Effect of a positive cosmological constant on cosmic strings

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We study cosmic Nielsen-Olesen strings in space-times with a positive cosmological constant. For the free cosmic string in a cylindrically symmetric space-time, we calculate the contribution of the cosmological constant to the angle deficit, and to the bending of null geodesics. For a cosmic string in a Schwarzschild–de Sitter space-time, we use Kruskal patches around the inner and outer horizons to show that a thin string can pierce them.

I. INTRODUCTION

In this paper, we study cosmic strings in space-times with a positive cosmological constant. By cosmic string we will mean a vortex line in the Abelian Higgs model [1]. There are several reasons for studying these. Recent observations suggest a strong possibility that the Universe is endowed with a positive cosmological constant, $\Lambda > 0$ [2,3]. This in turn implies that any observer will find a cosmic horizon at a length scale of $\Lambda^{-(1/2)}$. Since Λ is very small, and equivalently the length scale of the horizon is very large, one might be tempted to neglect the effect of Λ on local physics. However, there are situations in which local physics is affected by global topology.

An example comes from black hole no-hair statements. These say that a static or stationary black hole is characterized by a small number of parameters like its mass, angular momentum and charges corresponding to long-range fields. These statements about the uniqueness of black hole solutions were originally proven for asymptotically flat space-times [4,5], but later extended to space-times with $\Lambda > 0$ [6]. It was found that the existence of a cosmic horizon introduced some subtleties in the proofs of these statements. For an example which motivates us in this paper, consider the Abelian Higgs model coupled minimally to gravity. The only asymptotically flat, static, spherically symmetric black hole has the Higgs field fixed at the minimum of the potential, and the black hole is uncharged. On the other hand, in the presence of a cosmic horizon, there is an additional black hole solution in which the Higgs field is fixed at the maximum of the potential and the black hole is charged [6]. The black hole looks like the Reissner–Nördstrom–de Sitter solution with Higgs field in the false vacuum. This is of course the opposite of the usual no-hair statement.

This new solution exists because of different boundary conditions in the two cases—in the asymptotically flat case these are imposed at infinity, while for a positive cosmological constant it is both convenient and sufficient to impose them at the cosmic horizon. In general, given some asymptotically flat solution (corresponding to $\Lambda = 0$) of matter coupled to gravity, we may find additional solutions, or qualitatively different ones, when there is a cosmic horizon (corresponding to $\Lambda > 0$).

We are motivated by these arguments to look at infinitely long, straight cosmic strings in space-times with $\Lambda > 0$. While the role of such cosmic strings in cosmological perturbations and structure formation is ruled out and the contribution of these strings to the primordial perturbation spectrum must be less than 9% (for a review and references see [7]), such strings could exist in small numbers. How does a positive cosmological constant or a cosmic horizon affect the physics of the string? It is known that a string produces a conical geometry, or a deficit angle [8]. It is also known that a cosmological constant affects the bending of light [9,10]. Both effects should be present in a string space-time with $\Lambda > 0$.

We present analytical results for a cosmic string in two kinds of space-time with a positive cosmological constant. The first is one with only an infinite straight string, so that the space-time is cylindrically symmetric with a string on the axis. We calculate the angle deficit and the bending of light for this space-time. The other space-time we consider is the Schwarzschild–de Sitter space-time, and a cosmic string stretched between the inner and outer horizons. For this we consider maximally extended (Kruskal) coordinate patches near the horizons and find that the string can extend beyond the horizons, i.e. pierce them.

II. FREE COSMIC STRING AND ANGLE DEFICIT

We start with the Ansatz for a cylindrically symmetric static metric [11]

$$ds^2 = e^{A(\rho)}[-dt^2 + dz^2] + \rho^2 e^{B(\rho)}d\phi^2 + d\rho^2, \quad (1)$$

and first solve for the cosmological constant vacuum $R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0$, with $\Lambda > 0$. There are three Killing vector fields here, the timelike Killing field $(\partial_t)^a$ and two spacelike Killing fields $(\partial_z)^a$, $(\partial_\phi)^a$. The orbits of $(\partial_\phi)^a$ are closed spacelike curves which shrink to a point as $\rho \to 0$. We regard the set of points $\rho = 0$ as the axis of the space-

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time, and foliate this space-time with (ρ, ϕ) planes (orthogonal to $(\partial_t)^a$, $(\partial_z)^a$). A convenient coordinatization of these planes is by setting the metric to be locally flat on the axis, i.e.

$$ds^2 \xrightarrow{\rho \to 0} - dt^2 + dz^2 + \rho^2 d\phi^2 + d\rho^2. \qquad (2)$$

We can always do this as long as there is no curvature singularity on the axis. The vacuum solution subject to this boundary condition is given by [12–14]

$$ds^2 = \cos^{4/3} \frac{\rho \sqrt{3\Lambda}}{2} (-dt^2 + dz^2)$$
$$+ \frac{4}{3\Lambda} \sin^2 \frac{\rho \sqrt{3\Lambda}}{2} \cos^{-(2/3)} \frac{\rho \sqrt{3\Lambda}}{2} d\phi^2 + d\rho^2. \quad (3)$$

The metric is singular at $\rho = \frac{n\pi}{\sqrt{3\Lambda}}$, where n are integers. Of these points, those corresponding to even n are flat, with $n = 0$ being the axis. The points corresponding to odd n are curvature singularities. The quadratic invariant of Riemann tensor behaves there as

$$R_{abcd}R^{abcd} \approx \frac{\Lambda^2}{(\frac{n\pi}{2} - \frac{\rho \sqrt{3\Lambda}}{2})^4}, \qquad n \text{ odd.} \qquad (4)$$

The curvature singularity at $n = 1$ is not protected by a horizon, so it must be a naked singularity. The singularities for higher n thus appears to be unphysical, and will not concern us further.

Our region of interest will be near the axis and far from the naked singularity at $n = 1$. In this region, we consider Einstein's equations with energy-momentum tensor, $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$. The energy-momentum tensor T_{ab} corresponds to that of a string solution of the Abelian Higgs model, which has the Lagrangian

$$\mathcal{L} = -(D_a \Phi)^\dagger (D^a \Phi) - \frac{1}{4} \tilde{F}_{ab} \tilde{F}^{ab} - \frac{\lambda}{4} (\Phi^\dagger \Phi - \eta^2)^2. \qquad (5)$$

Here $D_a = \nabla_a + ieA_a$ is the gauge covariant derivative, $\tilde{F}_{ab} = \nabla_a A_b - \nabla_b A_a$ is the electromagnetic field strength tensor and Φ is a complex scalar. For convenience of calculations we will parametrize Φ and A_a as

$$\Phi = \eta X e^{i\chi}, \qquad A_a = \frac{1}{e}[P_a - \nabla_a \chi]. \qquad (6)$$

The Abelian Higgs model has string like solutions in flat space-time [1]. For these solutions, the phase χ is multiple valued outside the string,

$$\oint \nabla_a \chi dx^a = \oint d\chi = 2n\pi, \qquad (7)$$

where the integral is done over a closed loop around the string and n is some nonzero integer called the winding number. On the other hand, χ is single valued inside the core, so the Lagrangian inside the core becomes

$$\mathcal{L} = -\eta^2 \nabla_a X \nabla^a X - \eta^2 X^2 P_a P^a - \frac{1}{4e^2} F_{ab} F^{ab}$$
$$- \frac{\lambda \eta^4}{4} (X^2 - 1)^2, \qquad (8)$$

where $F_{ab} = \nabla_a P_b - \nabla_b P_a$. We will write ρ_0 for the core radius. Because of the cylindrical symmetry of the space-time we can take the following Ansätze for X and P_a:

$$X = X(\rho), \qquad P_a = P(\rho) \nabla_a \phi. \qquad (9)$$

The energy-momentum tensor is taken to be nonzero only inside the string core, and zero outside. The various non-vanishing components of energy-momentum tensor T_{ab} for the Abelian Higgs model in cylindrical coordinates (1) are

$$T_{tt} = \left[\eta^2 X'^2 + \frac{\eta^2 X^2 P^2 e^{-B}}{\rho^2} + \frac{P'^2 e^{-B}}{2e^2 \rho^2} + \frac{\lambda \eta^4}{4} (X^2 - 1)^2 \right] e^A,$$

$$T_{\rho\rho} = \left[\eta^2 X'^2 - \frac{\eta^2 X^2 P^2 e^{-B}}{\rho^2} + \frac{P'^2 e^{-B}}{2e^2 \rho^2} - \frac{\lambda \eta^4}{4} (X^2 - 1)^2 \right],$$

$$T_{\phi\phi} = \left[-\eta^2 X'^2 + \frac{\eta^2 X^2 P^2 e^{-B}}{\rho^2} + \frac{P'^2 e^{-B}}{2e^2 \rho^2} - \frac{\lambda \eta^4}{4} (X^2 - 1)^2 \right]$$
$$\times \rho^2 e^B,$$

$$T_{zz} = -\left[\eta^2 X'^2 + \frac{\eta^2 X^2 P^2 e^{-B}}{\rho^2} + \frac{P'^2 e^{-B}}{2e^2 \rho^2} + \frac{\lambda \eta^4}{4} (X^2 - 1)^2 \right]$$
$$\times e^A. \qquad (10)$$

Since $\Phi = \eta X e^{i\chi}$, we have along a closed loop of $X =$ constant outside the string core

$$\oint d\chi = 2n\pi = \frac{1}{i} \oint \frac{d\Phi}{\Phi}. \qquad (11)$$

It is clear that $\Phi = 0$ somewhere inside the loop and hence $X = 0$ somewhere inside the loop. For the string solution the Higgs field should vanish as we approach the axis, and should approach its vacuum expectation value outside the string. The gauge field A_ϕ should accordingly approach $-\frac{1}{e} \partial_\phi \chi$ away from the string and a constant on the axis. In other words, $X \to 0$, $P \to 1$ as we approach the axis, while $X \to 1$, $P \to 0$ outside the string core.

We now return to Einstein equations $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$. The variation of X and P, and thus of the energy-momentum tensor, near the "string surface" at $\rho = \rho_0$ is a problem of considerable interest and has been studied numerically in various papers. However, in this paper we are concerned about the existence of the string and its behavior near the horizons. Accordingly, we will fix boundary conditions by assuming $X = 0$, $P = 1$ inside the string core and $X = 1$, $P = 0$ outside. This guarantees that the energy-momentum tensor (10) is identically zero out-

side the core. The fields are assumed to be smoothed out at the string surface at $\rho = \rho_0$ such that the local conservation law $\nabla_a T^{ab} = 0$ remains valid. Then we can solve the Einstein equations to find inside the core ($0 \le \rho < \rho_0$)

$$ds^2 \approx \cos^{4/3} \frac{\rho \sqrt{3\Lambda'}}{2} (-dt^2 + dz^2)$$
$$+ \frac{4}{3\Lambda'} \sin^2 \frac{\rho \sqrt{3\Lambda'}}{2} \cos^{-(2/3)} \frac{\rho \sqrt{3\Lambda'}}{2} d\phi^2 + d\rho^2. \tag{12}$$

This solution inside the string is the same as the vacuum solution of Eq. (3), but with a modified cosmological constant

$$\Lambda' = \Lambda + 2\pi G \lambda \eta^4. \tag{13}$$

The solution for the metric in the vacuum region outside the string is given by

$$ds^2 = \cos^{4/3} \frac{\rho \sqrt{3\Lambda}}{2} (-dt^2 + dz^2)$$
$$+ \delta^2 \frac{4}{3\Lambda} \sin^2 \frac{\rho \sqrt{3\Lambda}}{2} \cos^{-(2/3)} \frac{\rho \sqrt{3\Lambda}}{2} d\phi^2 + d\rho^2. \tag{14}$$

This solution differs from the vacuum solution by the presence of a number δ, which is related to the angle deficit. In [15], where vortices in de Sitter space were studied perturbatively, the authors argued for the existence of this δ, but did not estimate it. Here we evaluate δ in the following way. We first compute

$$\frac{1}{2\pi} \iint \sqrt{g^{(2)}} d\rho d\phi (G_t^{\ t} + \Lambda) \tag{15}$$

on (ρ, ϕ) planes. Here $g^{(2)}$ is the determinant of the induced metric on these planes. It is clear that δ appears due to the energy-momentum tensor which is confined to the region $\rho \le \rho_0$. Then calculating $G_t^{\ t}$ from the general Ansatz (1), we have

$$\int_0^{\rho_0} \sqrt{g^{(2)}} d\rho (G_t^{\ t} + \Lambda) = \int_0^{\rho_0} d\rho \left[\rho e^{B/2} \left(\frac{A'^2}{4} + \Lambda \right) \right.$$
$$\left. + \left(\rho e^{B/2} \frac{A'}{2} \right)' + (\rho e^{B/2})'' \right], \tag{16}$$

where a prime denotes differentiation with respect to ρ. But according to the Einstein equation, $G_t^{\ t} + \Lambda = 8\pi G T_t^{\ t}$. Substituting the value of $G_t^{\ t} + \Lambda$ in Eq. (16),

we get

$$\frac{d}{d\rho} (\rho e^{B/2}) \Big|_0^{\rho_0} + \left(\rho e^{B/2} \frac{A'}{2} \right) \Big|_0^{\rho_0}$$
$$= -4G\mu - \int_0^{\rho_0} d\rho \rho e^{B/2} \left(\Lambda + \frac{A'^2}{4} \right), \tag{17}$$

where

$$\mu := -\int_0^{2\pi} \int_0^{\rho_0} d\phi d\rho \rho e^{B/2} T_t^{\ t}$$
$$\approx \frac{\pi \lambda \eta^4}{\Lambda'} \left[1 - \cos^{2/3} \frac{\rho_0 \sqrt{3\Lambda'}}{2} \right] \tag{18}$$

is the string mass per unit length. To get the approximate expression for μ in Eq. (18) we have used $T_t^{\ t} = -\frac{\lambda \eta^4}{4}$ which is due to our approximation $X = 0$ and $P = 1$ inside the core. Outside the core $T_t^{\ t} = 0$ identically, so we have used the metric functions in Eq. (12). In evaluating the total derivative terms in the left-hand side of Eq. (17), we will use the interior metric of Eq. (12) at $\rho = 0$, but the vacuum metric of Eq. (14) at the string "surface" $\rho = \rho_0$. This requires an explanation. If we assume the energy-momentum to be nonvanishing only within the string, the right-hand side of Eq. (16) will have contributions only from $\rho \le \rho_0$, as we have written. The integrand on the left-hand side of Eq. (16) also vanishes outside the string according to vacuum Einstein equations. When we integrate the left-hand side, we should do so to the surface of the string, i.e., where the energy-momentum tensor vanishes. But at that point we have the vacuum solution of Eq. (14), so that is what we should use at the upper limit of integration. Thus we find

$$1 - \delta \left(\cos^{2/3} \frac{\rho_0 \sqrt{3\Lambda}}{2} - \frac{1}{3} \cos^{-(4/3)} \frac{\rho_0 \sqrt{3\Lambda}}{2} \sin^2 \frac{\rho_0 \sqrt{3\Lambda}}{2} \right)$$
$$= 4G\mu + \int_0^{\rho_0} d\rho \rho e^{B/2} \left(\Lambda + \frac{A'^2}{4} \right). \tag{19}$$

The integrals on the right-hand side of Eq. (19) cannot be evaluated explicitly, since the integrand cannot be written as a total derivative, nor do we know the detailed behavior of the metric near the string surface at $\rho = \rho_0$. However, we can make an estimate of these integrals using the value of the metric coefficients inside the core. This means that we ignore the details of the falloff of the energy-momentum tensor near $\rho = \rho_0$. Then using Eq. (12) we get from Eq. (19) an expression for δ

$$\delta = \frac{1 - 4G\mu - \frac{2\Lambda}{\Lambda'}(1 - \cos^{2/3} \frac{\rho_0 \sqrt{3\Lambda'}}{2}) + \frac{1}{3}(1 - \cos^{-(4/3)} \frac{\rho_0 \sqrt{3\Lambda'}}{2}) + \frac{2}{3}(1 - \cos^{2/3} \frac{\rho_0 \sqrt{3\Lambda'}}{2})}{(\cos^{2/3} \frac{\rho_0 \sqrt{3\Lambda}}{2} - \frac{1}{3} \cos^{-(4/3)} \frac{\rho_0 \sqrt{3\Lambda}}{2} \sin^2 \frac{\rho_0 \sqrt{3\Lambda}}{2})}. \tag{20}$$

This result may be compared with one obtained in [16] where the authors considered point particles as source and solved Einstein equations in $2 + 1$ dimensional de Sitter space. The "particles" may be thought of as punctures created in the plane by infinitely thin long strings. A conical singularity was found, with an angle deficit $\delta = (1 - 4Gm)$ where m is the

mass of the particle. Our result includes corrections dependent on Λ, which we may think of as coming from the finite thickness of the string. The size of the string ρ_0 is of the order of $(\sqrt{\lambda}\eta)^{-1}$, at least when the winding number is small [17]. This is essentially because the metric is flat on the axis, so we can approximate ρ_0 by its value in flat space. Further, the scale of symmetry breaking η is small compared to the Planck scale in theories of particle physics in which cosmic strings appear. For example, the grand unified scale is about 10^{16} GeV, so that $G\eta^2 \sim 10^{-3}$. It is also reasonable to assume that the string size is small compared to the cosmic horizon. In other words, we assume $\rho_0^2\Lambda \ll 1$. Thus we find that $\rho_0^2\Lambda' \ll 1$ as well, where Λ' is given by Eq. (13).

Then by expanding $G\mu$ using the expression in Eq. (18), we find $\mu = \frac{\pi}{4}\lambda\eta^4\rho_0^2$ approximately, and thus $G\mu \ll 1$. We can also find an approximate expression for δ from Eq. (20) under these assumptions,

$$\delta \approx 1 - 4G\mu(1 + \tfrac{3}{4}\rho_0^2\Lambda + G\mu). \qquad (21)$$

The leading correction to δ due to the cosmological constant is of a higher order of smallness, as we can see from this. The meaning of δ is obvious in space-times with vanishing cosmological constant, for which Eq. (19) was worked out in [18] to find $\delta \approx 1 - 4G\mu$, where $G\mu \ll 1$ as before, and $\mathcal{O}(G^2\mu^2)$ corrections were ignored. Then asymptotically we get the conical space-time

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2\delta^2 d\phi^2. \qquad (22)$$

In this space-time the azimuthal angle runs from 0 to $2\pi\delta$. So Eq. (22) is Minkowski space-time minus a wedge which corresponds to a deficit $2\pi(1 - \delta)$ in the azimuthal angle. The difference of initial and final azimuthal angles of a null geodesic (i.e., light ray in the geometrical optics approximation) at $\rho \to \infty$ is $\frac{\pi}{\delta}$ [8,11]. Light bends towards the string even though the curvature of space-time is zero away from the axis.

For a positive cosmological constant, the metric in the exterior of the string is

$$ds^2 = \cos^{4/3}\frac{\rho\sqrt{3\Lambda}}{2}(-dt^2 + dz^2)$$
$$+ \delta^2\frac{4}{3\Lambda}\sin^2\frac{\rho\sqrt{3\Lambda}}{2}\cos^{-(2/3)}\frac{\rho\sqrt{3\Lambda}}{2}d\phi^2 + d\rho^2, \qquad (23)$$

with δ given in Eq. (20) or Eq. (21). Comparing with the string-free vacuum solution of Eq. (3) we find that, similar to the asymptotically flat space-time, the deficit in the azimuthal angle in space-time with a positive cosmological constant is also $2\pi(1 - \delta)$, but now with δ given by Eq. (21). However the bending of null geodesics will be quite different in the cosmic string space-time of (23) from that in asymptotically flat cosmic string space-time.

Since our space-time (23) has a translational isometry along z, for the sake of simplicity we can consider null geodesics on the $z = 0$ plane. It is well known that if χ^a is a Killing field, then for any geodesic with tangent u^a, the quantity $g_{ab}u^a\chi^b$ is conserved along the geodesic. Thus the conserved angular momentum of a future directed null test particle in the space-time (23) is

$$L = g_{ab}(\partial_\phi)^a u^b = \delta^2\frac{4}{3\Lambda}\sin^2\frac{\rho\sqrt{3\Lambda}}{2}\cos^{-(2/3)}\frac{\rho\sqrt{3\Lambda}}{2}\dot{\phi}, \qquad (24)$$

while its conserved energy is

$$E = -g_{ab}(\partial_t)^a u^b = \cos^{4/3}\frac{\rho\sqrt{3\Lambda}}{2}\dot{t}. \qquad (25)$$

The dot denotes differentiation with respect to an affine parameter and $(\partial_\phi)^a$ and $(\partial_t)^a$ are rotational and time translational Killing fields, respectively. Then for null geodesics on the $z = 0$ plane it is straightforward to obtain

$$\frac{d\phi}{d\rho} = \frac{3\Lambda L}{4E\delta^2}\frac{\cos^{4/3}\frac{\rho\sqrt{3\Lambda}}{2}}{\sin^2\frac{\rho\sqrt{3\Lambda}}{2}[1 - \frac{3\Lambda L^2}{4E^2\delta^2}\cot^2\frac{\rho\sqrt{3\Lambda}}{2}]^{1/2}}. \qquad (26)$$

Since (ρ, ϕ) are smooth functions of the affine parameter the derivative on the left-hand side of Eq. (26) is well defined. From the null geodesic equation on the $z = 0$ plane we have the distance of closest approach to the string

$$\rho_c = \frac{2}{\sqrt{3\Lambda}}\tan^{-1}\frac{\sqrt{3\Lambda}L}{2E\delta}. \qquad (27)$$

Let us consider a null geodesic which starts from some point $(\rho_{\max}, \phi_1)$ between the string surface and the singularity at $\rho = \frac{\pi}{\sqrt{3\Lambda}}$. We will look at till it reaches a point $(\rho_{\max}, \phi_2)$. For simplicity of interpretation, we have chosen the "initial" and "final" radial distances to be equal. Since the trajectory of the geodesic will be symmetric about the distance of closest approach ρ_c, the change in the azimuthal angle is

$$\Delta\phi = \phi_2 - \phi_1$$
$$= \frac{3\Lambda L}{2E\delta^2}\int_{\rho_c}^{\rho_{\max}}\frac{\cos^{4/3}\frac{\rho\sqrt{3\Lambda}}{2}}{\sin^2\frac{\rho\sqrt{3\Lambda}}{2}[1 - \frac{3\Lambda L^2}{4E^2\delta^2}\cot^2\frac{\rho\sqrt{3\Lambda}}{2}]^{1/2}}d\rho. \qquad (28)$$

Equation (28) along with the expression for ρ_c determines the change of ϕ with ρ. The full expression for the integral in Eq. (28) is rather messy and we will look at two special cases only. First, when ρ is much smaller than the radius of the cosmological singularity ($\rho \ll \frac{\pi}{\sqrt{3\Lambda}}$), we have approximately

$$\Delta\phi \approx \frac{2}{\delta}\sec^{-1}\left(\sqrt{1+k^2}\frac{\rho E\delta}{L}\right)\Bigg|_{\rho_c}^{\rho_{\max}}$$
$$-\frac{4k}{3\delta\sqrt{1+k^2}}\left(\frac{\rho^2 3\Lambda}{4}-\frac{k^2}{1+k^2}\right)^{1/2}\Bigg|_{\rho_c}^{\rho_{\max}}, \quad (29)$$

where $k = \frac{\sqrt{3}\Lambda L}{2E\delta}$. The second term in Eq. (29) is negative and the repulsive effect of positive Λ is manifest in this term. In the $\Lambda \to 0$ limit only the first term survives. In that case the limit $\rho_{\max} \to \infty$ recovers the well-known formula $\Delta\phi = \frac{\pi}{\delta}$. Next, near the singularity $\rho \to \frac{\pi}{\sqrt{3\Lambda}}$, we can approximately write $\cos\frac{\rho\sqrt{3\Lambda}}{2} \approx (\frac{\pi}{2} - \frac{\rho\sqrt{3\Lambda}}{2})$ and integrate Eq. (28) to get

$$\Delta\phi \approx -\frac{6k}{\delta}\left[\frac{1}{7}\left(\frac{\pi}{2}-\frac{\rho\sqrt{3\Lambda}}{2}\right)^{7/3}\right.$$
$$\left. +\frac{k^2}{26}\left(\frac{\pi}{2}-\frac{\rho\sqrt{3\Lambda}}{2}\right)^{13/3}+\ldots\right]_{\rho_c}^{\rho_{\max}}. \quad (30)$$

III. BLACK HOLE PIERCED BY A STRING

If a cosmic string pierces a Schwarzschild black hole, the resulting space-time has a conical singularity [19] as well. The authors of [20] showed by considering the equations of motion of the matter fields that an Abelian Higgs string (for both self-gravitating and non-self-gravitating matter) can pierce a Schwarzschild black hole. We will adapt in this section the method described in [20] to establish that a Schwarzschild–de Sitter black hole can be similarly pierced by a Nielsen-Olesen string.

Inside the core of a string, we can derive the equations of motion for the fields from (8),

$$\nabla_a\nabla^a X - XP_aP^a - \frac{\lambda\eta^2}{2}X(X^2-1)=0, \quad (31)$$

$$\nabla_a F^{ab} - 2e^2\eta^2 X^2 P^b = 0. \quad (32)$$

Consider for a moment flat space cylindrical coordinates (t, ρ, ϕ, z) and take the scalar field X to be cylindrically symmetric, $X = X(\rho)$. Also assume that the gauge field P_a can be written as $P_a = P(\rho)\nabla_a\phi$. Then the equations of motion (31) and (32) become

$$\frac{d^2X}{d\rho^2}+\frac{1}{\rho}\frac{dX}{d\rho}-\frac{XP^2}{\rho^2}-\frac{X}{2}(X^2-1)=0, \quad (33)$$

$$\frac{d^2P}{d\rho^2}-\frac{1}{\rho}\frac{dP}{d\rho}-\frac{2e^2}{\lambda}X^2P=0. \quad (34)$$

Here we have scaled ρ by $(\sqrt{\lambda}\eta)^{-1}$ to convert it to a dimensionless radial coordinate. These are the usual equations which were shown in [1] to have stringlike solutions. We wish to show that these equations hold also in the Schwarzschild–de Sitter background space-time if the string thickness is small compared to the black hole event

horizon, and if we neglect the backreaction of the string on the metric.

The Schwarzschid–de Sitter metric in the usual spherical polar coordinates reads

$$ds^2 = -\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)dt^2+\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)^{-1}dr^2$$
$$+ r^2 d\Omega^2. \quad (35)$$

There are three horizons in this space-time—the black hole event horizon at $r = r_H$, the cosmological horizon at $r = r_C$ and an unphysical horizon at $r = r_U$ with $r_U < 0$. If we assume $2M \ll \frac{1}{\sqrt{\Lambda}}$, which we will throughout, we find that the approximate sizes of the horizons are $r_H \approx 2M$, $r_C \approx \sqrt{\frac{3}{\Lambda}}$ and $r_U \approx -\sqrt{\frac{3}{\Lambda}}$. The string we are looking for is thin compared to the horizon size, i.e., we will assume also that

$$\frac{1}{\sqrt{\lambda}\eta} \ll 2M \ll \frac{1}{\sqrt{\Lambda}}. \quad (36)$$

We now expand the field equations in the Schwarzschild–de Sitter background; in other words, we will neglect the backreaction on the metric due to the string. Then Eq. (31) becomes

$$\frac{1}{r^2}\partial_r\left[r^2\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)\partial_r X\right]+\frac{1}{r^2\sin^2\theta}\partial_\theta(\sin\theta\partial_\theta X)$$
$$-\frac{XP^2}{r^2\sin^2\theta}-\frac{\lambda\eta^2}{2}X(X-1)=0. \quad (37)$$

For the string solution the matter distribution will be cylindrically symmetric. For convenience we take the string along the axis $\theta = 0$, although our arguments are clearly valid for $\theta = \pi$ as well. We define as before a dimensionless cylindrical radial coordinate $\rho = r\sqrt{\lambda}\eta\sin\theta$. For cylindrically symmetric matter distribution both (X, P) will be functions of ρ only. With this we can rewrite (37) as

$$\left(\sin^2\theta-\frac{2M\sqrt{\lambda}\eta\sin^3\theta}{\rho}-\frac{\bar{\Lambda}\rho^2}{3}\right)\left[\frac{d^2X}{d\rho^2}+\frac{2}{\rho}\frac{dX}{d\rho}\right]$$
$$+\left(\frac{2M\sqrt{\lambda}\eta\sin^3\theta}{\rho^2}-\frac{2\bar{\Lambda}\rho}{3}\right)\frac{dX}{d\rho}+\left[\frac{1}{\rho}\frac{dX}{d\rho}\cos^2\theta\right.$$
$$\left.-\frac{1}{\rho}\frac{dX}{d\rho}\sin^2\theta+\frac{d^2X}{d\rho^2}\cos^2\theta\right]-\frac{XP^2}{\rho^2}-\frac{1}{2}X(X-1)=0,$$
$$(38)$$

where $\bar{\Lambda} = \frac{\Lambda}{\lambda\eta^2}$ is a dimensionless number. Under the approximations of Eq. (36) and because $\sin\theta \ll 1$ inside the string core, Eq. (38) reduces to Eq. (33), i.e., the flat space equation of motion for Abelian Higgs model. Outside the string core ($\rho > 1$), we can as before set $X = 1$. Hence we can say that Eq. (38), and hence Eq. (37), gives rise to a configuration of scalar field X similar to that

of the Nielsen-Olesen string. A similar calculation for Eq. (32) in the Schwarzschild–de Sitter background shows that it reduces to Eq. (34) under the same approximations. Thus we conclude that the Schwarzschild–de Sitter space-time allows a uniform Nielsen-Olesen string along the axis $\theta = 0$ in the region $r_H < r < r_C$.

From the calculations so far we cannot conclude how the string behaves at or near the horizons. The reason is the following. The two horizons at (r_H, r_C) appear as two coordinate singularities in the chart described in Eq. (35). Clearly we cannot expand the field equations in this coordinate system at or around the horizons. To do the expansion we need to use maximally extended coordinates which will be free from coordinate singularities for the Schwarzschild–de Sitter space-time, and has only the curvature singularity at $r = 0$. Let us construct Kruskal like patches at the two horizons to remove the two coordinate singularities.

First we consider radial null geodesics in the vicinity of the black hole event horizon r_H. For these geodesics

$$\frac{dt}{dr} = \pm \frac{1}{(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3})} = \pm \frac{3r}{\Lambda(r - r_H)(r - r_U)(r - r_C)}. \tag{39}$$

Equation (39) can be easily integrated to give

$$t = \pm r_* + \text{constant}, \tag{40}$$

where r_* is the tortoise coordinate given by

$$r_* = \alpha \ln \left| \frac{r}{r_H} - 1 \right| + \beta \ln \left| \frac{r}{r_C} - 1 \right| + \gamma \ln \left| \frac{r}{r_U} - 1 \right|. \tag{41}$$

The three constants (α, β, γ) are given by

$$\alpha = \frac{3r_H}{\Lambda(r_C - r_H)(r_H - r_U)},$$

$$\beta = -\frac{3r_C}{\Lambda(r_C - r_H)(r_C - r_U)}, \tag{42}$$

$$\gamma = -\frac{3r_U}{\Lambda(r_C - r_U)(r_H - r_U)}.$$

In (t, r_*) coordinates the radial metric becomes

$$ds_{\text{radial}}^2 = \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)(-dt^2 + dr_*^2). \tag{43}$$

Defining null coordinates (u, v) such that

$$u = t - r_* \quad \text{and} \quad v = t + r_*, \tag{44}$$

we can write the radial metric (43) as

$$ds_{\text{radial}}^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)du dv. \tag{45}$$

Now we define timelike and spacelike coordinates (T, Y) by

$$T := \frac{e^{v/2\alpha} - e^{-(u/2\alpha)}}{2}, \qquad Y := \frac{e^{v/2\alpha} + e^{-(u/2\alpha)}}{2}. \tag{46}$$

(T, Y) satisfy the following relations:

$$Y^2 - T^2 = \left[\frac{r}{r_H} - 1\right]e^{(\beta/\alpha)\ln|(r/r_C)-1|+(\gamma/\alpha)\ln|(r/r_U)-1|}, \tag{47}$$

$$\frac{T}{Y} = \tanh\left(\frac{t}{2\alpha}\right). \tag{48}$$

In terms of (T, Y), the full space-time metric of Eq. (35) becomes

$$ds^2 = \frac{8M\alpha^2}{r}\left|\frac{r}{r_U} - 1\right|^{1-(\gamma/\alpha)}\left|\frac{r}{r_C} - 1\right|^{1-(\beta/\alpha)} \times (-dT^2 + dY^2) + r^2 d\Omega^2. \tag{49}$$

The metric (49) is nonsingular at $r = r_H$. Thus (T, Y) define a well-behaved coordinate system around the event horizon. When $r \rightarrow r_H(\approx 2M)$, from Eq. (47) we have approximately (after scaling $r \rightarrow \sqrt{\lambda}\eta r$),

$$r \approx 2M\sqrt{\lambda}\eta + 2M\sqrt{\lambda}\eta$$
$$\times \left[e^{-(\beta/\alpha)\ln|(2M/r_C)-1|-(\gamma/\alpha)\ln|(2M/r_U)-1|}\right]$$
$$\times (Y^2 - T^2). \tag{50}$$

Let us expand Eq. (31) in the vicinity of the black hole event horizon. We use Eq. (50) to get the following expressions for derivatives of the scalar field $X(\rho)$:

$$\partial_T X = -4M\sqrt{\lambda}\eta[e^{-(\beta/\alpha)\ln|(2M/r_C)-1|-(\gamma/\alpha)\ln|(2M/r_U)-1|}]$$
$$\times (Y^2 - T^2)T\partial_\rho X \sin\theta, \tag{51}$$

$$\partial_Y X = 4M\sqrt{\lambda}\eta[e^{-(\beta/\alpha)\ln|(2M/r_C)-1|-(\gamma/\alpha)\ln|(2M/r_U)-1|}]$$
$$\times (Y^2 - T^2)Y\partial_\rho X \sin\theta. \tag{52}$$

Here $\rho = r\sqrt{\lambda}\eta \sin\theta$ as before. Equation (31) now can be written on the background metric of Eq. (49) as

$$(1 - \sin^2\theta)\frac{d^2X}{d\rho^2} + \frac{1}{\rho}(1 - 2\sin^2\theta)\frac{dX}{d\rho} + \frac{A}{8M\alpha^2\rho\lambda\eta^2}$$
$$\times \left|\frac{r}{r_C} - 1\right|^{(\beta/\alpha)-1}\left|\frac{r}{r_U} - 1\right|^{(\gamma/\alpha)-1}$$
$$\times \left[2\rho^2\frac{dX}{d\rho} + 4\rho \sin\theta\frac{dX}{d\rho}(r - 2M\sqrt{\lambda}\eta)\right.$$
$$\left. + 2\rho^2\frac{d^2X}{d\rho^2}\sin\theta(r - 2M\sqrt{\lambda}\eta)\right] = 0, \tag{53}$$

where $A = 4M\sqrt{\lambda}\eta e^{-(\beta/\alpha)\ln|(2M/r_C)-1|-(\gamma/\alpha)\ln|(2M/r_U)-1|}$. Under the approximations of Eq. (36), the fact that $|r_U| \approx \sqrt{\frac{3}{\Lambda}}$ and $(r - 2M\sqrt{\lambda}\eta)$ is an infinitesimal quantity, Eq. (53) reduces to Eq. (33). Similar arguments hold for Eq. (32).

For calculations at the cosmological horizon, we have to use the following chart which is nonsingular as $r \to r_C$,

$$ds^2 = \frac{8M\beta^2}{r} \left| \frac{r}{r_U} - 1 \right|^{1-(\gamma/\beta)} \left| \frac{r}{r_H} - 1 \right|^{1-(\alpha/\beta)}$$
$$\times (-dT'^2 + dY'^2) + r^2 d\Omega^2. \quad (54)$$

T' and Y' are timelike and spacelike coordinates, respectively, well defined around $r = r_C$. They can be derived exactly in the same manner as for $r \approx r_H$. Following a similar procedure as before one can show that Eqs. (31) and (32), reduce to flat space Eqs. (33) and (34) respectively. Thus the flat space equations of motion hold on both the horizons. Since the coordinate system described in Eqs. (49) and (54) is well behaved around the respective horizons, we can also use them to expand the field equations in regions infinitesimally beyond the horizons. For $r \to r_H - 0$ the scalar field Eq. (53) still holds. The only difference is that the quantity $(r - 2M\sqrt{\lambda}\eta)$ is negative infinitesimal. But it can be neglected as before. Similar arguments can be given for the region $r \to r_C + 0$ using the chart of Eq. (54).

Thus with the boundary conditions on X and P and the approximations of Eq. (36), the configuration of matter fields are like the Nielsen-Olesen string within, at, even slightly beyond the horizons. Hence we conclude that a Schwarzschild–de Sitter black hole can be pierced by a thin Nielsen-Olesen string if the backreaction of the matter distribution to the background space-time can be neglected.

We end with a brief remark about the backreaction of the string on the metric. For a Schwarzschild–de Sitter space-time with a string along the z axis, the metric functions must be z dependent. If the cosmological constant were zero, the (Schwarzschild) space-time would be asymptotically flat, and we could use Weyl coordinates [21] to write the metric in an explicitly axisymmetric form,

$$ds^2 = -B^2 dt^2 + \rho^2 B^{-2} d\phi^2 + A^2 [d\rho^2 + dz^2], \quad (55)$$

where the coefficients A, B are functions of (ρ, z) only. It would be relatively easy to determine the existence of cosmic strings from the equations of motion of the gauge and Higgs fields written in these coordinates, as was done in [20]. On the other hand, when the cosmological constant is nonvanishing, it is no longer possible to write the metric in this form. We were unable to find an appropriate generalization of the Weyl coordinates, which are needed to solve Einstein equations coupled to the gauge and Higgs fields.

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