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Noncommutative nonlinear sigma models and integrability

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We first review the result that the noncommutative principal chiral model has an infinite tower of conserved currents and discuss the special case of the noncommutative $\mathbb{C}P^1$ model in some detail. Next, we focus our attention to a submodel of the $\mathbb{C}P^1$ model in the noncommutative spacetime $\mathcal{A}_\theta(\mathbb{R}^{2+1})$. By extending a generalized zero-curvature representation to $\mathcal{A}_\theta(\mathbb{R}^{2+1})$ we discuss its integrability and construct its infinitely many conserved currents. A supersymmetric principal chiral model with and without the Wess-Zumino-Witten term and a supersymmetric extension of the $\mathbb{C}P^1$ submodel in noncommutative spacetime [i.e., in superspaces $\mathcal{A}_\theta(\mathbb{R}^{1+1|2})$, $\mathcal{A}_\theta(\mathbb{R}^{2+1|2})$] are also examined in detail and their infinitely many conserved currents are given in a systematic manner. Finally, we discuss the solutions of the aforementioned submodels with or without supersymmetry.

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I. INTRODUCTION

Principal chiral models and several of its subfamilies, such as the $O(N)$ and the $\mathbb{C}P^N$ models, are important examples of classically integrable field theories [1–4]. These nonlinear systems possess many interesting features due to their integrability [5,6]. Among these, the existence of a linear system of equations and of an infinite number of conservation laws associated with nonlocal charges are two central properties from which others (such as the Bäcklund transformations) can be obtained. Making use of the conserved, curvature free connections present in these models, an infinite number of conserved currents can be explicitly constructed by an inductive procedure due to Brézin *et al.* [7], and a linear system of equations can thereby be easily obtained via introducing a spectral parameter. It can be verified that the latter imply the field equations as well as the zero-curvature condition on the appropriate connection. Nonlocal charges, if conserved at the quantum level, play a crucial role in finding the S matrix and proving its factorizability, and hence the quantum integrability of a given model. It is known that $O(N)$ and $\mathbb{C}P^1$ models [8,9] and principal chiral models based on certain classical groups [10,11] are quantum integrable, while $\mathbb{C}P^N$ ($N \geq 2$) is not [12]. More generally, sigma models on compact symmetric spaces G/H with H simple are known to be quantum integrable [13].

Supersymmetric (SUSY) extensions of these nonlinear systems both at the classical and at the quantum level have also been extensively studied in the past few decades [14–20]. At the classical level, conserved currents of the supersymmetric $O(N)$ and $\mathbb{C}P^N$ models were derived in component formalism in [19]. Later on, a much simpler superfield formulation with or without the SUSY Wess-Zumino-Witten (WZW) term was given in [20]. In [16], it was shown that supersymmetry renders the $\mathbb{C}P^N$ model quantum integrable.

Noncommutative (NC) field theories have been under investigation for about a decade now. (See, for instance, [21,22] for comprehensive reviews.) Among them, field theories defined on the Groenewold-Moyal (GM)–type deformations of spacetime [i.e., the noncommutative algebra $\mathcal{A}_\theta(\mathbb{R}^{(d+1)})$] hold a considerably large part of the literature. Formulation of instantons and solitons in GM spacetime and other noncommutative spaces, such as the noncommutative tori and fuzzy spaces, has been extensively studied and found to present very rich mathematical structures [21–24]. It has been found out that such noncommutative deformations of extended field configurations may be useful in studying the physics of D-branes, as certain low energy limits in string theory in the presence of background magnetic fields lead to noncommutative Yang-Mills (YM) theories [25–27].

Integrability properties of noncommutative nonlinear theories have been under investigation in the past decade as well. In [28], Dimakis and Müller-Hoissen have studied the existence and construction of conserved currents in nonlinear sigma models on noncommutative spaces where an appropriate notion of the Hodge operator can be prescribed, including the GM plane. Formulations of nonlinear sigma models on noncommutative 2-torus with two-point target space and construction of its conserved currents along the lines of [7] were given in [29].

In [30], a linear system of equations for noncommutative YM theory has been presented and it has been employed to discuss the construction of the NC 't Hooft instantons using the splitting approach. Later on, in [31] the presence of this linear system was used to study the formulation of YM instantons via the dressing and splitting methods, and in [32] that of monopoles by solving the appropriate

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Riemann-Hilbert problem, after a dimensional reduction. Another example of an integrable noncommutative theory is the $U(N)$ Ward model studied in Ref. [33]. This model is formulated in $\mathcal{A}_\theta(\mathbb{R}^{2+1})$ and it too explicitly exhibits a linear system implying the equation of motion, and applying the dressing method gives a systematic way to construct its solitonic solutions. It is worthwhile to note that particular noncommutative extensions of WZW and sine-Gordon models are obtained from this system via dimensional reduction. The latter possess several attractive features as discussed in [34,35]. Supersymmetric extensions of the noncommutative Ward model and its solitonic solutions are recently considered in [36].

In this paper, our purpose is to discuss the integrability properties of nonlinear sigma models defined on the GM spacetime. In particular, we will focus on the construction of an infinite number of conserved currents of the principal chiral model in $\mathcal{A}_\theta(\mathbb{R}^{1+1})$, the $\mathbb{C}P^1$ model, and a certain $\mathbb{C}P^1$ submodel in $\mathcal{A}_\theta(\mathbb{R}^{2+1})$. We will also treat their supersymmetric extensions. In Sec. II, we start by describing the integrability properties of the principal chiral model in $\mathcal{A}_\theta(\mathbb{R}^{(1+1)})$. Our presentation in Sec. II A has overlaps with the previous investigations in [28]. Then we specialize to the NC $\mathbb{C}P^1$ model [37], discuss its relevant properties, and present its Noether currents explicitly.

In Sec. III, we focus our attention to a certain $\mathbb{C}P^1$ submodel in $\mathcal{A}_\theta(\mathbb{R}^{2+1})$. A novel approach to exploring integrability in $d+1$ dimensions was introduced by Alvarez *et al.* in [38], and it essentially consists of formulating a generalized zero-curvature condition by introducing a d-form connection. Quite interestingly, this new formulation helps to reveal the existence of an infinite number of conserved quantities in a variety of models, such as those found for a submodel of $\mathbb{C}P^1$ model in 2 + 1 dimensions. By extending this approach and a parallel one developed by Fujii *et al.* [39] to noncommutative spacetime, we discuss the integrability properties of the aforementioned $\mathbb{C}P^1$ submodel and construct an infinite number of conserved currents for it in a systematic manner. We also discuss the solitonic solutions of the submodel in some detail and show that Bogomol'nyi-Prasad-Sommerfield (BPS) solutions of the NC $\mathbb{C}P^1$ model are solutions of the submodel too.

In Sec. IV, we examine the supersymmetric principal chiral model in $\mathcal{A}_\theta(\mathbb{R}^{1+1|2})$ with and without the WZW term in some detail. We discuss the integrability properties of these models and derive their conserved currents in the superfield formalism, using the methods of [20]. This is followed by a study of the SUSY extension of the $\mathbb{C}P^1$ submodel in $\mathcal{A}_\theta(\mathbb{R}^{(2+1)|2})$ and construction of its conserved currents. Solitonic configurations of this model are also given. We conclude by summarizing our results and stating some directions we are going to be exploring in the near future.

Until Sec. IV, we will be working on the noncommutative spacetimes $\mathcal{A}_\theta(\mathbb{R}^{1+1})$ and $\mathcal{A}_\theta(\mathbb{R}^{2+1})$, which are defined by the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{1.1}$$

and the indices run over 0, 1 and 0, 1, 2, respectively. We use the Minkowski metric with signature $(+,-,-)$. From Sec. IV onward, appropriate Grassmann variables will be introduced to obtain the superspaces $\mathcal{A}_\theta(\mathbb{R}^{1+1|2})$ and $\mathcal{A}_\theta(\mathbb{R}^{2+1|2})$, where only the bosonic coordinates do not commute.

II. NONLINEAR MODELS AND INTEGRABILITY

A. Principal chiral model in $\mathcal{A}_\theta(\mathbb{R}^{1+1})$

Let us start our discussion by considering the principal chiral model in $\mathcal{A}_\theta(\mathbb{R}^{1+1})$. It is defined by the action

$$S_{PC} = \frac{1}{4}\pi\theta \,\mathrm{Tr}\partial_\mu g \partial^\mu g^{-1}, \tag{2.1}$$

where g is a nonsingular matrix whose entries are operators in $\mathcal{A}_\theta(\mathbb{R}^{1+1})$ acting on the standard Heisenberg-Weyl Hilbert space $\mathcal{H}$.[1] For definiteness, we take $g \in U(N)$, thus it satisfies $gg^\dagger = g^\dagger g = 1$. We have that $\mathrm{Tr} = \mathrm{Tr}_{\mathcal{H}} \otimes \mathrm{Tr}_N$, where Tr_N is the trace in $Mat(N)$.

The equation of the motion following from S_{PC} is

$$\partial^\mu(g^{-1}\partial_\mu g) = 0, \tag{2.2}$$

and readily implies

$$A_\mu^{\mathrm{Noether}} = g^{-1}\partial_\mu g, \tag{2.3}$$

as the conserved Noether currents of the model under the global $U(N)$ symmetry.

To construct the conserved tower of currents, we closely follow the inductive procedure of [7]. Let us first define the covariant derivative $D_\mu = \partial_\mu + A_\mu$. Because of (2.3), it satisfies

$$[D_\mu, D_\nu] = 0, \tag{2.4}$$

and due to (2.2), we further have

$$\partial_\mu D^\mu = D^\mu \partial_\mu. \tag{2.5}$$

Let us now suppose that we have found the conserved current $J_\mu^{(n)}$ at level n. By Hodge decomposition of differential forms, which applies in the present NC spacetime $\mathcal{A}_\theta(\mathbb{R}^{(1+1)})$ as the algebra of derivatives are not deformed (i.e., derivatives commute), this implies that we can find $\chi^{(n)} \in \mathcal{A}_\theta(\mathbb{R}^{1+1}) \otimes Mat(N)$ such that

$$J_\mu^{(n)} = -\epsilon_{\mu\nu}\partial^\nu \chi^{(n)}, \qquad n \geq 1. \tag{2.6}$$

Then, the $(n+1)$th current is

$$J_\mu^{(n+1)} = D_\mu \chi^{(n)}, \qquad n \geq 0. \tag{2.7}$$

[1] Note that $\mathcal{H}$ cannot be taken in the Fock basis due to the Minkowski signature.

The construction starts with $\chi^{(0)} = 1$ and $J_\mu^{(1)} = A_\mu^{\text{Noether}}$. We can see that $J_\mu^{(n+1)}$ is conserved since

$$\begin{aligned}\partial^\mu J_\mu^{(n+1)} &= D_\mu \partial^\mu \chi^{(n)}, \qquad n \geq 1 \\ &= \epsilon_{\mu\nu} D_\mu J_\nu^{(n)} \\ &= \epsilon_{\mu\nu} D_\mu D_\nu \chi^{(n-1)} = 0,\end{aligned} \tag{2.8}$$

where we have used (2.4), (2.5), and (2.6). Thus, the construction of [7] works for the noncommutative principal chiral model too. As we have already stated in the Introduction, this result overlaps with that of [28].

The form of the conserved currents allows us to define the linear system of equations for this model. Introducing a spectral parameter λ via $\chi = \sum_0^\infty \lambda^{-n}\chi^n$, we can write using (2.6) and (2.7) that

$$-\epsilon_{\mu\nu}\partial^\nu \chi = \lambda^{-1} D_\mu \chi. \tag{2.9}$$

The last equation can be brought into the form

$$-\partial_1 \chi = \frac{\lambda A_0 + A_1}{1-\lambda^2}\chi, \qquad -\partial_0 \chi = \frac{\lambda A_1 + A_0}{1-\lambda^2}\chi. \tag{2.10}$$

Obviously, the system of equations in (2.10) is of the same form as that of the commutative model. However, we note that $A_\mu(\hat{x}_\mu)$, $\chi(\hat{x}_\mu, \lambda)$ are operators in $\mathcal{A}_\theta(\mathbb{R}^{1+1}) \otimes Mat(N)$ acting on the Hilbert space $\mathcal{H} \otimes \mathbb{C}^N$. Solvability of the system implies the equation of motion (2.2) and the zero-curvature condition (2.4).

The explicit form of the currents $J_\mu^{(n)}$ do indeed differ from those of the commutative model. In the following section, we present an example, namely, the Noether currents of the $\mathbb{C}P^1$ model to emphasize this point.

B. NC $\mathbb{C}P^1$ model

We can now focus on the NC $\mathbb{C}P^1$ model [37]. Restricting to the subset of operators of the form

$$g = g^{-1} = e^{i\pi P} = 1 - 2P, \tag{2.11}$$

where P is a projector in $\mathcal{A}_\theta(\mathbb{R}^{1+1}) \otimes Mat(2)$:

$$P^2 = P, \qquad P^\dagger = P, \qquad P \in \mathcal{A}_\theta(\mathbb{R}^{(1+1)}) \otimes Mat(2), \tag{2.12}$$

leads to the $\mathbb{C}P^1$ model action

$$S = \pi\theta \,\mathrm{Tr}\, \partial_\mu P \partial^\mu P, \qquad \mu = 0, 1. \tag{2.13}$$

The Noether currents take the form

$$J_\mu^{\text{Noether}} = [P, \partial_\mu P]. \tag{2.14}$$

Let us parametrize the projector as

$$P = \begin{pmatrix} \frac{1}{u^\dagger u+1} & \frac{1}{u^\dagger u+1}u^\dagger \\ u\frac{1}{u^\dagger u+1} & u\frac{1}{u^\dagger u+1}u^\dagger \end{pmatrix}, \tag{2.15}$$

then the conservation of J_μ^{Noether} implies the field equation for u

$$\partial_\mu \partial^\mu u - 2\partial_\mu u \frac{1}{u^\dagger u + 1} u^\dagger \partial_\mu u = 0. \tag{2.16}$$

Using (2.15), the Noether currents associated with the global $SU(2)$ symmetry take the form

$$\begin{aligned} J_{\mu,3} &= \frac{1}{2}\mathrm{Tr}_2 \lambda_3 [P, \partial_\mu P] \\ &= \frac{1}{2}\bigg(\frac{1}{u^\dagger u+1}(u^\dagger\partial_\mu u - \partial_\mu u^\dagger u)\frac{1}{u^\dagger u+1} \\ &\quad - u\frac{1}{(u^\dagger u+1)^2}\partial_\mu u^\dagger + \partial_\mu u \frac{1}{(u^\dagger u+1)^2}u^\dagger \\ &\quad - u\left[\frac{1}{u^\dagger u+1}, \partial_\mu \frac{1}{u^\dagger u+1}\right]u^\dagger \\ &\quad - \left[\frac{u^\dagger u}{u^\dagger u+1}, \partial\mu\left(u\frac{1}{u^\dagger u+1}u^\dagger\right)\right]\bigg), \end{aligned} \tag{2.17}$$

$$\begin{aligned} J_{\mu,+} &= \frac{1}{2}\mathrm{Tr}_2 \lambda_+ [P, \partial_\mu P] \\ &= -\frac{1}{2}\bigg(\partial_\mu u \frac{1}{(u^\dagger u+1)} \\ &\quad + u\frac{1}{u^\dagger u+1}(\partial_\mu u^\dagger u - u^\dagger \partial_\mu u)\frac{1}{u^\dagger u+1}\bigg), \end{aligned} \tag{2.18}$$

$$J_{\mu,-} = \tfrac{1}{2}\mathrm{Tr}_2 \lambda_- [P, \partial_\mu P] = -J_{\mu,+}^\dagger, \tag{2.19}$$

where λ_i $(i = 1, 2, 3)$ are the Pauli matrices and $\lambda_\pm = \lambda_1 \pm i\lambda_2$. Unlike the commutative $\mathbb{C}P^1$ model, the Noether current associated with the global $U(1)$ symmetry of the action is not zero, but it is given by

$$\begin{aligned} J_{\mu,0} &= \frac{1}{2}\mathrm{Tr}_2 [P, \partial_\mu P] \\ &= \frac{1}{2}\bigg(\frac{1}{u^\dagger u+1}(u^\dagger\partial_\mu u - \partial_\mu u^\dagger u)\frac{1}{u^\dagger u+1} \\ &\quad + u\frac{1}{(u^\dagger u+1)^2}\partial_\mu u^\dagger - \partial_\mu u \frac{1}{(u^\dagger u+1)^2}u^\dagger \\ &\quad + u\left[\frac{1}{u^\dagger u+1}, \partial_\mu \frac{1}{u^\dagger u+1}\right]u^\dagger \\ &\quad + \left[\frac{u^\dagger u}{u^\dagger u+1}, \partial_\mu\left(u\frac{1}{u^\dagger u+1}u^\dagger\right)\right]\bigg). \end{aligned} \tag{2.20}$$

In the commutative limit the standard expressions for the Noether currents are recovered. In particular, $J_{\mu,0}$ becomes zero in this limit.

III. A $\mathbb{C}P^1$ SUBMODEL IN $\mathcal{A}_\theta(\mathbb{R}^{2+1})$

A valuable approach to exploring integrability in 2 + 1 and higher dimensional theories is due to Alvarez *et al.* [38]. In this article, a generalized zero-curvature representation consisting of an appropriate curvature free connection together with a covariantly conserved vector field has been formulated. The generalized zero-curvature representation implies the presence of conserved currents which may be obtained in a systematic manner. In several diverse models admitting this representation, it has been found that the conserved currents are infinite in number leading to their integrability. For instance, in certain submodels of the principal chiral models and $\mathbb{C}P^N$ models in $2+1$ dimensions, which are determined by the requirement of additional equations to be satisfied by the fields over and above the equations of motions of their respective parent models, an infinite tower of conserved currents has been obtained explicitly using the generalized zero-curvature representation [38,40–42]. In another example in $3+1$ dimensions considered by Aratyn *et al.* [43], a full field theory possessing toroidal solitonic solutions has been shown to be integrable using the generalized zero-curvature representation and its conserved currents have been constructed.

A parallel approach to that of [38] has been developed by Fujii *et al.* [39]. In this formulation, for instance, the $\mathbb{C}P^N$ submodels are studied by implementing their defining conditions as additional equations to be satisfied by the projectors of the $\mathbb{C}P^N$ models, rather than on their particular parametrizations. This approach appears to be better suited for adapting to the present setting of noncommutative theories and will be followed in this section. However, before doing so, it seems instructive to briefly sketch how the ideas of [38] fit into the current framework, and state the type of limitation it faces, in providing explicit expressions for the conserved quantities.

Suppose that we have a finite-dimensional non-semisimple Lie algebra $\underline{\hat{G}}$. Then we can write $\underline{\hat{G}} = \underline{G} + \mathcal{I}$ where $\underline{G}$ is a semisimple Lie subalgebra of $\underline{\hat{G}}$ and $\mathcal{I}$ is its maximal solvable ideal (i.e., radical). We can consider now a connection one-form A_μ on $\mathcal{A}_\theta(\mathbb{R}^{2+1})$ valued in $\underline{G}$, and an antisymmetric tensor $B_{\mu\nu}$ valued in $\mathcal{I}$. In $2+1$ dimensions we can write the dual of $B_{\mu\nu}$ as

$$\tilde{B}^\mu = \tfrac{1}{2}\varepsilon^{\mu\nu\rho}B_{\nu\rho}. \tag{3.1}$$

A generalized set of integrability conditions can then be given as [38]

$$F_{\mu\nu} = [D_\mu, D_\nu] = 0, \qquad D_\mu \tilde{B}^\mu = 0, \qquad D_\mu = \partial_\mu + A_\mu. \tag{3.2}$$

Since A_μ is a flat connection we can write

$$A = g^{-1}\partial_\mu g, \qquad g \in G, \tag{3.3}$$

where G is the Lie group whose Lie algebra is $\underline{G}$. From these considerations, it is easy to verify that the currents

$$J_\mu = g^{-1}\tilde{B}_\mu g \tag{3.4}$$

are conserved. To construct these currents explicitly in a model with say $G \equiv SU(2)$, one essentially needs a suitable local parametrization of $SU(2)$. (See, for instance, the construction of the $\mathbb{C}P^1$ submodel currents in commutative space given in [38].) However, such a parametrization of $SU(2)$ does not exist in the noncommutative setting, and thus the above construction remains implicit for the currents.

Let us now turn to applying the methods of [39], and to be more concrete consider a $\mathbb{C}P^1$ submodel in $\mathcal{A}_\theta(\mathbb{R}^{2+1})$. With $P \in \mathcal{A}_\theta(\mathbb{R}^{2+1}) \otimes Mat(2)$, we observe that the tensor product [over $\mathcal{A}_\theta(\mathbb{R}^{2+1})$] $P \otimes P$ is a projector in $\mathcal{A}_\theta(\mathbb{R}^{2+1}) \otimes Mat(2^2)$. Then the submodel we are interested in may be specified by the equation [39]

$$[P \otimes P, \partial_\mu \partial^\mu P \otimes P] = 0, \qquad \mu = 0, 1, 2. \tag{3.5}$$

In (3.5) and what follows the derivatives on k-fold tensor products are given via

$$\partial_\mu \equiv \sum_i^{k-1} \underbrace{1 \otimes 1 \otimes \cdots \otimes}_{i} \partial_\mu \otimes \underbrace{1 \otimes 1 \cdots \otimes 1}_{k-1-j}, \tag{3.6}$$

and the same symbol is used in the tensor product space, as there is no risk of confusion.

It is easy to find that (3.5) can be expressed as the two equations

$$[P, \partial_\mu \partial^\mu P] = 0, \tag{3.7}$$

$$\partial^\mu P \otimes [P, \partial_\mu P] + [P, \partial_\mu P] \otimes \partial^\mu P = 0. \tag{3.8}$$

Clearly, the first of these is the equation of motion for the $\mathbb{C}P^1$ model, while (3.8) puts further restrictions on the projector P and thereby specifies a submodel. Using (2.15), we may also express these conditions as

$$\partial_\mu \partial^\mu u - 2\partial_\mu u \frac{1}{u^\dagger u + 1} u^\dagger \partial_\mu u = 0, \qquad \partial_\mu u \frac{1}{(u^\dagger u + 1)^2} u^\dagger \partial_\mu u = 0. \tag{3.9}$$

In the commutative limit these equations collapse to $\partial^\mu \partial_\mu u = 0$ and $\partial_\mu u \partial^\mu u = 0$, which define the submodel in the commutative space [38].

A. Conserved currents

In close analogy to the commutative model [39], the conserved matrix currents in this model can now be constructed. They are given by

$$J^k_\mu = \sum_{i=0}^{k-1} \underbrace{P \otimes P \cdots \otimes P}_{i} \otimes [P, \partial_\mu P] \otimes \underbrace{P \otimes P \cdots \otimes P}_{k-1-i}. \tag{3.10}$$

It follows from (3.7) and (3.8) that J^k_μ is conserved:

$$\partial^\mu J^k_\mu = 0. \tag{3.11}$$

For instance, at level $k = 3$ we have

$$\partial^\mu J_\mu^{k=3} = \partial^\mu[P, \partial_\mu P] \otimes P \otimes P + \partial^\mu P \otimes [P, \partial_\mu P] \otimes P + [P, \partial_\mu P] \otimes \partial^\mu P \otimes P + P \otimes \partial^\mu[P, \partial_\mu P] \otimes P + P \otimes \partial^\mu P \otimes [P, \partial_\mu P] + P \otimes [P, \partial_\mu P] \otimes \partial^\mu P + P \otimes P \otimes \partial^\mu[P, \partial_\mu P] + [P, \partial_\mu P] \otimes P \otimes \partial_\mu P + \partial^\mu P \otimes P \otimes [P, \partial_\mu P] = 0 \tag{3.12}$$

upon using (3.7) and (3.8).

A few simple comments are in order. Clearly, level $k = 1$ in the above construction corresponds to the NC $\mathbb{C}P^1$ model and from (3.10) we recover the Noether currents of the model, as given in (2.17), (2.18), (2.19), and (2.20), where now the index μ in these equations runs from 0 to 2. Next, we observe that all the results above go through for the NC $\mathbb{C}P^N$ model, once $Mat(2^2)$ is replaced by $Mat((N+1)^2)$. We can ask, how many conserved currents are there at a given level k? For the $\mathbb{C}P^1$ model, we have four conserved currents at level $k = 1$, and $2^k \times 2^k$ conserved current at level k, and for the $\mathbb{C}P^N$ model we have $(N+1)^k \times (N+1)^k$ conserved currents at level k. Clearly, the number of conserved currents tends to infinity as k does so.

A fast way to compute the component currents is to take the trace of the product of J_μ^k with elements of a suitably chosen basis. Let us illustrate this for the simplest case $k = 2$. In this case the tensor product space is $Mat(4)$ and it can be spanned by the basis

$$\Lambda_{ab} = \lambda_a \otimes \lambda_b, \qquad \lambda_a = (1_2, \lambda_+, \lambda_-, \lambda_3). \tag{3.13}$$

Using the identity $\mathrm{Tr} A \otimes B = \mathrm{Tr} A \, \mathrm{Tr} B$, we can write

$$\begin{aligned}(J_\mu^{k=2})_{ab} &= \mathrm{Tr}_4 \Lambda_{ab} J_\mu^{k=2} \\ &= \mathrm{Tr}_4 \lambda_a \otimes \lambda_b([P, \partial_\mu P] \otimes P + P \otimes [P, \partial_\mu P]) \\ &= \mathrm{Tr}_2 \lambda_a [P, \partial_\mu P] \mathrm{Tr}_2 \lambda_b P \\ &\quad + \mathrm{Tr}_2 \lambda_a P \, \mathrm{Tr}_2 \lambda_b [P, \partial_\mu P].\end{aligned} \tag{3.14}$$

The 16 conserved currents present at this level can be obtained from (3.14). We list a few examples for concreteness:

$$\begin{aligned}(J_\mu^{k=2})_{++} = &-\Big(\partial_\mu u \frac{1}{(u^\dagger u + 1)} + u \frac{1}{u^\dagger u + 1}(\partial_\mu u^\dagger u - u^\dagger \partial_\mu u)\frac{1}{u^\dagger u + 1}\Big) u \frac{1}{u^\dagger u + 1} \\ &- u \frac{1}{u^\dagger u + 1}\Big(\partial_\mu u \frac{1}{(u^\dagger u + 1)} + u \frac{1}{u^\dagger u + 1}(\partial_\mu u^\dagger u - u^\dagger \partial_\mu u)\frac{1}{u^\dagger u + 1}\Big),\end{aligned} \tag{3.15}$$

$$\begin{aligned}(J_\mu^{k=2})_{+-} = &-\Big(\partial_\mu u \frac{1}{(u^\dagger u + 1)} + u \frac{1}{u^\dagger u + 1}(\partial_\mu u^\dagger u - u^\dagger \partial_\mu u)\frac{1}{u^\dagger u + 1}\Big)\frac{1}{u^\dagger u + 1} u^\dagger \\ &+ u \frac{1}{u^\dagger u + 1}\Big(\frac{1}{u^\dagger u + 1}\partial_\mu u^\dagger - \frac{1}{u^\dagger u + 1}(\partial_\mu u^\dagger u - u^\dagger \partial_\mu u)\frac{1}{u^\dagger u + 1} u^\dagger\Big),\end{aligned} \tag{3.16}$$

$$\begin{aligned}(J_\mu^{k=2})_{+3} = &-\Big(\partial_\mu u \frac{1}{(u^\dagger u + 1)} + u \frac{1}{u^\dagger u + 1}(\partial_\mu u^\dagger u - u^\dagger \partial_\mu u)\frac{1}{u^\dagger u + 1}\Big)\Big(\frac{1}{u^\dagger u + 1} - u \frac{1}{u^\dagger u + 1} u^\dagger\Big) \\ &+ u \frac{1}{u^\dagger u + 1}\Big(\frac{1}{u^\dagger u + 1}(u^\dagger \partial_\mu u - \partial_\mu u^\dagger u)\frac{1}{u^\dagger u + 1} - u \frac{1}{(u^\dagger u + 1)^2}\partial_\mu u^\dagger + \partial_\mu u \frac{1}{(u^\dagger u + 1)^2} u^\dagger \\ &- u\Big[\frac{1}{u^\dagger u + 1}, \partial_\mu \frac{1}{u^\dagger u + 1}\Big] u^\dagger - \Big[\frac{u^\dagger u}{u^\dagger u + 1}, \partial_\mu\Big(u \frac{1}{u^\dagger u + 1} u^\dagger\Big)\Big]\Big).\end{aligned} \tag{3.17}$$

B. Solutions of the submodel

Static solitonic solutions of the noncommutative $\mathbb{C}P^1$ model are given by the BPS configurations [37]. In the complex coordinates $z = (\hat{x}_1 + i\hat{x}_2)/\sqrt{2}$ satisfying $[z, \bar{z}] = \theta$ the BPS configurations are specified by the equations

$$\begin{aligned}\partial_{\bar{z}} P P &= 0 \qquad \text{(self-dual)}, \\ \partial_z P P &= 0 \qquad \text{(anti-self-dual)},\end{aligned} \tag{3.18}$$

where the derivatives are given by the adjoint actions

$$\partial_z = -ad\bar{z} = -[\bar{z}, \cdot], \qquad \partial_{\bar{z}} = adz = [z, \cdot]. \tag{3.19}$$

In view of the fact that $\partial P = \partial P P + P \partial P$, (2.15) can also be expressed in the form

$$\begin{aligned}(1 - P)\partial_{\bar{z}} P &= 0 \qquad \text{(self-dual)}, \\ (1 - P)\partial_z P &= 0 \qquad \text{(anti-self-dual)}.\end{aligned} \tag{3.20}$$

Parametrizing the projector as in (2.15), it can be inferred that these equations are fulfilled by the functions $u = u(z)$ (self-dual) and $u = u(\bar{z})$ (anti-self-dual) analytic in their arguments.

Let us now show that these configurations are also solutions of the $\mathbb{C}P^1$ submodel. Equation (3.7), being the quadratic field equation for the $\mathbb{C}P^1$ model, is automatically satisfied by P fulfilling either of the two equations in (3.18). As for (3.8) taking for instance the anti-self-dual configurations we have

$$(3.8) = -\partial_z P \otimes \partial_{\bar{z}} PP + \partial_{\bar{z}} P \otimes P\partial_z P + P\partial_z P \otimes \partial_{\bar{z}} P - \partial_{\bar{z}} PP \otimes \partial_z P, \tag{3.21}$$

and it vanishes identically upon using the second equation in (3.18) and its Hermitian conjugate. Clearly, a similar calculation holds for the self-dual solution too.

IV. SUPERSYMMETRIC NONLINEAR MODELS

A. Noncommutative SUSY principal chiral model

Let us now focus our attention to the $\mathcal{N} = 1$ superspace $\mathcal{A}_\theta(\mathbb{R}^{1+1|2})$ with Moyal-type noncommutativity, i.e.,

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \qquad \{\theta_\alpha, \theta_\beta\} = 0, \qquad [\hat{x}_\mu, \theta_\alpha] = 0, \qquad \mu, \nu = 0, 1, \qquad \alpha, \beta = 1, 2. \tag{4.1}$$

The supersymmetric principal chiral model is given by the action

$$S = \frac{1}{4}\pi\theta \int d^2\theta \,\mathrm{Tr}\bar{D}G^\dagger DG, \tag{4.2}$$

where the SUSY covariant derivative is

$$D_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} + i(\gamma^\mu\theta)_\alpha\partial_\mu \tag{4.3}$$

and $G = G(x_\mu, \theta_\alpha)$ is a matrix valued superfield in NC space with $GG^\dagger = 1 = G^\dagger G$. For definiteness we will assume that $G \in U(N)$.

For the γ matrices we take

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^5 = \gamma^1\gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.4}$$

It may be noted that $\{D_1, D_2\} = 0$, and the commutators of D_α with the generators of Poincaré algebra are the same as those without noncommutativity, therefore the full SUSY algebra is present and is undeformed.

We will now demonstrate that this model satisfies a zero-curvature condition and is therefore integrable at the classical level and construct its conserved nonlocal currents. Our approach is the superspace generalization of that of [7] and was used by Chau and Yen [20] to construct the nonlocal charges in SUSY principal chiral models with or without the WZW term.

The equation of motion that follows from the variation of (4.3) is

$$\bar{D}(G^\dagger DG) = 0. \tag{4.5}$$

Let us define a gauge superfield as $\mathcal{A}_\alpha = G^\dagger D_\alpha G$. Then (4.5) becomes

$$D_1\mathcal{A}_2 - D_2\mathcal{A}_1 = 0. \tag{4.6}$$

Furthermore, we have the gauge covariant derivative $\mathcal{D}_\alpha = D_\alpha + \mathcal{A}_\alpha$, which immediately leads to zero curvature for $\mathcal{A}_\alpha$:

$$\{\mathcal{D}_1, \mathcal{D}_2\} = D_1\mathcal{A}_2 + D_2\mathcal{A}_1 + \{\mathcal{A}_1, \mathcal{A}_2\} = 0. \tag{4.7}$$

This condition together with (4.5) implies that the model is integrable. As a consequence of the equation of motion the identity

$$\{D_\alpha, \bar{\mathcal{D}}_\alpha\} = 0 \tag{4.8}$$

holds.

It is now easy to construct the nonlocal conserved currents. Suppose that we have found the conserved current $\mathcal{J}_\alpha^{(n)}$ at level n. This implies that we can find $\xi^{(n)} \in \mathcal{A}_\theta(\mathbb{R}^{1+1|2}) \otimes Mat(N)$ such that

$$\mathcal{J}_1^{(n)} = -D_1\xi^{(n)}, \qquad \mathcal{J}_2^{(n)} = D_2\xi^{(n)}. \tag{4.9}$$

Then, the $(n+1)$th current is

$$\mathcal{J}_\alpha^{(n+1)} = \mathcal{D}_\alpha\xi^{(n)}, \qquad n \geq 0. \tag{4.10}$$

The construction starts with $\xi^{(0)} = 1$ and $\mathcal{J}_\alpha^{(1)} = \mathcal{A}_\alpha$. We can see that $\mathcal{J}_\alpha^{(n+1)}$ is conserved

$$\begin{aligned} D_1\mathcal{J}_2^{(n+1)} - D_2\mathcal{J}_1^{(n+1)} &= D_1\mathcal{D}_2\xi^{(n)} - D_2\mathcal{D}_1\xi^{(n)} \\ &= -\mathcal{D}_2 D_1\xi^{(n)} + \mathcal{D}_1 D_2\xi^{(n)} \\ &= \mathcal{D}_2\mathcal{J}_1^{(n)} + \mathcal{D}_1\mathcal{J}_2^{(n)} \\ &= \mathcal{D}_2\mathcal{D}_1\xi^{(n-1)} + \mathcal{D}_1\mathcal{D}_2\xi^{(n-1)} \\ &= \{\mathcal{D}_1, \mathcal{D}_2\}\xi^{(n-1)} = 0. \end{aligned} \tag{4.11}$$

Introducing a spectral parameter κ and writing $\xi = \sum_n \kappa^n\xi^{(n)}$ with $\xi^{(0)} = 1$, we find from (4.9) and (4.10) that

$$D_1\xi = -\frac{\kappa}{1+\kappa}\mathcal{A}_1\xi, \qquad D_2\xi = \frac{\kappa}{1-\kappa}\mathcal{A}_2\xi \tag{4.12}$$

which is precisely of the same form as in the commutative space, but now $\mathcal{A}_\alpha$ and ξ are operators in $\mathcal{A}_\theta(\mathbb{R}^{1+1|2}) \otimes Mat(N)$.

B. Addition of the WZW term

The supersymmetric WZW term is of the form [44,45]

$$S_{\mathrm{WZW}} = \frac{k}{16\pi}2\pi\theta \int d^2\theta dt \,\mathrm{Tr}G^\dagger\frac{dG}{dt}\bar{D}G^\dagger\gamma_5 DG, \tag{4.13}$$

where $k \in \mathbb{Z}$. The variation of the total action $S = S_{PC} + S_{\mathrm{WZW}}$ yields

$$\bar{D}\left(\left(1 + \frac{k}{\pi}\gamma_5\right)G^\dagger DG\right) = 0. \tag{4.14}$$

We observe that all the results of the previous section hold, if we make the substitution

$$\mathcal{A}_\mu \to \left(1 - \frac{k}{\pi}\right)\mathcal{A}_\mu. \tag{4.15}$$

Thus, we conclude that all the classical integrability properties are possessed by the NC supersymmetric WZW model too.

C. SUSY $\mathbb{C}P^1$ model

The SUSY $\mathbb{C}P^1$ on $\mathcal{A}_\theta(\mathbb{R}^{1+1|2})$ model is specified by

$$G = e^{i\pi\mathcal{P}} = 1 - 2\mathcal{P}, \qquad \mathcal{P}^2 = \mathcal{P},$$
$$\mathcal{P} \equiv \mathcal{P}(\hat{x}_\mu, \theta_\alpha) \in \mathcal{A}_\theta(\mathbb{R}^{1+1|2}) \otimes Mat(2). \tag{4.16}$$

Its equation of motion is then

$$\tfrac{1}{2}(D + \bar{D})[\mathcal{P}, (D - \bar{D})\mathcal{P}] = [\mathcal{P}, \bar{D}D\mathcal{P}] = 0, \tag{4.17}$$

and the associated conserved currents are given via the spinorial superfield

$$\mathcal{J}_\alpha = [\mathcal{P}, (D_\alpha - \bar{D}_\alpha)\mathcal{P}]. \tag{4.18}$$

It is instructive to present the Noether currents precisely. These are obtained through the $(\gamma^\mu)_{\alpha\beta}\theta_\beta$ component j_μ of $\mathcal{J}_\alpha$. The remaining components of $\mathcal{J}_\alpha$ in the Grassmann expansion do not imply any further conservation laws in general. Expanding $\mathcal{P}$ in powers of θ we have

$$\mathcal{P} = P + i\theta_1\psi_2 - i\theta_2\psi_1 + i\theta_1\theta_2 F, \tag{4.19}$$

with $\mathcal{P}^2 = \mathcal{P}$ implying $P^2 = P$, $P\psi_\alpha P = 0$, and $F = i[\psi_1, \psi_2]$. Using (4.19), we find

$$j_\mu = [P, \partial_\mu P] + i\bar{\psi}\gamma_\mu\psi. \tag{4.20}$$

We recognize the bosonic part as the Noether currents of the NC $\mathbb{C}P^1$ model, and the fermionic part as those of the NC Gross-Neveu model.

D. A SUSY $\mathbb{C}P^1$ submodel

We now consider a SUSY $\mathbb{C}P^1$ submodel in $\mathcal{A}_\theta(\mathbb{R}^{2+1|2})$. Extending the discussion of Sec. III by including the supersymmetry, we consider the condition

$$[\mathcal{P} \otimes \mathcal{P}, \bar{D}D\mathcal{P} \otimes \mathcal{P}] = 0, \tag{4.21}$$

as the defining relation for the SUSY $\mathbb{C}P^1$ submodel.

On k-fold tensor products D is given by

$$D = \sum_i^{k-1} \underbrace{1 \otimes 1 \otimes \cdots \otimes}_{i} D \otimes \underbrace{1 \otimes 1 \cdots \otimes 1}_{k-1-j} \tag{4.22}$$

and likewise for $\bar{D}$. We further have that

$$(\bar{D} \otimes 1 + 1 \times \bar{D})(D \otimes 1 + 1 \otimes D)$$
$$= \bar{D}D \otimes 1 + \bar{D} \otimes D - D \otimes \bar{D} + 1 \otimes \bar{D}D, \tag{4.23}$$

the minus sign in the third term is due to the odd gradings of D and $\bar{D}$.

A short calculation shows that (4.21) is equivalent to the two equations

$$[\mathcal{P}, \bar{D}D\mathcal{P}] = 0,$$
$$\bar{D}\mathcal{P} \otimes [P, D\mathcal{P}] + [\mathcal{P}, \bar{D}\mathcal{P}] \otimes D\mathcal{P} = 0. \tag{4.24}$$

Following the steps of Sec. III, we define

$$\mathcal{J}_\alpha^k = \sum_{i=0}^{k-1} \underbrace{\mathcal{P} \otimes \mathcal{P} \cdots \otimes \mathcal{P}}_{i} \otimes [\mathcal{P}, (D_\alpha - \bar{D}_\alpha)\mathcal{P}]$$
$$\otimes \underbrace{\mathcal{P} \otimes \mathcal{P} \cdots \otimes \mathcal{P}}_{k-1-i}. \tag{4.25}$$

Because of (4.24), $\mathcal{J}_\alpha^k$ are conserved:

$$(D + \bar{D})\mathcal{J}^k = 0, \tag{4.26}$$

as can be checked explicitly for any given k. In components, the conserved currents are given by

$$j_\mu^k = \sum_{i=0}^{k-1} \underbrace{P \otimes P \cdots \otimes P}_{i} \otimes ([P, \partial_\mu P] + i\bar{\psi}\gamma_\mu\psi)$$
$$\otimes \underbrace{P \otimes P \cdots \otimes P}_{k-1-i}. \tag{4.27}$$

Conservation of j_μ^k is implied by the $\theta_1\theta_2$ component of (4.26). The matrix components of j_μ^k may also be obtained using the simple procedure outlined in (3.13) and (3.14).

The remaining components of $\mathcal{J}_\alpha^k$ do not in general imply any new conservation laws.

E. Solutions to the submodel

The static solitonic solutions of the SUSY $\mathbb{C}P^1$ model are well known [46]. We can obtain their noncommutative versions in a straightforward manner. They are given by the BPS configurations fulfilling

$$\mathcal{P}D_-\mathcal{P} = 0 \qquad \text{(self-dual)},$$
$$\mathcal{P}D_+\mathcal{P} = 0 \qquad \text{(anti-self-dual)}, \tag{4.28}$$

where the supersymmetric covariant derivatives $D_\pm = (D_1 \pm iD_2)/\sqrt{2}$ are given as[2]

$$D_+ = \partial_{\theta_-} + i\sqrt{2}\theta_-\partial_{\bar{z}}, \qquad D_- = \partial_{\theta_+} + i\sqrt{2}\theta_+\partial_z, \tag{4.29}$$

with $\theta_\pm = (\theta_1 \pm i\theta_2)/\sqrt{2}$. They fulfil

[2]In this section, we are using the Euclidean gamma matrices

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\gamma^5 = \gamma^1\gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$D_+^2 = i\sqrt{2}\partial_{\bar{z}}, \qquad D_-^2 = i\sqrt{2}\partial_z, \qquad \{D_+, D_-\} = 0. \tag{4.30}$$

In powers of the Grassmann variables, $\mathcal{P}$ expands to

$$\mathcal{P} = P - \theta_+\psi_- + \theta_-\psi_+ - \theta_+\theta_- F, \tag{4.31}$$

and $\mathcal{P}^2 = \mathcal{P}$ implies

$$P^2 = P, \qquad P\psi_\pm P = 0, \qquad F = -[\psi_+, \psi_-]. \tag{4.32}$$

After using the constraints (4.32), the component form of the self-dual equation in (4.28) can be cast into the equations:

$$P\partial_z P = 0, \qquad P\psi_- = 0, \qquad PF\psi_- = 0, \qquad P\partial_z\psi_+ - \psi_+\partial_z P = 0. \tag{4.33}$$

From (4.33) it is readily observed that the bosonic part of the solution is the BPS solution of the NC $\mathbb{C}P^1$ model (2.15). It is then easy to see that the self-dual solutions are given by

$$\mathcal{P} = \chi\chi^\dagger, \qquad \chi = \begin{pmatrix} 1 \\ u(z) - \theta_+\varphi(z) \end{pmatrix} \times \frac{1}{\sqrt{u^\dagger u - \theta_+ u^\dagger\varphi - i\theta_- u\varphi^\dagger + i\theta_+\theta_-\varphi^\dagger\varphi + 1}}, \qquad \chi^\dagger\chi = 1. \tag{4.34}$$

The remaining component matrices $\psi_\pm$ and F can be read off by differentiating $\mathcal{P}$ with respect to $\theta_\pm$.

We can see that these configurations solve our submodel. Clearly, the first of the equations in (4.24) is automatically satisfied by the BPS equations (4.28). As for the second equation in (4.24), picking the self-dual configuration we have

$$-D_+\mathcal{P}\otimes D_-\mathcal{P}\mathcal{P} - D_-\mathcal{P}\otimes\mathcal{P}D_+\mathcal{P} + \mathcal{P}D_+\mathcal{P}\otimes D_-\mathcal{P} + D_-\mathcal{P}\mathcal{P}\otimes D_+\mathcal{P}, \tag{4.35}$$

which vanishes identically, after using $D_\pm\mathcal{P} = D_\pm\mathcal{P}\mathcal{P} + \mathcal{P}D_\pm\mathcal{P}$ together with the self-duality equation. A similar calculation holds for the anti-self-dual case. Thus (4.34) constitutes a set of solutions for the submodel under investigation.

V. CONCLUSIONS AND OUTLOOK

In this paper, classical integrability properties of nonlinear field theories on the Groenewold-Moyal–type noncommutative spaces have been studied. We have obtained the infinite tower of conserved currents in the noncommutative principal chiral model and $\mathbb{C}P^1$ model and their supersymmetric extensions by employing an inductive procedure, which is well known in the corresponding commutative theories. In particular, the explicit expressions for the Noether currents of the noncommutative $\mathbb{C}P^1$ model, which differ from those of the commutative model, have been presented. We have also constructed noncommutative extensions of a $\mathbb{C}P^1$ submodel [on $\mathcal{A}_\theta(\mathbb{R}^{2+1})$], as well as its SUSY extension [on $\mathcal{A}_\theta(\mathbb{R}^{2+1|2})$], and proved their classical integrability by systematically obtaining their infinitely many conserved currents. In the $\mathbb{C}P^1$ submodel, a simple method to work out the explicit forms of the higher degree currents is given and it is applied on a few examples to reveal their structure. The solitonic solutions of the submodels are also studied, and they are shown to be the same as the BPS configurations of their parent models. We think that it may be worthwhile to explore the possible connections of the $\mathbb{C}P^1$ submodel to the $U(2)$ Ward model [33] and their SUSY extensions. It is also interesting to note that there is yet another integrable $\mathbb{C}P^1$ submodel, which is defined through a weaker integrability condition [47]. (Similar results in the context of the $\mathbb{C}P^N$ model in four dimensions are also known [48].) It would be desirable to study its noncommutative extension as well. Progress on these topics will help us to further enhance our understanding of integrability in $\mathcal{A}_\theta(\mathbb{R}^{2+1})$ and $\mathcal{A}_\theta(\mathbb{R}^{2+1|2})$. We hope to report on the developments on these and related topics in the near future.

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