PHYSICAL REVIEW D **78,** 065025 (2008)

Fermions on thick branes in the background of sine-Gordon kinks

Yu-Xiao Liu,[1,*] Li-Da Zhang,[1,+] Li-Jie Zhang,[2,‡] and Yi-Shi Duan[1,§]

[1]*Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, People's Republic of China*

[2]*Department of Physics, Shanghai University, Shanghai 200444, People's Republic of China*

(Received 21 June 2008; published 12 September 2008)

A class of thick branes in the background of sine-Gordon kinks with a scalar potential $V(\phi) = p(1 + \cos\frac{2\phi}{q})$ was constructed by R. Koley and S. Kar [Classical Quantum Gravity **22**, 753 (2005)]. In this paper, in the background of the warped geometry, we investigate the issue of localization of spin-1/2 fermions on these branes in the presence of two types of scalar-fermion couplings: $\eta\bar{\Psi}\phi\Psi$ and $\eta\bar{\Psi}\sin\phi\Psi$. By presenting the mass-independent potentials in the corresponding Schrödinger equations, we obtain the lowest Kaluza-Klein modes and a continuous gapless spectrum of Kaluza-Klein states with $m^2 > 0$ for both types of couplings. For the Yukawa coupling $\eta\bar{\Psi}\phi\Psi$, the effective potential of the right chiral fermions for positive q and η is always positive; hence only the effective potential of the left chiral fermions could trap the corresponding zero mode. This is a well-known conclusion which is discussed extensively in the literature. However, for the coupling $\eta\bar{\Psi}\sin\phi\Psi$, the effective potential of the right chiral fermions for positive q and η is no longer always positive. Although the value of the potential at the location of the brane is still positive, it has a series of wells and barriers on each side, which ensures that the right chiral fermion zero mode could be trapped. Thus we may draw the following remarkable conclusion: for positive η and q, the potentials of both the left and right chiral fermions could trap the corresponding zero modes under certain restrictions.

DOI: 10.1103/PhysRevD.78.065025 PACS numbers: 11.10.Kk

I. INTRODUCTION

The suggestion that extra dimensions may not be compact [1–8] or large [9,10] can provide new insights for solving the gauge hierarchy problem [10], the cosmological constant problem [4,7,11], etc. In the framework of brane scenarios, gravity is free to propagate in all dimensions, while all the matter fields are confined to a 3-brane [4,6,10,12,13]. In Ref. [1], an alternative scenario of the compactification was proposed. In this scenario, the internal manifold does not need to be compactified to the Planck scale anymore, which is one of the reasons why this new compactification scenario has attracted so much attention. Among all of the brane world models, there is an interesting and important model in which extra dimensions comprise a compact hyperbolic manifold [8]. The model is known to be free of the usual problems that plague the original Arkani-Hamed-Dimopoulos-Dvali models and share many common features with Randall-Sundrum (RS) models.

In the brane world scenario, an important question is the localization of various bulk fields on a brane by a natural mechanism. It is well known that massless scalar fields [14] and gravitons [1] can be localized on branes of different types, and that spin-1 Abelian vector fields can not be localized on the RS brane in five dimensions, but can be localized in some higher-dimensional cases [15]. Spin-1/2 fermions do not have normalizable zero modes and hence cannot be localized in five and six dimensions [14–20].

Recently, an increasing interest has been focused on the study of thick brane scenarios based on gravity coupled to scalars in higher-dimensional space-time [21–27]. A virtue of these models is that the branes can be obtained naturally rather than introduced by hand [21]. Besides, these scalar fields provide the "material" of which the thick branes are made. In Ref. [28], exact solutions of the Einstein-scalar equations with a sine-Gordon potential and a negative cosmological constant were constructed. In this system the scalar field configuration is in fact a kink, which provides a thick brane realization of the brane world as a domain wall in the bulk. The warped background space-time has a nonconstant but asymptotically negative Ricci curvature. Such a configuration was also illustrated in several examples in the literature [17,18].

The localization problem of spin-1/2 fermions on thick branes is interesting and important. Localization of fermions in general space-times has been studied, for example, in [29]. In five dimensions, with the scalar-fermion coupling, there may exist a single bound state and a continuous gapless spectrum of massive fermion Kaluza-Klein (KK) states [30–33], while for some other brane models, there exist finite discrete KK states (mass gap) and a continuous gapless spectrum starting at a positive m^2 [34,35]. In Ref. [36], it was found that fermions can escape into the bulk by tunneling, and the rate depends on the parameters of the scalar potential. In Ref. [28], the authors obtained trapped discrete massive fermion states on the brane,

*Corresponding author.
liuyx@lzu.edu.cn
+zhangld04@lzu.cn
‡lijzhang@shu.edu.cn
§ysduan@lzu.edu.cn

which in fact are quasibound and have a finite probability of escaping into the bulk. It is also interesting to note that in Ref. [37] fermion modes in sine-Gordon kink and kink-antikink systems were also studied in some $1+1$-dimensional scalar field theories. It was shown that there exist discrete bound states. However, when the wall and antiwall approach each other, the system cannot support fermion bound states and the discrete states merge into the continuous spectrum of the Dirac equation.

It is known that, under the Yukawa coupling $\eta\bar{\Psi}\phi\Psi$, only one of the effective potentials of the left and right chiral fermions could trap the corresponding zero mode for positive q and η. In this paper, we will reinvestigate the localization issues of fermions on the branes obtained in Ref. [28] in the presence of different types of scalar-fermion couplings, by presenting the mass-independent potentials in the corresponding Schrödinger equations. We will show that, for the coupling $\eta\bar{\Psi}\sin\phi\Psi$, not only the potential of the left chiral fermions but also the potential of the right ones could trap the corresponding zero modes for positive η and q under certain different restrictions for each case. Besides, instead of a discrete massive KK mode, there exists a continuous gapless spectrum of KK states with $m^2>0$. The shapes of the potentials also suggest that the massive KK modes asymptotically turn into plane waves, which represent delocalized massive KK fermions.

The paper is organized as follows: In Sec. II, we first give a brief review of the thick brane arising from a sine-Gordon potential in a 5-dimensional space-time. Then, in Sec. III, we study localization of spin-1/2 fermions on the thick brane with two different types of scalar-fermion interactions by presenting the shapes of the potentials of the corresponding Schrödinger problem. Finally, a brief conclusion and discussion are presented.

II. REVIEW OF THE SINE-GORDON KINK AND THE THICK BRANES

Let us consider thick branes arising from a real scalar field with a sine-Gordon potential

$$V(\phi)=p\left(1+\cos\frac{2\phi}{q}\right). \tag{1}$$

This special potential was considered in Ref. [28], and different choices of $V(\phi)$ can be found elsewhere [22]. In this model, the bulk sine-Gordon potential provides a thick brane realization of the Randall-Sundrum scenario, and the soliton configuration of the scalar field dynamically generates the domain wall configuration with warped geometry. The action for such a system is given by

$$S=\int d^5x\sqrt{-g}\left[\frac{1}{2\kappa_5^2}(R-2\Lambda)-\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi-V(\phi)\right], \tag{2}$$

where $\kappa_5^2=8\pi G_5$ with G_5 the 5-dimensional Newton constant, and Λ is the 5-dimensional cosmological constant. The line element which results in a 4-dimensional Poincaré invariance of the action (2) is assumed as

$$ds^2=\mathrm{e}^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu+dy^2, \tag{3}$$

where $\mathrm{e}^{2A(y)}$ is the warp factor and y stands for the extra coordinate. The scalar field is considered to be a function of y only. The field equations, which are derivable from (2) with the ansatz (3), reduce to the following coupled nonlinear differential equations:

$$A''=-\frac{\kappa_5^2}{3}\phi'^2, \tag{4}$$

$$A'^2=-\frac{\kappa_5^2}{12}(\phi'^2-2V)+\frac{\Lambda}{6}, \tag{5}$$

$$\phi''+4A'\phi'=\frac{dV}{d\phi}. \tag{6}$$

For the sine-Gordon potential (1), the solution can be calculated [28] as

$$A(y)=-\tau\ln\cosh ky, \tag{7}$$

$$\phi(y)=2q\arctan(\exp ky)-\frac{\pi q}{2}, \tag{8}$$

where τ and k are given by

$$\tau=\frac{1}{3}\kappa_5^2q^2,\qquad k=\frac{\sqrt{6|\Lambda|}}{6\tau}. \tag{9}$$

The parameters q and Λ are free to choose, and p is given by

$$p=\frac{|\Lambda|}{2\kappa_5^4}\left(\frac{\kappa_5^2}{3}+\frac{1}{4q^2}\right). \tag{10}$$

Observing the forms of (7) and (8), one can find that the configuration of $\phi(y)$ is a kink for positive q, and the warp factor $A(y)$ is a smooth function. Besides these properties, more detailed discussions can be found in Ref. [28].

Extensive work has been done on nonsupersymmetric [21,23,24,38] as well as supersymmetric [39–41] domain walls in different models. In Ref. [40], Maru *et al.* constructed an analytic non-Bogomol'nyi-Prasad-Sommerfeld solution of the sine-Gordon domain wall in a 4-dimensional global supersymmetric model. In Ref. [41], this sine-Gordon domain wall solution was extended to a solution in 4-dimensional supergravity and its stability was examined. In these papers, although the sine-Gordon domain wall had been considered in supersymmetric theories, the properties of the solution itself are also valid in the purely bosonic sector. In the following, we will reconsider the issue of localization of spin-1/2 fermions on the 3-brane in the presence of two types of kink-fermion couplings in the background of the sine-Gordon kink (8) and the corresponding warped geometry.

III. LOCALIZATION OF FERMIONS ON THE THICK BRANES

Now, let us investigate whether spin-$1/2$ fermions can be localized on the brane. We will analyze the spectrum of fermions for the thick branes by presenting the mass-independent potentials in the corresponding Schrödinger equations. In order to get the mass-independent potentials, we will follow Ref. [1] and change the metric given in (3) to a conformally flat one,

$$ds_5^2 = \mathrm{e}^{2A}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2), \tag{11}$$

by performing the coordinate transformation

$$dz = \mathrm{e}^{-A(y)}dy. \tag{12}$$

In five dimensions, fermions are four-component spinors, and their Dirac structure is described by $\Gamma^M = e^M_{\bar{M}}\Gamma^{\bar{M}}$ with $\{\Gamma^M, \Gamma^N\} = 2g^{MN}$. In this paper, $\bar{M}, \bar{N}, \cdots$ denote the local Lorentz indices, and $\Gamma^{\bar{M}}$ are the flat gamma matrices in five dimensions. In our setup, $\Gamma^M = (\mathrm{e}^{-A}\gamma^\mu, \mathrm{e}^{-A}\gamma^5)$, where γ^μ and γ^5 are the usual flat gamma matrices in the Dirac representation. The Dirac action of a massless spin-$1/2$ fermion coupled to the scalar is

$$S_{1/2} = \int d^5x\sqrt{-g}(\bar{\Psi}\Gamma^M D_M\Psi - \eta\bar{\Psi}F(\phi)\Psi), \tag{13}$$

where the covariant derivative D_M is defined as $D_M\Psi = (\partial_M + \frac{1}{4}\omega_M^{\bar{M}\bar{N}}\Gamma_{\bar{M}}\Gamma_{\bar{N}})\Psi$ with the spin connection $\omega_M = \frac{1}{4}\omega_M^{\bar{M}\bar{N}}\Gamma_{\bar{M}}\Gamma_{\bar{N}}$ and

$$\omega_M^{\bar{M}\bar{N}} = \tfrac{1}{2}e^{N\bar{M}}(\partial_M e_N^{\bar{N}} - \partial_N e_M^{\bar{N}}) - \tfrac{1}{2}e^{N\bar{N}}(\partial_M e_N^{\bar{M}} - \partial_N e_M^{\bar{M}}) - \tfrac{1}{2}e^{P\bar{M}}e^{Q\bar{N}}(\partial_P e_{Q\bar{R}} - \partial_Q e_{P\bar{R}})e_M^{\bar{R}}. \tag{14}$$

With the metric (11), the nonvanishing components of the spin connection ω_M are

$$\omega_\mu = \tfrac{1}{2}(\partial_z A)\gamma_\mu\gamma_5. \tag{15}$$

Then the equation of motion is given by

$$\{\gamma^\mu\partial_\mu + \gamma^5(\partial_z + 2\partial_z A) - \eta\mathrm{e}^A F(\phi)\}\Psi = 0, \tag{16}$$

where $\gamma^\mu\partial_\mu$ is the Dirac operator on the brane.

Now we study the above 5-dimensional Dirac equation. From the equation of motion (16), we will search for the solutions of the general chiral decomposition

$$\Psi(x, z) = \sum_n \psi_{Ln}(x)\alpha_{Ln}(z) + \sum_n \psi_{Rn}(x)\alpha_{Rn}(z) \tag{17}$$

with $\psi_{Ln}(x) = -\gamma^5\psi_{Ln}(x)$ and $\psi_{Rn}(x) = \gamma^5\psi_{Rn}(x)$ the left-handed and right-handed components of a 4-dimensional Dirac field; the sum over n can be both discrete and continuous. Here, we assume that $\psi_L(x)$ and $\psi_R(x)$ satisfy the 4-dimensional massive Dirac equations $\gamma^\mu\partial_\mu\psi_{Ln}(x) = m_n\psi_{R_n}(x)$ and $\gamma^\mu\partial_\mu\psi_{Rn}(x) = m_n\psi_{L_n}(x)$. Then $\alpha_L(z)$ and $\alpha_R(z)$ satisfy the following coupled equations:

$$\{\partial_z + 2\partial_z A + \eta\mathrm{e}^A F(\phi)\}\alpha_{Ln}(z) = m_n\alpha_{Rn}(z), \tag{18a}$$

$$\{\partial_z + 2\partial_z A - \eta\mathrm{e}^A F(\phi)\}\alpha_{Rn}(z) = -m_n\alpha_{Ln}(z). \tag{18b}$$

In order to obtain the standard 4-dimensional action for the massive chiral fermions, we need the following orthonormality conditions for α_{L_n} and α_{R_n}:

$$\int_{-\infty}^{\infty} \mathrm{e}^{4A}\alpha_{Lm}\alpha_{Rn}dz = \delta_{LR}\delta_{mn}. \tag{19}$$

Defining $\tilde{\alpha}_L = \mathrm{e}^{2A}\alpha_L$, we get the Schrödinger-like equation for the left chiral fermions,

$$[-\partial_z^2 + V_L(z)]\tilde{\alpha}_{Ln} = m_n^2\tilde{\alpha}_{Ln}, \tag{20}$$

where the effective potential is given by

$$V_L(z) = \mathrm{e}^{2A}\eta^2F^2(\phi) - \mathrm{e}^A\eta\partial_z F(\phi) - (\partial_z A)\mathrm{e}^A\eta F(\phi). \tag{21}$$

For the right chiral fermions, the corresponding potential can be written out easily by replacing $\eta \to -\eta$ from the above potential,

$$V_R(z) = \mathrm{e}^{2A}\eta^2F^2(\phi) + \mathrm{e}^A\eta\partial_z F(\phi) + (\partial_z A)\mathrm{e}^A\eta F(\phi). \tag{22}$$

It can be seen clearly that, for the left (right) chiral fermion localization, there must be some kind of Yukawa coupling. This situation can be compared with the one in the RS framework [14], where an additional localization method [42] was introduced for spin 1/2 fields. Furthermore, $F(\phi(z))$ must be an odd function of $\phi(z)$ when we demand that $V_L(z)$ or $V_R(z)$ is Z_2 even with respect to the extra dimension z. In this paper, we consider two cases, $F(\phi(z)) = \phi(z)$ and $F(\phi(z)) = \sin\phi(z)$, as examples. For each case, we get a continuous spectrum of KK modes with positive $m^2 > 0$. However, it is shown that only the massless chiral modes could be localized on the brane.

A. $F(\phi) = \phi$

Here, we face the difficulty that for general τ we cannot obtain the function $y(z)$ in an explicit form. But we can write the potentials as a function of y:

$$V_L(z(y)) = \frac{1}{4}q\eta\cosh^{-2\tau}(ky)\bigg(-\frac{8\mathrm{e}^{ky}k}{1+\mathrm{e}^{2ky}} + q\eta(\pi - 4\arctan\mathrm{e}^{ky})^2 - 2k\tau(\pi - 4\arctan\mathrm{e}^{ky}) \times \tanh ky\bigg), \tag{23}$$

$$V_R(z(y)) = V_L(z(y))|_{\eta\to-\eta}. \tag{24}$$

This potential for the left chiral fermions has the asymptotic behavior $V_L(y = \pm\infty) = 0$ and $V_L(y = 0) = -kq\eta$, where $k > 0$. For $q\eta > 0$, this is in fact a volcano-type potential [43,44]. The effective potential for the left chiral fermions is shown in Fig. 1. For positive τ, $z(y)$ is a monotonous function, which means that the potential for arbitrary, positive τ provides no mass gap to separate the zero mode from the excited KK modes.

In the following, without loss of generality, we mainly discuss the case $\tau = 1$, for which one can invert the coordinate transformation $dz = e^{-A(y)}dy$, namely,

$$y = \mathrm{arcsinh}(kz)/k, \tag{25}$$

and get the explicit forms of the potentials and the kink $\phi(z)$,

$$\begin{aligned} V_L(z) = \eta\bigg(&\frac{q^2\eta(\pi - 4\arctan e^{\mathrm{arcsinh}kz})^2}{4(1+k^2z^2)} \\ &- \frac{2kqe^{\mathrm{arcsinh}kz}}{(1+k^2z^2)(1+e^{2\mathrm{arcsinh}kz})} \\ &- \frac{k^2qz(\pi - 4\arctan e^{\mathrm{arcsinh}kz})}{2(1+k^2z^2)^{3/2}}\bigg), \end{aligned} \tag{26}$$

$$V_R(z) = V_L(z)|_{\eta\to-\eta}, \tag{27}$$

$$\phi(z) = 2q\arctan e^{\mathrm{arcsinh}(kz)} - \frac{\pi q}{2}. \tag{28}$$

The values of the potentials for the left chiral and right chiral fermions at $y = 0$ are given by

$$V_R(0) = -V_L(0) = kq\eta. \tag{29}$$

Both potentials have the asymptotic behavior $V_{L,R}(z = \pm\infty) = 0$. But for a given coupling constant η, the values of the potentials at $z = 0$ are the opposite. The shape of the kink $\phi(z)$ and the above two potentials are shown in Fig. 2 for given values of positive η and q. It can be seen that $V_L(z)$ is indeed a volcano-type potential. Hence, the potential provides no mass gap to separate the fermion zero mode from the excited KK modes, and there exists a continuous gapless spectrum of the KK modes for both the left chiral and right chiral fermions.

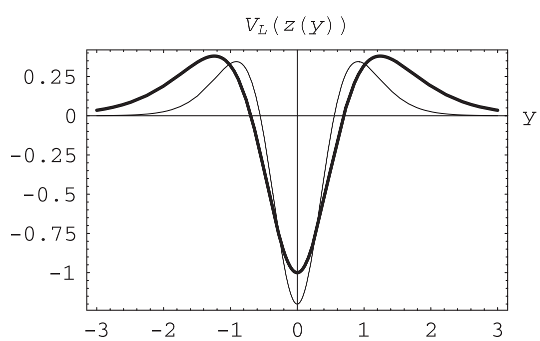


FIG. 1. The shape of the potential $V_L(z(y))$ for the case $F(\phi) = \phi$. The parameters are set to $k = 1$, $q = 1$, $\eta = 1$, and $\tau = 1$ for the thick line, and $\eta = 1.2$ and $\tau = 2$ for the thin line.

For positive q and η, only the potential for left chiral fermions has a negative value at the location of the brane, which could trap the left chiral fermion zero mode solved in (18a) by setting $m_0 = 0$:

$$\tilde{\alpha}_{L0}(z) = e^{2A}\alpha_{L0}(z) \propto \exp\bigg(-\eta\int^z dz' e^{A(z')}\phi(z')\bigg). \tag{30}$$

In order to check the normalization condition (19) for the zero mode (30), we need to check whether the inequality

$$\int dz \exp\bigg(-2\eta\int^z dz' e^{A(z')}\phi(z')\bigg) < \infty \tag{31}$$

is satisfied. For the integral $\int dz e^A\phi$, we only need to consider the asymptotic characteristic of the function $e^A\phi$ for $z \to \infty$. Noting that $\arctan z \to \pi/2$ when $z \to \infty$, we have

$$e^A\phi = \frac{4q\arctan e^{\mathrm{arcsinh}kz} - q\pi}{2\sqrt{1+k^2z^2}} \to \frac{q\pi}{2\sqrt{1+k^2z^2}}, \tag{32}$$

$$\int dz e^A\phi \to \frac{q\pi}{2k}\mathrm{arcsinh}kz. \tag{33}$$

Now, the normalization condition (31) is changed to $\int dz \exp(-\frac{\eta q\pi}{k}\mathrm{arcsinh}kz) < \infty$. Hence the condition on the free parameters η and q is

$$\eta q > \frac{k}{\pi}. \tag{34}$$

In fact, we can solve the problem in the y coordinate easily. In this coordinate, the condition (31) becomes

$$\int dy \exp\bigg(-A(y) - 2\eta\int^y dy'\phi(y')\bigg) < \infty. \tag{35}$$

When $y \to \infty$, we have $A(y) \to -ky$ and $\phi(y) \to q\pi/2$, and so $(-A(y) - 2\eta\int^y dy'\phi(y')) \to (k - \eta q\pi)y$. Then we can get the restriction condition (34) for localizing the zero mode of the left chiral fermions.

The zero mode (30) represents the lowest energy eigenfunction (ground state) of the Schrödinger equation (20) since it has no zeros, and it is the only bound state. Since the ground state has the lowest mass square $m_0^2 = 0$, there is no tachyonic left chiral fermion mode. The potential (26) provides no mass gap to separate the fermion zero mode from the excited KK modes. In Fig. 3, we plot the left chiral fermion potential $V_L(z)$, the corresponding zero mode, and the massive KK modes. We see that the zero mode is bound on the brane, while the massive modes propagate along the extra dimension. Those massive modes with lower energy experience an attenuation due to the presence of the potential barriers near the location of the brane.

In the case $q\eta > 0$, the potential for the right chiral fermions is always positive, which shows that it cannot

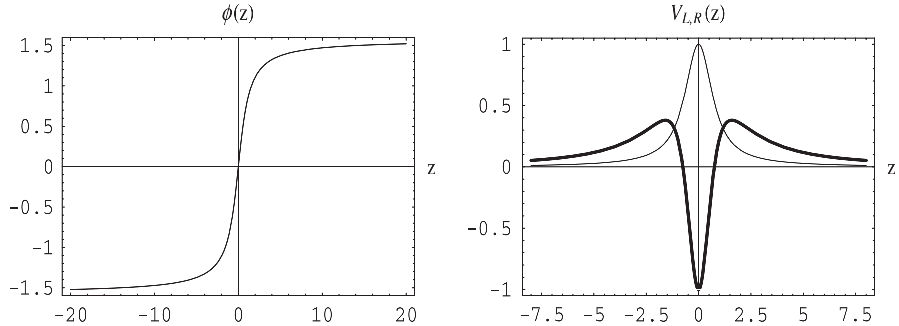


FIG. 2. The shape of the kink [$\phi(z)$ with positive k] and the potentials $V_L(z)$ (thick line) and $V_R(z)$ (thin line) for the left and right chiral fermions for the case $F(\phi) = \phi$ and $\tau = 1$ in the z coordinate. The parameters are set to $k = 1$, $q = 1$, and $\eta = 1$.

trap the right chiral zero mode. But for the case of negative $q\eta$, things are the opposite and only the right chiral zero mode can be trapped on the brane. For arbitrary $q\eta \neq 0$, the two potentials suggest that there is no mass gap but a continuous spectrum of KK modes with $m^2 > 0$.

B. $F(\phi) = \sin\phi$

For the case $F(\phi) = \sin\phi$, the potential as a function of y for the left chiral fermions is

$$V_L(z(y)) = -\tfrac{1}{2}\eta\cosh^{-1-2\tau}(ky)[\eta(\cos(2\phi(y)) - 1)\cosh ky + 2kq\cos\phi(y) - \tau q\sin\phi(y)\sinh ky]. \tag{36}$$

For different values of q, $F(\phi(y))$ has different behaviors, which should result in different types of the potential V_L. According to the expression of $F(\phi)$,

$$F(\phi(y)) = \sin\left(2q\arctan e^{ky} - \frac{\pi q}{2}\right), \tag{37}$$

we have $F|_{y\to\infty} \to \sin\frac{q\pi}{2}$. So, when $y \to \infty$, $F(\phi(y))$ has different limits for different values of q:

$$F(y \to \infty) > 0 \quad \text{for } 4n < q < 4n + 2, \tag{38}$$

$$F(y \to \infty) = 0 \quad \text{for } q = 2n, \tag{39}$$

$$F(y \to \infty) < 0 \quad \text{for } 4n + 2 < q < 4n + 4, \tag{40}$$

where n is an arbitrary integer. The shapes of $F(\phi(y)) = \sin\phi(y)$ for various values of q are shown in Fig. 4.

Considering that τ does not change the characteristics of the effective potentials acting on the left and right chiral fermions, we mainly focus on the case $\tau = 1$, for which one can get the explicit forms of the potentials in the z coordinate,

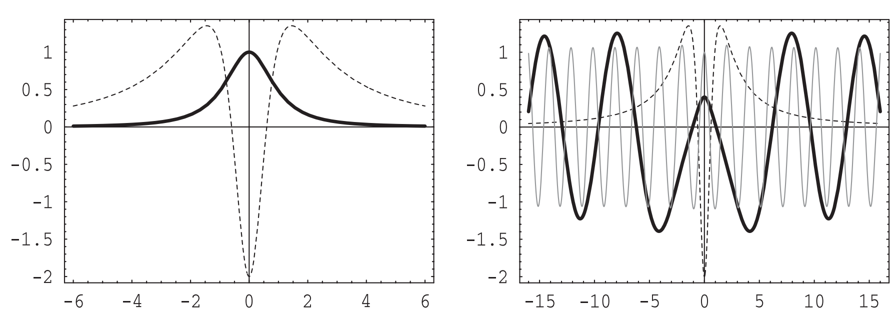
$$V_L(z) = \eta\left(\frac{k^2 z\sin\phi(z)}{(1 + k^2z^2)^{3/2}} + \frac{\eta\sin^2\phi(z)}{1 + k^2z^2} - \frac{2kqe^{\mathrm{arcsinh}kz}\cos\phi(z)}{(1 + e^{2\mathrm{arcsinh}kz})(1 + k^2z^2)}\right), \tag{41}$$

$$V_R(z) = V_L(z)|_{\eta\to-\eta}, \tag{42}$$

where $\phi(z)$ is given by Eq. (28). The values at $y = 0$ are given by

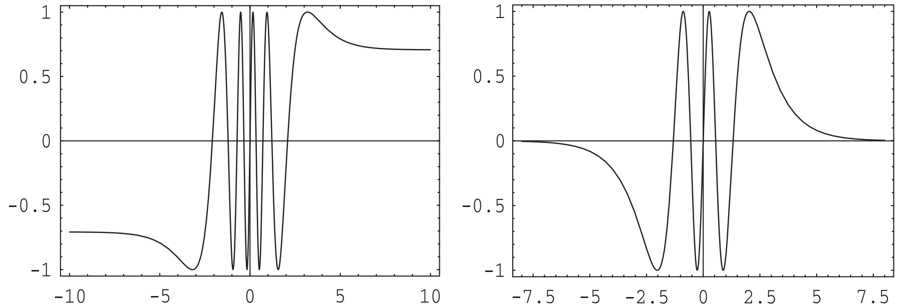
$$V_L(0) = -V_R(0) = -kq\eta. \tag{43}$$

Both potentials have the asymptotic characteristic $V_{L,R}|_{z\to\pm\infty} \to 0$. But for a given coupling constant η, the values of the potentials at $z = 0$ are the opposite. It can be seen from (41) and (42) that the shapes of the potentials are



(a) Zero mode ($m^2 = 0$) (b) Massive modes ($m^2 = 1, 10$)

FIG. 3. The shape of the potentials $V_L(z)$ (26) (dashed lines), the zero mode (30), and the massive modes for the left chiral fermions for the case $F(\phi) = \phi$. The parameters are set to $k = 1$, $q = 2$, and $\eta = 1$. In the right figure, we set $m^2 = 1$ and 10 for the thick black line and the thin gray line, respectively.



(a) $q = 9.5$ (b) $q = 6$

FIG. 4. The shape of $F(\phi) = \sin\phi$ for various values of q in the y coordinate. For $q = 9.5$, which is between $4n$ and $4n + 2$, F tends to a positive constant when $z \to \infty$ and tends to a negative constant when $z \to -\infty$. For $q = 6$, which is an even number, F tends to zero when $z \to \pm\infty$.

determined by $\sin\phi(z)$ and $\cos\phi(z)$, which depend closely on the value of q.

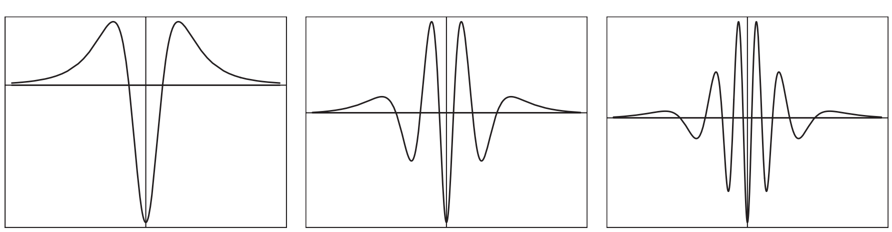
For general q, V_L is not a volcano-type potential anymore. Here we only discuss the case of positive η and q, which results in a negative potential at the location of the brane since k is positive. In order to localize fermions on the brane, we also need at least a potential barrier on each side. In fact, for $4n < q \le 4n + 2$ or $q = 4n + 3$, the potential for the left chiral fermions has $n + 1$ finite positive barriers on each side, in which the last one on each side vanishes asymptotically from above. The potential is always positive at long distances, so it can trap the zero mode. The shapes of the potential for this case are shown in Fig. 5. For $4n + 2 < q \le 4n + 4$ but $q \neq 4n + 3$, the potential for the left chiral fermions has also $n + 1$ finite positive barriers on each side, but the last barrier on each side vanishes asymptotically from below. The potential is always negative at long distances, which indicates that it cannot trap the zero mode. See Fig. 6 for the shapes of the potential. Hence, in order to get a potential for the left chiral fermions that can trap some fermion KK modes, we first need the following condition:

$$4n < q \le 4n + 2(n \ge 0) \quad \text{or} \quad q = 4n + 3(n \ge 0). \tag{44}$$

Next we discuss the relation of the potential $V_R(z)$ to the parameter q. We also limit our discussion on positive η and q, which results in a positive potential at the location of the brane for $V_R(z)$. This seems to show that the potential could not trap any KK modes of the right chiral fermions. But the result is the opposite. For $4n < q \le 4n + 2$ but $q \neq 4n + 1$, the potential for the right chiral fermions has $2n + 1$ finite barriers and $2n + 2$ finite wells; among them, a positive barrier is at the location of the brane. The potential vanishes asymptotically from below at long distances, which indicates that it cannot trap the zero mode. For $4n + 2 < q \le 4n + 4$, the potential has $2n + 3$ finite barriers and $2n + 2$ finite wells. For $q = 4n + 1$, the potential has $2n + 1$ finite barriers and $2n$ finite wells. For both cases, the potential vanishes asymptotically from above at long distances, which indicates that it cannot trap the zero mode. Hence, in order to get a potential for the right chiral fermions that can trap some fermion KK modes, we need the following condition:

$$4n + 2 < q \le 4n + 4(n \ge 0) \quad \text{or} \quad q = 4n + 1(n > 0). \tag{45}$$

This is a remarkable result which is very different from the case considered in the previous subsection, where the potential for the right chiral fermions with positive η and q cannot trap any KK modes because it is always positive.



(a) $0 < q \le 2,\ q = 3$ (b) $4 < q \le 6,\ q = 7$ (c) $8 < q \le 10,\ q = 11$

FIG. 5. The shapes of the potential $V_L(z)$ (41) for $4n < q \le 4n + 2$ or $q = 4n + 3$. The potential has a negative value at the location of the thick brane, and $n + 1$ finite positive barriers on each side which vanish asymptotically from above when far away from the brane.

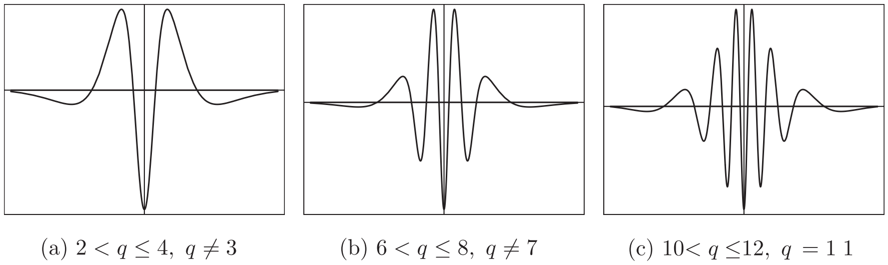


FIG. 6. The shapes of the potential $V_L(z)$ (41) for $4n + 2 < q \leq 4n + 4$ but $q \neq 4n + 3$. The potential has a negative value at the location of the thick brane, and $n + 1$ finite positive barriers on each side which vanish asymptotically from below when far away from the brane.

The shapes of the potential $V_R(z)$ for various values of q are shown in Figs. 7 and 8.

Now we examine the zero modes for the left and right chiral fermions. By setting $m_0 = 0$ and $F(\phi) = \sin\phi$, from Eq. (18) we find that the left and right zero modes have the following formalized solutions:

$$\tilde{\alpha}_{L0}(z) \propto \exp\left(-\eta \int^z dz' e^{A(z')} \sin\phi(z')\right), \tag{46}$$

$$\tilde{\alpha}_{R0}(z) \propto \exp\left(+\eta \int^z dz' e^{A(z')} \sin\phi(z')\right). \tag{47}$$

Using the same method as in the previous subsection, we can obtain the restriction on the free parameters η and q from the normalization condition (19) for the zero modes (46) and (47). In the y coordinate, the restriction condition for the zero modes is

$$\int dy \exp\left(-A(y) - (\pm)2\eta \int^y dy' \sin\phi(y')\right) < \infty. \tag{48}$$

When $y \to \infty$, we have $A(y) \to -ky$ and $\sin\phi(y) \to \sin\frac{q\pi}{2}$, and so $(-A(y) - (\pm)2\eta \int^y dy' \sin\phi(y')) \to (k - (\pm)2\eta \sin\frac{q\pi}{2})y$. Thus, the restriction condition reduces to

$$\pm 2\eta \sin(q\pi/2) > k \tag{49}$$

with "+" for the left fermions and "−" for the right ones. For the left chiral fermions, combining (49) with the constraint (44) coming from the effective potential V_L, the condition for localizing the zero mode turns out to be

$$4n < q < 4n + 2 \quad \text{or} \quad q = 4n + 3 (n \geq 0), \qquad \eta > \frac{k}{2\sin(q\pi/2)}. \tag{50}$$

For the right chiral fermions, the localization condition of the zero mode is

$$4n + 2 < q < 4n + 4 \quad \text{or} \quad q = 4n + 1 (n \geq 0), \qquad \eta > \frac{k}{-2\sin(q\pi/2)}. \tag{51}$$

Note that the values $q = 2n$ do not appear in (50) and (51). The reason is that the value of $F(\sin\phi)$ will tend to zero at long distances for $q = 2n$, which results in the non-normalizable zero modes. In Ref. [45], Melfo *et al.* studied the localization of fermions on various scalar thick branes. They showed that only one massless chiral mode is localized in double walls and branes interpolating between different AdS_5 space-times whenever the wall thickness is kept finite, while chiral fermion modes cannot be localized in dS_4 walls embedded in an M_5 space-time. In Ref. [46], Bietenholz *et al.* investigated fermions in the brane world of the 3D Gross-Neveu model and addressed, in particular, the question of whether approximate chiral symmetry can come about in a natural way under brane-type dimensional reduction. They found that a left-handed 2D fermion localized on the domain wall and a right-handed fermion localized on the antiwall communicate with each other through the 3D bulk, and the two 2D fermions are bound together to form a Dirac fermion of

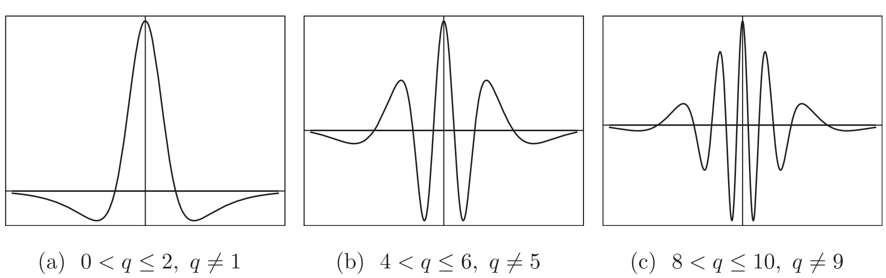
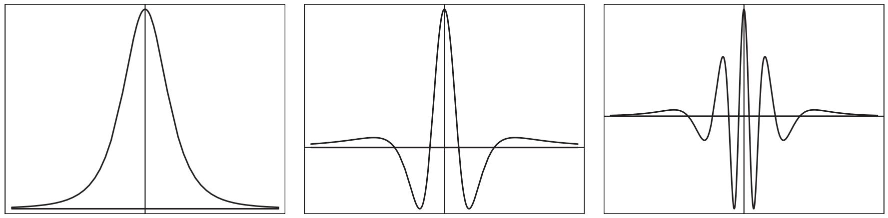


FIG. 7. The shapes of the potential $V_R(z)$ (42) for $4n < q \leq 4n + 2$ but $q \neq 4n + 1$. The potential has a positive value at the location of the thick brane, and n finite positive barriers on each side which vanish asymptotically from below when far away from the brane.



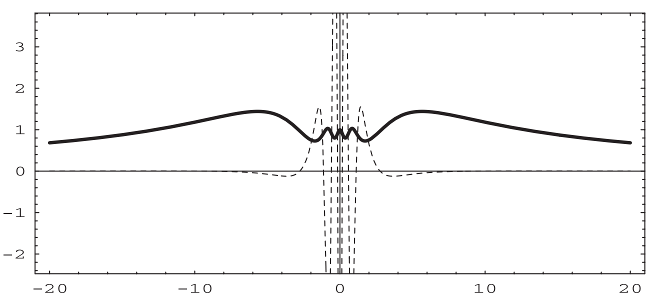
(a) $q = 1$ (b) $2 < q \leq 4,\ q = 5$ (c) $6 < q \leq 8,\ q = 9$

FIG. 8. The shapes of the potential $V_R(z)$ (42) for $4n + 2 < q \leq 4n + 4$ and $q = 4n + 1$. The potential has a positive value at the location of the thick brane, as well as $n + 1$ and n finite positive barriers for $4n + 2 < q \leq 4n + 4$ and $q = 4n + 1$, respectively, on each side which vanish asymptotically from above when far away from the brane.

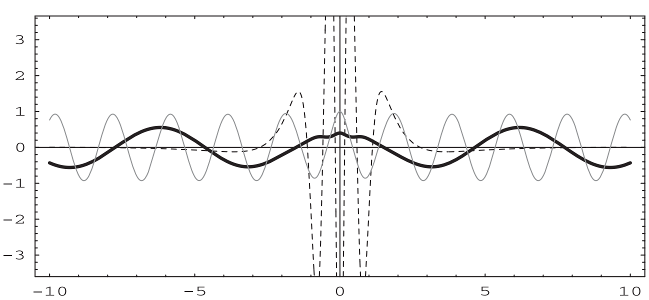
mass m. This involves a hierarchy problem with respect to the fermion mass.

For arbitrary $q\eta > 0$, the two potentials suggest that there is no mass gap but a continuous spectrum of KK modes. In Fig. 9, we plot the massless and massive KK modes for the left chiral fermions. It can be seen that the zero mode is bound on the brane if the condition (50) is satisfied. The massive modes with lower energy (especially the zero mode) experience an attenuation due to the series of potential barriers near the location of the brane.

To close this section, we make some comments on the issue of the localization of fermions. Localizing the fermions on branes or defects requires us to introduce other interactions besides gravity. More recently, Volkas *et al.* had extensively analyzed localization mechanisms on a domain wall. In particular, in Ref. [47], they proposed a well-defined model for localizing the SM, or something close to it, on a domain wall brane. There are some other backgrounds, for example, gauge field [48], supergravity [49,50] and vortex backgrounds [51–54], that could be considered. The topological vortex coupled to fermions may result in chiral fermion zero modes [55].



(a) Zero mode ($m^2 = 0,\ q = 9$)



(b) Massive modes ($m^2 = 1$ for the thick black line, $m^2 = 10$ for the thin gray line, $q = 9$)

FIG. 9. The shape of the potentials $V_L(z)$ (41) (dashed lines), the zero mode (46), and the massive modes for the left chiral fermions for the case $F(\phi) = \sin\phi$. The parameters are set to $k = 1$, $q = 9$, and $\eta = 1$.

IV. DISCUSSIONS

In this paper, by presenting the shapes of the mass-independent potentials in the corresponding Schrödinger equations, we have reinvestigated the possibility of localizing spin-$1/2$ fermions on a thick brane for two kinds of kink-fermion couplings. It is shown that, without scalar-fermion coupling, there is no bound state for both the left and right chiral fermions. Hence, in order to localize the massless and massive left or right chiral fermions on the brane, some kind of Yukawa coupling should be introduced.

For the Yukawa coupling $\eta\bar{\Psi}\phi\Psi$, only one of the potentials for the left and right chiral fermions has a finite well at the location of the brane and a finite barrier on each side, which vanishes asymptotically. It is shown that there is only one single bound state (zero mode) which is just the lowest energy eigenfunction of the Schrödinger equation for the corresponding chiral fermions. When the condition $\eta q \pi > k$ is satisfied, the zero mode is normalizable.

For the scalar-fermion coupling $\eta\bar{\Psi}\sin\phi\Psi$ with $q > 0$ and $\eta > 0$, the potential for the left chiral fermions has a finite well at the location of the brane as well as a series of finite positive barriers on each side, and vanishes asymptotically from above or below when it is far away from the brane. Under the condition (50), there exists a bound and normalizable left chiral fermion zero mode.

It is worth pointing out that, under the condition $q > 0$ and $\eta > 0$, for the usual coupling $\eta\bar{\Psi}\phi\Psi$, the potentials for the left and right chiral fermions have very different shapes and only the left fermion zero mode could be localized. However, for the coupling $\eta\bar{\Psi}\sin\phi\Psi$, the potentials for the left and right chiral fermions have similar shapes, and the right fermion zero mode could also be localized on the brane under the condition (51). The reason is that, although the potential for the right chiral fermions has a positive value at the location of the brane, it has some wells and a series of positive barriers near the brane, which ensures that it can trap the right chiral fermion zero mode on the brane.

Since the potentials for both scalar-fermion couplings vanish asymptotically when far away from the brane, all values of $m^2 > 0$ are allowed, and there exists no mass gap but a continuous gapless spectrum of KK states with $m^2 > 0$. The massive KK modes asymptotically turn into continuous plane waves when far away from the brane [2,21], and represent delocalized massive KK fermions.

ACKNOWLEDGMENTS

Y. X. Liu is very grateful to Professor Jian-Xin Lu and Professor Huan-Xiong Yang for their invitation to visit USTC-ICTS and interesting discussions. The authors are also thankful to Dr. Wolfgang Bietenholz, Xin-Hui Zhang, and Shao-Wen Wei for valuable recommendations and discussions. L. J. Zhang acknowledges financial support from the Shanghai Education Commission. This work was supported by the National Natural Science Foundation of China (No. 502-041016 and No. 10705013), the Doctor Education Fund of the Educational Department of China (No. 20070730055), and the Fundamental Research Fund for Physics and Mathematics of Lanzhou University (No. Lzu07002).

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).

[2] J. Lykken and L. Randall, J. High Energy Phys. 06 (2000) 014.

[3] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **125B**, 136 (1983).

[4] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **125B**, 139 (1983).

[5] K. Akama, in *Proceedings of the Symposium on Gauge Theory and Gravitation*, edited by K. Kikkawa, N. Nakanishi, and H. Nariai (Springer-Verlag, Japan, 1983).

[6] M. Visser, Phys. Lett. **159B**, 22 (1985).

[7] S. Randjbar-Daemi and C. Wetterich, Phys. Lett. **166B**, 65 (1986).

[8] G. D. Starkman, D. Stojkovic, and M. Trodden, Phys. Rev. D **63**, 103511 (2001); Phys. Rev. Lett. **87**, 231303 (2001).

[9] I. Antoniadis, Phys. Lett. B **246**, 377 (1990).

[10] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **436**, 257 (1998).

[11] A. Kehagias, Phys. Lett. B **600**, 133 (2004).

[12] E. J. Squires, Phys. Lett. **167B**, 286 (1986).

[13] M. Gogberashvili, Mod. Phys. Lett. A **14**, 2025 (1999); Int. J. Mod. Phys. D **11**, 1635 (2002).

[14] B. Bajc and G. Gabadadze, Phys. Lett. B **474**, 282 (2000).

[15] I. Oda, Phys. Lett. B **496**, 113 (2000).

[16] Y. Grossman and N. Neubert, Phys. Lett. B **474**, 361 (2000); R. Koley and S. Kar, Phys. Lett. B **623**, 244 (2005); **631**, 199(E) (2005).

[17] S. Ichinose, Phys. Rev. D **66**, 104015 (2002).

[18] C. Ringeval, P. Peter, and J. P. Uzan, Phys. Rev. D **65**, 044016 (2002).

[19] T. Gherghetta and M. Shaposhnikov, Phys. Rev. Lett. **85**, 240 (2000).

[20] I. P. Neupane, J. High Energy Phys. 09 (2000) 040; I. P. Neupane, Classical Quantum Gravity **19**, 5507 (2002).

[21] O. DeWolfe, D. Z. Freedman, S. S. Gubser, and A. Karch, Phys. Rev. D **62**, 046008 (2000).

[22] M. Gremm, Phys. Lett. B **478**, 434 (2000); Phys. Rev. D **62**, 044017 (2000); K. Ghoroku and M. Yahiro, arXiv:hep-th/0305150; A. Kehagias and K. Tamvakis, Mod. Phys. Lett. A **17**, 1767 (2002); Phys. Lett. B **504**, 38 (2001); M. Giovannini, Phys. Rev. D **64**, 064023 (2001); **65**, 064008 (2002); S. Kobayashi, K. Koyama, and J. Soda, Phys. Rev. D **65**, 064014 (2002).

[23] C. Csaki, J. Erlich, T. Hollowood, and Y. Shirman, Nucl. Phys. **B581**, 309 (2000).

[24] A. Campos, Phys. Rev. Lett. **88**, 141602 (2002).

[25] R. Emparan, R. Gregory, and C. Santos, Phys. Rev. D **63**, 104022 (2001); A. Wang, Phys. Rev. D **66**, 024024 (2002); R. Guerrero, A. Melfo, and N. Pantoja, Phys. Rev. D **65**, 125010 (2002); A. Melfo, N. Pantoja, and A. Skirzewski, Phys. Rev. D **67**, 105003 (2003); K. A. Bronnikov and B. E. Meierovich, Gravitation Cosmol. **9**, 313 (2003); O. Castillo-Felisola, A. Melfo, N. Pantoja, and A. Ramirez, Phys. Rev. D **70**, 104029 (2004); M. Minamitsuji, W. Naylor, and M. Sasaki, Nucl. Phys. **B737**, 121 (2006); T. R. Slatyer and R. R. Volkas, J. High Energy Phys. 04 (2007) 062.

[26] V. Dzhunushaliev, V. Folomeev, D. Singleton, and S. Aguilar-Rudametkin, Phys. Rev. D **77**, 044006 (2008); V. Dzhunushaliev, V. Folomeev, K. Myrzakulov, and R. Myrzakulov, arXiv:0705.4014.

[27] D. Bazeia, F. A. Brito, and J. R. Nascimento, Phys. Rev. D **68**, 085007 (2003); D. Bazeia, C. Furtado, and A. R. Gomes, J. Cosmol. Astropart. Phys. 02 (2004) 002; D. Bazeia, F. A. Brito, and A. R. Gomes, J. High Energy Phys. 11 (2004) 070; D. Bazeia and A. R. Gomes, J. High Energy Phys. 05 (2004) 012; D. Bazeia, F. A. Brito, and L. Losano, J. High Energy Phys. 11 (2006) 064; D. Bazeia, A. R. Gomes, and L. Losano, arXiv:0708.3530.

[28] R. Koley and S. Kar, Classical Quantum Gravity **22**, 753 (2005).

[29] S. Randjbar-Daemi and M. Shaposhnikov, Phys. Lett. B **492**, 361 (2000).

[30] O. Arias, R. Cardenas, and I. Quiros, Nucl. Phys. **B643**,

187 (2002).
[31] N. Barbosa-Cendejas and A. Herrera-Aguilar, J. High Energy Phys. 10 (2005) 101.
[32] N. Barbosa-Cendejas and A. Herrera-Aguilar, Phys. Rev. D **73**, 084022 (2006).
[33] Y.-X. Liu, X.-H. Zhang, L.-D. Zhang, and Y.-S. Duan, J. High Energy Phys. 02 (2008) 067.
[34] N. Barbosa-Cendejas, A. Herrera-Aguilar, M. A. Reyes, and C. Schubert, Phys. Rev. D **77** 126013 (2008); N. Barbosa-Cendejas, A. Herrera-Aguilar, U. Nucamendi, and I. Quiros, arXiv:0712.3098.
[35] Y.-X. Liu, L.-D. Zhang, S.-W. Wei, and Y.-S. Duan, J. High Energy Phys. 08 (2008) 041.
[36] S. L. Dubovsky, V. A. Rubakov, and P. G. Tinyakov, Phys. Rev. D **62**, 105011 (2000).
[37] Y. Brihaye and T. Delsate, Phys. Rev. D **78**, 025014 (2008).
[38] S. Ichinose, Phys. Rev. D **65**, 084038 (2002); Classical Quantum Gravity **18**, 5239 (2001).
[39] M. Cvetic and H. Soleng, Phys. Rep. **282**, 159 (1997); P. M. Cowdall, H. Lu, C. N. Pope, K. S. Stelle, and P. K. Townsend, Nucl. Phys. **B486**, 49 (1997); K. Behrndt, Nucl. Phys. **B573**, 127 (2000); K. Behrndt and M. Cvetic, Phys. Lett. B **475**, 253 (2000); K. Behrndt, C. Herrmann, J. Louis, and S. Thomas, J. High Energy Phys. 01 (2001) 011; A. H. Chamseddine and W. A. Sabra, Nucl. Phys. **B630**, 326 (2002); L. Anguelova and C. I. Lazaroiu, J. High Energy Phys. 09 (2002) 053; S. L. Cacciatori, D. Klemm, and W. A. Sabra, J. High Energy Phys. 03 (2003) 023; M. Eto, N. Maru, and N. Sakai, Nucl. Phys. **B696**, 3 (2004).
[40] N. Maru, N. Sakai, Y. Sakamura, and R. Sugisaka, Nucl. Phys. **B616**, 47 (2001).
[41] M. Eto, N. Maru, N. Sakai, and T. Sakata, Phys. Lett. B **553**, 87 (2003); M. Eto, N. Maru, and N. Sakai, Nucl. Phys. **B673**, 98 (2003).
[42] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).
[43] E. Caliceti, V. Grecchi, and M. Maioli, Commun. Math. Phys. **176**, 1 (1996); J. Zamastil, J. Cizek, and L. Skala, Phys. Rev. Lett. **84**, 5683 (2000); F. J. Gomez and L. Sesma, Phys. Lett. A **301**, 184 (2002); M. M. Nieto, Phys. Lett. B **486**, 414 (2000).
[44] H. Davoudiasl, J. L. Hewett, and T. G. Rizzo, Phys. Lett. B **473**, 43 (2000).
[45] A. Melfo, N. Pantoja, and J. D. Tempo, Phys. Rev. D **73**, 044033 (2006).
[46] W. Bietenholz, A. Gfeller, and U. J. Wiese, J. High Energy Phys. 10 (2003) 018.
[47] T. R. Slatyer and R. R. Volkas, J. High Energy Phys. 04 (2007) 062; R. Davies, D. P. George, and R. R. Volkas, Phys. Rev. D **77**, 124038 (2008).
[48] Y. X. Liu, L. Zhao, and Y. S. Duan, J. High Energy Phys. 04 (2007) 097.
[49] G. de Pol, H. Singh, and M. Tonin, Int. J. Mod. Phys. A **15**, 4447 (2000).
[50] S. L. Parameswaran, S. Randjbar-Daemi, and A. Salvio, Nucl. Phys. **B767**, 54 (2007).
[51] Y. X. Liu, L. Zhao, X. H. Zhang, and Y. S. Duan, Nucl. Phys. **B785**, 234 (2007).
[52] Y. Q. Wang, T. Y. Si, Y. X. Liu, and Y. S. Duan, Mod. Phys. Lett. A **20**, 3045 (2005); Y. S. Duan, Y. X. Liu, and Y. Q. Wang, Mod. Phys. Lett. A **21**, 2019 (2006); Y. X. Liu, Y. Q. Wang, and Y. S. Duan, Commun. Theor. Phys. **48**, 675 (2007).
[53] S. Rafael and S. Torrealba, arXiv:0803.0313.
[54] G. Starkman, D. Stojkovic, and T. Vachaspati, Phys. Rev. D **65**, 065003 (2002); **63**, 085011 (2001); D. Stojkovic, Phys. Rev. D **63**, 025010 (2000).
[55] R. Jackiw and P. Rossi, Nucl. Phys. **B190**, 681 (1981).