, was its ease of implementation

and renormalization, which is no different from those of the standard perturbation theory. One recalls that similar studies in the context of the 2PI and related methods typically face difficulties in the renormalization procedure, making their applicability a nontrivial task. In addition, the LDE method, differently from other methods, like SPT (or OPT—optimized perturbation theory), makes no assumption on the introduced mass parameter η , thus we do not need to adjust the order of the loop expansion accordingly.

By using the LDE, we have studied the thermodynamics of the $\lambda\phi^4$ scalar field theory in the symmetric and broken phases. The LDE is used with two popular optimization procedures, known as PMS and FAC. There are two major differences with the work we have done here and previous ones, like e.g. in Refs. [31,32]. First, while in general the PMS procedure is the favorite optimization criterion in the literature related to the LDE and similar methods, we here have shown that the FAC procedure leads to numerically indistinguishable results from the ones obtained with the PMS. In addition, while there may be cases where the PMS procedure leads to trivial results only, the same may not be the case for the FAC (here we have shown this to be the case in the broken phase). In this sense they can be used in a complementary way, when PMS fails, one can try FAC, or vice versa. Second, unlike in Refs. [31,32], where the quantity optimized is the self-energy, here we choose to optimize the free energy. One advantage of this is that, while there is no solution for the LDE at first order when optimizing the self-energy, we do find solutions when optimizing the free energy already at first order in δ . Furthermore, as shown in Ref. [30], the optimization of the free energy can be shown to immediately lead to the solution of the gap equation (here verified numerically through the results for the optimum η), while in optimizing the self-energy further constraints must be employed, as, for example, renormalization group equations. In the numerical studies performed in the present work, we have shown that the optimum η carries both temperature and

coupling constant contributions. Thus, the LDE with optimization of the free energy implements automatically a nonperturbative resummation of the thermal corrections, in conformity with analytical results produced when this method was used to study the Gross-Neveu model in Ref. [30], from which exact solutions are available and a close comparison with the LDE results is possible.

By studying the behavior of the pressure and contrasting the results obtained with perturbation theory and the 2PI method, we have shown that the LDE leads to convergent results already at lowest order in the LDE expansion parameter δ , with the first and second order results changing only slightly and producing results consistent with the 2PI nonperturbative method. In addition, as already mentioned above, we have shown that both optimization procedures, FAC and PMS, lead to equivalent results for the pressure.

Another important result of our work is that the LDE is shown to be adequate for studying phase transitions at high temperatures. In particular, when applied to the phase transition in the $\lambda\phi^4$, the LDE predicts the correct order of the phase transition, which is second order, in agreement with general results of statistical mechanics. Besides this, since the LDE method automatically produces an infrared cutoff, the results are shown to be valid and applicable *below*, *above*, and *at* the critical temperature T_c , showing that the LDE circumvents the usual problem seen in perturbation theory, namely, the appearance of infrared divergences close to critical points.

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