

Center-stabilized Yang-Mills theory: Confinement and large N volume independenceMithat Ünsal^{1,*} and Laurence G. Yaffe^{2,+}¹SLAC and Physics Department, Stanford University, Stanford, California 94305, USA²Department of Physics, University of Washington, Seattle, Washington 981951560, USA

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We examine a double trace deformation of $SU(N)$ Yang-Mills theory which, for large N and large volume, is equivalent to unmodified Yang-Mills theory up to $O(1/N^2)$ corrections. In contrast to the unmodified theory, large N volume independence is valid in the deformed theory down to arbitrarily small volumes. The double trace deformation prevents the spontaneous breaking of center symmetry which would otherwise disrupt large N volume independence in small volumes. For small values of N , if the theory is formulated on $\mathbb{R}^3 \times S^1$ with a sufficiently small compactification size L , then an analytic treatment of the nonperturbative dynamics of the deformed theory is possible. In this regime, we show that the deformed Yang-Mills theory has a mass gap and exhibits linear confinement. Increasing the circumference L or number of colors N decreases the separation of scales on which the analytic treatment relies. However, there are no order parameters which distinguish the small and large radius regimes. Consequently, for small N the deformed theory provides a novel example of a locally four-dimensional pure-gauge theory in which one has analytic control over confinement, while for large N it provides a simple fully reduced model for Yang-Mills theory. The construction is easily generalized to QCD and other QCD-like theories.

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I. LARGE N VOLUME INDEPENDENCE

The large N limits of $SU(N)$ Yang-Mills (YM) theories, when formulated on toroidal compactifications of \mathbb{R}^d , are independent of volume provided the $(\mathbb{Z}_N)^d$ center symmetry is not spontaneously broken [1–3].¹ However, above two dimensions, center symmetry *does* break spontaneously when the (smallest) compactification circumference L is less than a critical size L_c [4]. (If just one dimension is compactified, then this center-symmetry breaking transition is the usual thermally induced deconfinement transition.) In four dimensions, the critical size L_c is approximately Λ^{-1} , where Λ is the $\overline{\text{MS}}$ strong scale of the theory [4,5].

Notwithstanding the limitation to $L \geq L_c$, the volume independence of large N Yang-Mills theory (“partial reduction”) has practical utility for lattice studies, because simulations on lattices of size $(L_c)^d$ are sufficient to extract infinite volume properties of large N Yang-Mills theory [4,6–8]. But it would be even more helpful to have a formulation of the theory in which volume independence holds for arbitrarily small volumes—since this allows one to reduce the lattice all the way down to a single site.

Several schemes for preserving volume independence in arbitrarily small volumes have been proposed. In so-called

quenched reduced models, one constrains the eigenvalues of link variables (or Wilson lines) in a manner which prevents Wilson lines from acquiring expectation values [3].² In the $N \rightarrow \infty$ limit, quenched reduced models correctly reproduce properties of infinite volume Yang-Mills theory. However, corrections to the $N = \infty$ limit scale as $1/N$, not $1/N^2$, in quenched reduced models. This makes extracting large N properties from numerical simulations of quenched reduced models quite challenging. An alternative proposal, known as twisted reduced models, involves modifying the Wilson action of a single-site model so that the action explicitly disfavors configurations in which Wilson lines in different directions mutually commute [10,11]. Unfortunately, this clever scheme fails to work sufficiently close to the continuum limit [12,13]. In essence, the penalty imposed by the twisting of the action is insufficient to overcome entropic effects which favor breaking of the center symmetry.

If light adjoint representation fermions are added to an $SU(N)$ Yang-Mills theory, and periodic (not antiperiodic) boundary conditions imposed on the fermions, then the fermion contribution to the Wilson line effective potential stabilizes the unbroken center-symmetry phase. Hence these QCD-like theories satisfy large N volume independence for arbitrarily small volumes [14]. (Moreover, in the large N limit, charge-conjugation even observables coincide between these theories and corresponding theories, in sufficiently large volume, with fermions in the rank-

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¹Center-symmetry transformations are gauge transformations which are periodic only up to an element of the center of the gauge group. Volume independence applies to the leading large N behavior of expectation values and connected correlators of topologically trivial Wilson loops.

²See Ref. [9] for an extended discussion of quenched and twisted reduced models.

two symmetric or antisymmetric tensor representations [15–17].)

Motivated by this fermion-induced stabilization of center symmetry, in this paper we introduce a simple scheme for preserving volume independence in pure Yang-Mills theory. We add double trace terms to the action which prevent spontaneous breaking of the center symmetry, while simultaneously perturbing the dynamics of the unbroken symmetry phase only by $O(1/N^2)$ corrections. This leads to a “stabilized reduced model” which reproduces the dynamics of infinite volume Yang-Mills theory up to relative corrections which scale as $1/N^2$. The construction may be easily generalized to other QCD-like theories with matter fields in rank-one or rank-two representations.

In addition to providing a simple large N reduced model, the deformed Yang-Mills theory is interesting in its own right when N is not large [compared to $(\Lambda L)^{-1}$].³ When formulated on $\mathbb{R}^3 \times S^1$, we show that the large distance dynamics of the theory are analytically tractable provided $N\Lambda L \ll 1$. In this regime, a semiclassical analysis [closely related to Polyakov’s classic treatment of $3d$ $SU(2)$ adjoint Higgs theory [24]] reveals the existence of a mass gap and area law behavior of spatial Wilson loops. It is noteworthy that our compactified, deformed Yang-Mills theory is an analytically tractable confining theory with no fundamental scalar fields or supersymmetry, in contrast to other instructive models of confinement [24–27]. The confinement mechanism involves the formation of a dilute plasma of magnetic monopoles (and antimonopoles) carrying topological charge $\pm 1/N$.⁴

Large N volume independence of our stabilized reduced model holds in the $N \rightarrow \infty$ limit for any fixed compactification radius L . The existence of the analytically tractable volume *dependent* regime, $N\Lambda L \ll 1$, in which one has analytic control over confinement, does not conflict with large N volume independence since, as $N \rightarrow \infty$, one must scale $L \propto (N\Lambda)^{-1}$ to remain within this regime. One may, of course, consider a double scaling limit in which $N\Lambda L$ is held fixed as $N \rightarrow \infty$. We find that the low-lying mass spectrum has a smooth large N limit when $N\Lambda L$ is fixed at some value small compared to one. But large N volume independence does not apply to this double scaling limit with small $N\Lambda L$.

II. DEFORMED YANG-MILLS THEORY

We consider pure Yang-Mills theory with gauge group $SU(N)$ defined on the four manifold $\mathbb{R}^3 \times S^1$, with the S^1

having circumference L . The extension to multiple compactified dimensions will be discussed below, but we begin with a single compactified dimension to simplify the exposition. We start with the usual continuum action,

$$S^{\text{YM}} = \int_{\mathbb{R}^3 \times S^1} \frac{1}{2g^2} \text{tr} F_{\mu\nu}^2(x), \quad (2.1)$$

or a lattice formulation with the Wilson action,

$$S^{\text{YM}} = -\frac{\beta}{2} \sum_{p \in \Lambda_4} \text{tr}(U[\partial p] + U[\partial p]^\dagger), \quad (2.2)$$

where the sum is over all oriented plaquettes, Λ_4 is the four-dimensional spacetime lattice, and $U[\partial p]$ denotes the usual product of link matrices around the boundary of plaquette p . The lattice coupling $\beta \equiv 2/g^2$. In our discussion, we will use both continuum and lattice formulations, and benefit from both perspectives. As usual, a key virtue of the lattice formulation is that it provides an explicit nonperturbative definition of the theory.

Let $\Omega(\mathbf{x}) \equiv \mathcal{P}(e^{i \int_{dx_4} A_4(\mathbf{x}, x_4)})$ denote the Wilson line (or Polyakov loop) operator—the holonomy of the gauge field around a circle wrapping the S^1 and sitting at the point $\mathbf{x} \in \mathbb{R}^3$. We will construct a deformation of the Yang-Mills action on our compactified geometry by adding terms, respecting all symmetries of the unmodified theory, built from the Wilson line operator. The deformed action is given by

$$S^{\text{deformed}} = S^{\text{YM}} + \Delta S, \quad (2.3)$$

with

$$\Delta S \equiv \int_{\mathbb{R}^3} \frac{1}{L^3} P[\Omega(\mathbf{x})] \quad (2.4a)$$

in the continuum, or

$$\Delta S \equiv \frac{1}{N_t^3} \sum_{\mathbf{x} \in \Lambda_3} P[\Omega(\mathbf{x})] \quad (2.4b)$$

on the lattice. In the lattice form, $N_t \equiv L/a$ denotes the size of the lattice in the compactified direction and $\Lambda_3 \subset \Lambda_4$ is a three-dimensional sublattice of the four-dimensional lattice corresponding to a fixed Euclidean time slice. We want the deformation potential $P[\Omega]$ to guarantee the stability of the phase with unbroken center symmetry (at small volume). It will be chosen to have the form

$$P[\Omega] \equiv \sum_{n=1}^{\lfloor N/2 \rfloor} a_n |\text{tr}(\Omega^n)|^2, \quad (2.5)$$

with positive coefficients $\{a_n\}$ (and $\lfloor N/2 \rfloor$ denoting the integer part of $N/2$). In other words, $P[\Omega]$ is a sum of the double trace operators $\text{tr}(\Omega^n) \text{tr}(\Omega^n)^\dagger$. When consider-

³See also related recent work in Refs. [18–23].

⁴Confinement due to such topological objects has been previously discussed in, for example, Refs. [28,29] and references therein. What is novel about our deformed theory at small $N\Lambda L$ is that this confinement mechanism operates in a regime in which one has analytic control over the long-distance dynamics.

ing the large N limit, the coefficients $\{a_n\}$ will be held fixed as $N \rightarrow \infty$.⁵

Under a center-symmetry transformation by some element $z \in \mathbb{Z}_N$, the Wilson loop $\text{tr}(\Omega^p)$ is multiplied by z^p . The value of $p \bmod N$, which determines the \mathbb{Z}_N representation, is referred to as the N -ality. It will be important that the deformation potential (2.5) only contains, by construction, absolute squares of Wilson loops with nonzero N -ality.

If $P[\Omega]$ were simply proportional to $|\text{tr}\Omega|^2$, with a sufficiently large positive coefficient, then this would prevent breaking of the center symmetry with $\langle \text{tr}(\Omega) \rangle$ as an order parameter. But if $N > 3$, then this single term is not sufficient to prevent any spontaneous breaking of center symmetry, as this term alone does nothing to prevent $\text{tr}(\Omega^2)$ from developing an expectation value. In other words, a stabilizing term proportional to $|\text{tr}(\Omega)|^2$ cannot prevent \mathbb{Z}_N breaking to \mathbb{Z}_2 (assuming N is even) with $\text{tr}(\Omega^2)$ as an order parameter.⁶ Adding an additional stabilizing term proportional to $|\text{tr}(\Omega^2)|^2$ could prevent such a breaking to \mathbb{Z}_2 , but does not prevent breaking to \mathbb{Z}_3 (if $N \bmod 3 = 0$), or to any larger discrete subgroup of \mathbb{Z}_N . This is why we have allowed $P[\Omega]$ to include terms up to $|\text{tr}\Omega|^{[N/2]}|^2$.

We will argue that the deformed theory satisfies the following:

- (i) For suitable choices of the deformation parameters a_n (i.e., each coefficient sufficiently large and positive) the stabilizing potential (2.5) will prevent the \mathbb{Z}_N center symmetry from breaking to any subgroup.
- (ii) In the $N \rightarrow \infty$ limit, pure Yang-Mills theory on \mathbb{R}^4 is equivalent to the deformed theory formulated on any $\mathbb{R}^3 \times S^1$ (for choices of the $\{a_n\}$ satisfying point i). This equivalence applies to expectation values of Wilson loops (on \mathbb{R}^4), or the leading large N behavior of their connected correlators. In the lattice formulation, the number of sites in the compactified direction may be reduced to one.

⁵Since Wilson loop traces divided by N are $O(1)$ in the large N limit, this scaling of the coefficients makes each term in deformation potential $O(N^2)$ as $N \rightarrow \infty$. This $O(N^2)$ scaling is the same as the Wilson action, provided the 't Hooft coupling $\lambda \equiv g^2 N$ is held fixed as $N \rightarrow \infty$. In terms of N counting, the sum over winding numbers in the deformation potential is innocuous and should be regarded as analogous to a character expansion of a class function on a group manifold. As will be seen explicitly below, the coefficients a_n may be chosen so that the sum over winding numbers remains finite and bounded even if the sum extends to infinity. It should be noted that since the deformation to the action is $O(N^2)$, large N factorization does *not* apply to the exponential of these double trace terms, $\lim_{N \rightarrow \infty} \langle \exp(\Delta S) \rangle \neq \lim_{N \rightarrow \infty} \exp(\Delta S)$.

⁶Properties of such unusual phases with partially broken center symmetry have been recently studied in Refs. [20–22]. Although interesting in their own right, our goal is to craft deformations of the action which prevent the appearance of these exotic phases.

(iii) When $N\Lambda L \ll 1$ limit, the deformed Yang-Mills theory is solvable in the same sense as the Polyakov model. The existence of a mass gap and linear confinement can be shown analytically. One can regard this regime as having spontaneous breaking of the $SU(N)$ gauge symmetry down to $U(1)^{N-1}$, but this is a perturbative gauge-dependent description with no well-defined invariant content.

(iv) There exist no order parameters which can distinguish the $N\Lambda L \ll 1$ “Higgs” regime from the $N\Lambda L \gg 1$ regime in which gauge symmetry is “restored.”

As noted earlier, pure Yang-Mills theory on $\mathbb{R}^3 \times S^1$ satisfies volume independence (in the large N limit) so long as the \mathbb{Z}_N center symmetry remains unbroken [2–4]. We will show that this is equally true for the deformed theory; the additional terms in the action do not affect the proof that volume independence, at $N = \infty$, is an automatic consequence of unbroken center (and translation) symmetry. In the undeformed theory, the unbroken center-symmetry phase is the low temperature confined phase, $L > L_c$ with $L_c \sim \Lambda^{-1}$ the inverse deconfinement temperature. But in the deformed Yang-Mills theory, the stability of the unbroken center-symmetry phase is enforced by hand for all values of L .

Large N volume independence, and the large N equivalence between ordinary Yang-Mills theory on \mathbb{R}^4 and the deformed theory on $\mathbb{R}^3 \times S^1$, may be demonstrated by comparing Dyson-Schwinger equations (i.e., Migdal-Makeenko loop equations) for expectation values and correlators of Wilson loops, or alternatively by comparing the large N classical dynamics that may be derived by using appropriate large N coherent states [2,30–32]. Figure 1 summarizes the relation between the large N limits of ordinary and deformed Yang-Mills theories. As long as the \mathbb{Z}_N center symmetry is not spontaneously broken, the dynamics of the theories defined by S^{YM} and S^{deformed} are indistinguishable at leading order in the $1/N$ expansion. In particular, glueball spectra of the two theories can differ only by order $1/N^2$ effects. Agreement up to $O(1/N^2)$ terms also applies to the string tension which characterizes the area law behavior of large Wilson loops.

The large N equivalence between ordinary Yang-Mills theory and the deformed theory in large volume, combined with the large N volume independence of the deformed theory, circumvents the problems with previous formulations of reduced models for pure Yang-Mills theory. Unlike the original Eguchi-Kawai model [1], its twisted variant [10,11], and the partial reduction of Refs. [4,5], the equivalence to deformed YM theory remains valid in the limit of zero compactification radius, irrespective of the value of the (bare) gauge coupling. And for finite N , corrections to the large N limit scale as $1/N^2$, not $1/N$ as in quenched reduced models [3]. Consequently, it should be possible to study the deformed theory, for relatively modest values of

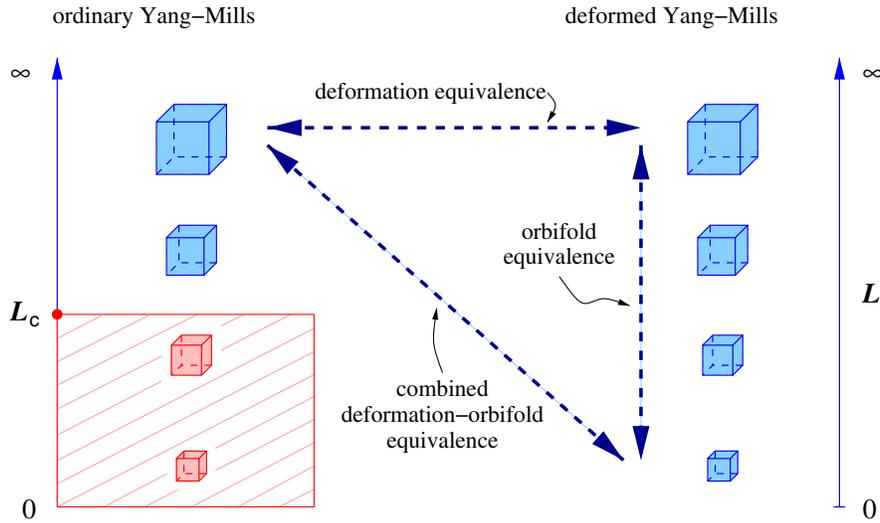


FIG. 1 (color online). Large N equivalences relating ordinary and deformed $SU(N)$ Yang-Mills theories, as a function of the size L of the periodic volume in which the theories are defined. In deformed YM volume independence holds for all L , while in ordinary YM volume independence fails below a critical size, $L < L_c$, (shaded region) due to spontaneous breaking of center symmetry. This prevents reduction all the way down to a single-site matrix model for ordinary YM. Large N equivalence holds between ordinary and deformed YM theories as long as center symmetry is unbroken in ordinary YM. Large N volume independence is a type of orbifold equivalence, as discussed in Ref. [14]. The combination of volume changing orbifold projections in the deformed theory, along with the deformation equivalence in sufficiently large volume, provides a useful equivalence between deformed YM in small volume and ordinary YM in large volumes. In particular, a single-site matrix model of the deformed theory will reproduce properties of ordinary Yang-Mills theory in infinite volume. The construction can be generalized to QCD, with the deformed theory providing a fully reduced matrix model for QCD.

N and vanishingly small volume, and obtain accurate results for properties of ordinary Yang-Mills theory, in the large N limit, on \mathbb{R}^4 . As we discuss below, it is also instructive to study the deformed theory for small values of N . In this regime, it will be seen to provide a novel example of a confining theory, only involving an $SU(N)$ gauge field, which is analytically soluble.

A. Stabilization of center symmetry

The possibility of preventing spontaneous breaking of center symmetry through the addition of a deformation potential of the form (2.5) is largely self-evident. A positive coefficient a_n suppresses configurations in which $\text{tr}(\Omega^n)$ is nonzero. Although the pure-gauge dynamics of the undeformed theory, in small volume, lead to an effective potential for the Wilson line which is minimized when Ω is an element of \mathbb{Z}_N , adding the deformation potential $P[\Omega]$ changes the shape of the Wilson line effective potential. For sufficiently large values of the coefficients $\{a_n\}$, the effective potential will be minimized by configurations in which $\text{tr}(\Omega^n) = 0$ for $1 \leq n \leq [N/2]$. This implies that $\text{tr}(\Omega^n) = 0$ for any integer n which is nonzero modulo N because, for $SU(N)$ -valued matrices, $\text{tr}(\Omega^n)$ is not independent of lower order traces when $n > [N/2]$. Vanishing of these traces implies that the eigenvalues of Ω are uniformly spaced around the unit circle, so that the set of eigenvalues is invariant under \mathbb{Z}_N transformations (which

multiply every eigenvalue by $e^{2\pi i/N}$). This shows that the center symmetry is not spontaneously broken. Henceforth, we assume that the coefficients $\{a_n\}$ of the deformation potential $P[\Omega]$ are suitably chosen so as to enforce unbroken center symmetry for all compactification radii.

This argument may be made much more explicit if one considers small compactifications, $L \ll \Lambda^{-1}$, so that (due to asymptotic freedom) the gauge coupling at the scale of the compactification is small and the theory is amenable to a perturbative treatment. Quantum fluctuations generate a nontrivial potential for the Wilson line [33]. In ordinary Yang-Mills theory, integrating out the gauge field (and Faddeev-Popov ghosts) produces the functional determinants

$$[\det_+(-D_{\text{adj}}^2 \delta_{\mu\nu})]^{-1/2} [\det_+(-D_{\text{adj}}^2)] = [\det_+(-D_{\text{adj}}^2)]^{-1}, \tag{2.6}$$

where \det_+ denotes a determinant in the space of periodic functions with period L . Therefore, the effective potential for the Wilson line is

$$V[\Omega] = L^{-1} \ln \det_+(-D_{\text{adj}}^2). \tag{2.7}$$

For constant (or slowly varying) configurations, the evaluation of the functional determinant is straightforward and yields [33]

$$V[\Omega] = \int_{\mathbb{R}^3} \frac{1}{L^4} \mathcal{V}[\Omega(x)], \quad (2.8)$$

with

$$\mathcal{V}[\Omega] \equiv -\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^4} |\text{tr}\Omega^n|^2. \quad (2.9)$$

For sufficiently small L , corrections to this one-loop result are negligible. The effective potential (2.9) is minimized when the Wilson line is an element of the center, $\Omega = e^{2\pi i k/N}$, $k = 0, \dots, N-1$, demonstrating the spontaneous breaking of \mathbb{Z}_N symmetry, in ordinary Yang-Mills theory, for sufficiently small compactifications.

To force unbroken center symmetry in the deformed theory, the deformation potential $P[\Omega]$ must overcome the effect of the one-loop potential (2.9). A simple specific choice for the deformation coefficients which, in the continuum limit and for sufficiently large N , accomplishes this is $a_n = 4/(\pi^2 n^4)$. For this choice, the deformation potential $\int_{\mathbb{R}^3} L^{-4} P[\Omega]$ is minus twice the one-loop Wilson line effective potential (2.9) of the undeformed theory, so the net effect of the deformation is to flip the sign of the effective potential for the Wilson line. The resulting combined potential is minimized when $\text{tr}\Omega^n = 0$ for all n which are nonzero modulo N , indicating unbroken center symmetry.⁷

It should be noted that in the weak coupling expression for the induced potential (2.9), and likewise in the stabilizing potential (2.5), the loops should be interpreted as renormalized Polyakov loops, which are multiplicatively renormalized in a perimeter-law fashion, $\text{tr}\Omega^n|_{\text{ren}} = Z^n \text{tr}\Omega^n|_{\text{bare}}$, with Z a function of the bare gauge coupling. In general, in the coefficient space of all the stabilizing terms, the boundary between the stabilized and nonstabilized regions will depend on the lattice spacing (or lattice

⁷One might wonder if the need for stabilizing terms involving loops wrapping the torus multiple times could be avoided by simply restricting attention to prime values of N . After all, one can choose to use an infinite sequence of primes to define the $N \rightarrow \infty$, and for N prime the \mathbb{Z}_N symmetry must be either unbroken or completely broken. However, this strategy does not work. To have volume independence at $N = \infty$, one needs stabilizing terms which have the effect of preventing breaking of the center symmetry for all possible values of N . One way to see this is to note that, for large N , expectation values of Wilson loops are smooth in N . Although one could take a large N limit via a sequence of primes, one is also free to use a sequence involving values of N which are composite. A valid stabilizing deformation must work for either choice. More concretely, for large but prime values of N if one omits stabilizing terms involving N -ality k loops, then for sufficiently small values of L the Wilson line eigenvalue distribution will develop k clumps, with some clumps having one more eigenvalue than other clumps. This will not reproduce the desired large volume physics. To keep the eigenvalue distribution uniform and prevent such eigenvalue clumping, all the terms in the stabilizing potential (2.5) are needed.

coupling β). The origin (corresponding to no deformation potential) lies inside the stabilized region for sufficiently large lattice spacing or strong coupling, and we know from the weak coupling analysis where the boundary lies near the continuum limit. In between, however, when the bare coupling is neither very large nor very small, the precise location of the boundary can only be determined numerically.

B. Multiple compactified dimensions

Instead of compactifying just one dimension, one may consider the gauge theory on $\mathcal{M} \equiv \mathbb{R}^{4-d} \times T^d$, where T^d is a $d > 1$ dimensional torus. For simplicity, we will discuss the case of a symmetric torus having size L in each dimension. Two particularly interesting cases are $d = 3$, where the $L \rightarrow 0$ limit will reduce to $SU(N)$ matrix quantum mechanics, and $d = 4$, where the zero size limit will reduce to a matrix model.

With multiple compactified dimensions, the center symmetry is $(\mathbb{Z}_N)^d$. The preceding discussion generalizes in a straightforward fashion, but there is one significant change: with multiple compactified dimensions there is a richer set of possible symmetry realizations in which the center symmetry is partially broken. One needs to craft the deformation potential in such a fashion that all unwanted symmetry realizations are suppressed.

With multiple compactified directions, one may define distinct Wilson line operators $\Omega_1, \Omega_2, \dots, \Omega_d$, each of which wraps one elementary cycle of the torus. The classical Yang-Mills action suppresses configurations in which Wilson lines in different directions are noncommuting. (In other words, configurations with vanishing field strength must have Wilson lines which commute and are covariantly constant.) A deformation potential with the form (2.5), involving just one Wilson line Ω_i , can prevent symmetry realizations in which the \mathbb{Z}_N center symmetry associated with direction i spontaneously breaks in a manner which is independent of the \mathbb{Z}_N symmetries associated with other directions. However, it is also possible for the center symmetry to break in a manner which leaves an unbroken subgroup involving correlated symmetry transformations in different directions. (This phenomenon has been studied in the context of quenched large N reduction in the recent paper [34].) With, for example, $d = 2$, suppose that the $(\mathbb{Z}_N)^2$ symmetry breaks to the diagonal \mathbb{Z}_N subgroup. Such a symmetry realization does not allow expectations of $\text{tr}\Omega_1^k$ or $\text{tr}\Omega_2^k$ to be nonvanishing (for any k which is nonzero modulo N). But this realization does allow $\text{tr}\Omega_1\Omega_2^\dagger$ to have a nonzero expectation value. To prevent such a symmetry realization, it may be necessary to add a $|\text{tr}\Omega_1\Omega_2^\dagger|^2$ term to the deformation potential.

To see that this concern is not moot, it is instructive to examine the one-loop effective potential for Wilson lines when there are multiple compactified directions. Once again, if the physical size of the d torus is much smaller

than the inverse strong scale Λ^{-1} , then ‘‘Kaluza-Klein’’ modes with momenta at and above the compactification scale can be integrated out perturbatively. This produces a one-loop effective potential for mutually commuting Wilson lines given by [35,36]

$$\begin{aligned} \mathcal{V}[\Omega_1, \dots, \Omega_d] \\ \equiv -\frac{1}{\pi^2} \sum_{(n_1, \dots, n_d) \in (\mathbb{Z}^{d-d})}^{\infty} \frac{|\text{tr}(\Omega_1^{n_1} \dots \Omega_d^{n_d})|^2}{(n_1^2 + \dots + n_d^2)^2}. \end{aligned} \quad (2.10)$$

This effective potential has N^d degenerate global minima at which each Wilson line is an independent element of the center. This demonstrates the spontaneous breaking of $(\mathbb{Z}_N)^d$ symmetry (down to the identity) in ordinary Yang-Mills theory on $\mathcal{M} = \mathbb{R}^{4-d} \times T^d$ for sufficiently small compactifications.⁸ Adding a deformation to the Yang-Mills action which is the direct generalization of the previously discussed deformation (2.4), namely,

$$\Delta S = \sum_i \int_{\mathcal{M}} \frac{1}{L^4} P[\Omega_i], \quad (2.11)$$

with $P[\Omega]$ having the form (2.5), will deform the effective potential and can prevent this symmetry realization in which the $(\mathbb{Z}_N)^d$ symmetry breaks down to nothing.

However, the fact that all terms in the effective potential (2.10) have negative coefficients demonstrates that the center symmetry *will* partially break spontaneously even in the presence of stabilizing terms of the form (2.11). In particular, even when all traces wrapping a single compactified direction vanish, the potential (2.10) favors configurations in which traces wrapping multiple cycles (such as $\text{tr} \Omega_1 \Omega_2^\dagger$) are nonzero relative to the center-symmetry preserving configuration we are trying to stabilize (for which $\text{tr} \Omega_1^{n_1} \dots \Omega_d^{n_d} = 0$ for all n_i that are nonzero modulo N).

Therefore, to prevent any spontaneous breaking of the center symmetry when there are multiple compactified directions, one must allow the deformation of the action

⁸More precisely, this demonstrates the spontaneous breaking of $(\mathbb{Z}_N)^d$ symmetry down to the identity when $d = 1$ or 2 . If $d > 2$, then there are fewer than two noncompactified directions. In this case, the center symmetry cannot break spontaneously for any finite value of N , due to the nonzero probability of fluctuations (or tunneling events) which effectively average over all degenerate minima of the Wilson line effective potential. However, the probability of such fluctuations vanishes exponentially with increasing N . Consequently, in the $N \rightarrow \infty$ limit one can have spontaneous breaking of the $(\mathbb{Z}_N)^d$ center symmetry even when all directions are compactified. In order to have a large N equivalence between the compactified theory and Yang-Mills theory on \mathbb{R}^4 , one must prevent spontaneous breaking of the center symmetry in the large N limit. Therefore, the more complicated form of the stabilizing potential discussed below is necessary for all $d > 1$.

to include absolute squares of order parameters which wrap multiple cycles of the torus. In other words, we need to have

$$\Delta S = \int_{\mathcal{M}} \frac{1}{L^4} P[\Omega_1, \dots, \Omega_d], \quad (2.12)$$

where the deformation potential has the form

$$P[\Omega_1, \dots, \Omega_d] = \sum_{n_1, \dots, n_d = -[N/2]}^{[N/2]} a_{n_1, \dots, n_d} |\text{tr}(\Omega_1^{n_1} \dots \Omega_d^{n_d})|^2, \quad (2.13)$$

with coefficients a_{n_1, \dots, n_d} which are sufficiently large and positive.

A simple specific choice for the coefficients which, in the continuum limit and for sufficiently large N , accomplishes this is $a_{n_1, \dots, n_d} = \frac{2}{\pi^2} (n_1^2 + \dots + n_d^2)^{-2}$. For this choice, the deformation potential is minus twice the one-loop effective potential (2.10) of the undeformed theory, so once again the net effect of the deformation is to flip the sign of the effective potential for the Wilson line. The resulting combined potential has a unique center-symmetry preserving minimum.⁹

In the following sections we will, for simplicity, largely focus on the case of just one compactified direction.

C. Large N equivalence between ordinary and deformed YM

The most direct way to demonstrate equivalence between ordinary Yang-Mills theory and our deformed theory is to compare the Schwinger-Dyson (or loop) equations for gauge invariant observables. As usual, for a rigorous treatment it is appropriate (and convenient) to work with lattice regulated formulations of both theories. It is also conve-

⁹In the case of $\mathbb{R} \times T^3$, the undeformed theory was studied in detail by Lüscher, van Baal, and others in the mid 1980s. (See the reviews [37,38] and references therein.) The goal of this ‘‘QCD in a box’’ program was to use asymptotic freedom combined with the absence of the phase transitions in finite volumes to extract lessons about QCD on \mathbb{R}^4 . This approach confines the theory to its short distance perturbative regime where it is perturbatively solvable. However, features of this small universe are very different from QCD on \mathbb{R}^4 . In particular, when the physical box size becomes comparable to the inverse strong scale Λ^{-1} , there is a crossover from a world of hadrons to a world of quarks and gluons. As noted in footnote 8, this crossover becomes a sharp phase transition in the large N limit. In our deformed Yang-Mills theory, as will be discussed below, there are no phase transitions as a function of box size even in the $N = \infty$ limit. For the deformed theory we will find that it is not $1/L$ which acts as an infrared cutoff, but rather $1/LN$. If $LN\Lambda \gg 1$ then the deformed theory reproduces the dynamics of Yang-Mills theory on \mathbb{R}^4 .

nient to consider $U(N)$ gauge theories instead of $SU(N)$; the difference in gauge groups only affects subleading $O(1/N^2)$ relative corrections to Wilson loop expectation values or connected correlators.

Let δ_ℓ^a denote an operator which varies individual link fields according to $\delta_\ell^a(U[\ell']) \equiv \delta_{\ell\ell'} t^a U[\ell]$, where $\{\ell^a\}$ is a set of $U(N)$ Lie algebra basis matrices satisfying $\text{tr} t^a t^b = \frac{1}{2} \delta^{ab}$. Invariance of the Haar measure implies that the integral of any variation vanishes,

$$\int d\mu_0 \delta_\ell^a(\text{anything}) = 0, \quad (2.14)$$

where $d\mu_0 \equiv \prod_{\ell'} dU[\ell']$. Choose (anything) to be $e^{-S} \delta_\ell^a W[C]$, where $W[C] \equiv \frac{1}{N} \text{tr} U[C]$ is the Wilson loop around some closed contour C . Summing over the Lie algebra index a and the link ℓ yields

$$\int d\mu_0 e^{-S} (-\delta S \cdot \delta W[C] + \delta^2 W[C]) = 0, \quad (2.15)$$

where the dot product is shorthand for the sum over a and ℓ . Dividing by the partition function $Z \equiv \int d\mu_0 e^{-S}$ yields relations among expectation values,

$$-\langle \delta S \cdot \delta W[C] \rangle + \langle \delta^2 W[C] \rangle = 0. \quad (2.16)$$

These are Schwinger-Dyson equations for Wilson loop expectation values.

After working out the action of the variations, the result may be expressed in a purely geometric form. For lattice gauge theory with the Wilson action one finds [39]

$$\begin{aligned} \frac{1}{2} |C| \langle W[C] \rangle &= \sum_{\ell \subset C} \sum_{p | \ell \subset \partial p} \frac{\beta}{4N} [\langle W[(\partial \bar{p})C] \rangle - \langle W[(\partial p)C] \rangle] \\ &+ \sum_{\text{self-intersections}} \mp \langle W[C'] W[C''] \rangle. \end{aligned} \quad (2.17)$$

Here $|C|$ is the length of the loop C (i.e., the number of links in the loop), $W[(\partial p)C]$ denotes a Wilson loop which goes around the boundary of plaquette p (which contains a link contained in the contour C) and then around the contour C , and $\partial \bar{p}$ denotes the oppositely oriented plaquette boundary. The sum over self-intersections runs over all ways of decomposing a loop C which multiply traverses some link ℓ into two separate loops, $C = C' C''$, with the associated sign determined by whether C' and C'' traverse the link ℓ in the same or opposite directions. See Ref. [40] for more detailed discussion.

In the large N limit, with $\tilde{\beta} \equiv \frac{\beta}{N} = \frac{2}{\lambda}$ held fixed (where $\lambda \equiv g^2 N$ is the 't Hooft coupling), all N dependence dis-

appears. Fluctuations in the values of Wilson loops vanish in this limit (their distributions become arbitrarily sharply peaked). This is a reflection of the classical nature of the large N limit [2], and implies that the expectation value of a product of loops factorizes, up to $1/N^2$ corrections,

$$\langle W[C'] W[C''] \rangle = \langle W[C'] \rangle \langle W[C''] \rangle + O(1/N^2). \quad (2.18)$$

[The $O(1/N^2)$ remainder is the connected correlator.] Consequently, in the large N limit Wilson loop expectation values satisfy a closed set of nonlinear algebraic equations,

$$\begin{aligned} \frac{1}{2} |C| \langle W[C] \rangle &= \sum_{\ell \subset C} \sum_{p | \ell \subset \partial p} \frac{\tilde{\beta}}{4} [\langle W[(\partial \bar{p})C] \rangle - \langle W[(\partial p)C] \rangle] \\ &+ \sum_{\text{self-intersections}} \mp \langle W[C'] \rangle \langle W[C''] \rangle. \end{aligned} \quad (2.19)$$

The loop equations in the concise form (2.16) are equally valid for the deformed theory. The only difference is that S now includes the double trace deformation ΔS , and this generates new terms in the loop equations given by $\langle \delta(\Delta S) \cdot \delta W[C] \rangle$. Just as the usual Wilson action leads to terms in which a plaquette is inserted into the loop C , the piece of ΔS proportional to $|\text{tr} \Omega^k|^2$ generates terms in the loop equation in which the topologically nontrivial loop Ω^k (or its inverse) is ‘‘sewn’’ into the loop C (if C contains links pointing in the compactified direction). But because ΔS contains absolute squares of traces, each such term is multiplied by the complex conjugate of the trace of the inserted loop. Hence,

$$\langle \delta(\Delta S) \cdot \delta W[C] \rangle = \sum_{k \neq 0} b_k[C] \langle W[\Omega^k C] W[\Omega^{-k}] \rangle, \quad (2.20)$$

where $\Omega^k C$ denotes a loop obtained by concatenating Ω^k and C at their intersection links, and the coefficients $b_k[C]$ are proportional to $a_{|k|}$ but also depend on the number of links in C which point in the compactified direction. The essential point is that the variation acts on one of the two traces comprising the double trace deformation, leaving the other trace unchanged. In the large N limit, due to factorization,

$$\langle W[\Omega^k C] W[\Omega^{-k}] \rangle = \langle W[\Omega^k C] \rangle \langle W[\Omega^{-k}] \rangle + O(1/N^2). \quad (2.21)$$

But for any k which is nonzero modulo N , $W[\Omega^{-k}]$ transforms nontrivially (acquiring a phase $e^{-2\pi i k/N}$) under a \mathbb{Z}_N center-symmetry transformation. Hence its expectation value is an order parameter for the center symmetry and $\langle W[\Omega^{-k}] \rangle$ must vanish in any phase with unbroken center symmetry. As discussed above, the deformed theory is

constructed so as to ensure unbroken center symmetry for all compactification radii. Consequently, all additional terms in the loop equations generated by the deformation of the action vanish in the large N limit,

$$\langle \delta(\Delta S) \cdot \delta W[C] \rangle = O(1/N^2), \quad (2.22)$$

implying that Wilson loops in the original and deformed Yang-Mills theory satisfy identical large N Schwinger-Dyson equations.

Ordinary Yang-Mills theory has unbroken center symmetry only for sufficiently large compactifications, $L > L_c$. The coinciding large N loop equations in ordinary and deformed Yang-Mills theories imply that Wilson loop expectation values in these two theories have identical large N limits when $L > L_c$.¹⁰

The same approach may be used to compare the Schwinger-Dyson equations satisfied by connected correlators of two or more Wilson loops, with exactly the same conclusion: the leading large N behavior of connected correlators coincides between ordinary and deformed Yang-Mills theories, provided $L > L_c$. Thus, in sufficiently large volume the net effect of the double trace deformation on the dynamics of the theory is $O(1/N^2)$, and vanishes in the large N limit. In other words, the physics of the deformed Yang-Mills theory depends on the deformation parameters $\{a_i\}$ only in the combination a_i/N^2 which vanishes at $N = \infty$. This demonstrates the nonperturbative equivalence of ordinary Yang-Mills theory and the deformed YM theory, formulated on $\mathbb{R}^3 \times S^1$ (or more generally, on any toroidal compactification of flat space), provided the compactification size is above the critical size for center symmetry breaking in the undeformed theory.

D. Large N volume independence of deformed YM theory

Unbroken center symmetry is necessary and sufficient for the validity of the large N volume independence of Yang-Mills theory (or more general gauge theories containing adjoint representation matter fields). This may be demonstrated by comparing large N loop equations, or the $N = \infty$ classical dynamics generated by suitable coherent

¹⁰This argument, that coinciding loop equations imply coinciding expectation values, glosses over the possibility that the infinite set of loop equations may have multiple solutions which respect center symmetry, with different theories potentially corresponding to different solutions of the same set of equations. The alternative approach of comparing the $N = \infty$ classical dynamics generated by appropriate large N coherent states, discussed in Ref. [30], eliminates this loophole and demonstrates equivalence in any phase of the theories which satisfy the necessary and sufficient symmetry realization conditions.

states [14]. Corrections to this equivalence for finite- N scale as $1/N^2$. The loop equation analysis is very similar to that sketched above. In the large N loop equations for topologically trivial Wilson loops, one finds that the only volume-dependent terms (arising from self-intersections) automatically vanish as long as the center symmetry is not spontaneously broken.

The analysis of large N volume independence in Ref. [14] applies equally well to the deformed theory which, by construction, has unbroken center symmetry for any compactification radius. Because the double trace operators in $P[\Omega]$ are squares of loops with nonzero N -ality, the presence of the deformation potential $P[\Omega]$ has no effect on the large N classical dynamics of center-symmetry symmetric states. Consequently, deformed Yang-Mills theory, in the large N limit, is completely volume independent.

In the lattice formulation, if one compactifies all directions then one may reduce the lattice size all the way down to a single site, in which case the theory becomes a simple matrix model of Wilson lines $\{\Omega_i\}$ running in each lattice direction with action,

$$\begin{aligned} S_{\text{single-site}}^{\text{deformed}} = & -\frac{\beta}{2} \sum_{i>j=1}^d \text{tr}(\Omega_i \Omega_j \Omega_i^\dagger \Omega_j^\dagger + \Omega_j \Omega_i \Omega_j^\dagger \Omega_i^\dagger) \\ & + P[\Omega_1, \dots, \Omega_d]. \end{aligned} \quad (2.23)$$

The large N limit of this matrix model will reproduce the leading large N behavior of expectation values and connected correlators of Wilson loops in uncompactified Yang-Mills theory. As discussed in Sec. I, the single-site deformed Yang-Mills theory (2.23) provides a simple generalization of Eguchi-Kawai reduction which is valid for any value of the lattice coupling β .^{11,12}

E. Addition of matter fields

Consider adding N_f species of matter fields (either fermions or scalars) in the fundamental representation to $SU(N)$ Yang-Mills theory, either ordinary or deformed, with one dimension compactified. The addition of funda-

¹¹We recently learned that the idea of adding stabilizing terms to the action of the single-site Eguchi-Kawai model, to prevent unwanted spontaneous symmetry breaking, was previously suggested (back in 1982) in Ref. [41].

¹²For a fixed finite value of N , the single-site model (2.23) can, and will, have nonzero expectation values for loops whose winding numbers are multiples of N . This does not mean that this finite- N reduced model reproduces properties of a normal lattice gauge theory on a periodic lattice of size N^d . (Suggestions to this effect appear in some discussions of quenched and twisted large N reductions.) Rather, the reduced model reproduces properties of a larger volume $SU(N)$ gauge theory only in the $N \rightarrow \infty$ limit.

mental representation matter explicitly breaks the \mathbb{Z}_N center symmetry. However, if N_f is held fixed as $N \rightarrow \infty$, then the fundamental representation matter fields have only a subleading $O(N_f/N)$ effect on the gauge field dynamics. As a result, everything discussed above remains valid. That is, the leading large N behavior of expectation values or connected correlators of Wilson loops in the undeformed theory, in sufficiently large volume, coincide with the corresponding observables in the theory, in arbitrary volume, deformed by the addition of the stabilizing potential $P[\Omega]$. In addition, one may also show that the same equivalence applies to the leading large N behavior of mesonic expectation values and connected correlators.¹³ Note, however, that these large N equivalences cease to apply if N_f/N is held fixed as $N \rightarrow \infty$ [14].

If adjoint representation matter fields are added to the theory (ordinary or deformed), then the large N equivalences discussed above also remain valid. Adding adjoint representation fields enlarges the natural set of gauge invariant observables from simple Wilson loops to Wilson loops decorated by arbitrary numbers of insertions of adjoint matter fields. But the presence of adjoint matter fields preserves the center symmetry of the underlying Yang-Mills theory. As a result, the above-described comparison of large N loop equations (or large N classical dynamics) between the ordinary and deformed theories immediately generalizes to the case of Yang-Mills theories with adjoint matter, with exactly the same conclusions.¹⁴

Finally, one may also consider the addition of matter fields in rank-two antisymmetric or symmetric tensor representations [yielding theories we will refer to as QCD(AS) or QCD(S), respectively]. The presence of fields in these representations reduces the $U(1)$ center symmetry of $U(N)$ Yang-Mills theory down to \mathbb{Z}_2 . Given the central role the center symmetry played in the above large N equivalences, one might think this reduction in center symmetry would destroy these large N equivalences. This is not the case. One way to see this is to note, as discussed in Ref. [14], that volume-dependent terms in the $N = \infty$ loop equations only appear if loops with nonzero winding number around the compactified direction acquire nonzero expectation values. The addition of the deformation potential $P[\Omega]$ prevents topologically nontrivial Wilson loops from acquiring nonzero large N expectation values, even in small volumes. Consequently, the situation is analogous to the pure-gauge case: the leading large N behavior of expectation values or connected correlators of single trace observables in QCD(AS/S) in sufficiently large volume coincides

¹³This is easiest to understand by considering the equivalent gluonic observables produced by integrating out the matter fields. For the case of large N volume independence in the undeformed theory, see Ref. [14] for details. The presence of the deformation potential does not affect this analysis.

¹⁴For a detailed discussion of loop equations in theories with adjoint matter, see Ref. [40].

with the corresponding observables in the theory modified by the addition of the deformation potential in arbitrary volume, as depicted in Fig. 1.¹⁵

III. CONFINEMENT AT SMALL RADIUS AND FIXED N

When compactified on a small circle, $L \ll \Lambda^{-1}$, the gauge coupling of the deformed $SU(N)$ theory is small at the compactification scale, $g^2(1/L) \ll 1$. As discussed earlier, the combined potential $\mathcal{V}[\Omega] + P[\Omega]$ is minimized when

$$\Omega = \text{Diag}(1, e^{2\pi i/N}, e^{4\pi i/N}, \dots, e^{2\pi i(N-1)/N}), \quad (3.1)$$

up to conjugation by an arbitrary $SU(N)$ matrix. Working in a gauge in which Ω is diagonal, and using (temporarily) gauge-dependent language, this configuration may be regarded as breaking the gauge symmetry down to the maximal Abelian subgroup,¹⁶

$$SU(N) \rightarrow U(1)^{(N-1)}. \quad (3.2)$$

Modes of the diagonal components of the $SU(N)$ gauge field with no momentum along the compactified \hat{x}_4 direc-

¹⁵Another way to understand this is to note the existence of a large N equivalence (so-called orientifold equivalence) between theories with rank-two symmetric or antisymmetric representation matter and corresponding theories with adjoint representation matter [“QCD(adj)”] [15–17]. This large N equivalence applies to the charge-conjugation even sectors of the two theories, and only holds if charge-conjugation symmetry is not spontaneously broken. When, for example, the matter fields are fermions with periodic boundary conditions, examination of the Wilson line effective potential shows that QCD(AS/S) *does* spontaneously break both charge-conjugation and center symmetry when compactified with sufficiently small size [15]. But the addition of a deformation potential of the form (2.5) (with sufficiently positive coefficients) will prevent this spontaneous symmetry breaking, just as it does in the pure Yang-Mills case. Since QCD(adj) satisfies large N volume independence (as long as its center symmetry is not spontaneously broken), the same large N volume independence must also apply to QCD(AS/S) (as long as charge conjugation is not broken). In sufficiently large volumes, there is no reason to believe that charge-conjugation symmetry breaks spontaneously in QCD-like theories with rank-two tensor representation matter. Therefore, large N orientifold equivalence combines with large N volume independence of QCD(adj) to imply volume independence in QCD(AS/S) as long as center and charge-conjugation symmetries are not spontaneously broken—which is what the deformation potential ensures.

¹⁶The infrared dynamics of the small S^1 center-symmetric gauge theory are very different from those of the undeformed theory for which a small S^1 corresponds to a high temperature T . In the latter case, eigenvalues of Ω clump at one of the N minima of the thermal one-loop potential, spontaneously breaking the \mathbb{Z}_N center symmetry, and the dynamics are nonperturbative and remain non-Abelian on wavelengths larger than the magnetic scale $(\lambda T)^{-1}$. No analog of this magnetic scale appears for small S^1 compactifications in center-symmetry respecting theories, which instead Abelianize at large distances.

tion describe photons associated with the Cartan subgroup of $SU(N)$. Modes of the diagonal components of the gauge field with nonzero momentum in the compactified direction form a Kaluza-Klein tower and receive masses which are integer multiples of $2\pi/L$. The off-diagonal components of the $SU(N)$ gauge field describe Kaluza-Klein towers of W bosons which are charged under the unbroken $U(1)^{N-1}$ gauge group. The nonzero value of $A_4 \equiv -(i/L)\ln\Omega$ shifts the masses of these off-diagonal components by multiples of $2\pi/(NL)$. The net effect is that there are charged W bosons with masses

$$m_{W_k} = \frac{2\pi k}{NL}, \quad k = 1, 2, \dots, \quad (3.3)$$

For later convenience, we define m_W to be the mass of the lightest W bosons,

$$m_W \equiv \frac{2\pi}{NL}. \quad (3.4)$$

This is the mass scale below which the dynamics are effectively Abelian.¹⁷ Non-Abelian running of the gauge coupling ceases below this scale—which is why the long-distance dynamics will be under analytic control. Note that, at fixed L , the lightest W bosons have masses which become small when $N \rightarrow \infty$. This will be important in the discussion of the large N behavior of the deformed YM theory. But first, in this section, we consider the dynamics of the deformed theory when N is fixed.

The $N - 1$ photons of the Cartan subgroup do not couple (directly) to the Wilson line and remain massless to all orders in perturbation theory. Thus, a strictly perturbative analysis would lead one to expect that the deformed theory, for sufficiently small L , would have a nonconfining Coulomb phase. We will see that this is incorrect—nonperturbative effects lead to the generation of a mass gap and produce confining long-distance physics.

The analysis of nonperturbative properties in our compactified deformed Yang-Mills theory is very similar to Polyakov's treatment of the $3d$ Georgi-Glashow model [24]. But instead of a three-dimensional theory with a noncompact Higgs field, we have a compactified four-dimensional theory with the group-valued Wilson line Ω serving as a compact Higgs field. For theories involving massless complex fermions, the difference between compact and noncompact Higgs systems can be major [42]. However, in our case, the differences relative to Polyakov's classic discussion are rather minimal.

¹⁷Fluctuations in the eigenvalues of Ω away from the minimum (3.1) correspond to neutral ‘‘Higgs bosons.’’ The masses of these fluctuations depend on the coefficients $\{a_n\}$ of the deformation potential but parametrically are of order $\sqrt{\lambda}/L \sim \sqrt{\lambda N} m_W$. This scale will be large compared to the mass scale of the nonperturbative $3d$ dynamics, and these fluctuations will play no role in the following discussion.

Because of the $SU(N) \rightarrow U(1)^{N-1}$ gauge symmetry ‘‘breaking,’’ there exist topologically stable, semiclassical field configurations, namely, monopoles [33]. At the center of a monopole, one pair of eigenvalues of the Wilson line become degenerate. For fundamental (i.e., minimal action) monopoles, this will be a pair of eigenvalues which are nearest neighbors at infinity. If the adjoint Higgs field was noncompact, then there would be $N - 1$ species of fundamental monopoles. This follows from the topological considerations: the second homotopy group $\pi_2[SU(N)/U(1)^{N-1}] = \pi_1[U(1)^{N-1}] = \mathbb{Z}^{N-1}$, implying that fundamental monopoles come in $N - 1$ varieties. However, with a compact Higgs field there is an extra fundamental (‘‘Kaluza-Klein’’) monopole which arises due to the fact that the underlying theory is formulated on a cylinder, $\mathbb{R}^3 \times S^1$, or equivalently that the configuration space of Ω is compact.

The monopoles may be characterized by their magnetic charges, topological charge, and action. The magnetic charges of the N different types of fundamental monopoles are proportional to the simple roots and affine root of the Lie algebra of the unbroken $U(1)^N$ gauge group.¹⁸ The simple roots are given by¹⁹

$$\alpha_1 = (1, -1, 0, \dots, 0) = \hat{e}_1 - \hat{e}_2, \quad (3.5a)$$

$$\alpha_2 = (0, 1, -1, \dots, 0) = \hat{e}_2 - \hat{e}_3, \quad (3.5b)$$

⋮

$$\alpha_{N-1} = (0, \dots, 0, 1, -1) = \hat{e}_{N-1} - \hat{e}_N, \quad (3.5c)$$

and the affine root is

$$\alpha_N \equiv - \sum_{j=1}^{N-1} \alpha_j = (-1, 0, 0, \dots, 1) = \hat{e}_N - \hat{e}_1. \quad (3.6)$$

For later convenience, let Δ_{aff}^0 denote the affine (extended) root system of the associated Lie algebra,

$$\Delta_{\text{aff}}^0 \equiv \{\alpha_1, \alpha_2, \dots, \alpha_{N-1}, \alpha_N\}. \quad (3.7)$$

It is the affine root system which is relevant for compact Yang-Mills Higgs systems. The roots $\alpha_i \in \Delta_{\text{aff}}^0$ obey

$$\alpha_i \cdot \alpha_j = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}, \quad i, j = 1, \dots, N. \quad (3.8)$$

¹⁸Even though the gauge symmetry breaking is $SU(N) \rightarrow U(1)^{N-1}$, for ease of presentation it is convenient to add an extra photon to the original theory and discuss $U(N) \rightarrow U(1)^N$. This simplifies the discussion of charge assignments of monopoles, and the affine roots of the associated Lie algebra. In the continuum limit of the theory this extra photon completely decouples from the other degrees of freedom and may simply be ignored. It should not be confused with the $N - 1$ photons which have nontrivial nonperturbative dynamics.

¹⁹This set of simple roots corresponds to choosing Lie algebra generators normalized to satisfy $\text{tr} t^a t^b = \delta^{ab}$, instead of $\frac{1}{2} \delta^{ab}$ as in the previous section.

The form (3.8) of these inner products will translate into self- and nearest-neighbor interactions between monopoles in the Dynkin space. The above choice of basis is natural due to its visual simplicity, but the inner products (3.8) of the roots of the associated Lie algebra are basis independent.

Let $F^\mu \equiv \frac{1}{2g} \epsilon^{\mu\nu\lambda} F_{\nu\lambda}$ denote the $U(1)^N$ -valued $3d$ magnetic field, with conventional perturbative normalization. (In a gauge where Ω is diagonal, F^μ is just the list of diagonal elements of the original non-Abelian field strength, multiplied by $1/g$.) The magnetic charges of a monopole of type $i = 1, \dots, N$ are given by the root α_i (up to a factor of $2\pi/g$),

$$\int_{S^2} d\Sigma \cdot \mathbf{F} = \frac{2\pi}{g} \alpha_i \quad [\text{type } (i) \text{ monopole}]. \quad (3.9)$$

(The S^2 is an arbitrarily large sphere in \mathbb{R}^3 . The flux is independent of the value of x^4 at which the integral is performed, as the long-distance monopole fields are independent of x^4 .)

The topological charge is correlated with the magnetic charge of the monopole. For fundamental monopoles with magnetic charges $\alpha_i \in \Delta_{\text{aff}}^0$, the topological charge is

$$\nu \equiv \int_{\mathbb{R}^3 \times S^1} \frac{1}{16\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} = \frac{1}{N}. \quad (3.10)$$

For antimonopoles with magnetic charges $-\alpha_i$, the topological charge $\nu = -1/N$.

The electric charges of W bosons may also be simply expressed in terms of the affine roots Δ_{aff}^0 . The lightest W bosons, with mass m_W , may be labeled by a single root which gives their electric charges (up to a factor of g),

$$Q_{W_{\alpha_i}} = g\alpha_i. \quad (3.11)$$

W bosons in the next heavier multiplet are labeled by a pair of neighboring roots, and have charges

$$Q_{W_{\alpha_i + \alpha_{i+1}}} = g(\alpha_i + \alpha_{i+1}), \quad (3.12)$$

etc. Dot products of the W -boson charges and monopole charges obey the Dirac quantization condition,

$$\begin{aligned} Q_{W_{\alpha_i}} \cdot Q_{M_{\alpha_j}} &= g\alpha_i \cdot \frac{2\pi}{g} \alpha_j = 2\pi(2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}) \\ &= \begin{cases} 4\pi, & \text{for } i = j; \\ -2\pi, & \text{for } i = j \pm 1; \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (3.13)$$

Conjugation by a \mathbb{Z}_N ‘‘shift’’ matrix, which is part of the global gauge symmetry, cyclically permutes the Wilson line eigenvalues and hence cyclically permutes the N different species of fundamental monopoles. The presence of this symmetry (which is one of the features which distinguishes compact and noncompact Higgs systems) guarantees that the N different types of fundamental mono-

poles have identical values of the action. Monopole solutions are self-dual,

$$F_{\mu\nu} = \tilde{F}_{\mu\nu}, \quad (3.14)$$

and hence the Yang-Mills action of a fundamental monopole (or antimonopole) is

$$\begin{aligned} S_{\text{YM}} &= \int_{\mathbb{R}^3 \times S^1} \frac{1}{2g^2} \text{tr} F_{\mu\nu}^2 = \left| \int_{\mathbb{R}^3 \times S^1} \frac{1}{2g^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \right| \\ &= \frac{8\pi^2}{g^2} |\nu| = \frac{8\pi^2}{g^2 N}. \end{aligned} \quad (3.15)$$

After adding the contributions of the deformation potential $P[\Omega]$ and the induced one-loop effective potential $\mathcal{V}[\Omega]$, the complete monopole action will differ from this value. But the deviation is perturbative in g^2 , so the monopole action

$$S_0 \equiv S_{\text{YM}} + \Delta S + S_{1\text{-loop}} = \frac{8\pi^2}{g^2 N} + O(1). \quad (3.16)$$

The correct infrared description of the deformed Yang-Mills theory on $\mathbb{R}^3 \times S^1$ at small radius is generated by a dilute gas of monopoles (and antimonopoles) of N different types, interacting via the species-dependent long range Coulomb potential,

$$\begin{aligned} V_{\pm i, \pm j}(\mathbf{r}) &= L \left(\frac{2\pi}{g} \right)^2 \frac{(\pm \alpha_i) \cdot (\pm \alpha_j)}{4\pi |\mathbf{r}|} \\ &= \pm L \left(\frac{2\pi}{g} \right)^2 \frac{2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}}{4\pi |\mathbf{r}|}. \end{aligned} \quad (3.17)$$

(The overall sign is plus for monopole-monopole, and minus for monopole-antimonopole.) Hence, we are dealing with a multicomponent classical plasma, with nearest-neighbor interactions in the Dynkin space. As with any classical plasma, this system will exhibit Debye screening. The field due to a static external magnetic charge will fall exponentially with distance, $|\mathbf{F}| \sim e^{-m_D r}/r^2$, with m_D^{-1} the characteristic Debye screening length. This implies that external fields cannot propagate coherently over distances large compared to the Debye length, which will be the longest correlation length in the system. The Debye mass m_D will appear as a dynamically generated photon mass. This will be shown explicitly.

For momentum scales small compared to the lightest W mass, the equilibrium dynamics are correctly represented by a grand canonical ensemble of all types of monopoles and antimonopoles. Consider a configuration in which, for each type $i = 1, \dots, N$, there is an arbitrary number $n^{(i)}$ of monopoles and $\bar{n}^{(i)}$ antimonopoles, located at positions $\mathbf{r}_k^{(i)}$, $k = 1, \dots, n^{(i)}$, and $\bar{\mathbf{r}}_l^{(i)}$, $l = 1, \dots, \bar{n}^{(i)}$, respectively. The magnetic field generated by this ensemble of magnetic charges is

$$\mathbf{B}(\mathbf{x}) = \sum_{i=1}^N \frac{2\pi}{g} \alpha_i \left[\sum_{k=1}^{n^{(i)}} \frac{\mathbf{x} - \mathbf{r}_k^{(i)}}{4\pi|\mathbf{x} - \mathbf{r}_k^{(i)}|^3} - \sum_{l=1}^{\bar{n}^{(i)}} \frac{\mathbf{x} - \bar{\mathbf{r}}_l^{(i)}}{4\pi|\mathbf{x} - \bar{\mathbf{r}}_l^{(i)}|^3} \right]. \quad (3.18)$$

The action of such a monopole configuration is the sum of the monopole self-energies plus their potential energy due to Coulomb interactions,

$$S_{\text{monopole-gas}} = S_0 \sum_{i=1}^N (n^{(i)} + \bar{n}^{(i)}) + S_{\text{int}}, \quad (3.19)$$

with

$$S_{\text{int}} = \frac{2\pi^2 L}{g^2} \sum_{i,j=1}^N \alpha_i \cdot \alpha_j \left[\sum_{k=1}^{n^{(i)}} \sum_{l=1}^{n^{(j)}} G(\mathbf{r}_k^{(i)} - \mathbf{r}_l^{(j)}) + \sum_{k=1}^{\bar{n}^{(i)}} \sum_{l=1}^{\bar{n}^{(j)}} G(\bar{\mathbf{r}}_k^{(i)} - \bar{\mathbf{r}}_l^{(j)}) - 2 \sum_{k=1}^{n^{(i)}} \sum_{l=1}^{\bar{n}^{(j)}} G(\mathbf{r}_k^{(i)} - \bar{\mathbf{r}}_l^{(j)}) \right], \quad (3.20)$$

and

$$G(\mathbf{r}) \equiv \frac{1}{4\pi|\mathbf{r}|}. \quad (3.21)$$

The grand canonical partition function of this multicomponent Coulomb gas is

$$Z = \prod_{i=1}^N \left\{ \sum_{n^{(i)}=0}^{\infty} \frac{\zeta^{n^{(i)}}}{n^{(i)}!} \sum_{\bar{n}^{(i)}=0}^{\infty} \frac{\zeta^{\bar{n}^{(i)}}}{\bar{n}^{(i)}!} \times \int_{\mathbb{R}^3} \prod_{k=1}^{n^{(i)}} d\mathbf{r}_k^{(i)} \int_{\mathbb{R}^3} \prod_{l=1}^{\bar{n}^{(i)}} d\bar{\mathbf{r}}_l^{(i)} \right\} e^{-S_{\text{int}}}, \quad (3.22)$$

where

$$\zeta \equiv C e^{-S_0} = A m_W^3 (g^2 N)^{-2} e^{-\Delta S} e^{-8\pi^2/Ng^2(m_W)} \quad (3.23)$$

is the monopole fugacity. The prefactor C represents the one-loop functional determinant in the monopole background. Extracting the zero modes of the small fluctuation operator via the usual collective coordinate procedure leads to factors of $(g^2 N)^{-2} m_W^3$. (See the appendix for details.) If the coupling is evaluated at the scale m_W , which is natural for this problem, then the nonzero mode part of the one-loop determinant merely gives rise to an overall dimensionless (and N -independent) coefficient A . In the final form of (3.23), ΔS denotes the deformation term in the action (2.4) evaluated in the background of a fundamental monopole. This is an $O(1)$ number, independent of the coupling g^2 , whose explicit value depends, of course, on the deformation parameters $\{a_n\}$. [For large N , ΔS scales as $O(1/N)$.]

Using the fact that $G(\mathbf{r})$ is the Green's function for the $3d$ Laplacian, this partition function can be exactly transformed into a $3d$ scalar field theory with an N -component real scalar field,

$$Z = \int \prod_{i=1}^N \mathcal{D}\sigma_i e^{-S^{\text{dual}}[\boldsymbol{\sigma}]}, \quad (3.24)$$

where

$$S^{\text{dual}} = \int_{\mathbb{R}^3} \left[\frac{1}{2L} \left(\frac{g}{2\pi} \right)^2 (\nabla \boldsymbol{\sigma})^2 - \zeta \sum_{i=1}^N \cos(\alpha_i \cdot \boldsymbol{\sigma}) \right]. \quad (3.25)$$

To verify this, it is easiest to start with the functional integral (3.24), rewrite the cosines in terms of exponentials of $\boldsymbol{\sigma}$, expand the exponential of each of the resulting interaction terms in a power-series in $e^{-S_0} e^{\pm i\alpha_i \cdot \boldsymbol{\sigma}}$, and then perform the functional integral over $\boldsymbol{\sigma}$. The scalar fields σ_i appearing in this representation are dual fields for the $3d$ Abelian gauge fields A_μ^i . The dual description (3.25) is valid for distances large compared to m_W^{-1} .²⁰

The fields $\{\sigma_i\}$ should be regarded as compact scalar fields defined modulo 2π . In addition to invariance under 2π shifts in any component of $\boldsymbol{\sigma}$, note that the monopole induced interaction vertex has the additional shift symmetry

$$\boldsymbol{\sigma} \rightarrow \boldsymbol{\sigma} + 2\pi\boldsymbol{\mu}_i, \quad i = 1, \dots, N-1, \quad (3.26)$$

where $\{\boldsymbol{\mu}_i\}$ are the $N-1$ fundamental weights of the $SU(N)$ algebra. These are defined by the reciprocity relation with the simple roots

$$\boldsymbol{\mu}_i \cdot \boldsymbol{\alpha}_j = \frac{1}{2} \delta_{ij} \alpha_j^2 = \delta_{ij} \quad (3.27)$$

for $i = 1, \dots, N-1$, which implies that the fundamental weights $\{\boldsymbol{\mu}_i\}$ form a basis which is dual to the fundamental roots $\{\boldsymbol{\alpha}_j\}$. The presence of the symmetry (3.26) is related to the fact that the vacuum of the original theory can be

²⁰In three dimensions, Abelian duality relates a photon to a compact scalar. With $\sigma^j(\mathbf{x})$ the compact scalar dual to the photon $A_\mu^{(j)}(\mathbf{x})$ of the j th $U(1)$ subgroup, the Abelian duality relations are

$$* d\sigma^j = \frac{1}{2} L \text{Im}(\tau) F^{(j)}, \quad \tau(L^{-1}) = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi},$$

$$F_{\mu\nu}^{(j)} = \frac{g^2}{2\pi L} \epsilon_{\mu\nu\rho} \partial_\rho \sigma^j.$$

The $3d$ Maxwell action becomes $\frac{L}{4g^2} (F_{\mu\nu}^{(j)})^2 = \frac{L \text{Im}(\tau)}{16\pi} (F_{\mu\nu}^{(j)})^2 = \frac{1}{2\pi L \text{Im}(\tau)} (\partial_\mu \sigma^j)^2$. The path integral of the Abelian gauge theory in the presence of a monopole with charge $\pm\alpha_j$ located at position \mathbf{x} is equivalent to the insertion of $e^{\pm i\alpha_j \cdot \boldsymbol{\sigma}(\mathbf{x})}$ into the path integral over the dual scalar fields [24,27]. The complete partition function of the long-distance effective theory is a sum over all topological sectors, each of which may contain an arbitrary number of monopoles and antimonopoles (whose charges sum to give the appropriate topological class). Summing over all numbers and locations of monopoles (and antimonopoles), weighted with the appropriate fugacity, directly yields the result (3.25).

probed by $N - 1$ different types of external charges, distinguished by their (nonzero) values of N -ality. This will be discussed below.

Including a nonzero theta parameter in the original Yang-Mills action,

$$S^{\text{YM}} \rightarrow S^{\text{YM}} + i\theta \int_{\mathbb{R}^3 \times S^1} \frac{1}{16\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (3.28)$$

has the effect, in the grand canonical partition function (3.22), of multiplying the monopole fugacity by $e^{i\theta/N}$ and antimonopole fugacity by $e^{-i\theta/N}$. In the dual representation (3.25), this amounts to shifting the argument of the cosine by θ/N , so that the interaction becomes

$$- \zeta \sum_{i=1}^N \cos(\alpha_i \cdot \boldsymbol{\sigma} + i\theta/N). \quad (3.29)$$

In this form, 2π periodicity of the theory with respect to θ is not manifest. However, a shift of the dual scalar fields, $\boldsymbol{\sigma} \rightarrow \boldsymbol{\sigma} + (\theta/N)\boldsymbol{\beta}$ with $\boldsymbol{\beta} \equiv (0, 1, 2, \dots, N - 1)$, converts the interaction term to the manifestly 2π periodic form

$$- \zeta \left[\sum_{i=1}^{N-1} \cos(\alpha_i \cdot \boldsymbol{\sigma}) + \cos(\alpha_N \cdot \boldsymbol{\sigma} + \theta) \right], \quad (3.30)$$

in which theta dependence only appears in the term involving the affine root. For simplicity, in the following subsections we will focus on the case of $\theta = 0$.

A. Mass gap

The cosine potential in the dual action (3.25) generates a mass term for the photons. Rescaling $\boldsymbol{\sigma}$ to put the kinetic term into canonical form and expanding the potential to quadratic order around the minimum at $\boldsymbol{\sigma} = 0$ gives

$$V(\sigma_i) = (\text{const}) + \frac{1}{2} m_\gamma^2 \sum_{i=1}^N (\sigma_{i+1} - \sigma_i)^2, \quad (3.31)$$

with $\sigma_{N+1} \equiv \sigma_1$ and

$$m_\gamma^2 \equiv \frac{(2\pi)^2}{g^2} L\zeta = Am_W^2 \left(\frac{2\pi}{g^2 N} \right)^3 e^{-\Delta S} e^{-8\pi^2/(Ng^2(m_w))}. \quad (3.32)$$

A \mathbb{Z}_N Fourier transform,

$$\tilde{\sigma}_p \equiv \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{-2\pi i p j/N} \sigma_j, \quad p = 0, \dots, N - 1, \quad (3.33)$$

diagonalizes this mass term and yields

$$V(\sigma_i) = (\text{const}) + \frac{1}{2} \sum_{p=0}^{N-1} m_p^2 |\tilde{\sigma}_p|^2, \quad (3.34)$$

with

$$m_p \equiv m_\gamma \sin \frac{\pi p}{N}. \quad (3.35)$$

Expressing m_γ in terms of the renormalization group invariant scale Λ , defined by

$$\Lambda^{b_0} = \mu^{b_0} (Ng^2(\mu))^{-b_1/b_0} e^{-8\pi^2/(Ng^2(\mu))} \quad (3.36)$$

(with $b_0 = 11/3$ and $b_1 = 17/3$), yields

$$m_\gamma = \tilde{A} \Lambda (\Lambda N L)^{5/6} |\ln \Lambda L|^{9/11}, \quad (3.37)$$

where \tilde{A} is an $O(1)$ coefficient. Relative corrections suppressed by powers of $g^2(m_w) \sim 1/|\ln \Lambda L|$ have, of course, been neglected. The result (3.35), for $p = 1, \dots, N - 1$, shows that the $N - 1$ photons of the ‘‘unbroken’’ $U(1)^{N-1}$ gauge group receive nonzero masses due to nonperturbative effects.²¹

B. String tensions

Let us first examine the vacuum structure of the dual theory in more detail. The dual scalars are defined to be periodic with period 2π . This implies that shifting $\boldsymbol{\sigma}$ by 2π times any root vector is an identity, $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma} + 2\pi\alpha_i$ for all $\alpha_i \in \Delta_{\text{aff}}^0$. As noted earlier, the dual action (3.25) is also invariant under $\boldsymbol{\sigma} \rightarrow \boldsymbol{\sigma} + 2\pi\mu_j$, where $\{\mu_j\}$ are the fundamental weights of the $SU(N)$ gauge group, defined by the reciprocity relations (3.27). The simple roots $\{\alpha_i\}$ generate the root lattice Λ_r . Its dual, the weight lattice Λ_w , is generated by the fundamental weights $\{\mu_j\}$. The root lattice is a sublattice of the weight lattice and their quotient is

$$\Lambda_w/\Lambda_r = \mathbb{Z}_N. \quad (3.38)$$

This implies that the dual theory potential, $V(\boldsymbol{\sigma}) \equiv -\zeta \sum_i \cos(\alpha_i \cdot \boldsymbol{\sigma})$, has N isolated minima lying within the unit cell of Λ_r . These minima are located at $\boldsymbol{\sigma} = 0$ and

$$\boldsymbol{\sigma} = 2\pi\mu_j, \quad j = 1, \dots, N - 1. \quad (3.39)$$

(Equivalently, one may describe the minima as lying at $2\pi j\mu_1$, for $j = 0, 1, \dots, N$, since $\mu_j = j\mu_1 + \alpha$ for some $\alpha \in \Lambda_r$.)

Let \mathcal{R} be some chosen irreducible representation of $SU(N)$. The expectation value of the Wilson loop $W_{\mathcal{R}}(C)$ characterizes the response of the system to external test charges in the representation \mathcal{R} . In a confining phase with a nonzero mass gap, if external charges in representation \mathcal{R} cannot be screened by gluons, then expectation values of large Wilson loops in this representation are expected to decrease exponentially with the area of the minimal spanning surface,

$$\langle W_{\mathcal{R}}(C) \rangle \sim e^{-T(\mathcal{R})\text{Area}(\Sigma)}. \quad (3.40)$$

Here Σ denotes the minimal surface with boundary C , and

²¹The vanishing $p = 0$ mass corresponds to the extra decoupled photon which was added to simplify the duality transformation but is not present in the original theory. It should be ignored.

$T(\mathcal{R})$ is the string tension for representation \mathcal{R} . Such area law behavior implies the presence of an asymptotically linear confining potential between static charges in representation \mathcal{R} and anticharges in representation $\bar{\mathcal{R}}$, $V_{\mathcal{R}}(\mathbf{x}) \sim T(\mathcal{R})|\mathbf{x}|$ as $|\mathbf{x}| \rightarrow \infty$.

The irreducible representation \mathcal{R} may be associated with its highest weight vector $w \in \Lambda_w$. Identifying weight vectors which differ by elements of the root lattice produces a \mathbb{Z}_N grading of representations which corresponds to their N -ality (the charge of the representation under the \mathbb{Z}_N center). In particular, if this equivalence associates the highest weight vector w with k times the fundamental weight μ_1 ,

$$w = k\mu_1 + \alpha, \quad \text{for some } \alpha \in \Lambda_r, \quad (3.41)$$

then the representation \mathcal{R} has N -ality k .²²

As discussed in Refs. [24,27], the insertion of a Wilson loop $W_{\mathcal{R}}(C)$ in a representation \mathcal{R} with nonzero N -ality k corresponds, in the low-energy dual theory, to the requirement that the dual scalar fields have nontrivial monodromy,

$$\int_{C'} d\sigma = 2\pi\mu_k, \quad (3.42)$$

where C' is any closed curve whose linking number with C is one. In other words, in the presence of the Wilson loop $W_{\mathcal{R}}(C)$ the dual scalar fields must have a discontinuity of $2\pi\mu_k$ across some surface Σ which spans the loop C . One way to see this is to go back to the duality relation. For simplicity, consider the case of a large planar loop lying in the xy plane. As the size of the loop grows, the spanning surface Σ approaches an infinite flat plane. In the presence of the Wilson loop, the Abelian duality relation $F \sim *d\sigma$ is replaced by $F \sim *d\sigma + \mu_k \delta(z) dx \wedge dy$. Therefore the dual scalars σ must be discontinuous across Σ in order for the field strength F to be continuous.

The fact that dual low-energy theory depends on the representation \mathcal{R} of the Wilson loop only through its N -ality k shows that there are only $N - 1$ distinct string tensions, referred to as k -string tensions, $\{T_k\}$. (Charge-conjugation symmetry implies that $T_k = T_{N-k}$.) The dual theory representation of Wilson loops also shows that external charges in representations with zero N -ality will not be confined. These are precisely the representations which can be screened by adjoint representation gluons.

To evaluate a Wilson loop expectation value, one must minimize the dual action in the space of field configurations satisfying the monodromy condition (3.42). To extract the string tension,

$$T_k \equiv - \lim_{\text{area}(\Sigma) \rightarrow \infty} \frac{\ln \langle W_{\mathcal{R}}(C) \rangle}{\text{area}(\Sigma)}, \quad (3.43)$$

²²Representations contained in the product of m powers of the fundamental representation with n powers of the antifundamental have N -ality $m - n$.

it is sufficient to consider the limit where Σ fills the xy plane. In this case, the field $\sigma(\mathbf{x})$ will only depend on z . It must approach some minimum of the dual potential at infinity, $\lim_{z \rightarrow \pm\infty} \sigma(z) = 2\pi\mu_l$, and must be discontinuous across $z = 0$ with a jump given by the prescribed fundamental weight, $\lim_{z \rightarrow 0^+} \sigma(z) - \lim_{z \rightarrow 0^-} \sigma(z) = 2\pi\mu_k \pmod{2\pi}$. Because shifts by $2\pi\mu_k$ are an invariance of the dual potential $V(\sigma)$, one may equally well minimize the action for field configurations $\sigma(z)$ which are continuous but whose asymptotic values differ,

$$T_k = \min_{\sigma(z)} \frac{\Delta S(\sigma)}{\text{area}(\mathbb{R}^2)} \Big|_{\Delta\sigma = 2\pi\mu_k \pmod{2\pi}}, \quad (3.44)$$

where $\Delta\sigma \equiv \sigma(\infty) - \sigma(-\infty)$, and $\Delta S(\sigma)$ is the dual action minus its vacuum value. Explicitly,

$$T_k = \min_{\sigma(z)} \int dz \left\{ \frac{1}{2L} \left(\frac{g}{2\pi} \right)^2 \left(\frac{\partial\sigma}{\partial z} \right)^2 + \zeta \sum_i [1 - \cos(\sigma_i - \sigma_{i+1})] \right\} \Big|_{\Delta\sigma = 2\pi\mu_k \pmod{2\pi}}. \quad (3.45)$$

In other words, the k -string tension T_k equals the action of a kink solution with topological charge k in this one-dimensional theory.

The width of the kink solution must be of order of the inverse photon mass m_γ^{-1} . Consequently, the k -string tension will have the form $T_k = f_k T$, where

$$T \equiv \zeta/m_\gamma \sim \Lambda^2 (\Lambda LN)^{-1/6} |\log(\Lambda LN)|^{-3/11}, \quad (3.46)$$

and f_k is an $O(1)$ coefficient. Even without finding the minimizing kink solutions explicitly, it is apparent that the resulting k -string tension T_k will be nonzero (for $k = 1, \dots, N - 1$), and must satisfy the convexity relation $T_{k+l} \leq T_k + T_l$.

We were unable to solve the kink equations of motions analytically for general N , but when $N = 2$ the equations of motion reduce to the Sine-Gordon model. In this case, one finds

$$T \equiv T_1 = 4\sqrt{2}\zeta/m_\gamma. \quad (3.47)$$

C. Larger size or larger N

The above semiclassical analysis of the deformed Yang-Mills theory is reliable provided there is a parametrically large separation of scales between the lightest W -boson mass, $m_W = 2\pi/(NL)$, and the nonperturbatively induced dual photon mass m_γ . Their ratio scales as

$$\frac{m_W}{m_\gamma} \sim (LN\Lambda)^{-11/6} |\log(LN\Lambda)|^{-9/11}, \quad (3.48)$$

and hence there is a large separation of mass scales pro-

vided $L N \Lambda \ll 1$. In this regime, the monopole gas is highly dilute and a semiclassical analysis is justified. Increasing $L N \Lambda$, by increasing N , L , or both, decreases the separation of scales; the heaviest photon mass, m_γ , grows while the lightest W mass, m_W , drops. When $L N \Lambda \approx 1$, the scale separation is entirely lost, the effective 't Hooft coupling $\lambda \equiv g^2 N$ at the scale of m_W ceases to be small, and the long-distance dynamics can no longer be described by a weakly coupled $U(1)^{N-1}$ effective theory.

One can consider sending N to infinity while staying within the analytically tractable regime. This is a double scaling limit in which $g^2 N$ and $L N \Lambda$ both are held fixed (and both are much less than unity) as $N \rightarrow \infty$. Taking a large N limit in this fashion allows monopole effects to survive and to continue dictating the nonperturbative physics of the deformed Yang-Mills theory. However, this region shrinks to a vanishingly small window in the large N limit, since the double scaling implies that $0 < L \ll L_{\max}$ with $L_{\max} \Lambda \sim 1/N$. For any fixed compactification size L , if one sends $N \rightarrow \infty$ then the deformed YM theory ceases to possess a monopole dominated, Abelian long-distance regime.²³

D. Connection to integrable Toda theory

The \mathbb{Z}_N symmetric model (3.25) is a deformation of a complex affine Toda theory with action

$$S^{\text{affine Toda}} = \int_{\mathbb{R}^3} \left[\frac{1}{2L} \left(\frac{g}{2\pi} \right)^2 (\nabla \sigma)^2 - \zeta \sum_{i=1}^N e^{i(\sigma_i - \sigma_{i+1})} \right]. \quad (3.49)$$

This complex (*CPT* noninvariant) action describes a plasma which is composed solely of monopoles with no antimonopoles. (Because of the existence of the affine root, one can have a neutral plasma composed solely of monopoles!) Interestingly, the soliton spectrum of the affine Toda theory is exactly computable. When reduced to one dimension, this theory is an integrable system as shown by Hollowood [43], using techniques due to Hirota [44,45].

²³An analog of this double scaled limit was previously discussed by Douglas and Shenker in mass deformed $SU(N)$ $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on \mathbb{R}^4 [26]. This deformed $\mathcal{N} = 1$ supersymmetric theory, just like our deformed Yang-Mills theory, possesses a regime in which the long-distance gauge dynamics reduce to the Abelian subgroup $U(1)^{N-1}$. Reference [26] shows that in the $N \rightarrow \infty$ limit of the mass deformed theory, the Abelian long-distance regime is preserved only if the mass deformation m is sent to zero in a correlated fashion, $m/\Lambda \sim 1/N^4$. In particular, at any fixed nonzero m , if one takes $N \rightarrow \infty$ first then there is no regime of the supersymmetric gauge theory in which the long-distance dynamics remain Abelian. Although this phenomenon only appeared previously in the context of supersymmetric gauge theories, it is generic in deformed Yang-Mills and other deformed QCD-like theories.

As discussed above, the k -string tension T_k is equal to the action of the kink solution with topological charge k . Borrowing the exact soliton spectrum from Ref. [43], one finds that the k -string tensions in the affine Toda theory are given by

$$T_k^{\text{affine Toda}} = TN \sin \frac{\pi k}{N}, \quad k = 1, \dots, N-1, \quad (3.50)$$

with T given above in Eq. (3.46).

The long-distance effective theory (3.25) for our deformed Yang-Mills theory (when $L N \Lambda \ll 1$) is a deformation of the affine Toda system by complex conjugation. Unfortunately, unlike the integrable affine Toda system, when $N > 2$ the resulting *CPT* invariant system is no longer exactly integrable according to Hirota's criteria.²⁴ Consequently, we do not expect k -string tensions in the deformed Yang-Mills theory to have the sine-law form (3.50).

Recently, there have been attempts [46], to model the *strongly* coupled confined regime of Yang-Mills theory assuming the Wilson line has the center-symmetric form (3.1). (See also earlier related work in Refs. [47–50].) A few remarks concerning the connection with Ref. [46] may be in order. First, our results for the k -string tensions do not support the claim of Ref. [46], which asserts that k -string tensions will have the sine-law form (3.50). As just noted, sine-law string tensions are a property of the affine Toda subsystem, whereas the center-stabilized Yang-Mills in a weak coupling regime is dual to a real deformation of the affine Toda theory. We see no reason to believe that the k dependence of the string tensions will be unaffected by the deformation. Second, it should be emphasized that the deformation (2.5) stabilizes the center-symmetric vacuum in the *weakly* coupled regime, and thereby provides a window in which a semiclassical analysis is reliable. Many earlier discussions of center-symmetric backgrounds do not clearly distinguish the weakly coupled “Higgs” regime, in which fluctuations of the Wilson line eigenvalues are small, from the strong coupling regime in which the eigenvalues have large fluctuations and are essentially randomized over the unit circle. In our deformed Yang-Mills theory, both regimes exist. As the compactification size L increases, the theory moves from the weakly coupled regime to the strongly coupled regime. These two regimes are expected to be smoothly connected—no physical order parameter sharply distinguishes the two regimes. Nevertheless, the long-distance physics of the weak coupling Higgs regime is effectively Abelian, while in the strong coupling regime there is no length scale

²⁴In the absence of the complex conjugate term in the potential, there is a change of variables which converts the soliton equation of motion into “Hirota bilinear type,” which is synonymous with solvability [43]. The presence of the complex conjugate term spoils the bilinearity.

beyond which the dynamics can be described accurately in terms of Abelian degrees of freedom.

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APPENDIX: MONOPOLE MEASURE

The appropriate one-loop measure for integrating over configurations of a single monopole (of any type) may be expressed as²⁵

$$d\mu_{\text{monopole}} = \mu^4 e^{-(S_{\text{YM}} + \Delta S)} \frac{d^3 \mathbf{a}}{(2\pi)^{3/2}} J_a \frac{d\phi}{(2\pi)^{1/2}} \times J_\phi [\det'(-D_{\text{adj}}^2)]^{-1}, \quad (\text{A1})$$

where $\mathbf{a} \in \mathbb{R}^3$ is the monopole position, $\phi \in [-\pi, \pi]$ is the internal $U(1)$ angle of the monopole, and μ is the (Pauli-Villars) renormalization scale. Global $U(1)$ gauge transformations [in the $U(1)$ subgroup associated with the given type of monopole] shift the angle ϕ . Fluctuations in the position and $U(1)$ angle of the monopole represent the

²⁵The following summary is an adaptation of the appendix of Ref. [49], which treats the monopole measure in supersymmetric Yang-Mills theory.

²⁶For comparison, recall that an instanton in $SU(N)$ Yang-Mills theory has $4N$ zero modes. (See, for example, Ref. [51].) For $SU(2)$ gauge theory, these are the four zero modes corresponding to changes in the instanton position—one for its size, and three global gauge rotations. For $SU(N)$ with $N > 2$, there are in addition $4N - 8$ zero modes associated with changes in the embedding of $SU(2)$ within $SU(N)$. When compactified in one dimension, with nontrivial holonomy Ω , one may regard an instanton as being composed of N independent monopole constituents [47,48], each of which carries four zero modes.

four zero modes in the monopole small fluctuation operator;²⁶ the factor of μ^4 can be viewed as the contributions of the Pauli-Villars regulator fields associated with these bosonic zero modes. The exponential factor is, of course, the exponential of minus the classical action of the monopole. The collective coordinate Jacobians are given by [49]

$$J_a = S_{\text{YM}}^{3/2}, \quad J_\Omega = \frac{2\pi S_{\text{YM}}^{1/2}}{\alpha_j \cdot \lambda} = NLS_{\text{YM}}^{1/2}, \quad (\text{A2})$$

where $\lambda = \{0, 2\pi, 4\pi, \dots, 2\pi(N-1)\}/(NL)$ are the eigenvalues of $-i \ln \Omega$. The primed determinant represents the result of Gaussian integrals over all fluctuations other than zero modes; the prime on the determinant denotes omission of the zero modes. The contributions from gauge bosons and ghosts,

$$\underbrace{[\det'(-D^2 \delta_{\mu\nu} - 2F_{\mu\nu})_{\text{adj}}]^{-1/2}}_{\text{gaugebosons}} \times \underbrace{\det(D^2)_{\text{adj}}}_{\text{ghosts}}, \quad (\text{A3})$$

combine to give this simple form because

$$[\det'(-D^2 \delta_{\mu\nu} - 2F_{\mu\nu})_{\text{adj}}]^{-1/2} = [\det(-D^2)_{\text{adj}}]^{-2} \quad (\text{A4})$$

in any self-dual background. These functional determinants may be regularized using the Pauli-Villars scheme.

The fields of fundamental monopoles reside entirely within an $SU(2)$ subgroup of $SU(N)$, and the characteristic size of these monopoles is given by the inverse of the lightest W -boson mass, $m_W^{-1} \sim NL$. (This is the only scale which appears in the classical equations for the monopole.) The regularized scalar determinant depends on the cube root of the renormalization scale, $\det(-D^2) \sim \mu^{1/3}$. Since the determinant is dimensionless, it must have the form

$$[\det(-D^2)]^{-1} = 2\pi C(\mu NL)^{-1/3}, \quad (\text{A5})$$

where C is a pure number (N independent). Consequently, the one-loop monopole measure equals

$$d\mu_{\text{monopole}} = C \mu^{11/3} (NL)^{2/3} (S_{\text{YM}})^2 e^{-S_{\text{YM}} + \Delta S} d^3 \mathbf{a} d\Omega. \quad (\text{A6})$$

Performing the trivial integral over the angle Ω , the result is the $\zeta d^3 \mathbf{a}$, with ζ the monopole fugacity. Choosing to use m_W as the value of the renormalization point yields the expression (3.23) for the fugacity.

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