Casimir energy for a double spherical shell: A global mode sum approach

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In this work we study the configuration of two perfectly conducting spherical shells. This is a problem of basic importance to make possible development of experimental apparatuses that they make possible to measure the spherical Casimir effect, an open subject. We apply the mode sum method via cutoff exponential function regularization with two independent parameters: one to regularize the infinite order sum of the Bessel functions; other, to regularize the integral that becomes related, due to the argument theorem, with the infinite zero sum of the Bessel functions. We obtain a general expression of the Casimir energy as a quadrature sum. We investigate two immediate limit cases as a consistency test of the expression obtained: that of a spherical shell and that of two parallel plates. In the approximation of a thin spherical shell we obtain an expression that allows to relate our result with that of the proximity-force approximation, supplying a correction to this result.

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I. INTRODUCTION

The Casimir attractive force between conducting parallel plates [1] is one of the most striking demonstrations of the quantum nature of the electromagnetic field. Such effect was experimentally confirmed by Sparnaay in 1958 [2] and with great accuracy by Lamoreaux [3] and Mohideen and Roy [4,5]. Casimir himself proposed to build a model for the electron as an application of this effect [6]. He suggested two models: a solid sphere and a thin spherical shell. If the Casimir pressure turned out to point inwards (as could be guessed from a crude analogy with the parallel plates setup), it would stabilize the electron against the electrostatic repulsion, and it would provide a theoretical value for the fine structure constant. Boyer was the first to actually calculate the Casimir pressure for a spherical shell [7]. Contrary to the expectations, the pressure is repulsive (i.e., it points outwards), and so, invalidating Casimir's electron model.

Since Boyer's pioneer work, independent calculations have confirmed that the Casimir pressure is repulsive for a perfectly reflecting spherical shell. Different methods have been employed: the Schwinger source theory [8], the Green function technique [9], the multiple scattering approach [10], and the zeta function formalism [11]. However, a simple physical explanation of why the pressure is repulsive is still lacking.

Being the first realistic approach to investigate the physical systems of the nature, the spherical symmetry can be applied for the study of cosmological systems or bag models of hadrons, to quote two examples of great relevance [12–15].

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In this paper, we compute the Casimir energy for two concentric perfectly reflecting spherical shells (internal and external radii are a and b). This type of setup was analyzed with the help of the Green function technique for material media with particular electric and magnetic properties (satisfying the so-called uniform velocity of light condition) [16–19] and for more general dieletric media [20]. A quantum statistical approach was employed for ideal metals [21] as well as for dieletric media [22]. Recently, the measurement of the Casimir force between two concentric spherical surfaces was proposed [23].

Here, we compute separately the contributions of field modes from each of the three spatial regions (internal, between the shells and external) directly from the field zero-point energy. This ''mode sum'' approach [24–26], when combined with the introduction of suitable cutoff functions [27], allows for a physical interpretation of the several terms contributing to the Casimir energy, which correspond to different field polarizations and spatial regions. All series and integrals are properly regularized by exponential cutoff functions, so that we only deal with well-defined quantities. We keep the cutoff dependent terms and analyze their dependence on radius, area, and internal volume of the spherical shells, before considering the cancellations resulting from the sum over the three spatial regions. As suggested by Barton [28], these cutoff dependent terms might be physically relevant and provide for a net attractive pressure in the context of a more realist model for the material medium.

We are particularly interested in the limit of a thin intershell region $(b - a)/a \ll 1$. In this case, we should recover the Casimir *attractive* force between parallel plates according to the proximity-force approximation (PFA) [29], which replace the surfaces by tangent planes.

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Hence, there must be a crossover between the single shell repulsive regime and the thin intershell attractive regime. Our model also allows for evaluation of the accuracy of PFA in a problem for which an exact solution is available [30].

The paper is organized in the following way: section II presents the derivation of the formal results for the Casimir energy in the form of a multipole series. The limit $(b - a)/a \ll 1$ is considered in Sec. III, where corrections to the PFA result are obtained. In Sec. IV, we present some final remarks and a conclusion.

II. CASIMIR EFFECT BETWEEN TWO SPHERICAL SHELLS

Since we consider perflectly reflecting shells, each electromagnetic field mode is confined in one of the three spatial regions, which are labeled by the index τ, with $\tau = 1, 2, 3$ denoting the inner, intershell and outer spatial regions, respectively. Moreover, due to spherical symmetry, we may decompose the vectorial boundary value problem into two independent problems by defining the usual transverse electric (TE, electric field perpendicular to the radial direction) and transverse magnetic (TM) polarizations. Hence, we have 6 independent classes of field modes, which we represent by the indexes τ and $p =$ TE, TM for polarization.

The Casimir energy for the double shell is given by the modification of the zero-point energy due to the boundary conditions

$$\mathcal{E}(a,b) = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\tau=1}^{3}\sum_{p}(j+1/2)\hbar[\omega_{njp}^{(\tau)}(a,b) - \omega_{njp}^{(\tau)}(\text{ref})], \tag{1}$$

where $\omega_{njp}^{\tau}(a,b)$ is a mode frequency for a given angular momentum j, polarization p, and spatial region τ. We have taken into account the degeneracy factor $2j+1$ and subtracted the reference frequencies $\omega_{jn}^{\tau}(\text{ref})$, corresponding to the free-space limit.

In order to have discrete spectra for the outer region, we consider a third "auxiliary" spherical surface of radius R, and take the limit $R \to \infty$ (see Fig. 1).

To obtain the zero-point energy for the free-space case, we consider a similar configuration [7], taking the two innermost shells to have radius R/ξ and R/ζ, with $\xi > \zeta > 1$. We evaluate the sum over n in Eq. (1) with the help of Cauchy's theorem for analytic functions in the complex plane of frequency [31,32], taking the contour $C(\Lambda, \phi)$ indicated in Fig. 2.

We define analytic functions $f_{jp}^{(\tau)}(a,b;z)$ such that their zeros (when considered as functions of z) correspond to the eigenfrequencies $\omega_{njp}^{(\tau)}(a,b)$, $n = 1, 2, \ldots$ They are all contained within the contour $C(\Lambda, \varphi)$ in the limit $\Lambda \to \infty$.

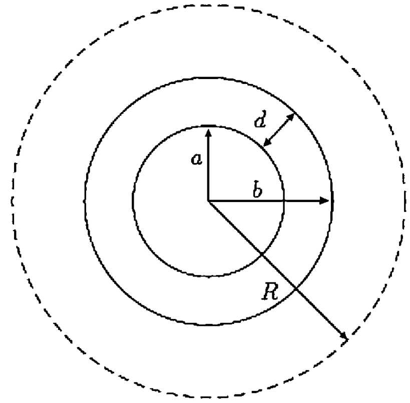

FIG. 1. Boundary conditions of two spherical shells.

We find, by using the prescription proposed in [27], with $\nu = j + 1/2$,

$$\mathcal{E}(a,b) = \lim_{\Xi\to\Xi_0}\frac{\hbar c}{2\pi i}\sum_{j=1}^{\infty}\nu\exp(-\varepsilon\nu)\lim_{\Lambda\to\infty}\oint_{C(\Lambda,\varphi)}dzz \times\exp(-\sigma z)\frac{d}{dz}\sum_{\tau=1}^{3}\sum_{p}\Big\{\log[f_{jp}^{(\tau)}(a,b;z)] - \log\Big[f_{jp}^{(\tau)}\Big(\frac{R}{\xi},\frac{R}{\zeta};z\Big)\Big]\Big\}, \tag{2}$$

where Ξ denotes the set R, ξ, ζ together with the cutoff parameters σ and ε. Note that we employ two exponential cutoff functions to regularize our expression (including one for the sum over angular momentum).

Equation (2) is an application of the approach proposed in [27], which presents a cutoff method of calculating the Casimir effect for a spherical shell including both the interior and exterior modes in a new calculation method that has not been used in this problem before. Basically, as it is known in the literature, generally, the j and n sum do not converge, and the procedure to regularize them uses one cutoff exponential function $e^{-\sigma\omega_{jn}^{\tau}}$. However, when we

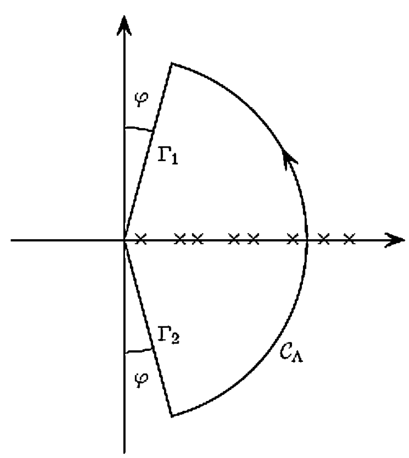

FIG. 2. Path integration in complex plan.

use the argument theorem to bypass the problem of evaluating the implicity frequency, the cutoff exponential function $e^{-\sigma\omega_{jn}^{\tau}}$ renounces to protect the j series of the usual divergence. To prevent this divergence we have used an additional cutoff exponential function $e^{-\varepsilon\nu}$. Thus, we can write the equality in (2) that represents a well-defined expression of the mathematical point of view. Therefore, taking this prescription, the expression to Casimir energy becomes well defined, and the manipulations can be done without difficulty before the limits are realized.

The positive aspect of the approach proposed rests on the fact that (i) as remarkable in literature [24,25], the mode sum method presents a great advantage in relation to other techniques due its simplicity and visualization of the diverse stages involved in calculation, as its global version determines without difficulty the total Casimir energy; (ii) we work exclusively with regularized expressions, preventing cancellation of any possible divergence without previous justification—a procedure that is not explicit in regularizations as the generalized zeta function which, in general, does not use the renormalization process [33], as well as in procedures similar to those used in Refs. [34,35] due to the appearance of divergent series in intermediate steps. In relation to the subtraction procedure, we followed the usual one initially considered by Casimir [1] for parallel plates and by Boyer [7] for spherical effect.

The limit in Eq. (2) is taken in the following sequence: we first take $R\to\infty$, then $\zeta\to 1$, $\xi\to 1$, and finally $\sigma, \varepsilon\to 0$.

The functions $f_{jp}^{(1)}$ actually do not depend on b, because they correspond to the modes in the inner region $r < a$

$$f_{j\mathrm{TE}}^{(1)}(a; z) = S_j(az), \tag{3}$$

where $S_j(x) = \sqrt{\pi x/2}J_{j+1/2}(x)$ is the Riccati-Bessel function of the first kind [36,37], and

$$f_{j\mathrm{TM}}^{(1)}(a; z) = S_j'(az), \tag{4}$$

where the prime denotes the derivative.

Likewise, the functions $f_{jp}^{(3)}$ do not depend on a, because they correspond to the outer region

$$f_{j\mathrm{TE}}^{(3)}(b; z) = C_j(Rz)S_j(bz) - S_j(Rz)C_j(bz), \tag{5}$$

where $C_j(x) = -\sqrt{\pi x/2}N_{j+1/2}(x)$ is the Riccati-Bessel of the second kind, and

$$f_{j\mathrm{TM}}^{(3)}(b; z) = C_j'(Rz)S_j'(bz) - S_j'(Rz)C_j'(bz). \tag{6}$$

The functions $f_{jp}^{(2)}$ corresponding to the intershell region have the same form, with the replacements $b\to a$ and $R\to b$

$$f_{j\mathrm{TE}}^{(2)}(a, b; z) = C_j(bz)S_j(az) - S_j(bz)C_j(az), \tag{7}$$

$$f_{j\mathrm{TM}}^{(2)}(a, b; z) = C_j'(bz)S_j'(az) - S_j'(bz)C_j'(az). \tag{8}$$

Since we take the limit $R\to\infty$, we may replace $S_j(Rz)$, $C_j(Rz)$, and their derivatives by the corresponding asymptotic expansions for large arguments. We find

$$f_{j\mathrm{TE}}^{(3)}(b; z) = \cos[\delta_j(z)]S_j(bz) - \tan[\delta_j(z)]C_j(bz), \tag{9}$$

with $\delta_j(z) = Rz - \frac{j\pi}{2}$, and

$$f_{j\mathrm{TM}}^{(3)}(b; z) = \sin[\delta_j(z)]S_j'(bz) + \cos[\delta_j(z)]C_j'(bz). \tag{10}$$

We may also take the asymptotic expansions for large arguments when computing $f_{jp}^{(\tau)}(R/\xi, R/\zeta; z)$ in Eq. (2), which account for the free-space zero-point energy.

In the contour of Fig. 2, only the segments Γ_1 and Γ_2 contribute in the limit $\Lambda\to\infty$. For convenience, we take $\varphi\to 0$, allowing us to take $z = i\rho$ (with real ρ) everywhere in Eq. (2) except for the exponential cutoff $\exp(-\sigma z) = \exp[\mp i\sigma\rho\exp(\mp i\varphi)]$ for segment $\Gamma_{1,2}$, which provides a damping term $\exp(-\sigma\rho\sin\varphi)$ [31]. The integrand is then written in terms of the modified Bessel functions of the first and second kinds I_ν and K_ν. It is also convenient to rescale the integration variable by multiplying by ν.

As expected, the R dependent terms are canceled due to the subtraction of the free-space energy in Eq. (2). The resulting expression is written as

$$\mathcal{E}(a, b) = \mathcal{E}(a) + \mathcal{E}(b) + \mathcal{E}_c(a, b), \tag{11}$$

where $\mathcal{E}(a)$ [and likewise for $\mathcal{E}(b)$] is the Casimir energy for a single spherical shell of radius a. Besides the single-shell energies, the Casimir energy for the double shell configuration contains a nontrivial term representing the joint effect of the two shells:

$$\mathcal{E}_c(a, b) = -\frac{\hbar c}{\pi}\,\mathrm{Re}\sum_{j=1}^{\infty}\nu^2\int_0^{\infty}d\rho\,\rho\frac{d}{d\rho}\bigg\{\log\bigg[1 - \frac{K_\nu(\nu b\rho)/I_\nu(\nu b\rho)}{K_\nu(\nu a\rho)/I_\nu(\nu a\rho)}\bigg] + \log\bigg[1 - \frac{[\frac{1}{2}K_\nu(\nu b\rho) + \nu b\rho K_\nu'(\nu b\rho)]/[\frac{1}{2}I_\nu(\nu b\rho) + \nu b\rho I_\nu'(\nu b\rho)]}{[\frac{1}{2}K_\nu(\nu a\rho) + \nu a\rho K_\nu'(\nu a\rho)]/[\frac{1}{2}I_\nu(\nu a\rho) + \nu a\rho I_\nu'(\nu a\rho)]}\bigg]\bigg\}. \tag{12}$$

In expression (11) of the Casimir energy, the terms $\mathcal{E}(a)$ and $\mathcal{E}(b)$ are free of regularization and, of course, are well defined. In the case of the remaining contribution, it is also well defined without regularization. This fact results from the sum-integration functional form as can be seen when using the Debye expansion of Bessel functions that appears there. The result of this expansion produces attenuation exponential functions that are inherent to the physical system and dispense the regularization functions. So we can take the limits $\sigma \to 0$, $\varepsilon \to 0$, $\varphi \to 0$, $R \to \infty$, $\zeta \to 1$, and $\xi \to 1$ understood in (2).

Let us consider the "interference" parcel up to the second term of the Debye expansion [36] in the ν parameter of the functions in the argument of the two logarithms in (12). We will be considering the expansion until the $1/\nu$ order because it will be enough to make a small annular region approximation. In the expansion up to the $1/\nu$ order for the TE mode parcel we have

$$\log\left[1-\frac{\frac{K_\nu(\nu b\rho)}{I_\nu(\nu b\rho)}}{\frac{K_\nu(\nu a\rho)}{I_\nu(\nu a\rho)}}\right] \cong \log\left\{1-\exp(-2\nu(\eta_2-\eta_1))\left[1+\frac{\Gamma_1^{(TE)}(t)}{\nu}\right]\right\}$$
$$= -\sum_{n=1}^{\infty}\frac{\exp(-2n\nu(\eta_2-\eta_1))}{n}\times\left\{1+\sum_{m=1}^{n}\frac{n!}{m!(n-m)!}\left[\frac{\Gamma_1^{(\mathrm{TE})}(t)}{\nu}\right]^m\right\}$$
$$\cong -\sum_{n=1}^{\infty}\frac{\exp(-2n\nu(\eta_2-\eta_1))}{n}\left[1+\frac{O^{(\mathrm{TE})}(1)}{\nu}\right], \tag{13}$$

where in the last line we only consider the first term $m = 1$. The η_1, η_2, t_1, and t_2 amounts associated to the Debye expansion are given by [36]

$$\eta_1 = \sqrt{1+a^2\rho^2} + \log\left(\frac{a\rho}{1+\sqrt{1+a^2\rho^2}}\right), \tag{14}$$

$$\eta_2 = \sqrt{1+b^2\rho^2} + \log\left(\frac{b\rho}{1+\sqrt{1+b^2\rho^2}}\right), \tag{15}$$

$$t_1 = \frac{1}{\sqrt{1+a^2\rho^2}}, \tag{16}$$

$$t_2 = \frac{1}{\sqrt{1+b^2\rho^2}}, \tag{17}$$

and $\Gamma_1^{(\mathrm{TE})}(t)$ has the form

$$\Gamma_1^{(\mathrm{TE})}(t) = \left(\frac{5}{12}t_2^3 + \frac{1}{4}t_1 - \frac{1}{4}t_2 - \frac{5}{12}t_1^3\right), \tag{18}$$

so that

$$O^{(\mathrm{TE})}(1) = n\left(\frac{5}{12}t_2^3 + \frac{1}{4}t_1 - \frac{1}{4}t_2 - \frac{5}{12}t_1^3\right). \tag{19}$$

In the expansion up to the $1/\nu$ order for the TM mode parcel we have

$$\log\left[1-\frac{\frac{[(1/2)K_\nu(\nu b\rho)+\nu b\rho K'_\nu(\nu b\rho)]}{[(1/2)I_\nu(b\rho)+\nu b\rho I'_\nu(\nu b\rho)]}}{\frac{[(1/2)K_\nu(\nu a\rho)+\nu a\rho K'_\nu(\nu a\rho)]}{[(1/2)I_\nu(\nu a\rho)+\nu a\rho I'_\nu(\nu a\rho)]}}\right] \cong \log\left\{1-\exp(-2\nu(\eta_2-\eta_1))\left[1+\frac{\Gamma_1^{(\mathrm{TM})}(t)}{\nu}\right]\right\}$$
$$= -\sum_{n=1}^{\infty}\frac{\exp(-2n\nu(\eta_2-\eta_1))}{n}\times\left\{1+\sum_{m=1}^{n}\frac{n!}{m!(n-m)!}\left[\frac{\Gamma_1^{(\mathrm{TM})}(t)}{\nu}\right]^m\right\}$$
$$\cong -\sum_{n=1}^{\infty}\frac{\exp(-2n\nu(\eta_2-\eta_1))}{n}\left[1+\frac{O^{(\mathrm{TM})}(1)}{\nu}\right], \tag{20}$$

where $\Gamma_1^{(\mathrm{TM})}(t)$ and $O^{(\mathrm{TM})}(1)$ amounts have the forms

$$\Gamma_1^{(\mathrm{TM})}(t) = \left(-\frac{7}{12}t_2^3 + \frac{1}{4}t_1 - \frac{1}{4}t_2 + \frac{7}{12}t_1^3\right), \tag{21}$$

$$O^{(\mathrm{TM})}(1) = n\left(-\frac{7}{12}t_2^3 + \frac{1}{4}t_1 - \frac{1}{4}t_2 + \frac{7}{12}t_1^3\right). \tag{22}$$

Emphasize that, in the limit where $\rho \to 0$, the logarithms in (12) are given by

$$\log\left[1-\frac{\frac{K_\nu(\nu b\rho)}{I_\nu(\nu b\rho)}}{\frac{K_\nu(\nu a\rho)}{I_\nu(\nu a\rho)}}\right] \cong \log\left[1-\frac{\frac{[(1/2)K_\nu(\nu b\rho)+\nu b\rho K'_\nu(\nu b\rho)]}{[(1/2)I_\nu(b\rho)+\nu b\rho I'_\nu(\nu b\rho)]}}{\frac{[(1/2)K_\nu(\nu a\rho)+\nu a\rho K'_\nu(\nu a\rho)]}{[(1/2)I_\nu(\nu a\rho)+\nu a\rho I'_\nu(\nu a\rho)]}}\right]$$
$$\cong \log\left(1-\left(\frac{a}{b}\right)^{2\nu}\right), \tag{23}$$

and do not cause problems in the calculation of that expression in the inferior limit of the integral that appears in (12).

In order to obtain an expression equivalent to (11), however, more appropriate to approximations and numerical calculations we can add and subtract the approximate quantities (13) and (20) in the integrand of (12) of the Casimir energy getting

$$\begin{aligned}\mathcal{E}(a,b) &= \mathcal{E}(a) + \mathcal{E}(b) + \frac{\hbar c}{\pi}\Re\sum_{j=1}^{\infty}\nu^2\int_0^{\infty}d\rho\left\{\log\left[1-\frac{\frac{K_\nu(\nu b\rho)}{I_\nu(\nu b\rho)}}{\frac{K_\nu(\nu a\rho)}{I_\nu(\nu a\rho)}}\right]+\sum_{n=1}^{\infty}\frac{\exp(-2n\nu(\eta_2-\eta_1))}{n}\left[1+\frac{O^{(\mathrm{TE})}(1)}{\nu}\right]\right\}\\ &+\frac{\hbar c}{\pi}\Re\sum_{j=1}^{\infty}\nu^2\int_0^{\infty}d\rho\left\{\log\left[1-\frac{\frac{[(1/2)K_\nu(\nu b\rho)+\nu b\rho K'_\nu(\nu b\rho)]}{[(1/2)I_\nu(\nu b\rho)+\nu b\rho I'_\nu(\nu b\rho)]}}{\frac{[(1/2)K_\nu(\nu a\rho)+\nu a\rho K'_\nu(\nu a\rho)]}{[(1/2)I_\nu(\nu a\rho)+\nu a\rho I'_\nu(\nu a\rho)]}}\right]+\sum_{n=1}^{\infty}\frac{\exp(-2n\nu(\eta_2-\eta_1))}{n}\left[1+\frac{O^{(\mathrm{TM})}(1)}{\nu}\right]\right\}\\ &-\frac{\hbar c}{\pi}\Re\sum_{j=1}^{\infty}\nu^2\int_0^{\infty}d\rho\sum_{n=1}^{\infty}\frac{\exp(-2n\nu(\eta_2-\eta_1))}{n}\left[1+\frac{O^{(\mathrm{TE})}(1)}{\nu}\right]\\ &-\frac{\hbar c}{\pi}\Re\sum_{j=1}^{\infty}\nu^2\int_0^{\infty}d\rho\sum_{n=1}^{\infty}\frac{\exp(-2n\nu(\eta_2-\eta_1))}{n}\left[1+\frac{O^{(\mathrm{TM})}(1)}{\nu}\right],\end{aligned} \tag{24}$$

where we have realized partial integration.

We rewrite (24) in the form

$$\mathcal{E}(a,b) = \mathcal{E}(a) + \mathcal{E}(b) + \mathcal{E}_{(01)}^{(\mathrm{TE\ inter}f)} + \mathcal{E}_{(01)}^{(\mathrm{TM\ inter}f)} + \mathcal{E}_{(\mathrm{num})}^{(\mathrm{inter}f)}, \tag{25}$$

where we have used the definitions

$$\mathcal{E}_{(01)}^{(\mathrm{TE\ inter}f)} = -\frac{\hbar c}{\pi}\Re\sum_{j=1}^{\infty}\nu^2\int_0^{\infty}d\rho\sum_{n=1}^{\infty}\frac{\exp(-2n\nu(\eta_2-\eta_1))}{n}\left[1+\frac{O^{(\mathrm{TE})}(1)}{\nu}\right], \tag{26}$$

$$\mathcal{E}_{(01)}^{(\mathrm{TM\ inter}f)} = -\frac{\hbar c}{\pi}\Re\sum_{j=1}^{\infty}\nu^2\int_0^{\infty}d\rho\sum_{n=1}^{\infty}\frac{\exp(-2n\nu(\eta_2-\eta_1))}{n}\left[1+\frac{O^{(\mathrm{TM})}(1)}{\nu}\right], \tag{27}$$

and

$$\begin{aligned}\mathcal{E}_{(\mathrm{num})}^{(\mathrm{inter}f)} &= \frac{\hbar c}{\pi}\Re\sum_{j=1}^{\infty}\nu^2\int_0^{\infty}d\rho\left\{\log\left[1-\frac{\frac{K_\nu(\nu b\rho)}{I_\nu(\nu b\rho)}}{\frac{K_\nu(\nu a\rho)}{I_\nu(\nu a\rho)}}\right]+\sum_{n=1}^{\infty}\frac{\exp(-2n\nu(\eta_2-\eta_1))}{n}\left[1+\frac{O^{(\mathrm{TE})}(1)}{\nu}\right]\right\}\\ &+\frac{\hbar c}{\pi}\Re\sum_{j=1}^{\infty}\nu^2\int_0^{\infty}d\rho\left\{\log\left[1-\frac{\frac{[(1/2)K_\nu(\nu b\rho)+\nu b\rho K'_\nu(\nu b\rho)]}{[(1/2)I_\nu(\nu b\rho)+\nu b\rho I'_\nu(\nu b\rho)]}}{\frac{[(1/2)K_\nu(\nu a\rho)+\nu a\rho K'_\nu(\nu a\rho)]}{[(1/2)I_\nu(\nu a\rho)+\nu a\rho I'_\nu(\nu a\rho)]}}\right]+\sum_{n=1}^{\infty}\frac{\exp(-2n\nu(\eta_2-\eta_1))}{n}\left[1+\frac{O^{(\mathrm{TM})}(1)}{\nu}\right]\right\}.\end{aligned} \tag{28}$$

In expression (25), the Casimir energy is given by the sum of the Casimir energies that would have the shells separately, with two contributions $\mathcal{E}_{(01)}^{(\mathrm{TE\ inter}f)}$ and $\mathcal{E}_{(01)}^{(\mathrm{TM\ inter}f)}$, of TE and TM mode from the region between the two shells, plus a remaining energy $\mathcal{E}_{(\mathrm{num})}^{(\mathrm{inter}f)}$. Since $\mathcal{E}_{(01)}^{(\mathrm{TE\ inter}f)}$ and $\mathcal{E}_{(01)}^{(\mathrm{TM\ inter}f)}$ are contributions proceeding from the two first terms in the Debye expansion, the zero and first orders, the remaining energy $\mathcal{E}_{(\mathrm{num})}^{(\mathrm{inter}f)}$ must take into consideration all the other terms of the Debye expansion and will be in such a way lesser, as the convergence of this expansion increases. In this way, we can characterize $\mathcal{E}_{(\mathrm{num})}^{(\mathrm{inter}f)}$ as the remaining portion of the Casimir energy's expansion up to the second term of the Debye expansion.

We know that the Debye expansion is of fast convergence in the case of great angular moment waves. As we are dealing with contributions proceeding from the region between the two shells of a and b radii ($a < b$), this condition of great angular moment is given by k wave numbers with $ak \gg 1$. But in the annular region, we have wave lengths of the order of $d = b - a$ separation between the shells or minors, that is, $k \sim (1/d)$ (or greater); soon, we must have $a \gg d$, or, the fast convergence of the Debye expansion occurs for small separations between the shells relative to the radii of the shells, which is small annular region. In this case, the remaining portion $\mathcal{E}_{(\mathrm{num})}^{(\mathrm{inter}f)}$ of the expansion (11) is small.

We then got a general expression (24) for the Casimir effect between two concentric conducting spherical shells of radii a and $b(a < b)$ that is accurate as the original expression (11), and is ready to be applied to the case of the small annular region. It will be used in two applications. The first one consists of carrying through the waited verification of consistency that occurs in the limit where the radius of the external shell goes to the infinite, and so the Casimir energy of the two shells tends to the Casimir energy of one shell, the intern. The second application consists in obtaining the Casimir energy of the two shells when the annular region is small, and for consistency, the dominant term of this approximation must be the Casimir energy of two parallel plates. This second application is the subject of the next section.

To study the limit case $b \to \infty$ let us consider the expression of the Casimir energy given by (24) and rewritten in (25). In the limit where $b \to \infty$, we desire that it remain only as the first term $\mathcal{E}(a)$, the Casimir energy of the internal spherical shell. We must, therefore, demonstrate that the limit of the other terms is null.

The second term in (25) is the Casimir energy of only one spherical shell of radius b. We have evidently

$$\lim_{b\to\infty}\mathcal{E}(b) = \lim_{b\to\infty} 0,09234738972\frac{\hbar c}{2b} = 0. \tag{29}$$

The third and fourth terms in (25), the "interference" contributions of TE and TM modes are given by (26) and (27). In these expressions, the exponential in the integrand are given by

$$\begin{aligned}\exp[-2\nu(\eta_2-\eta_1)] &= \exp\Big\{-2\nu\Big[\sqrt{1+b^2\rho^2}-\sqrt{1+a^2\rho^2} \\ &\quad + \log\Big(\frac{b\rho}{a\rho}\frac{1+\sqrt{1+a^2\rho^2}}{1+\sqrt{1+b^2\rho^2}}\Big)\Big]\Big\},\end{aligned} \tag{30}$$

where we use the definitions in (14) and (15). This exponential goes to zero in the limit where $b \to \infty$. Moreover, in this limit the exponential (30) dominates on the $O^{(\mathrm{TE})}(1)$ and $O^{(\mathrm{TM})}(1)$ terms, as it is easy to verify though its definitions (19) and (22), given in terms of (16) and (17).

Therefore, we conclude that

$$\lim_{b\to\infty}\mathcal{E}^{(\mathrm{TE\ inter}f)}_{(01)} = 0, \tag{31}$$

$$\lim_{b\to\infty}\mathcal{E}^{(\mathrm{TM\ inter}f)}_{(01)} = 0. \tag{32}$$

In the last term in (25), in its expression given for (28) we must observe that the terms of integrand involving the exponential ones (30) are null in the limit $b \to \infty$ for the same reasons for which we demonstrate that the contributions (26) and (27) are null. As the further terms of the integrand have logarithms, we can use the properties of the Bessel functions [36]

$$\lim_{b\to\infty}K_\nu(\nu b\rho) = 0, \qquad \lim_{b\to\infty}K'_\nu(\nu b\rho) = 0, \tag{33}$$

$$\lim_{b\to\infty}I_\nu(\nu b\rho) = \infty \quad \text{and} \quad \lim_{b\to\infty}I'_\nu(\nu b\rho) = \infty, \tag{34}$$

and conclude that all the logarithms tend to zero in the limit $b \to \infty$. With this we get of (28),

$$\lim_{b\to\infty}\mathcal{E}^{(\mathrm{interf})}_{(\mathrm{num})} = 0. \tag{35}$$

Using the results (29), (31), (32), and (35) in (25), we obtain

$$\lim_{b\to\infty}\mathcal{E}(a,b) = \mathcal{E}(a) = 0,0923473\frac{\hbar c}{2a}, \tag{36}$$

that is, the Casimir energy of a spherical shell of radius a, as we waited for consistency reasons.

III. CASIMIR ENERGY TO SMALL ANNULAR REGION

To get the approximate Casimir energy to the small annular region, let us consider formula (24). Being the radii of the internal and external spherical shells given, respectively, for a and b, we have that the annular region between the shells has thickness equal to $d = b - a$, so that the excellent parameter to get the approximate energy is given by

$$\gamma = \frac{d}{a} = \frac{b-a}{a}. \tag{37}$$

With this parameter, we can express the radius of the external shell in terms of the internal one

$$b = a(1+\gamma), \tag{38}$$

and the condition to small annular region is given by $\gamma \ll 1$.

To first order in γ the amounts t_2 and η_2, defined in (15) and (17) are given by

$$t_2 = t_1 - \gamma a^2 z^2 t_1^3, \tag{39}$$

$$\eta_2 = \eta_1 + \gamma\sqrt{1+a^2z^2}, \tag{40}$$

where t_1 and η_1 are given by (14) and (16).

Considering negligible the interference term $\mathcal{E}^{(\mathrm{interf})}_{(\mathrm{num})}$ (28) for γ sufficiently small and taking into account the approximations (39) and (40) and considering $\mathcal{E}(a) + \mathcal{E}(b) = (2-\gamma)\mathcal{E}(a)$, the Casimir energy expression (24) reduces to

$$\mathcal{E}(a,b) = (2-\gamma)\mathcal{E}(a) + \mathcal{E}^{(TE,\mathrm{interf})}_{(01)} + \mathcal{E}^{(TM,\mathrm{interf})}_{(01)}, \tag{41}$$

where

$$\mathcal{E}_{(01)}^{(TE,\mathrm{interf})} = -\frac{\hbar c}{\pi}\sum_{j=1}^{\infty}\nu^2\int_0^{\infty}d\rho \times \sum_{n=1}^{\infty}\frac{\exp(-2n\nu\gamma\sqrt{1+a^2\rho^2})}{n}\left[1+\frac{O^{(TE)}(1)}{\nu}\right], \tag{42}$$

$$\mathcal{E}_{(01)}^{(TM,\mathrm{interf})} = -\frac{\hbar c}{\pi}\sum_{j=1}^{\infty}\nu^2\int_0^{\infty}d\rho \times \sum_{n=1}^{\infty}\frac{\exp(-2n\nu\gamma\sqrt{1+a^2\rho^2})}{n}\left[1+\frac{O^{(TM)}(1)}{\nu}\right], \tag{43}$$

and $O^{(TE)}(1)$ and $O^{(TM)}(1)$, defined by (19) and (22), are now given by

$$O^{(TE)}(1) = \left(-\frac{5}{4}na^2\rho^2{t_1}^5 + \frac{1}{4}na^2\rho^2{t_1}^3\right)\gamma, \tag{44}$$

$$O^{(TM)}(1) = \left(\frac{7}{4}na^2\rho^2{t_1}^5 + \frac{1}{4}na^2\rho^2{t_1}^3\right)\gamma. \tag{45}$$

We get for the zero order term in (42)

$$\mathcal{E}_{(0)}^{(TE,\mathrm{interf})} = -\frac{\hbar c}{\pi}\sum_{j=1}^{\infty}\nu^2\int_0^{\infty}d\rho\sum_{n=1}^{\infty}\frac{\exp(-2n\nu\gamma\sqrt{1+a^2\rho^2})}{n} = -\frac{\hbar c}{\pi}\sum_{j=1}^{\infty}\nu^2\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{dyy}{a\sqrt{y^2-1}}\exp(-2n\nu\gamma y)$$
$$= -\frac{\hbar c}{\pi a}\sum_{j=1}^{\infty}\nu^2\sum_{n=1}^{\infty}\frac{1}{n}K_1(2n\nu\gamma) = -\frac{\hbar c}{\pi a}\sum_{n=1}^{\infty}\left[\frac{1}{n^4}\frac{1}{4\gamma^3} - \frac{1}{n^2}\frac{11}{48\gamma}\right], \tag{46}$$

where we use an integral representation formula to K_1 [38] and the Euler-Maclaurin formula in the j sum. The series in n in the final expression are given by $\zeta(4)$ and $\zeta(2)$, so that we get

$$\mathcal{E}_{(0)}^{(TE,\mathrm{interf})} = \frac{\hbar c}{a}\left[-\frac{\pi^3}{360\gamma^3} + \frac{11\pi}{288\gamma}\right]. \tag{47}$$

The $1/\nu$ order term in (42) provides

$$\mathcal{E}_{(1)}^{(TE,\mathrm{interf})} = -\frac{\hbar c}{\pi a}\sum_{j=1}^{\infty}\nu\sum_{n=1}^{\infty}\int_0^{\infty}dy\exp(-2n\nu\gamma\sqrt{1+y^2})\left[-\frac{5}{4}y^2(1+y^2)^{-5/2}\gamma + \frac{1}{4}y^2(1+y^2)^{-3/2}\gamma\right], \tag{48}$$

where the expression (44) of $O^{(TE)}(1)$ was used and changed the integration variable to $y = a\rho$. To realize the integrations in the above expression various integrations by parts are utilized with the objective of getting integrals to be given in terms of Bessel functions K_ν [38]. By the end of this procedure, we got the following expression:

$$\mathcal{E}_{(1)}^{(TE,\mathrm{interf})} = -\gamma\frac{\hbar c}{\pi a}\sum_{j=1}^{\infty}\nu\sum_{n=1}^{\infty}\left[\left(\frac{5(2n\nu\gamma)^4}{24} - \frac{2(2n\nu\gamma)^2}{3}\right)\int_0^{\infty}dy\frac{e^{-2n\nu\gamma\sqrt{1+y^2}}y^2}{\sqrt{1+y^2}} + \left(-\frac{5(2n\nu\gamma)^4}{24} + \frac{7(2n\nu\gamma)^2}{8}\right)\right.$$
$$\left.\times\int_0^{\infty}dy\frac{e^{-2n\nu\gamma\sqrt{1+y^2}}y\arctan(y)}{\sqrt{1+y^2}} + \left(\frac{5(2n\nu\gamma)^3}{24} - \frac{(2n\nu\gamma)}{4}\right)\int_0^{\infty}dy\frac{e^{-2n\nu\gamma\sqrt{1+y^2}}y\,\mathrm{arcsinh}(y)}{\sqrt{1+y^2}}\right]. \tag{49}$$

Resolving the remaining integrals, the expression above can be written in terms of Bessel functions as follows:

$$\mathcal{E}_{(1)}^{(TE,\mathrm{interf})} = -\gamma\frac{\hbar c}{\pi a}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\left[-\frac{5}{3}\nu^4K_0(2n\nu\gamma)n^3\gamma^3 + \frac{5}{3}\nu^4K_1(2n\nu\gamma)n^3\gamma^3 - \frac{5}{6}\nu^3K_0(2n\nu\gamma)n^2\gamma^2 + \frac{7}{4}\nu^2K_0(2n\nu\gamma)n\gamma\right.$$
$$\left.-\frac{4}{3}\nu^2K_1(2n\nu\gamma)n\gamma + \frac{1}{4}\nu K_0(2n\nu\gamma)\right]. \tag{50}$$

To calculate the sums in j of each one of the six parcels of this expression we use the Euler-Maclaurin formula. Holding back only the dominant term of order γ^{-1} in which appears the n series that is given by $\zeta(2)$, we get

$$\mathcal{E}_{(1)}^{(TE,\mathrm{interf})} = -\frac{\hbar c}{a}\left(\frac{17}{288} - \frac{\pi}{48}\right)\frac{\pi}{\gamma}. \quad (51)$$

Adding the contributions of the terms of order ν^0 (47) and of order ν^{-1} (51), we get the "interference" energy (42) in the TE mode

$$\mathcal{E}_{(01)}^{(TE,\mathrm{interf})} = \frac{\hbar c}{a}\left[-\frac{\pi^3}{360}\frac{1}{\gamma^3} - \left(\frac{1}{48} - \frac{\pi}{48}\right)\frac{\pi}{\gamma}\right]. \quad (52)$$

The contributions to orders ν^0 and ν^{-1} that correspond to TM mode (43) are obtained by analogous procedures

$$\mathcal{E}_{(0)}^{(TM,\mathrm{interf})} = \frac{\hbar c}{a}\left[-\frac{\pi^3}{360\gamma^3} + \frac{11\pi}{288\gamma}\right], \quad (53)$$

$$\mathcal{E}_{(1)}^{(TM,\mathrm{interf})} = -\frac{\hbar c}{a}\left(-\frac{31}{288} + \frac{\pi}{24}\right)\frac{\pi}{\gamma}. \quad (54)$$

Adding these two contributions, we get the interference energy (43) in TM mode

$$\mathcal{E}_{(01)}^{(TM,\mathrm{interf})} = \frac{\hbar c}{a}\left[-\frac{\pi^3}{360}\frac{1}{\gamma^3} + \left(\frac{7}{48} - \frac{\pi}{24}\right)\frac{\pi}{\gamma}\right]. \quad (55)$$

Let us notice that accomplishment of the calculations separately for TE and TM modes allows us to evaluate, to each order of approximation, what mode supplies the preponderant contribution. Also, it allows to collect all the contributions in the TE mode to join to the $j = 0$ contribution of the scalar field and to get the Casimir energy of a scalar field subject to Dirichlet conditions in two concentric spherical surfaces.

Finally, substituting the "interference" contributions (52) to the TE mode and (55) to TM one in (41), and discarding the γ positive power in the $(2-\gamma)\mathcal{E}(a)$ contribution, we get the Casimir energy of concentric spherical shells in the case of small annular region

$$\mathcal{E}(a,b) = \frac{\hbar c}{a}\left[-\frac{\pi^3}{180}\frac{1}{\gamma^3} + \left(\frac{1}{8} - \frac{\pi}{48}\right)\frac{\pi}{\gamma} + 0.09234739002\right]. \quad (56)$$

To verify the consistency of the developed formalism let us consider the limit where the radii of the concentric shells go to the infinite, keeping constant the separation between them. In this limit the Casimir energy density must become equal to the case of two parallel plates. Dividing both the members of Eq. (56) by the area $A = 4\pi a^2$ of the internal shell and remembering that $\gamma = d/a$, we get

$$\frac{\mathcal{E}(a,b)}{A} = -\frac{\pi^2\hbar c}{720}\frac{1}{d^3} + \left(\frac{1}{8} - \frac{\pi}{48}\right)\frac{\hbar c}{4a^2 d} + 0.09234739002\frac{\hbar c}{4\pi a^3}. \quad (57)$$

In the limit where the radii of the spherical shells go to the infinite, with constant d separation, we get the previous equality

$$\lim_{\substack{a\to\infty\\ b-a=d}} \frac{\mathcal{E}(a,b)}{A} = -\frac{\pi^2\hbar c}{720}\frac{1}{d^3}. \quad (58)$$

When this limit is accurately the Casimir energy of the electromagnetic field in the presence of two conducting parallel plates, we have that this verification of consistency supplies a satisfactory result to the formalism of the two spherical shells here considered.

To relate our result given in (56) or (57) with that of proximity-force approximation (PFA), we consider the PFA result for the case of two concentric spherical shells.

Following [39], the PFA for the Casimir energy $\mathcal{E}_C$ of two arbitrary smooth surfaces *is given by the surface integral over the Casimir energy per area, which belongs to an equivalent parallel-plate system that locally follows the two surfaces* [39]

$$\mathcal{E}_{\mathrm{PFA}} = \iint_A d\sigma\,\epsilon[z(\sigma)], \quad (59)$$

where A is the area of one of the opposing surfaces, which are locally separated by the surface-dependence distance $z(\sigma)$, and $\epsilon[z(\sigma)]$ is the corresponding Casimir energy per area.

Considering that, in general, *the plate segment $d\sigma$ is tangential to only one of the surfaces, and therefore, the local distance vector $\vec{z}(\sigma)$ is perpendicular only to this surface and not to the other* [39], in the case of two concentric spherical shells, where the local distance vector $\vec{z}(\sigma)$ is perpendicular to both spheres and have constant modulus d, we have for the "inner-sphere-based PFA" that

$$\mathcal{E}^{\circledcirc}_{\mathrm{inner-sphere\ PFA}} = -\iint_{\mathrm{half-sphere}} \frac{\pi^2\hbar c}{720}\frac{dA}{|\vec{z}(\sigma)|^3} = -\frac{\pi^2\hbar c}{720 d^3}\iint_{\mathrm{half-sphere}} dA = -\frac{\pi^2\hbar c}{720 d^3}A. \quad (60)$$

The choice for concentric-spheres PFA and not for plate-sphere PFA has been made, because in this paper we calculated the Casimir effect for thin spherical shell. Moreover, nowadays the possibility of a measurement of spherical Casimir effect is concrete [23], which justifies this choice.

Comparing the result (57) with the result of the PFA (60) we have

$$\frac{\mathcal{E}(a,b)}{A} = \frac{\mathcal{E}^{\circledcirc}_{\mathrm{inner-sphere\ PFA}}}{A} + \left(\frac{1}{8} - \frac{\pi}{48}\right)\frac{\hbar c}{4a^2 d} + 0.09234739002\frac{\hbar c}{4\pi a^3}. \quad (61)$$

We see by (60) that the proximity-force approximation for the Casimir energy of two concentric spherical shells does not allow to evaluate the error committed when using it. This approach supplies the dominant term, the energy of two parallel plates and nothing more. In the approximation (61) that we call small annular region approximation, we have the same dominant term as that in the PFA, and more terms that supply corrections to this dominant term. Supposedly, the expression of the energy with these corrections better approaches the energy in the case of two concentric shells. Let us notice that the first correction in (61) to the term in γ^{-3} that gives the energy of two parallel plates is a term in γ^{-1} that originates a term with the $(a^2 d)^{-1}$ dependence; to follow comes the term in γ^0, which originates a term with the $(a^3)^{-1}$ dependence. It is noticed, then, the absence of the term in γ^{-2}. This absence is not a decurrent accident of the considered approaches, as it can be verified in a tedious inspection of higher order in the Debye expansion.

IV. FINAL CONSIDERATIONS

The expression (61) also allows to evaluate the error that occurs by using the PFA for the energy of two concentric spherical shells. The first term reveals that for each separation d between the shells, it has an error that diminishes with the square of the radius of the internal shell, that is, with its area. The second term shows that it has an error that diminishes with the cube of its radius, that is, with its volume. It is important to note the absence of a term in γ^{-2} that would originate a term with $(ad^2)^{-1}$ dependence and therefore, an error that would diminish with the proper radius of the internal spherical shell.

Deriving the expression from the energy (61) in relation to the separation d, we get the Casimir pressure in small annular region approximation

$$\mathcal{P}(a,b) = \mathcal{P}^{\circledcirc}_{\text{inner-sphere PFA}}(d) + \left(\frac{1}{8} - \frac{\pi}{48}\right)\frac{\hbar c}{2a^2 d^2}, \quad (62)$$

where $\mathcal{P}^{\circledcirc}_{\text{inner-sphere PFA}}(d)$ is the proximity-force approximation for the pressure between the spherical shells

$$\mathcal{P}^{\circledcirc}_{\text{inner-sphere PFA}}(d) = -\frac{\pi^2 \hbar c}{240}\frac{1}{d^4}, \quad (63)$$

that is, the pressure between the parallel plates.

For the calculation of the relative correction to the PFA pressure, we have

$$\left|\frac{\Delta\mathcal{P}(a,b)}{\mathcal{P}^{\circledcirc}_{\text{inner-sphere PFA}}(d)}\right| = \left(\frac{1}{8} - \frac{\pi}{48}\right)\frac{120}{\pi^2}\frac{d^2}{a^2} = 0.7240\left(\frac{d}{a}\right)^2. \quad (64)$$

For example, for $d/a = 0.1$, we have a relative error of 0.72%. For $a \to \infty$, we have a null relative error, corresponding to the situation of parallel plates. Fixing one determined radius a, the relative error (64) decreases with the quadratic power when the parameter d diminishes.

The pressure $\mathcal{P}(a,b)$ also was obtained previously by Brevik, Skurdal, and Sollie [20], using the Green function formalism. However, our result not only differs from that one in the coefficients' values for each term, as also for the terms itself. In fact, in Brevik and collaborators' result, it appears the dependence in $(ad^3)^{-1}$ proceeding from the contribution of the γ^{-2} order that is absent in our result. In the formalism that we have adopted, we could verify that this contribution does not appear even if we consider superior orders in $1/\nu$ (that is, $1/\nu^2$ or above) in the Debye expansion.

Concerning the qualitative different result obtained relative Ref. [20], which was obtained by a different method, after all, only when measurement of the energy can be performed will a definite answer be found.

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