Dynamics of Chern-Simons vortices

Benjamin Collie* and David Tong+

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, United Kingdom

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We study vortex dynamics in three-dimensional theories with Chern-Simons interactions. The dynamics is governed by motion on the moduli space $\mathcal{M}$ in the presence of a magnetic field. For Abelian vortices, the magnetic field is shown to be the Ricci form over $\mathcal{M}$; for non-Abelian vortices, it is the first Chern character of a suitable index bundle. We derive these results by integrating out massive fermions and following the fate of their zero modes.

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I. INTRODUCTION

The moduli space approximation provides an elegant description of the low-energy behavior of solitons [1]. Information about soliton interactions is packaged in a simple geometric form which has proven useful in extracting both the classical and quantum dynamics of the system. In this paper we use the moduli space approximation to study the motion of vortices in the presence of Chern-Simons interactions [2].

For vortices in the Abelian-Higgs model in $d = 2 + 1$ dimensions, a moduli space $\mathcal{M}$ of solutions exists only when the potential is tuned to critical coupling, meaning that the theory lies on the borderline between type I and type II superconductivity. For k vortices, the moduli space has dimension $\dim(\mathcal{M}) = 2k$, with the coordinates X^a, $a = 1, \ldots, 2k$, on $\mathcal{M}$ corresponding to the positions of the vortices on the plane [3,4]. At low energies, the scattering of vortices can be described as geodesic motion on $\mathcal{M}$ with respect to a metric g_{ab},

$$L_{\rm vortex} = \tfrac{1}{2} g_{ab}(X)\dot{X}^a\dot{X}^b. \tag{1.1}$$

Although the metric g_{ab} is not known explicitly for $k \geq 2$, its properties have been well studied [5–7]. Most notably, g_{ab} is Kähler.

One can ask how the dynamics of the vortices is affected by the addition of a Chern-Simons interaction [8–10]. On general grounds, one expects the low-energy dynamics of vortices to be governed by geodesic motion on $\mathcal{M}$, now in the presence of a magnetic field $\mathcal{F} \in \Omega^2(\mathcal{M})$. Locally we may write $\mathcal{F} = d\mathcal{A}$ and the Lagrangian takes the form

$$L_{\rm vortex} = \tfrac{1}{2} \tilde{g}_{ab}(X)\dot{X}^a\dot{X}^b - \kappa\mathcal{A}_a(X)\dot{X}^a, \tag{1.2}$$

where κ is the coefficient of the Chern-Simons term in three dimensions. Working perturbatively in κ, Kim and Lee found that to leading order $\tilde{g}_{ab} = g_{ab}$, while an expression for $\mathcal{A}$ was given in terms of the profile functions of the vortices [10]. However, the geometric meaning of $\mathcal{A}$ has remained mysterious. Here we remedy this. We show that $\mathcal{F}$ is the Ricci form on $\mathcal{M}$.

We further study the dynamics of non-Abelian $U(N)$ vortices introduced in [11,12] in the presence of Chern-Simons interactions. In this case the moduli space has dimension $\dim(\mathcal{M}) = 2kN$ and the dynamics is again given by (1.2). We show that $\mathcal{F}$ is the first Chern character of a particular index bundle over $\mathcal{M}$.

The technique we use to derive these results is simple yet indirect, and can be viewed as an application of the Goldstone-Wilczek method [13,14]. We make use of the well-known fact that the Chern-Simons terms can be induced by integrating out heavy fermions in three dimensions [15,16]. We follow the fate of these fermions from the perspective of the vortices. The fermi zero modes live in an index bundle over $\mathcal{M}$ and we show that, as their mass becomes large, they may be integrated out to reproduce the result (1.2). It is then simple to show that there is no further contribution from nonzero modes. We recently employed this method to derive the dynamics of instantons in five-dimensional Yang-Mills Chern-Simons theories [17].

The plan of the paper is as follows: in Sec. II we introduce the model of interest and describe its vortex solutions. It is a $U(N)$ Yang-Mills theory, with Chern-Simons interactions, coupled to matter fields. The Lagrangian admits $\mathcal{N} = 2$ supersymmetry and the vortices are Bogomol'nyi-Prasad-Sommerfield (BPS). In Sec. III we present our main results, analyzing the impact on the vortex dynamics as fermions are introduced, made heavy, and finally integrated out. Section IV is devoted to two examples. In the first example, we study the qualitative dynamics of two Abelian vortices and describe the bound orbits. We also show that our technique correctly reproduces the fractional statistics of Abelian vortices. The second example concerns a single vortex in the $U(N)$ theory for which the moduli space is $\mathbf{CP}^{N-1}$ and the appropriate magnetic field $\mathcal{F}$ is proportional to Ω, the Kähler form. We also show how to reproduce this magnetic field from a direct study of the vortex equations in the moduli space approximation.

*b.p.collie@damtp.cam.ac.uk
+d.tong@damtp.cam.ac.uk

II. THE VORTEX EQUATIONS

The literature contains a veritable smorgasbord of Chern-Simons models which admit vortex solutions. These include Abelian theories with [18] and without [19–21] a Maxwell term, non-Abelian theories [22,23], and theories with nonrelativistic kinetic terms for the matter fields [24–27]. The properties of many of these models are summarized in the excellent review [28].

Our interest in this paper lies in a $U(N)$ Yang-Mills-Chern-Simons theory coupled to a real adjoint scalar ϕ and N_f scalars q_i, $i = 1, \ldots, N_f$, each of which transforms in the fundamental representation of the gauge group. With suitable fermion content, the theory enjoys $\mathcal{N} = 2$ supersymmetry (i.e. four supercharges) which dictates the form of the bosonic interactions:

$$\mathcal{L} = -\frac{1}{2e^2}\,\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - \frac{\kappa}{4\pi}\,\mathrm{Tr}\epsilon^{\mu\nu\rho}\Big(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\Big) + \frac{1}{e^2}\,\mathrm{Tr}(\mathcal{D}_\mu\phi)^2 + |\mathcal{D}_\mu q_i|^2 - q_i^\dagger\phi^2 q_i - \frac{e^2}{4}\,\mathrm{Tr}(q_i q_i^\dagger - \kappa\phi/2\pi - v^2)^2. \quad (2.1)$$

Notice that we have not considered separate Chern-Simons coefficients for the $U(1)$ and $SU(N)$ parts of the gauge group, but instead have taken a specific combination in which they are packaged together in $U(N)$. For $N \geq 2$, invariance of the partition function under large gauge transformation requires that $\kappa \in \mathbf{Z}$. For the Abelian theory, there is no such constraint.

To make contact with the other models on the market, it is instructive to consider various limits of this Lagrangian.

(i) For the $U(1)$ gauge group, the Lagrangian reduces to the Maxwell-Chern-Simons-Higgs theory introduced in [18].

(ii) When $\kappa = 0$, the Lagrangian reduces to Yang-Mills theory coupled to a number of fundamental scalar fields. This theory is known to admit non-Abelian vortices, first introduced in [11,12] and since studied in some detail. (See, for example, [29–31] for reviews). We will make much use of this limit.

(iii) When $e^2 \to \infty$, the Yang-Mills term vanishes, and the scalar field ϕ becomes auxiliary. Integrating out ϕ reproduces the Chern-Simons-Higgs theory with sixth order scalar potential, first introduced in the Abelian case in [19–21], and studied more recently in the non-Abelian case in [23].

Two important ground states of the theory are the unbroken phase and the Higgs phase. The gauge symmetry is unbroken when the scalar fields take the vacuum expectation values

$$\text{unbroken phase}: \phi^a{}_b = -\frac{2\pi v^2}{\kappa}\delta^a{}_b, \qquad q_i = 0, \quad (2.2)$$

where $a, b = 1, \ldots, N$ is the color index. This state exists regardless of the number N_f of fundamental flavors. In contrast, a ground state with fully broken gauge symmetry only exists when $N_f \geq N$ and the rank N_f term $q_i q_i^\dagger$ in the potential can successfully cancel the rank N term v^2 (which comes with an implicit $N \times N$ unit matrix). For simplicity, in what follows we choose $N_f = N$. There is then a unique ground state with fully broken gauge symmetry given by

$$\text{Higgs phase}: \phi = 0, \qquad q_i{}^a = v\delta_i{}^a. \quad (2.3)$$

In this vacuum, both the $U(N)$ gauge symmetry and the $SU(N)$ flavor symmetry which rotates the Higgs fields q_i are spontaneously broken. However, the diagonal of the two survives: $U(N)_{\mathrm{gauge}} \times SU(N)_{\mathrm{flavor}} \to SU(N)_{\mathrm{diag}}$. The theory also has several ground states with partly broken gauge symmetry. For each such state, the vacuum expectation values of the fields have some diagonal entries equal to those in (2.2) and the rest equal to those in (2.3). We will not consider these partly broken phases further.

In the Higgs phase, the model admits topologically stable BPS vortices. First order equations of motion may be derived using the standard Bogomol'nyi trick, and read

$$B = \frac{e^2}{2}(q_i q_i^\dagger - \kappa\phi/2\pi - v^2), \qquad \mathcal{D}_z q_i \equiv \mathcal{D}_1 q_i - i\mathcal{D}_2 q_i = 0, \quad (2.4)$$

$$E_\alpha + \mathcal{D}_\alpha\phi = 0, \qquad \mathcal{D}_0\phi = 0, \qquad \mathcal{D}_0 q_i + i\phi q_i = 0. \quad (2.5)$$

Here $B = F_{12}$ and $E_\alpha = F_{0\alpha}$. Note however that, in contrast to vortices in $\kappa = 0$ theories, it is not enough to solve these first order equations alone: we must also solve Gauss' law. This is most simply written in static gauge $\partial_0 = 0$. Then the three equations in (2.5) may all be solved by setting $A_0 = \phi$, which is determined by Gauss' law

$$2\mathcal{D}^2\phi + \frac{\kappa}{2\pi}e^2 B - e^2\{\phi, q_i q_i^\dagger\} = 0. \quad (2.6)$$

Note that the presence of the Chern-Simons coupling ensures that ϕ is sourced at the core of the vortex where $B \neq 0$. The fact that the first order vortex equations (2.4) must be supplemented by the second order equation (2.6) is what makes the study of vortex dynamics somewhat more of a technical challenge in the presence of a Chern-Simons interaction.

Configurations satisfying (2.4) and (2.6) have energy,

$$E = -v^2\int d^2x\,\mathrm{Tr}\,B = 2\pi v^2 k, \quad (2.7)$$

where $k \in \mathbf{Z}^+$ is the topological charge of the vortex. It is expected that Eqs. (2.4) and (2.6) enjoy a moduli space of solutions of dimension $\dim(\mathcal{M}) = 2kN$. This is suggested by the counting of zero modes using index theorems

[3,11,19]. However, to our knowledge the only rigorous proof of this statement when $\kappa \neq 0$ holds in the Abelian theory in the limit $e^2 \to \infty$ [32]. To some extent, the method we propose in the next section circumvents this issue since our starting point will be the theory with $\kappa = 0$ where the existence of a moduli space has been rigorously proven [4].

Our theory has $\mathcal{N} = 2$ supersymmetry. Yet so far we have not mentioned the fermions. They consist of a single Dirac fermion λ in the adjoint representation of the gauge group (this is the superpartner of A_μ and ϕ) together with N_f Dirac fermions ψ_i in the fundamental representation (the superpartners of q_i). In the background of the vortex, these fermions carry zero modes. These zero modes will not be the focus of our discussion in the next section, although one should remember that they are present. Instead we will be interested in the zero modes of some extra, supplementary, fermions that we now introduce.

III. INTEGRATING OUT FERMIONS

Our strategy in this section is to replace the Chern-Simons interactions in the bosonic Lagrangian (2.1) with something that we understand better, namely, fermions. To this end, we start with the theory without Chern-Simons interactions by setting $\kappa = 0$ in (2.1). We now introduce $\tilde{N}$ chiral multiplets $\tilde{Q}$, each transforming in the antifundamental representation of the $U(N)$ gauge group. Each of these chiral multiplets may be given a mass m consistent with supersymmetry.[1] In the limit $m \to \infty$, the chiral multiplets may be happily integrated out. All of their effects decouple, except for a remnant $U(N)$ Chern-Simons term, with coefficient [15,16]

$$\kappa = -\frac{\tilde{N}}{2}\mathrm{sign}(m). \tag{3.1}$$

Importantly, the $\kappa\phi$ term in the potential in (2.1) is the supersymmetric partner of the Chern-Simons term. Supersymmetry is unbroken in this model (the Witten index is nonvanishing), so when we integrate out the chiral multiplets $\tilde{Q}$, the $\kappa\phi$ term must be generated together with the Chern-Simons term. In fact, there is one further, related, effect that is important: the scalar vacuum expectation value (vev) v^2 (which is a Fayet-Iliopoulos parameter in the language of supersymmetry) picks up a finite renormalization [33,34]:

$$v^2 \to v^2_{\mathrm{eff}} = v^2 + \frac{m\kappa}{2\pi} = v^2 - \frac{\tilde{N}|m|}{4\pi}. \tag{3.2}$$

Notice that for suitably large $|m|$, we have $v^2 < 0$, and the theory exits the Higgs phase where the vortices live. If we wish to stay in the Higgs phase, and keep the vortex mass fixed, we must scale v^2 so that v^2_{eff} remains constant as $m \to \infty$. If we perform such a scaling, we conclude that the Chern-Simons theory (2.1) is equivalent to the Yang-Mills theory coupled to $\tilde{N} = 2\kappa$ supplementary massive chiral multiplets $\tilde{Q}$ in the limit $m \to \pm\infty$.

A. The index bundle of Fermi zero modes

Let us now follow the effect of this procedure on the vortex dynamics, focussing first on a Dirac fermion $\tilde{\psi}$ in one of the chiral multiplets $\tilde{Q}$. The Dirac equation is given by

$$i\not{D}\tilde{\psi} - \tilde{\psi}\phi = m\tilde{\psi}. \tag{3.3}$$

We are interested in the solutions to this equation in the background of the vortex. Since we are working in the theory with $\kappa = 0$, the bosonic fields of the vortex are solutions to

$$B = \frac{e^2}{2}(q_i q_i^\dagger - v^2), \qquad \mathcal{D}_z q_i = 0, \qquad A_0 = \phi = 0. \tag{3.4}$$

Of the full spectrum of solutions to the Dirac equation (3.3) in the background of the vortex, only the zero modes will prove important. We discuss these first, returning to the nonzero modes shortly. We work with the basis of gamma matrices $\gamma^\mu = (\sigma^3, i\sigma^2, -i\sigma^1)$. The zero modes then take the form [35]

$$\tilde{\psi}(t, x_\alpha) = e^{-imt}\begin{pmatrix}\tilde{\psi}_-(x_\alpha)\\ 0\end{pmatrix} \quad \text{or} \quad \tilde{\psi}(t, x_\alpha) = e^{+imt}\begin{pmatrix}0\\ \tilde{\psi}_+(x_\alpha)\end{pmatrix}, \tag{3.5}$$

where $\mathcal{D}_{\bar{z}}\tilde{\psi}_- = \mathcal{D}_z\tilde{\psi}_+ = 0$. Standard index theorems state that the equation $\mathcal{D}_{\bar{z}}\tilde{\psi}_- = 0$ has k solutions in the background of the vortex, while $\mathcal{D}_z\tilde{\psi}_+ = 0$ has none.[2] For example, in the Abelian case this follows from the fact that there is no holomorphic line bundle of negative degree.

The space of zero modes of the Dirac equation defines a bundle over the vortex moduli space $\mathcal{M}$, with fiber $\mathbf{C}^k$. This is commonly referred to as the index bundle. As we move in moduli space by adiabatically changing the background vortex configuration, the Fermi zero modes undergo a holonomy described by a Hermitian $u(k)$ connection ω over $\mathcal{M}$. The index bundle can be defined by introducing a set of basis vectors $\Psi_l(x, X)$ with $l = 1, \ldots, k$

[1] This mass term is not possible in $d = 3 + 1$ dimensions, where it would break Lorentz invariance. It is allowed in $d = 2 + 1$, and was called a "real mass" in [33] to distinguish it from the more familiar complex mass that appears in the superpotential. For the present purposes, the important point is its effect on the fermions which is shown in the Dirac equation (3.3).

[2] Recall that $\tilde{\psi}$ transforms in the $\bar{\mathbf{N}}$ representation, while q_i transforms in the $\mathbf{N}$ representation—this is responsible for the fact that $\mathcal{D}_{\bar{z}}$ carries the zero modes in the background $\mathcal{D}_z q_i = 0$. More details on these Fermi zero modes in the context of vortex strings in related four-dimensional theories can be found in [36].

for the Fermi zero modes. The connection is defined by

$$i(\omega_a)^l{}_m = \int d^2x (\Psi^l)^\dagger \frac{\partial}{\partial X^a} \Psi_m. \quad (3.6)$$

For the case of Fermi zero modes in the background of a magnetic monopole, the connection on the index bundle has been studied in [37,38]. However, in the case of vortices, it appears to have received less attention in the literature. We will provide some explicit examples of the connection ω in Sec. IV of this paper.

We denote the Grassmann-valued coordinates of the $\mathbf{C}^k$ fiber as ξ^l, $l = 1, \ldots, k$. The low-energy dynamics of the vortex should now be augmented to include the Fermi zero modes, which are described by the kinetic terms[3]

$$L = \bar{\xi}^l (iD_t - m)\xi^l, \quad (3.7)$$

where the covariant derivative is defined by

$$D_t \xi^l = \partial_t \xi^l + i(\omega_a)^l{}_m \dot{X}^a \xi^m. \quad (3.8)$$

Let us pause briefly to discuss how one should quantize these zero modes. As usual, each complex fermionic zero mode gives rise to two states—occupied and unoccupied—whose energy differs by m. However, the question of the absolute ground state energy requires us to resolve the usual ordering ambiguities. Comparison with the renormalization of v^2 given in (3.2) shows that a single Fermi zero mode should cause the mass of the vortex to shift by $M_{\text{vortex}} \to M_{\text{vortex}} - |m|/2$. This strongly suggests that we should take the ground state of the Fermi zero modes to have energy $-|m|/2$, and the excited state to have energy $+|m|/2$. It would be interesting to understand better why this choice of ordering is forced upon us.

B. Integrating out the index bundle

Throughout this discussion, we have been referring to the relevant solutions of the Dirac equation as "zero modes." This is a slight misnomer because, as is clear from (3.7), they are excited at a cost of energy equal to $|m|$. They become true zero modes only in the $m \to 0$ limit which is, of course, to be expected since they arose from fermions with mass m. However, we are interested in the opposite limit $m \to \infty$. In this limit, the effect of the fermi zero modes *almost* decouples. They do not correct the metric g_{ab}. However, as we now show, they do give rise to new Chern-Simons terms for the moduli space dynamics.

Integrating out the fermion ξ in the path integral leads to the ratio of determinants

$$\det\left(\frac{iD_t - m}{i\partial_t - m}\right). \quad (3.9)$$

We can compute this ratio using standard methods. We work with compact Euclidean time $\tau = it$, with periodicity $\tau \in [0, \beta)$. We look for the eigenvalues λ of the operator

$$(-\partial_\tau - i\omega - m)\chi = \lambda\chi, \quad (3.10)$$

where $\omega = \omega_a \partial_\tau X^a$. The eigenfunctions are subject to periodic boundary conditions $\chi(0) = \chi(\beta)$. Solutions are given by the usual time-ordered product

$$\chi = e^{-(m+\lambda)\tau} V(\tau)\chi$$
$$\text{with } V(\tau) = T\exp\left(-i\int_0^\tau d\tau' \omega(\tau')\right) \in U(k). \quad (3.11)$$

Let us denote the eigenvalues of $V(\beta)$ as e^{v_l}, $l = 1, \ldots, k$. Then the periodicity requirement $\chi(0) = \chi(\beta)$ means that the eigenvalues λ are given by

$$\lambda = \frac{2\pi i n + v_l}{\beta} - m, \qquad n \in \mathbf{Z}, \qquad l = 1, \ldots, k. \quad (3.12)$$

From this we compute the ratio of determinants

$$\det\left(\frac{D_\tau + m}{\partial_\tau + m}\right) = \prod_{l=1}^k \prod_{n\in\mathbf{Z}} \left(\frac{2\pi i n/\beta + v_l/\beta - m}{2\pi i n/\beta - m}\right)$$
$$= \prod_{l=1}^k \left(1 - \frac{v_l}{m\beta}\right)\left(\frac{\sinh(\beta m/2 - v_l/2)}{\sinh \beta m/2}\right)$$
$$\xrightarrow{\beta\to\infty} \exp\left(-\frac{1}{2}\text{sign}(m)\sum_l v_l\right), \quad (3.13)$$

where we assume that ω has compact support in taking the limit in the last line. Translating back to Minkowski space, we can write this as a contribution to the effective Lagrangian involving the original $u(k)$ connection ω,

$$L_{\text{eff}} = \tfrac{1}{2}\text{sign}(m)(\text{Tr}\,\omega_a)\dot{X}^a. \quad (3.14)$$

This is the promised result. We see that, even in the limit $m \to \infty$, the zero modes leave a remnant of their existence by inducing an effective magnetic field $\mathcal{F} = d\mathcal{A}$ on the moduli space, where $\mathcal{A} = \text{Tr}\,\omega$ is defined in terms of the connection on the index bundle. $\mathcal{F}$ is proportional to the first Chern character of the index bundle while, conveniently enough, $\mathcal{A}$ is known as the Chern-Simons one-form. This is the worldline counterpart to the statement that the parent three-dimensional fermions induce a Chern-Simons term.

The result (3.14) holds for integrating out the zero modes associated to a single chiral multiplet fermion. As we saw in (3.1), we must integrate $\tilde{N} = 2\kappa$ chiral multiplets. Our

[3]There is an important caveat here: the zero modes under discussion are non-normalizable; they have a long-range $1/r$ tail, causing them to suffer from an infrared logarithmic divergence. In the context of four-dimensional theories, there are several examples where ignoring this fact, and treating these modes with kinetic terms of the form (3.7), leads to quantitatively and qualitatively correct physics [39,40]. This approach has been criticized in [41]. For the time being, we proceed by ignoring this issue. However, in Sec. III C we will present a slightly more involved construction that yields the same answer, but does not suffer from this technical problem.

final result is that the low-energy dynamics of vortices in the Chern-Simons theory (2.1) is given by

$$L = \tfrac{1}{2} g_{ab} \dot{X}^a \dot{X}^b - \kappa \mathcal{A}_a \dot{X}^a. \tag{3.15}$$

1. Why only zero modes matter

In deriving the Lagrangian (3.15), we have integrated out only the zero modes on the vortex worldline, while ignoring the infinite tower of higher solutions to the Dirac equation. We now show that this is consistent. The key point is that higher excitations of fermions come in pairs, with energy $\pm E$:

$$\begin{pmatrix} 0 & i\mathcal{D}_z \\ -i\mathcal{D}_{\bar{z}} & 0 \end{pmatrix}\begin{pmatrix} \tilde{\psi}_- \\ \tilde{\psi}_+ \end{pmatrix} = E\begin{pmatrix} \tilde{\psi}_- \\ \tilde{\psi}_+ \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 0 & i\mathcal{D}_z \\ -i\mathcal{D}_{\bar{z}} & 0 \end{pmatrix}\begin{pmatrix} \tilde{\psi}_- \\ -\tilde{\psi}_+ \end{pmatrix} = -E\begin{pmatrix} \tilde{\psi}_- \\ -\tilde{\psi}_+ \end{pmatrix}.$$

Contributions to the Chern-Simons term on the vortex worldline cancel between each pair. To see this, we write the general eigenfunction as $\tilde{\psi}^T = (\tilde{\psi}_- \zeta_-, \tilde{\psi}_+ \zeta_+)$ and promote $\zeta_\pm$ to time-dependent Grassmann fields. The action for these objects is schematically

$$L_{\text{nonzero modes}} = \bar{\zeta}_+(iD_t - m)\zeta_+ + \bar{\zeta}_-(iD_t + m)\zeta_- + E(\bar{\zeta}_+\zeta_- + \bar{\zeta}_-\zeta_+), \tag{3.16}$$

which is schematic only in the sense that we have dropped overall coefficients that arise from the overlap of the eigenfunctions. Integrating out the nonzero modes now gives us a determinant of the form,

$$\det\begin{pmatrix} iD_t - m & E \\ E & iD_t + m \end{pmatrix} = \det(iD_t + \sqrt{m^2 + E^2}) \times \det(iD_t - \sqrt{m^2 + E^2}). \tag{3.17}$$

We see that the effective mass of these objects is $\pm\sqrt{m^2 + E^2}$, leading to a cancellation due to the presence of the sign(m) term in (3.14). In the limit $m \to \infty$, these nonzero modes leave no trace of their existence on the vortex dynamics.

C. Abelian vortices and the tangent bundle

Our final answer (3.15) for the vortex dynamics is pleasingly simple and geometrical. Yet it suffers from two drawbacks. First, we have no concrete expression for the index bundle and its associated first Chern character. Second, as mentioned in footnote [3], there is a technical subtlety due to the non-normalizability of the zero modes. In this section we remedy both of these issues for Abelian vortices. In Sec. III D we shall also remedy the problem of non-normalizability for non-Abelian vortices.

Our strategy is a slightly more refined version of that described above. We again generate the Chern-Simons terms by integrating out supplementary matter multiplets. The only thing that differs from the previous discussion is the matter that we choose to integrate out. Our starting point this time will be the Abelian-Higgs model with $\mathcal{N} = 4$ supersymmetry (i.e. 8 supercharges). We set $\kappa = 0$ in (2.1), and introduce a neutral chiral multiplet A, containing the Dirac fermion η, together with a single chiral multiplet $\tilde{Q}$ of charge -1, containing the fermion $\tilde{\psi}$. The extended supersymmetry requires that these are coupled to the original chiral multiplet Q, containing the scalar q, through the superpotential,

$$\mathcal{W} = \sqrt{2}\tilde{Q}AQ. \tag{3.18}$$

The benefit of working in the $\mathcal{N} = 4$ model is that the geometry of the Fermi zero modes is well understood. Indeed, in the background of the Abelian vortex, the Dirac equations for η and $\tilde{\psi}$ reduce to[4]

$$i\mathcal{D}_{\bar{z}}\eta_- - \sqrt{2}q^\dagger\tilde{\psi}_-^\dagger = 0, \qquad -i\mathcal{D}_z\tilde{\psi}_-^\dagger - \sqrt{2}\eta_- q = 0. \tag{3.19}$$

The index theorem remains the same as before, and these equations again have k complex zero modes. However, the presence of the coupling to q—which has a nonzero vacuum expectation value—ensures that the zero mode profiles are localized exponentially near the vortex cores and are normalizable. This resolves the problem described in footnote [3].

Moreover, it can be shown that the k Fermi zero modes are proportional to the bosonic zero modes of the vortex: they are related by the extended supersymmetry. The upshot of this is that the Fermi zero modes live—like their bosonic counterparts—in the *tangent bundle* over $\mathcal{M}$. The appropriate covariant derivative for the k Grassmann collective coordinates ξ is now

$$(D_t\xi)^a = \partial_t\xi^a + \Gamma^a_{bc}\dot{Z}^b\xi^c, \tag{3.20}$$

where, in contrast to previous formulae, we have switched to complex notation, defining the holomorphic coordinates Z^a, $a = 1, \dots, k$ on a patch of the moduli space $\mathcal{M}$. The Γ^a_{bc} are the holomorphic components of the Levi-Civita connection.

The above is merely a review of well-known results about Fermi zero modes of vortices in theories with $\mathcal{N} = 4$ supersymmetry. As before, we now deform our theory by adding a real mass m for the chiral multiplets A and $\tilde{Q}$. We then integrate A and $\tilde{Q}$ out. The multiplet A is neutral and decouples in the $m \to \infty$ limit. In contrast, $\tilde{Q}$ induces a Chern-Simons interaction with coefficient $\kappa = -\frac{1}{2}\text{sign}(m)$.

[4]A recent detailed discussion of these issues, with an explicit demonstration of the relationship between fermionic and bosonic zero modes, can be found in [36].

Integrating out the Fermi zero modes on the worldline proceeds as before. But, since the zero modes live in the tangent bundle, locally we have $d\mathcal{A} = R$ where R is the Ricci form. This is defined in terms of the metric $g_{a\bar{b}}$ by

$$R = i\partial\bar{\partial}\ln\sqrt{g}. \tag{3.21}$$

In terms of local complex coordinates on $\mathcal{M}$, the vortex dynamics becomes

$$L = g_{a\bar{b}}\dot{Z}^a\dot{\bar{Z}}^{\bar{b}} - \kappa(\mathcal{A}_a\dot{Z}^a + \bar{\mathcal{A}}_{\bar{a}}\dot{\bar{Z}}^{\bar{a}}), \tag{3.22}$$

where the complex Chern-Simons one-form can be written locally as

$$\mathcal{A}_a = -\frac{i}{2}\frac{\partial}{\partial Z^a}\ln\sqrt{g}. \tag{3.23}$$

D. Non-Abelian vortices revisited

The discussion in Sec. III C was solely for Abelian vortices. What goes wrong if we try to repeat it for non-Abelian vortices? In order to build the non-Abelian theory with $\mathcal{N} = 4$ supersymmetry, we must augment the $\kappa = 0$ Lagrangian with N chiral multiplets $\tilde{Q}$ in the antifundamental representation, and a single chiral multiplet A in the adjoint representation. Integrating out the $\tilde{Q}$ results in a $U(N)$ Chern-Simons interaction of the type given in (2.1). However, integrating out the adjoint multiplet A contributes to the $SU(N)$ Chern-Simons term, but not the $U(1)$ Chern-Simons term. Thus the mass deformed $\mathcal{N} = 4$ theory does not yield the $U(N)$ $\mathcal{N} = 2$ theory of the form (2.1), but rather a theory with different Chern-Simons coefficients for the $SU(N)$ and $U(1)$ parts of the gauge group.

To make progress, we could instead augment the $\kappa = 0$ Lagrangian with N chiral multiplets $\tilde{Q}$ in the antifundamental representation, and a single neutral chiral multiplet A. The theory no longer admits $\mathcal{N} = 4$ supersymmetry, so we cannot use the above argument to show that the zero modes live in the tangent bundle. Nonetheless, adding a superpotential of the form (3.18) means that the Dirac equations are once more of the form (3.19), and the Fermi zero modes are rendered normalizable. Thus, although we cannot show that the magnetic field on the moduli space of non-Abelian vortices takes the simple form (3.23), any lingering worries caused by footnote 3 may now be left behind.

IV. EXAMPLES

In this section, we illustrate our result with two examples. We first examine the qualitative dynamics of two Abelian vortices and show that the moduli space dynamics correctly captures their fractional statistics. Second, we look at a single vortex in the $U(N)$ theory, for which the internal moduli space is $\mathbf{CP}^{N-1}$. We derive the dynamics both from the method described in Sec. III, and also from a direct moduli space computation.

A. Two Abelian vortices

The relative dynamics of two Abelian vortices takes place in the moduli space $\mathcal{M} \cong \mathbf{C}/\mathbf{Z}_2$. The metric is given by

$$ds^2 = f^2(\sigma)(d\sigma^2 + \sigma^2 d\theta^2), \tag{4.1}$$

where $\theta \in [0, \pi)$. Asymptotically, as $\sigma \to \infty$, we have $f^2(\sigma) \to 1 + \mathcal{O}(e^{-\sigma})$ [5,7] and the moduli space is a cone with deficit angle π. Although the function $f(\sigma)$ is not known analytically, it can be shown that $f^2(\sigma) \sim \sigma^2$ as $\sigma \to 0$, ensuring that the tip of the cone is smooth. The moduli space is sketched in Fig. 1, together with an example of the motion which we will describe shortly.

We work with the single valued holomorphic coordinate $z = \sigma^2 e^{2i\theta}$. Then the Chern-Simons one-form (3.23) on the vortex worldline is given by

$$L_{\rm CS} = -\kappa(\mathcal{A}\dot{z} + \bar{\mathcal{A}}\dot{\bar{z}}) = -\kappa\left(\frac{\sigma}{2}\frac{\partial}{\partial\sigma}\log f^2 - 1\right)\dot{\theta}. \tag{4.2}$$

A similar expression, expressed in slightly different variables, can be found in Eq. (85) of [10].

Although the explicit function $f(\sigma)$ is not known, we may still study the qualitative behavior of vortices. The conserved Noether charge associated to θ is given by

$$J = f^2\sigma^2\dot{\theta} + \kappa\left(1 - \frac{\sigma}{2}\frac{\partial\log f^2}{\partial\sigma}\right). \tag{4.3}$$

As explained in [10], this differs from the angular momentum of the two vortices by a constant. Meanwhile the conserved Hamiltonian is

$$H = \tfrac{1}{2}f^2\dot{\sigma}^2 + V_{\rm eff}(\sigma), \tag{4.4}$$

where the effective potential is due to the Chern-Simons term, together with the usual angular momentum barrier,

$$V_{\rm eff}(\sigma) = \frac{1}{2f^2\sigma^2}\left(J - \kappa + \frac{\kappa\sigma}{2}\frac{\partial\log f^2}{\partial\sigma}\right)^2. \tag{4.5}$$

The classical scattering of vortices depends on the form of

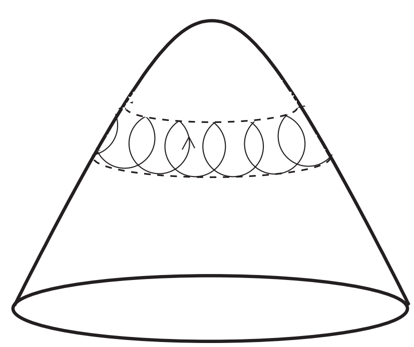

FIG. 1. The moduli space is a cone.

$V_{\rm eff}$ which, in turn, depends on the relative values of κ and J. Let us fix $\kappa > 0$. On physical grounds, the form of the effective potential is shown in Fig. 2:

(i) $J > \kappa$. In this regime, we have $\dot{\theta} > 0$ and $V_{\rm eff}$ is shown in Fig. 2(a). $V_{\rm eff}$ acts as an effective angular momentum barrier and the scattering of vortices is not qualitatively different from the case without a Chern-Simons term.

(ii) The regime $0 < J < \kappa$ is more interesting. The effective potential is shown in Fig. 2(b). The root of the effective potential corresponds to the static solution. We see that, as emphasized in [10], static solutions with different vortex separation σ carry different angular momentum J.
Small oscillations around the minimum of $V_{\rm eff}$ give rise to bound orbits of vortices. From the expression (4.3), we see that $\dot{\theta}$ oscillates from negative to positive in such orbits. The corresponding motion on the moduli space is drawn in Fig. 1. The two vortices trace Larmor circles, while orbiting one another. This moduli space motion can be understood using a standard argument involving adiabatic invariants: in the slowly varying magnetic field, a particle drifts along lines of constant field strength.

(iii) For $J < 0$, we have $\dot{\theta} < 0$. There are two distinct shapes of $V_{\rm eff}$. For suitably small $|J|$, the effective potential takes the form shown in Fig. 2(c). There are once again bound orbits, including one at fixed σ. For $J \ll 0$, the minimum of $V_{\rm eff}$ disappears and the potential once again takes the shape of Fig. 2(a), with only scattering trajectories.

Before we move on, we also note that there is a simple quantum effect that follows from (4.2). The first term vanishes as $\sigma \to \infty$, while the second survives. This ensures that as the particles orbit asymptotically, the wave function picks up a phase $\exp(\pm i\pi\kappa)$. For $\kappa \notin \mathbf{Z}$, this endows the vortices with fractional statistics in agreement with the analysis of [8–10].

B. One non-Abelian vortex

For our second example, we examine a single vortex in $U(N)$. We first review the dynamics of the vortex in the $\kappa = 0$ case. The vortex has an internal moduli space $\mathcal{M} \cong \mathbf{CP}^{N-1}$, describing its orientation in color and flavor space [11,12]. We introduce homogeneous coordinates on $\mathcal{M}$ by starting with a solution $B_\star$ for the magnetic field of a single Abelian vortex configuration. We can embed the Abelian solution into a non-Abelian configuration by writing

$$B^a{}_b = \frac{B_\star}{r}\varphi^a\bar{\varphi}_b \tag{4.6}$$

with a similar expression for the Higgs field which we will describe in more detail in Sec. IV C. The coordinates $\varphi_a \in \mathbf{C}$, $a = 1, \ldots, N$, satisfy the constraint

$$\sum_{a=1}^{N}|\varphi_a|^2 = r, \tag{4.7}$$

where r is a constant which is determined to be $r = 2\pi/e^2$ [11,12,42]. The solutions (4.6) are invariant under the simultaneous rotation

$$\varphi_a \to e^{i\vartheta}\varphi_a. \tag{4.8}$$

The φ_a, subject to the constraint (4.7) and identification (4.8), provide homogeneous coordinates on the moduli space $\mathcal{M} \cong \mathbf{CP}^{N-1}$. The low-energy dynamics of the vortex is described by a sigma-model on $\mathcal{M}$ endowed with the Fubini-Study metric and Kähler class r. There is a simple way to impose the identification (4.8) by introducing an auxiliary gauge field α on the worldline. The Lagrangian for the internal modes of the vortex takes the form

$$L_{\rm vortex} = \sum_{a=1}^{N}|\mathcal{D}_t\varphi_a|^2, \tag{4.9}$$

where the degrees of freedom are subject to the constraint (4.7), and the covariant derivative is given by $\mathcal{D}_t\varphi_a = \dot{\varphi}_a - i\alpha\varphi_a$.

Let us now ask how this dynamics is altered by the presence of the Chern-Simons term. The moduli space is compact and the cohomology is generated by Ω, the Kähler form. Thus the first Chern character $\mathcal{F}$ of the index bundle must be proportional to Ω. We need only determine the proportionality constant. In fact, this is simple to achieve in the language introduced above. Let $\tilde{\psi}_\star$ denote the solution to the Abelian Dirac equation (3.3). Then the solution to the non-Abelian Dirac equation, with gauge field given by (4.6), is

$$\tilde{\psi}^b = \tilde{\psi}_\star\xi\bar{\varphi}_b. \tag{4.10}$$

This is compatible with the symmetry (4.8) if the

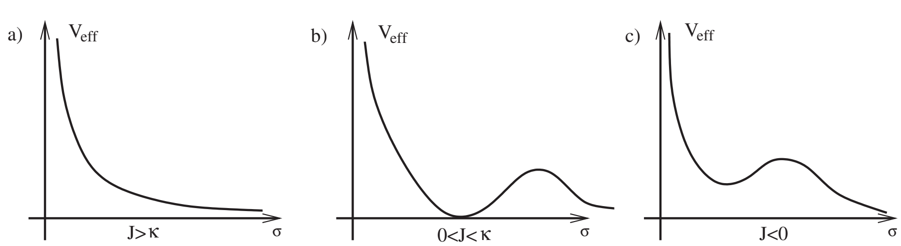

FIG. 2. The effective potential for different values of J.

Grassmann collective coordinate ξ is assigned charge,

$$\xi \to e^{i\vartheta}\xi. \tag{4.11}$$

This transformation rule determines the index bundle, for it fixes the kinetic term of the Grassmann variable to be given by the covariant derivative $D_t\xi = \dot{\xi} - i\alpha\xi$. We may now take $m \to \infty$, and integrate out ξ. The calculation is the same as that described in Sec. III, and yields

$$L_{1-\text{vortex}} = \sum_{a=1}^{N} |\mathcal{D}_t\varphi_a|^2 - \kappa\alpha. \tag{4.12}$$

An example of the example

For a single vortex in the $U(2)$ theory, the moduli space is $\mathbf{S}^2 \cong \mathbf{CP}^1$. We now provide a more explicit description of the dynamics in this case. The constraints (4.7) are simply solved by

$$\begin{aligned}\varphi_1 &= \sqrt{r}e^{i\psi - i\phi/2}\cos(\theta/2),\\ \varphi_2 &= \sqrt{r}e^{i\psi + i\phi/2}\sin(\theta/2),\end{aligned} \tag{4.13}$$

where the angles take ranges $\psi \in [0, 2\pi)$, $\phi \in [0, 2\pi)$, and $\theta \in [0, \pi)$. Expanding out the Lagrangian gives

$$\begin{aligned}L_{\text{vortex}} &= r\cos^2(\theta/2)(\dot{\psi} - \dot{\phi}/2 - \alpha)^2\\ &\quad + r\sin^2(\theta/2)(\dot{\psi} + \dot{\phi}/2 - \alpha)^2 + \frac{r}{4}\dot{\theta}^2 - \kappa\alpha.\end{aligned}$$

We now eliminate the gauge field α by its equation of motion. Ignoring an overall constant term and treating total derivatives carefully, the resulting dynamics is given by

$$L_{1-\text{vortex}} = \frac{r}{4}[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2] + \frac{\kappa}{2}(\cos\theta - 1)\dot{\phi}. \tag{4.14}$$

We recognize the first term as the familiar sigma-model on $\mathbf{S}^2$ with radius $R = \sqrt{r/2}$. The second term is the Dirac monopole connection of strength κ, expressed in a form which gives a well-defined potential everywhere except at the south pole.

C. One non-Abelian vortex: Explicit moduli space computation

In this final section, we show how to rederive the Dirac monopole connection (4.14) from an explicit moduli space calculation. As we shall see, the calculation requires that we take care with the topology of the moduli space.

Following [10], we work perturbatively both in the velocity of the vortices, and in κ. Practically, this means that we start with the Bogomol'nyi equations with $\kappa = 0$,

$$B = \frac{e^2}{2}(q_i q_i^\dagger - v^2), \qquad \mathcal{D}_z q_i = 0, \tag{4.15}$$

but with $\phi = A_0$ determined by Gauss' law (2.6).[5] Let us first quantify the price that we pay by working perturbatively in κ. Since the Chern-Simons term clearly plays a crucial role in this discussion, it is necessary to work with the Lagrangian instead of the energy functional. We evaluate the Lagrangian (2.1) on the solution to Eqs. (2.6) and (4.15), with $\partial_0 = 0$. This gives

$$L = \int d^2x\mathcal{L} = -2\pi v^2 k - \frac{e^2\kappa^2}{16\pi^2}\int d^2x \,\mathrm{Tr}\,\phi^2. \tag{4.16}$$

The last term is the correction to the Lagrangian due to the fact that we chose to work with the $\kappa = 0$ Bogomol'nyi equations, rather than the true Eqs. (2.4) and (2.5). The mass of the configuration is

$$M_{\text{vortex}} = 2\pi v^2 k\left(1 + \mathcal{O}\left(\frac{e^4\kappa^4}{v^4}\right)\right). \tag{4.17}$$

The extra term is the price we pay for our approximation. At our level of approximation, we neglect all terms of this order in what follows.

1. Zero modes

Let us now turn to the dynamics of the system. Here we see the advantage of our approximation, because we may deal with the familiar vortex equations (4.15). Denote the collective coordinates of this system by X^a, with $a = 1, \ldots, 2kN$. The zero modes of the solution are then given by differentiating, together with a gauge transformation:

$$\delta_a A_\alpha = \frac{\partial A_\alpha}{\partial X^a} - \mathcal{D}_\alpha w_a, \qquad \delta_a q_i = \frac{\partial q_i}{\partial X^a} - i w_a q_i. \tag{4.18}$$

The gauge transformation $w_a \in u(N)$ is dictated by the gauge fixing condition,

$$\mathcal{D}_\alpha \delta_a A_\alpha = -\frac{ie^2}{2}(\delta_a q_i q_i^\dagger - q_i \delta_a q_i^\dagger). \tag{4.19}$$

We next write $A_0 = w + \phi$, where $w \equiv w_a \dot{X}^a$, which ensures that the zero modes are related to the covariant time derivatives as follows:

$$\mathcal{D}_0 q_i = \delta_a q_i \dot{X}^a - i\phi q_i, \qquad E_\alpha = \delta_a A_\alpha \dot{X}^a - \mathcal{D}_\alpha \phi. \tag{4.20}$$

The presence of the ϕ terms on the right-hand side of these equations is what distinguishes the Chern-Simons dynamics from the case $\kappa = 0$. Notice that in our approximation, we have not needed to linearize the second order Gauss' law equation (2.6) since the terms $(\mathcal{D}_0\phi)^2$ are of order $\kappa^2\dot{X}^2$ and may be safely ignored. Substituting into the Lagrangian (2.1), and making use of the constraint (4.19),

[5]Since $\kappa \in \mathbf{Z}$, it does not seem like a good candidate for perturbation theory. A more careful study shows that $e^2\kappa^2/v^2 \ll 1$ is the small parameter.

we derive an expression for the Lagrangian governing the dynamics of the vortex,

$$L = g_{ab}\dot{X}^a\dot{X}^b - 2\pi v^2 k - \frac{\kappa}{4\pi}\int d^2x\,\mathrm{Tr}(2Bw_a\dot{X}^a - \epsilon_{\alpha\beta}A_\alpha\dot{A}_\beta). \quad (4.21)$$

This generalizes the result derived in [8–10] to the non-Abelian case. The first term in this expression is the usual metric on the vortex moduli space, given by

$$g_{ab} = \int d^2x\left(\frac{1}{e^2}\,\mathrm{Tr}\delta_a A_\alpha\delta_b A_\alpha + \delta_{(a}q_i^\dagger\delta_{b)}q_i\right). \quad (4.22)$$

The effect of the Chern-Simons interaction is shown in the last term of (4.21), which is of order $\kappa\dot{X}$.

2. Nonsingular gauge

We now apply this formula to the simple case of a single vortex in the $U(2)$ gauge theory. In this case, the moduli space is $\mathbf{CP}^1$. Previous field theoretic studies of this system have always employed singular gauge [12], in which the Higgs field q_i has no winding at infinity. While this gauge is perfectly adequate for studying the metric on moduli space (see, for example [42]), it hides the interesting topology of the moduli space and is not suitable for studying the effect of the Chern-Simons term. We therefore first describe the collective coordinates of the single $U(2)$ vortex in a gauge that does not suffer from singular behavior.

Consider the $U(1)$ vortex equations (4.15). We work in polar coordinates on the spatial plane: $x_1 = \rho\cos\chi$ and $x_2 = \rho\sin\chi$. Then the solution to the equations for the $k = 1$ vortex is given by

$$q = vq_\star(\rho)e^{i\chi} \quad \text{and} \quad A_\chi = 1 - f(\rho), \qquad A_\rho = 0, \quad (4.23)$$

where the profile functions satisfy the ordinary differential equations,

$$\rho q_\star' = -fq_\star \quad \text{and} \quad \frac{f'}{\rho} = -\frac{e^2v^2}{2}(q_\star^2 - 1) \quad (4.24)$$

subject to the boundary conditions $q_\star(\rho) \to 1,\ 0$ and $f(\rho) \to 0, 1$ as $\rho \to +\infty, 0$. Given these Abelian solutions, it is now a simple matter to embed them into the fields of the $U(2)$ theory to arrive at new solutions. There are two natural embeddings:

$$\text{(i)}\quad q_{(1)} = v\begin{pmatrix} q_\star e^{i\chi} & 0 \\ 0 & 1\end{pmatrix}, \quad A_\chi = \begin{pmatrix} (1-f) & 0 \\ 0 & 0\end{pmatrix}, \quad A_\rho = 0, \quad (4.25)$$

$$\text{(ii)}\quad q_{(2)} = v\begin{pmatrix} 1 & 0 \\ 0 & q_\star e^{i\chi}\end{pmatrix}, \quad A_\chi = \begin{pmatrix} 0 & 0 \\ 0 & (1-f)\end{pmatrix}, \quad A_\rho = 0. \quad (4.26)$$

Here the rows and columns of the q matrix correspond to color and flavor indices, respectively. However, these embeddings are not the only two. Given either of these solutions, one may act upon it with a diagonal combination of the $SU(2)_{\text{flavor}}$ symmetry and $SU(2)_{\text{gauge}}$ symmetry of the model in such a way that the diagonal structure of the vacuum remains invariant,

$$q \to UqV^\dagger, \qquad A \to UAU^\dagger - i(\partial U)U^\dagger, \quad (4.27)$$

where $V \in SU(2)_{\text{flavor}}$ is a constant matrix, and $U = U(\rho, \chi) \in SU(2)_{\text{gauge}}$. In singular gauge, we would impose the condition that $U \to V$ as $\rho \to \infty$. However, the presence of the winding scalar field in (4.25) and (4.26) means that cannot be quite right in the present case. Indeed, the only transformation such that $U \to V$ that is allowed is

$$U = V = \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}$$

which maps $q_{(1)}$ to $q_{(2)}$. For more general transformations, U must itself include some winding. The necessary condition is not difficult to determine. For

$$V = \begin{pmatrix} \hat{a}_1 & \hat{a}_2 \\ \hat{a}_3 & \hat{a}_4\end{pmatrix} \in SU(2)_{\text{flavor}},$$

we require

$$U_{(1)}(\rho, \chi) = \begin{pmatrix} a_1(\rho) & a_2(\rho)e^{i\chi} \\ a_3(\rho)e^{-i\chi} & a_4(\rho)\end{pmatrix} \quad \text{or} \quad U_{(2)}(\rho, \chi) = \begin{pmatrix} a_1(\rho) & a_2(\rho)e^{-i\chi} \\ a_3(\rho)e^{i\chi} & a_4(\rho)\end{pmatrix}, \quad (4.28)$$

where the matrix $U_{(1)}$ is to be used for transformations away from $q_{(1)}$, while the matrix $U_{(2)}$ is required for transformations away from $q_{(2)}$. In both cases, the profile functions in the gauge transformation satisfy the boundary conditions $a_i(\rho) \to \hat{a}_i$ as $\rho \to \infty$.

Perhaps unsurprisingly, the picture that emerges is that two patches are required to cover the moduli space. The solution $q_{(1)}$ can be thought of as the north pole of $\mathbf{CP}^1$, and combined gauge and flavor transformations given by $U_{(1)}$ cover nearly all the space, but cannot take us to $q_{(2)}$. Similarly, $q_{(2)}$ is thought of as the south pole of the moduli space and transformations using $U_{(2)}$ can reach the full moduli space, except for the north pole.

3. Finding the Dirac monopole connection

We now use these results to derive the Dirac monopole connection on moduli space. Let us start with the solution $q = q_{(1)}$. We look for zero modes corresponding to a simultaneous $SU(2)$ gauge and flavor rotation, with parameters Ω and $\hat{\Omega}$, respectively. The zero modes are given by

$$\delta q \equiv \delta_a q\dot{X}^a = i(\Omega q - q\hat{\Omega}), \qquad \delta A_\alpha \equiv \delta_a A_\alpha\dot{X}^a = \mathcal{D}_\alpha\Omega. \quad (4.29)$$

The requirement that the vacuum remains invariant fixes $\Omega_\infty \equiv \lim_{\rho\to\infty}\Omega(\rho,\chi)$ in terms of $\hat{\Omega}$. The remaining freedom in Ω is fixed by the constraint (4.19), which now reads

$$\mathcal{D}^2\Omega = \frac{e^2}{2}(\{\Omega, qq^\dagger\} - 2q\hat{\Omega}q^\dagger). \tag{4.30}$$

We demand that varying the fields with respect to the collective coordinates corresponds to the "large" part of the gauge and flavor rotation, with parameters Ω_∞ and $\hat{\Omega}$. This means that

$$\partial_0 q = \frac{\partial q}{\partial X^a}\dot{X}^a = i(\Omega_\infty q - q\hat{\Omega}) \quad \text{and} \quad \partial_0 A_\alpha = \frac{\partial A_\alpha}{\partial X^a}\dot{X}^a = \mathcal{D}_\alpha\Omega_\infty. \tag{4.31}$$

To achieve this and satisfy (4.29), we set $w = \Omega_\infty - \Omega$ in (4.18).

We choose our flavor transformation to be

$$\hat{\Omega} = \frac{\dot{\theta}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in su(2)_{\text{flavor}},$$

where we are using the coordinates (4.13), and the factor of $\theta/2$ in this expression follows directly from the same factor in (4.13). Then (4.30) is solved by [42]

$$\Omega = \frac{\dot{\theta}}{2}\begin{pmatrix} 0 & q_\star(\rho)e^{i\chi} \\ q_\star(\rho)e^{-i\chi} & 0 \end{pmatrix}, \tag{4.32}$$

where the boundary condition on Ω is inherited from $\hat{\Omega}$. The asymptotic winding in (4.32) results from working in nonsingular gauge as in Eq. (4.28).

To compute the terms in the low-energy dynamics of the Lagrangian, we substitute (4.31) and $w = \Omega_\infty - \Omega$ into our moduli dynamics (4.21) to get

$$L_{\text{CS}} = -\frac{\kappa}{4\pi}\int d^2x\, \text{Tr}(2B(\Omega_\infty - \Omega) - \epsilon_{\alpha\beta}A_\alpha\mathcal{D}_\beta\Omega_\infty). \tag{4.33}$$

Using (4.25) and (4.32), we see that $(\Omega_\infty - \Omega)$ and $\mathcal{D}_\rho\Omega_\infty$ are off diagonal, while B and A_χ are diagonal and A_ρ is zero. Hence (4.33) vanishes.

However, we should not be too hasty in concluding that the Chern-Simons term has no effect on the vortex dynamics. We should first compare with the expected Dirac monopole solution found in Sec. IV B. We have worked about the "north pole" solution (4.25). As discussed previously, this patch covers all but the south pole of $\mathbf{CP}^1$. If we were to write the Dirac monopole in these coordinates, the Dirac string would point along the direction of the south pole. The corresponding term on the worldline is given by

$$L_{\text{Dirac}} = \frac{\kappa}{2}(\cos\theta - 1)\dot{\phi}. \tag{4.34}$$

However, as shown in Fig. 3, the calculation that we have just done corresponds to moving downwards from the north pole. This is equivalent to looking for a $\dot{\theta}$ term in the effective action. It is not surprising that it gave a vanishing answer. Said another way, there is always a coordinate choice so that a given infinitesimal motion does not reveal a Dirac monopole connection in the Lagrangian. We have made that coordinate choice above; moreover, such a coordinate choice is always made implicitly if we work in singular gauge because this gauge disguises the presence of the Dirac string.

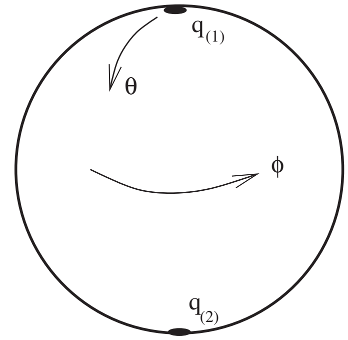

FIG. 3. Coordinates on the moduli space.

With this understanding of the topology of moduli space, it is a simple matter to perform a calculation that does see the Dirac monopole connection. Our first goal is to rotate the $q_{(1)}$ solution to a configuration corresponding to latitude θ on the moduli space. This is done by a flavor rotation of the form

$$V = \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \in SU(2)_{\text{flavor}} \tag{4.35}$$

together with a suitable gauge transformation $U_{(1)}$ with boundary conditions given in (4.28). We now search for zero modes around this new background. Our task is to solve for the infinitesimal gauge transformation Ω satisfying (4.30), subject to the appropriate boundary condition. This boundary condition comes from the requirement that the gauge transformation acts in the longitudinal ϕ direction, and returns us to our starting point after ϕ has increased by 2π. Using the coordinates (4.13), we see that this can be achieved if we supplement our gauge and flavor transformations by $U(1)$ rotations corresponding to motion in the ψ direction. An appropriate choice is

$$\hat{\Omega} = \dot{\phi}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since this is diagonal, we have $\Omega_\infty = \hat{\Omega}$ [see (4.28)]. Using the fact that $\partial_\alpha\Omega_\infty = 0$ and performing an integration by parts, we may write

$$\int d^2x\, \text{Tr}(-\epsilon_{\alpha\beta}A_\alpha\mathcal{D}_\beta\Omega_\infty) = \int d^2x\, \text{Tr}(-2B\Omega_\infty). \tag{4.36}$$

Once we substitute this into the Lagrangian (4.33), we are left with

$$L_{\rm CS} = -\frac{\kappa}{4\pi}\int d^2x\,{\rm Tr}(-2B\Omega). \tag{4.37}$$

We may make use of the gauge covariance of (4.30) to translate the task of finding Ω into something equivalent: solving (4.30) in the background of the original vortex solution (4.25) now subject to the boundary condition arising from

$$V^\dagger\hat{\Omega}V = \dot{\phi}V^\dagger\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}V = \frac{\dot{\phi}}{2}\begin{pmatrix} 1-\cos\theta & -i\sin\theta \\ i\sin\theta & 1+\cos\theta \end{pmatrix}. \tag{4.38}$$

It is straightforward to show that the solution is given by

$$U^\dagger\Omega(\rho,\chi)U = \frac{\dot{\phi}}{2}\begin{pmatrix} 1-\cos\theta & -ie^{i\chi}q_\star(\rho)\sin\theta \\ ie^{-i\chi}q_\star(\rho)\sin\theta & 1+\cos\theta \end{pmatrix}. \tag{4.39}$$

We now substitute our results into the expression (4.37) arising from moduli space dynamics. Noting that the magnetic field associated with (4.25) is given by $U^\dagger BU$, we have

$$L_{\rm CS} = \frac{\kappa}{2\pi}\int d^2x\,{\rm Tr}\,U^\dagger\Omega UU^\dagger BU = \frac{\kappa(1-\cos\theta)\dot{\phi}}{4\pi}\int d^2x\,{\rm Tr}\,B = \frac{\kappa}{2}(\cos\theta - 1)\dot{\phi}. \tag{4.40}$$

This reproduces the Dirac monopole connection as claimed.

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