Classification of BPS equations in higher dimensions

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We systematically classify all possible Bogomol'nyi-Prasad-Sommerfield (BPS) equations in Euclidean dimension $d \leq 8$. We discuss symmetries of BPS equations and their connection with the self-dual Yang-Mills equations. Also, we present a general method allowing to obtain the BPS equations in any dimension. In addition, we find all BPS equations in the Minkowski space of dimension $d \leq 6$ and apply the obtained results to the supersymmetric Yang-Mills theories. In conclusion, we discuss the possibility of using the classification to construct soliton solutions of the low-energy effective theory of the heterotic string.

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I. INTRODUCTION

Bogomol'nyi-Prasad-Sommerfield (BPS) states are the most important ingredients for recent developments in nonperturbative aspects of supersymmetric Yang-Mills theory, string theory, and M-theory. In dimensions higher than four, BPS configurations can be found as solutions to first-order equations, known as generalized self-duality or generalized self-dual Yang-Mills equations. Already more than 20 years ago such equations were proposed [1,2], and some of their solutions were found in [3–13]. In the low-energy effective theory, the BPS states were described by various classical solitonic solutions of various superstring theories [14–23]. More recently, various BPS solutions to the noncommutative Yang-Mills equations in higher dimensions have been investigated in [24–33].

The main purpose of this paper is to systematically classify possible BPS equations in Euclidean dimension $d \leq 8$. In particular, we consider the super Yang-Mills theories on Euclidean space, which may be obtained by a dimensional reduction of the $D = 10$ $N = 1$ super Yang-Mills theory. In Euclidean dimensions, these theories are realized as the field theoretic description of d branes. Note that d branes in a background of the Kalb-Ramond field (NS-NS B field) have been attracting much interest in the development of string theory. The constant magnetic B field on the d brane, in particular, gives a string theoretical realization of the noncommutative geometry [34–36] and the world-volume effective theory on it is described by the noncommutative Yang-Mills theory. Note also that the d-brane bound states with the B field are very interesting in the context of both brane dynamics and brane world-volume theory. In the past few years, their systems have been discussed from various points of view in [37–51].

This paper is organized as follows. In Sec II, we list the properties of some mathematical structures relevant to our work. In Sec. III, we formulate the classified theorem and prove it in the case of even dimensions. In Secs. IV and V, we prove the theorem for odd dimensions. In the next section, we present a general method allowing to obtain any systems of BPS equations and then construct these systems in dimension $d \leq 8$. The final section is devoted to discussions and comments.

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II. PRELIMINARIES

In this section, we collect the properties of spinors in various dimensions and over $\mathbb{R}$ for spaces of various signatures. We also give a brief summary of octonion algebra, Clifford algebra, and symmetric spaces. We list the features of the mathematical structure as far as they are of relevance to our work.

A. Spinors

There are essentially two frameworks for viewing the notion of a spinor. One representation is theoretic. In this point of view, one knows *a priori* that there are some representations of the Lie algebra of the orthogonal group that cannot be formed by the usual tensor constructions. These missing representations are then labeled the spin representations, and their constituents spinors. In this view, a spinor must belong to a representation of the double cover of the rotation group $SO(d)$, or more generally of the generalized special orthogonal group $SO(p, q)$ on spaces with metric signature (p, q). These double covers are Lie groups, called the spin groups $\mathrm{Spin}(p, q)$. All the properties of spinors, and their applications and derived objects, are manifested first in the spin group. The other point of view is geometrical. One can explicitly construct the spinors, and then examine how they behave under the action of the relevant Lie groups. This latter approach has the advantage of being able to say precisely what a spinor is, without invoking some nonconstructive theorem from representation theory. Representation theory must eventually supplement the geometrical machinery once the latter becomes too unwieldy. Therefore, we will use the representation theoretic frameworks for viewing the notion of a spinor.

Let $\mathbb{R}^{p,q}$ be a finite real space with the nondegenerate metric η of signature (p, q). We choose the orthogonal basis $\Gamma_1, \ldots, \Gamma_p, \Gamma_{p+1}, \ldots, \Gamma_{p+q}$ in $\mathbb{R}^{p,q}$, so as the quadratic form η has the standard diagonal form

$$\eta = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1). \tag{1.1}$$

Clifford algebra $\mathrm{Cl}_{p,q}(\mathbb{R})$ is a real associative algebra generated by elements of $\mathbb{R}^{p,q}$ and defined by the relations

$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2\eta_{ab}. \tag{1.2}$$

It follows from (1.2) that the matrices Γ_a are unitary if we impose the conditions

$$\Gamma_a^\dagger = \Gamma^a. \tag{1.3}$$

The algebra $\mathrm{Cl}_{p,q}(\mathbb{R})$ has dimension 2^{p+q}, and its element is a linear combination of the monomials

$$\Gamma_{a_1a_2\ldots a_k} = \Gamma_{a_1}\Gamma_{a_2}\ldots\Gamma_{a_k}, \tag{1.4}$$

where $1 \leq a_1 < a_2 < \ldots < a_k \leq p + q$. It is obvious that the set of all monomials (1.4) with the identity of $\mathrm{Cl}_{p,q}(\mathbb{R})$ form its basis. This basis is called canonical.

The subalgebra of $\mathrm{Cl}_{p,q}(\mathbb{R})$ generated by all monomials Γ_{ab} is called even and denoted by the symbol $\mathrm{Cl}^0_{p,q}(\mathbb{R})$. Since

$$[\Gamma_{ab}, \Gamma_{cd}] = \eta_{ad}\Gamma_{bc} + \eta_{bc}\Gamma_{ad} - \eta_{ac}\Gamma_{bd} - \eta_{bd}\Gamma_{ac}, \tag{1.5}$$

its commutator algebra contains the Lie algebra $so(p, q)$. The follows isomorphisms are true:

$$\mathrm{Cl}^0_{p,q}(\mathbb{R}) \simeq \mathrm{Cl}_{p,q-1}(\mathbb{R}), \qquad q > 0, \tag{1.6}$$

$$\mathrm{Cl}^0_{p,q}(\mathbb{R}) \simeq \mathrm{Cl}_{q,p-1}(\mathbb{R}), \qquad p > 0. \tag{1.7}$$

Complexifying the vector space $\mathrm{Cl}_{p,q}(\mathbb{R})$, we get the complex Clifford algebra $\mathrm{Cl}_d(\mathbb{C})$, where $d = p + q$. This algebra is isomorphic to the algebra $\mathbb{C}(2^n)$ of all complex $2^n \times 2^n$ matrices, if $d = 2n$, or the direct sum of such algebras, if $d = 2n + 1$, i.e.

$$\mathrm{Cl}_{2n}(\mathbb{C}) \simeq \mathbb{C}(2^n), \tag{1.8}$$

$$\mathrm{Cl}_{2n+1}(\mathbb{C}) \simeq \mathbb{C}(2^n) \oplus \mathbb{C}(2^n). \tag{1.9}$$

It therefore has a unique irreducible representation of dimension $2k$. Any such irreducible representation is, by definition, a space of spinors called a spin representation.

The Pin group $\mathrm{Pin}(p, q)$ is the subgroup of the multiplicative group of elements of norm 1 in $\mathrm{Cl}_{p,q}(\mathbb{R})$, and similarly the Spin group $\mathrm{Spin}(p, q)$ is the subgroup of even elements in $\mathrm{Pin}(p, q)$. It is obvious that any representation of $\mathrm{Cl}_{p,q}(\mathbb{C})$ induces a complex representation of $\mathrm{Spin}(p, q)$. One is called the Dirac representation. In odd dimensions, this representation is irreducible. In even dimensions, it is reducible when taken as a representation of $\mathrm{Spin}(p, q)$ and may be decomposed into two: the left-handed and right-handed Weyl spinor representations. In addition, sometimes the noncomplexified version of $\mathrm{Cl}_{p,q}(\mathbb{R})$ has a smaller real representation, the Majorana spinor representation. If this happens in an even dimension, the Majorana spinor representation will sometimes decompose into two Majorana-Weyl spinor representations. Of all these, only the Dirac representation exists in all dimensions. Dirac and Weyl spinors are complex representations, while Majorana spinors are real representations.

The irreducible representations of $\mathrm{Spin}(p, q)$ for $p + q < 8$ can be obtained from Table I , (in the table, $p + q$runs vertically, $p - q$ runs horizontally, and $\mathbb{A}^2 \equiv \mathbb{A} \oplus \mathbb{A}$), if we make use of the isomorphisms (1.6) and (1.7).

Table I continues with a periodicity of eight, that is, $\mathrm{Cl}_{p+8,q} \simeq \mathrm{Cl}_{p,q+8} \simeq \mathrm{Cl}_{p,q}(16)$, which is the 16×16 matrix algebra with entries in the Clifford algebra $\mathrm{Cl}_{p,q}(\mathbb{R})$. Therefore, in fact, we have spinor representations of $\mathrm{Spin}(p, q)$ any p and q. For example, the Dirac representation of $\mathrm{Spin}(2n + 1)$ is real, if $n \equiv 0$, 3 mod 4, and pseudoreal, if $n \equiv 1$, 2 mod 4. The Weyl representations of $\mathrm{Spin}(2n)$ are complex conjugates of one another as $n \equiv 1$ mod 2, real as $n \equiv 0$ mod 4, and pseudoreal as $n \equiv 2$ mod 4. These two representations are dual of one another, if n is odd, and self-dual, if n is even.

B. Octonions

We recall that the algebra of octonions $\mathbb{O}$ is a real linear algebra with the canonical basis $1, e_1, \ldots, e_7$ such that

TABLE I. Representations of the Clifford algebra $\mathrm{Cl}_{p,q}(\mathbb{R})$.

	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7
0								$\mathbb{R}$							
1							$\mathbb{C}$		$\mathbb{R}^2$						
2						$\mathbb{H}$		$\mathbb{R}(2)$		$\mathbb{R}(2)$					
3					$\mathbb{H}^2$		$\mathbb{C}(2)$		$\mathbb{R}^2(2)$		$\mathbb{C}(2)$				
4				$\mathbb{H}(2)$		$\mathbb{H}(2)$		$\mathbb{R}(4)$		$\mathbb{R}(4)$		$\mathbb{H}(2)$			
5			$\mathbb{C}(4)$		$\mathbb{H}^2(2)$		$\mathbb{C}(4)$		$\mathbb{R}^2(4)$		$\mathbb{C}(4)$		$\mathbb{H}^2(2)$		
6		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$		$\mathbb{R}(8)$		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$	
7	$\mathbb{R}^2(8)$		$\mathbb{C}(8)$		$\mathbb{H}^2(4)$		$\mathbb{C}(8)$		$\mathbb{R}^2(8)$		$\mathbb{C}(8)$		$\mathbb{H}^2(4)$		$\mathbb{C}(8)$

$$e_i e_j = -\delta_{ij} + c_{ijk} e_k, \tag{1.10}$$

where the structure constants c_{ijk} are completely antisymmetric and nonzero and equal to unity for the seven combinations (or cycles)

$$(ijk) = (123), (145), (167), (246), (275), (374), (365).$$

The algebra of octonions is not associative but alternative, i.e. the associator

$$(x, y, z) = (xy)z - x(yz) \tag{1.11}$$

is totally antisymmetric in x, y, z. Consequently, any two elements of $\mathbb{O}$ generate an associative subalgebra. The algebra of octonions satisfies the identity

$$((zx)y)x = z(xyx), \tag{1.12}$$

which is called the right Moufang identity. The algebra $\mathbb{O}$ permits the involution (anti-automorphism of period two) $x \longrightarrow \bar{x}$ such that the elements

$$t(x) = x + \bar{x}, \qquad n(x) = \bar{x}x \tag{1.13}$$

are in $\mathbb{R}$. In the canonical basis, this involution is defined by $\bar{e}_i = -e_i$. It follows that the bilinear form

$$(x, y) = \tfrac{1}{2}(\bar{x}y + \bar{y}x) \tag{1.14}$$

is positive definite and defines an inner product on $\mathbb{O}$. It is easy to prove that the quadratic form $n(x)$ permits the composition

$$n(xy) = n(x)n(y). \tag{1.15}$$

Since the quadratic form $n(x)$ is positive definite, it follows that $\mathbb{O}$ is a division algebra. Linearization of (1.15) to x and y gives

$$n(x)(y, z) = (xy, xz) = (yx, zx), \tag{1.16}$$

$$2(x, y)(z, t) = (xz, yt) + (xt, yz). \tag{1.17}$$

Finally, notice that the algebra of octonions is unique, to within isomorphism, alternative nonassociative simple real division algebra.

Now let $\Gamma_1, \ldots, \Gamma_7$ be generators of the Clifford algebra $\mathrm{Cl}_{0,7}(\mathbb{R})$ satisfying the relations (1.2). Further, let $x \in \mathbb{O}$. Denote by R_x the operator of right multiplication in $\mathbb{O}$

$$yR_x = yx, \qquad y \in \mathbb{O}. \tag{1.18}$$

Using the multiplication law (1.10) and antisymmetry of the associator (1.11), we prove the equalities

$$R_{e_i} R_{e_j} + R_{e_j} R_{e_i} = -2\delta_{ij} E, \tag{1.19}$$

where E is the identity 8×8 matrix. Comparing (1.19) with (1.2), we see that the correspondence $\Gamma_i \longrightarrow R_{e_i}$ can be extended to the homomorphism

$$\mathrm{Cl}_{0,7}(\mathbb{R}) \longrightarrow \mathrm{End}\mathbb{O}. \tag{1.20}$$

Using Table I, we prove that the mapping (1.20) is surjective and $\mathrm{End}\mathbb{O} \simeq \mathbb{R}(8)$. Since

$$\mathrm{Cl}_{0,7}(\mathbb{R}) \simeq \mathrm{Cl}^0_{8,0}(\mathbb{R}), \tag{1.21}$$

it follows that the homomorphism (1.20) induces the homomorphism $\mathrm{Spin}(8) \longrightarrow SO(8)$. We define the sets

$$\mathbb{S}^7 = \{a \in \mathbb{O} \mid n(a) = 1\}, \tag{1.22}$$

$$\mathbb{S}^6 = \{\boldsymbol{a} \in \mathbb{O} \mid n(\boldsymbol{a}) = 1\}, \tag{1.23}$$

where $\boldsymbol{a}$ is a vector part of the octonion $a = a_0 + \boldsymbol{a}$. It follows from (1.16), (1.20), and (1.21) that the sets

$$X = \{R_a \mid a \in \mathbb{S}^7\}, \tag{1.24}$$

$$Y = \{R_{\mathbf{a}} R_{\mathbf{b}} \mid \boldsymbol{a}, \boldsymbol{b} \in \mathbb{S}^6\} \tag{1.25}$$

generate the groups $SO(8)$ and Spin(7), respectively. Note also that the product

$$R_{e_1} R_{e_2} \ldots R_{e_7} = E. \tag{1.26}$$

The equality (1.26) follows from the simplicity of $\mathbb{R}(8)$ and the fact that the element $\Gamma_1 \Gamma_2 \ldots \Gamma_7$ lies in the center of $\mathrm{Cl}_{0,7}(\mathbb{R})$. It follows from (1.26) that restriction of the homomorphism (1.20) on Spin(7) is injection.

C. Symmetric spaces

We list the properties of symmetric spaces relevant to our work. Let G be a connected Lie group, σ an involutive automorphism of G, and G_σ a set of all fixed point of G under σ. Further, let H be a closed subgroup in G_σ containing the identity component of G_σ. The quotient space G/H is called a symmetric homogeneous space. If the subgroup H is compact, then the space G/H admits an G-invariant Riemannian metric. The symmetric space G/H equipped with such metric is called a globally symmetric Riemannian space.

Automorphism σ induces an involutive automorphism of the Lie algebra A of the group G. With respect to this automorphism the algebra A can be decomposable into the direct sum

$$A = A^+ \oplus A^- \tag{1.27}$$

of proper subspaces corresponding to the eigenvalues ± 1. We have obviously

$$[A^+, A^+] \subseteq A^+, \qquad [A^+, A^-] \subseteq A^-, \qquad [A^-, A^-] \subseteq A^+. \tag{1.28}$$

The space A^+ coincides with the Lie algebra of the group H, and the space A^- is closed under the composition $[x, y, z] = [[x, y], z]$. The vector space A^- equipped with this trilinear composition is called a triple Lie system.

A globally symmetric Riemannian space G/H is said to be irreducible if the algebra A is semisimple, the subalgebra A^+ is a maximal proper subalgebra in A, and A^+ contains no nonzero ideals of A. In particular, irreducible global symmetric Riemannian spaces are the spaces

$$M_{pq} = SO(p+q)/SO(p) \times SO(q), \tag{1.29}$$

$$N_{pq} = SU(p+q)/S(U(p) \times U(q)). \tag{1.30}$$

Note that M_{pq} and N_{pq} are compact simple connected spaces of dimension pq and $2pq$, respectively.

III. THE MAIN THEOREM

BPS states refer to field configurations that are invariant under some supersymmetries. In super Yang-Mills theories on the Euclidean space $\mathbb{R}^d$, a bosonic configuration is BPS if there exist a nonzero constant spinor ε in an unitary space V of dimension $2^{[d/2]}$, where $[d/2]$ is an integral part of $d/2$, such that the infinitesimal supersymmetric transformation of the fermion field vanishes

$$\delta\chi = F_{ab}\Gamma_{ab}\varepsilon = 0. \tag{2.1}$$

Such zero eigenspinors of the matrix $F_{ab}\Gamma_{ab}$ form the subspace $W \subseteq V$. The BPS field strength should satisfy certain conditions in order to have a given number of unbroken supersymmetries. These conditions can be written as a system of linear equations (BPS equations) connecting components of F_{ab}. We say that two systems of BPS equations are equivalent if either they are incompatible or they have the same solutions up to a nondegenerate transformation of $\mathbb{R}^d$. Otherwise, they are called nonequivalent. Since we consider a global supersymmetry, the conditions imposed on F_{ab} do not depend on a choice of basis in $\mathbb{R}^d$. Hence, we must find nonequivalent systems of BPS equations.

In order that to find such systems, we define the projection operator Ω mapping V onto W, as has been done previously in [24]. With a suitable orthonormal basis for V, this operator appears as $2^{[d/2]} \times 2^{[d/2]}$ matrix

$$\tilde{\Omega} = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.2}$$

where E_r is the identity $r \times r$ matrix, and $r = \dim W$. Obviously, the projection operator is diagonalizable in an orthonormal basis, and it has a real spectrum (its eigenvalues are 0 or 1). Therefore, it is Hermitian. Thus,

$$\Omega^2 = \Omega, \tag{2.3}$$

$$\Omega^\dagger = \Omega. \tag{2.4}$$

Now we can rewrite the Eq. (2.1) in the following equivalent form

$$F_{ab}\Gamma_{ab}\Omega = 0. \tag{2.5}$$

In order to get the system of BPS equations from (2.5), we must represent the projector Ω as a linear combination of the identity matrix and the monomials (1.4), and further use the identities (1.2). Note also that the constant ν, defined by

$$\operatorname{tr}\Omega = \nu \times 2^{[d/2]}, \tag{2.6}$$

gives the fraction of the unbroken supersymmetry, so $0 \le \nu \le 1$. The $\nu = 0$ or 1 cases are trivial, either meaning the non-BPS state or the vacuum, $F_{ab} = 0$. The following theorem contains the main result of the paper:

Theorem 1. Suppose the constant spinor ε satisfying (2.1) is Weyl as even d, Majorana as $d = 7$, and Majorana-Weyl as $d = 8$. Then there exists to within equivalence a unique system of BPS equations for every pair of values $d \le 8$ and $\nu = \nu(d)$.

Proof. Let $\{e_a\}$ and $\{e'_a\}$ be two orthonormal bases in $\mathbb{R}^d$. Then there exists an orthogonal transformation of $\mathbb{R}^d$ such that $e'_a = A_a^b e_b$. In this case, the components of F_{ab} are transformed by the rule

$$F^k_{ab} \to F^k_{cd} A^c_a A^d_b. \tag{2.7}$$

Denote by the symbol F^k a real skew-symmetric $d \times d$ matrix with the elements F^k_{ab}. Then the transformation (2.7) can be rewritten in the matrix form

$$F^k \to AF^kA^{-1}, \tag{2.8}$$

where A is an orthogonal matrix with the elements $A_{ab} = A_a^b$ such that $\det A = 1$. Obviously, the matrices A and F^k are elements of the group $SO(d)$ and the algebra $so(d)$, respectively. Since F^k is arbitrary real skew-symmetric matrix, it follows that the transformation (2.8) defines an inner automorphism of $so(d)$.

On the other hand, the antisymmetry matrices Γ_{ab} satisfy the commutation relations (1.5). Therefore, they generate a Lie algebra $\tilde{so}(d)$ that is isomorphic to $so(d)$. Denote by $\tilde{F}^k$ an image of F^k with respect to the isomorphism $so(d) \to \tilde{so}(d)$. Then we have the following diagram,

$$\begin{array}{ccc} F^k & \longrightarrow & AF^kA^{-1} \\ \downarrow & & \downarrow \\ \tilde{F}^k & \longrightarrow & B\tilde{F}^kB^{-1} \end{array}, \tag{2.9}$$

where the matrix $B \in \mathrm{Spin}(d)$. It is obvious that this diagram is commutative. In particular, any inner automorphism of $\tilde{so}(d)$ defined by the mapping

$$\tilde{F}^k \to B\tilde{F}^kB^{-1} \tag{2.10}$$

induces the transformation (2.7).

Further, the matrices Γ_{ab} make up a basis of $\tilde{so}(d)$. Therefore, any of its element $\tilde{F}^k$ can be represented in the form

$$\tilde{F}^k = \tilde{F}^k_{ab}\Gamma_{ab}. \tag{2.11}$$

Denote by $\tilde{F}_{ab}$ an antisymmetry tensor with the components $\tilde{F}^k_{ab}$, and consider the equation

$$\tilde{F}_{ab}\Gamma_{ab}(B^{-1}\Omega B) = 0. \tag{2.12}$$

It follows from commutativity of the diagram that the Eqs. (2.5) and (2.12) are equivalent. Thus, if we prove that by the transformation

$$\Omega \rightarrow \tilde{\Omega} = B^{-1}\Omega B, \tag{2.13}$$

where $B \in \mathrm{Spin}(d)$, the matrix Ω can be reduced to the form (2.2), then we prove the theorem.

We consider even dimension $d = 2n$. Without loss of generality, we can suppose that ε is a right-handed (chiral) spinor, i.e.

$$\Gamma_* \varepsilon = \varepsilon, \qquad \Gamma_* = (-i)^n \Gamma_1 \ldots \Gamma_{2n}. \tag{2.14}$$

We will seek representations of gamma matrices such that

$$\Gamma_a = \begin{pmatrix} 0 & \Lambda_a \\ \Lambda_a^\dagger & 0 \end{pmatrix}, \qquad \Gamma_* = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \tag{2.15}$$

where E is the identity matrix. It is obvious that in this representation, the Hermitian projection operator Ω and the element B of $\mathrm{Spin}(n)$ take the form

$$\Omega = \begin{pmatrix} \Omega_+ & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}, \tag{2.16}$$

where Ω and $B_\pm$ are $n \times n$ matrices. Now we consider the concrete values of n.

(1) In two Euclidean dimensions, the Weyl spinor is one-component and complex representation. Therefore, we choose the representation in terms of Pauli matrices

$$\Gamma_1 = \sigma_1, \qquad \Gamma_2 = \sigma_2. \tag{2.17}$$

It follows from (2.3) that $\Omega_+ = 0$ or 1.

(2) In four Euclidean dimensions, there are two inequivalent pseudoreal two-component Weyl spinor, and each of them transform under $SU(2)$. We choose the gamma matrices in the form

$$\Gamma_k = \sigma_1 \otimes \sigma_k, \qquad \Gamma_4 = \sigma_2 \otimes \sigma_0, \tag{2.18}$$

where σ_0 is the identity 2×2 matrix. In this representation, the generators of Spin(4) have the block diagonal form

$$\Gamma_{ij} = i\varepsilon_{ijk}(\sigma_0 \otimes \sigma_k), \qquad \Gamma_{k4} = i(\sigma_3 \otimes \sigma_k). \tag{2.19}$$

The matrices σ_k form a basis of $su(2)$. Therefore, B_+ is arbitrary unitary 2×2 matrix. Since the matrix Ω is Hermitian, if follows that it can be reduced to the form (2.2) by the transformation (2.13).

(3) In six Euclidean dimensions, the isomorphism $\mathrm{Spin}(6) \simeq SU(4)$ guarantees that there are two four-dimensional complex Weil representations that are complex conjugates of one another. We choose the gamma matrices in the form

$$\Gamma_k = \sigma_1 \otimes \sigma_k \otimes \sigma_0, \qquad \Gamma_{k+3} = \sigma_2 \otimes \sigma_0 \otimes \sigma_k, \tag{2.20}$$

where $k = 1, 2, 3$. In this representation, the generators of Spin(6) have the following form

$$\begin{aligned} \Gamma_{ij} &= i\varepsilon_{ijk}(\sigma_0 \otimes \sigma_k \otimes \sigma_0), \\ \Gamma_{i(j+3)} &= i(\sigma_3 \otimes \sigma_i \otimes \sigma_j), \\ \Gamma_{(i+3)(j+3)} &= i\varepsilon_{ijk}(\sigma_0 \otimes \sigma_0 \otimes \sigma_k). \end{aligned} \tag{2.21}$$

Noting that the matrices $\sigma_k \otimes \sigma_0$, $\sigma_i \otimes \sigma_j$, and $\sigma_0 \otimes \sigma_k$ form a basis of $su(4)$, we prove that B_+ is arbitrary unitary 4×4 matrix. Hence the Hermitian matrix Ω can be reduced to the form (2.2) by (2.13).

(4) In eight Euclidean dimensions, the Weyl-Majorana representation is eight dimensional and real. We choose the Γ matrices in the form

$$\Gamma_8 = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}, \qquad \Gamma_k = \begin{pmatrix} 0 & R_k \\ -R_k & 0 \end{pmatrix}, \tag{2.22}$$

where the real 8×8 matrices R_k ($k = 1, \ldots, 7$) are antisymmetric and satisfy

$$R_i R_j + R_j R_i = -2\delta_{ij} E. \tag{2.23}$$

Obviously, we can choose this matrices in the form of operators (1.18) of right multiplication on the basic elements e_i of $\mathbb{O}$, i.e. we suppose $R_i = R_{e_i}$. Since by (1.26) the product

$$R_1 R_2 \ldots R_7 = E, \tag{2.24}$$

the matrix Γ_* has the form (2.15). It follows from (2.22) that the generators of Spin(8) are

$$\Gamma_{i8} = \begin{pmatrix} R_i & 0 \\ 0 & -R_i \end{pmatrix}, \qquad \Gamma_{ij} = \begin{pmatrix} [R_j, R_i] & 0 \\ 0 & [R_j, R_i] \end{pmatrix}. \tag{2.25}$$

The elements R_i and $[R_j, R_i]$ make up a basis of $so(8)$. Therefore, B_+ is an arbitrary orthogonal 8×8 matrix. Since Ω is a real symmetric matrix, it can be reduced to the form (2.2) by the transformation (2.13).

IV. SEVEN DIMENSIONS

In seven Euclidean dimensions, the single spinor representation is eight dimensional and real. Therefore, the projection operator Ω is represented as an 8×8 real symmetric matrix. We must prove that

$$B^{-1}\Omega B = \tilde{\Omega} = \mathrm{diag}\{1, \ldots, 1, 0, \ldots, 0\} \tag{3.1}$$

for some $B \in \mathrm{Spin}(7)$. In the first place, we note that there exists an element $U \in SO(8)$ such that

$$\tilde{\Omega} = U\Omega U^{-1}. \tag{3.2}$$

Then it follows from (3.1) and (3.2) that

$$\tilde{\Omega}\tilde{B} = \tilde{B}\tilde{\Omega}, \tag{3.3}$$

where the matrix $\tilde{B} = UB$. Further, the general solution of the Eq. (3.3) has the form

$$\tilde{B} = \begin{pmatrix} \tilde{B}_1 & 0 \\ 0 & \tilde{B}_2 \end{pmatrix}, \tag{3.4}$$

where $\tilde{B}_1$ and $\tilde{B}_2$ are orthogonal matrices such that $\det\tilde{B}_i = 1$. Therefore,

$$\tilde{B} \in H_k \simeq SO(k) \times SO(8-k), \qquad 1 \leq k \leq 4. \tag{3.5}$$

On the other hand, $U = \tilde{B}B^{-1}$. Hence, the equality (3.1) is true if the group

$$SO(8) = H_k\mathrm{Spin}(7), \tag{3.6}$$

i.e. if any element $g \in SO(8)$ can be represented as the product $g = hf$, where $h \in H_k$ and $f \in \mathrm{Spin}(7)$. We will prove the equality (3.6).

A. The case $k = 1$

As stated above, the groups $SO(8)$ and Spin(7) are generated by the sets (1.24) and (1.25), respectively. We choose a basis in the algebra octonions $\mathbb{O}$ such that the subgroup $H_1 \in SO(8)$ is a stabilizer of the identity element of $\mathbb{O}$. It follows from the Moufang identity (1.12) that

$$R_{ab}R_b^{-1}R_a^{-1} \in H_1 \tag{3.7}$$

for any $a, b \in \mathbb{S}^7$. We consider a right coset $H_1 g$ of $SO(8)$. Since the set X in (1.24) generates $SO(8)$, the element

$$g = R_{a_1} \ldots R_{a_k}. \tag{3.8}$$

Multiplying (3.8) by suitable elements of the form (3.7), we get the element R_c as a representative of $H_1 g$.

On the other hand, it follows from (1.10) and (1.14) that the product

$$\boldsymbol{a}\,\boldsymbol{b} = -(\boldsymbol{a}, \boldsymbol{b}) + \boldsymbol{a} \times \boldsymbol{b}, \tag{3.9}$$

where $\boldsymbol{a} \times \boldsymbol{b} = \frac{1}{2}[\boldsymbol{a}, \boldsymbol{b}]$. Using properties of the algebra $\mathbb{O}$, we prove the equalities

$$\begin{aligned} -(\boldsymbol{b}, \boldsymbol{b})\boldsymbol{a} &= (\boldsymbol{a}\boldsymbol{b})\boldsymbol{b} \\ &= -(\boldsymbol{a}, \boldsymbol{b})\boldsymbol{b} - (\boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{b}. \end{aligned} \tag{3.10}$$

It follows from (3.10) that

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{b} = (\boldsymbol{a}, \boldsymbol{b})\boldsymbol{b} - (\boldsymbol{b}, \boldsymbol{b})\boldsymbol{a}, \tag{3.11}$$

$$(\boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{b}) = 0. \tag{3.12}$$

Using (3.11), we find a solution $\boldsymbol{b}$ of the system

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}, \qquad -(\boldsymbol{a}, \boldsymbol{b}) = c_0, \tag{3.13}$$

where the vectors $\boldsymbol{a}$ and $\boldsymbol{c}$ satisfy the equalities $(\boldsymbol{a}, \boldsymbol{a}) = 1$ and $(\boldsymbol{a}, \boldsymbol{c}) = 0$. This solution is

$$\boldsymbol{b} = -c_0\boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{a}. \tag{3.14}$$

Linearizing the identity (3.12), we find the scalar square

$$(\boldsymbol{b}, \boldsymbol{b}) = c_0^2 + (\boldsymbol{c}, \boldsymbol{c}). \tag{3.15}$$

Comparing (3.13) with (3.9) and taking into account (3.15), we see that any element $c \in \mathbb{S}^7$ can be represented as

$$c = \boldsymbol{a}\boldsymbol{b}. \tag{3.16}$$

As proven above, the coset $H_1 g = H_1 R_c$ for some $c \in \mathbb{S}^7$. We multiply R_c by the element

$$R_{\boldsymbol{a}}R_{\boldsymbol{b}}R_{\boldsymbol{ab}}^{-1} \in H_1. \tag{3.17}$$

Then, by (3.16) we get the element $R_{\boldsymbol{a}}R_{\boldsymbol{b}}$ as a representative of $H_1 g$. Since this element lies in Spin(7), it follows that the equality (3.6) is proved for $k = 1$.

B. The case $k \neq 1$.

We use below an explicit form of the operators R_{e_i} in the canonical basis of $\mathbb{O}$. Using the multiplication law (1.10), we can easily find the required expressions. We have

$$R_{e_i} = e_{i0} + \tfrac{1}{2}c_{ijk}e_{jk}, \tag{3.18}$$

where e_{mn} are skew-symmetric 8×8 matrices with the elements

$$(e_{mn})^\alpha_\beta = \delta_{m\beta}\delta^\alpha_n - \delta_{n\beta}\delta^\alpha_m. \tag{3.19}$$

Since the matrices R_{e_i} and $[R_{e_i}, R_{e_j}]$ are linearly independent over $\mathbb{R}$, they form a basis of a Lie algebra A that is isomorphic to $so(8)$. Suppose

$$I = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \tag{3.20}$$

where E is the identity 4×4 matrix. It is obvious that the transformation

$$R_{e_i} \to IR_{e_i}I \tag{3.21}$$

may be extended to an involutive automorphism of A. With respect to this automorphism the algebra A is decomposed into the direct sum (1.27) of proper subspaces A^+ and A^-. Using the representation (3.18), we prove that

$$\begin{aligned} IR_{e_i}I &= R_{e_i} \quad \text{for } i = 1, 2, 3, \\ IR_{e_i}I &= -R_{e_i} \quad \text{for } i = 4, 5, 6, 7. \end{aligned} \tag{3.22}$$

A simple calculation shows that $\dim A^+ = 12$ and $\dim A^- = 16$. Therefore, the corresponding symmetric space is isomorphic to $SO(8)/H_4$.

Now we consider the transformation

$$R_{e_i} \to JR_{e_i}J^{-1}. \tag{3.23}$$

Once again using (3.18), we prove that

$$\begin{aligned} JR_{e_i}J^{-1} &= R_{e_i} \quad \text{for } i = 1, 2, 4, 5, 6, \\ JR_{e_i}J^{-1} &= -R_{e_i} \quad \text{for } i = 3, 7. \end{aligned} \tag{3.24}$$

Extending (3.23) to an involutive automorphism of A, we get that $\dim A^+ = 16$ and $\dim A^- = 12$. Hence, the corresponding symmetric space is isomorphic to $SO(8)/H_2$.

Finally, we consider the transformation

$$R_{e_i} \to JR_{e_i}J. \tag{3.25}$$

Since the transformation (3.25) is a composition of (3.23) and the transformation $R_{e_i} \to -R_{e_i}$, we have the equalities

$$\begin{aligned} JR_{e_i}J &= R_{e_i} \quad \text{for } i = 3, 7, \\ JR_{e_i}J &= -R_{e_i} \quad \text{for } i = 1, 2, 4, 5, 6. \end{aligned} \tag{3.26}$$

Using (3.23), we easily prove that the transformation (3.25) may be extended to an involutive automorphism of A. It is obvious that $\dim A^+ = 13$ and $\dim A^- = 15$. Therefore, the corresponding symmetric space is isomorphic to $SO(8)/H_3$.

We extend the involutive automorphism of A defined by (3.22), (3.24), and (3.26) to an automorphism σ of the corresponding simply connected Lie group Spin(8). It follows from (1.21) that this group can be embedded into the Clifford algebra $\mathrm{Cl}_{0,7}(\mathbb{R})$. Suppose Γ_i is a prototype of R_{e_i} relative to the homomorphism (1.20). It is obvious that $\Gamma_i \in \mathrm{Spin}(8)$. On the other hand, it follows from (1.19) that the matrices Γ_i generate $\mathrm{Cl}_{0,7}(\mathbb{R})$. Hence, $\Gamma_i \notin \mathrm{Spin}(7)$. Now, let $\tilde{H}_k$ be a subgroup of Spin(8) that is invariant under σ. Then it follows from (3.22), (3.24), and (3.26) that $\Gamma_i \in \tilde{H}_k$ for some value of i.

Further, let the matrix $\Gamma_i \in \tilde{H}_k$, and let $\tilde{H}_k g$ be a coset of Spin(8). Since Spin(7) is a maximal subgroup in Spin(8), the element g can be represented by a product of Γ_i and elements of Spin(8). Now, note that the algebra End$\mathbb{O}$ satisfies the identity

$$R_x R_y R_x = R_{xyx}, \tag{3.27}$$

which is a direct corollary of (1.12). Since $R_{e_i}R_{\bar{e}_i} = 1$, it follows that

$$R_{e_i} R_{\boldsymbol{a}} R_{\boldsymbol{b}} = R_{e_i \boldsymbol{a} \bar{e}_i} R_{e_i \boldsymbol{b} \bar{e}_i} R_{e_i}, \tag{3.28}$$

where we do not sum on the recurring indexes. Obviously, the products $e_i\boldsymbol{a}\bar{e}_i$ and $e_i\boldsymbol{b}\bar{e}_i$ are vector octonions. Since a restriction of the homomorphism (1.20) to Spin(7) is injection, it follows from (1.25) and (3.28) that

$$\Gamma_i f \Gamma_i^{-1} \in \mathrm{Spin}(7) \tag{3.29}$$

for any $f \in \mathrm{Spin}(7)$. Hence, the element g can be represent in the form $g = \Gamma_i^p f$. Since $\Gamma_i \in \tilde{H}_k$, it follows that the element $g \in \mathrm{Spin}(7)$. Mapping Spin(8) onto $SO(8)$, we prove the equality (3.6) for $k \neq 1$.

V. THREE AND FIVE DIMENSIONS

In three Euclidean dimensions, the single spinor representation is two dimensional and pseudoreal. Therefore, the projection operator Ω may be represented as a 2×2 Hermitian matrix. Since the group $\mathrm{Spin}(3) \simeq SU(2)$, it follows that the matrix Ω can be reduced to the form (2.2) by the transformation (2.13).

Now, we consider five Euclidean dimensions. In these dimensions, the relevant isomorphism is $\mathrm{Spin}(5) \simeq Sp(2)$, which implies that the single spinor representation is four dimensional and pseudoreal. Hence, we must prove that

$$B^{-1}\Omega B = \tilde{\Omega} = \mathrm{diag}\{1, \ldots, 1, 0, \ldots, 0\} \tag{4.1}$$

for some $B \in Sp(2)$. Since the space of spinor representation of Spin(5) is a four-dimensional unitary space, the Hermitian matrix Ω can be reduced to the form (2.2) by the transformation

$$\tilde{\Omega} = U\Omega U^{-1}, \tag{4.2}$$

where $U \in SU(4)$. As above, it follows from (4.1) and (4.2) that

$$\tilde{\Omega}\tilde{B} = \tilde{B}\tilde{\Omega}, \tag{4.3}$$

where the matrix $\tilde{B} = UB$. The general solution of the Eq. (4.3) has the form

$$\tilde{B} = \begin{pmatrix} \tilde{B}_1 & 0 \\ 0 & \tilde{B}_2 \end{pmatrix}, \tag{4.4}$$

where $\tilde{B}_1$ and $\tilde{B}_2$ are unitary matrices such that $\det\tilde{B} = 1$. It is obvious that

$$\tilde{B} \in H_k \simeq S(U(k) \times U(4-k)), \qquad 1 \leq k \leq 2. \tag{4.5}$$

Since $U = \tilde{B}B^{-1}$, the equality (4.1) is true if the group

$$SU(4) = H_k Sp(2), \tag{4.6}$$

i.e. if any element $SU(4)$ can be represented as the product $g = hf$, where $h \in H_k$ and $f \in Sp(2)$. We will prove the equality (4.6).

A. The case $k = 1$

As before, we will use properties of the algebra $\mathbb{O}$. We fix first the field $\mathbb{C}$ in $\mathbb{O}$ by the condition $e_1 \in \mathbb{C}$. Further, any two elements of $\mathbb{O}$ generate an associative subalgebra. Therefore,

$$x(yz) = (xy)z \tag{4.7}$$

for any $x, y \in \mathbb{C}$, and $z \in \mathbb{O}$. It follows that we may consider $\mathbb{O}$ as a (left) vector space over $\mathbb{C}$ relative to the multiplication xz, where $x \in \mathbb{C}$ and $z \in \mathbb{O}$. Obviously, $\mathbb{O}$ is four dimensional over $\mathbb{C}$. For $x, y \in \mathbb{O}$ we define

$$\langle x, y\rangle = (x, y) - e_1(e_1x, y). \tag{4.8}$$

Then $\langle x, y\rangle \in \mathbb{C}$. Using the identities (1.16) and (1.17), we prove the equalities

$$\langle e_1x, y\rangle = e_1\langle x, y\rangle = -\langle x, e_1y\rangle. \tag{4.9}$$

Hence, $\langle x, y\rangle$ is a Hermitian form in $\mathbb{O}$ over $\mathbb{C}$. If $\langle x, y\rangle = 0$, then $(x, y) = 0$, since 1 and e_1 are independent over $\mathbb{R}$.

Since the form (1.14) is positive definite, it follows that the Hermitian form (5.9) is nondegenerate.

Further, let V be a linear span of the elements 1, e_1, e_2. Denote by $\mathbb{C}^{\perp}$ and $V^{\perp}$ the orthogonal complements to $\mathbb{C}$ and V in $\mathbb{O}$ and define the sets

$$\mathbb{S}^5 = \{\boldsymbol{a} \in \mathbb{C}^{\perp} \mid n(\boldsymbol{a}) = 1\}, \tag{4.10}$$

$$\mathbb{S}^4 = \{\boldsymbol{a} \in V^{\perp} \mid n(\boldsymbol{a}) = 1\}. \tag{4.11}$$

Now, note that the elements $\Gamma_2, \Gamma_3, \ldots, \Gamma_7$ of the Clifford algebra $\mathrm{Cl}_{0,7}(\mathbb{R})$ generate the subalgebra $\mathrm{Cl}_{0,6}(\mathbb{R})$. It follows from Table I that $\mathrm{Cl}_{0,6}(\mathbb{R})$ is isomorphic to the simple matrix algebra $\mathbb{R}(8)$. Therefore, the restriction of the homomorphism (1.20) to $\mathrm{Cl}_{0,6}(\mathbb{R})$ is injection. It is obvious that the restriction of this homomorphism to the algebra $\mathrm{Cl}_{0,5}(\mathbb{R})$ with the generators $\Gamma_3, \ldots, \Gamma_7$ is also injection. Hence, the sets

$$Z_1 = \{R_{\boldsymbol{a}} R_{\boldsymbol{b}} \mid \boldsymbol{a}, \boldsymbol{b} \in \mathbb{S}^5\}, \tag{4.12}$$

$$Z_2 = \{R_{\boldsymbol{a}} R_{\boldsymbol{b}} \mid \boldsymbol{a}, \boldsymbol{b} \in \mathbb{S}^4\} \tag{4.13}$$

generate the groups G_1 and G_2, which are isomorphic to Spin(6) and Spin(5), respectively. Further, if follows from (1.10) that the elements 1, e_2, e_4, e_6 form a basis of $\mathbb{O}$ over $\mathbb{C}$. We will prove that in this basis the groups G_1 and G_2 coincide with $SU(4)$ and $Sp(2)$. Indeed, for all $x \in \mathbb{O}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{S}^5$ the equality

$$(xR_{\boldsymbol{a}} R_{\boldsymbol{b}})e_1 = (xe_1)R_{\boldsymbol{a}} R_{\boldsymbol{b}} \tag{4.14}$$

is true. This equality can easily obtain with the help of the multiplication law (1.10). Using (4.14) and (4.9), we prove that the form (4.8) is invariant under elements of (4.12). Therefore, elements of G_1 may be represented as 4×4 unitary matrices. Our assertion follows then from the isomorphisms $\mathrm{Spin}(6) \simeq SU(4)$ and $\mathrm{Spin}(5) \simeq Sp(2)$. In addition, we note that

$$H = \{g \in G_1 \mid 1g = 1\} \tag{4.15}$$

is a group that isomorphic to $SU(3)$.

Now, suppose Hg is the right coset of G_1, where H is defined by (4.15). Obviously, the element

$$g_1 = R_{e_2} R_{e_4} \tag{4.16}$$

belong to G_1 but do not belong to G_2. On the other hand, the groups $SU(4)$ and $Sp(2)$ are the double cover of $SO(6)$ and $SO(5)$, respectively. Therefore, G_2 is a maximal subgroup of G_1. Hence, g can represent a product of elements of $G_2 \cup \{g_1\}$. Using (1.10) and (3.27), we prove that

$$R_{e_2} R_{\boldsymbol{a}} = R_{\bar{\boldsymbol{a}}} R_{e_2} \tag{4.17}$$

for all $\boldsymbol{a} \in \mathbb{S}^4$. Since G_2 is generated by (4.13) and $\overline{\boldsymbol{a}} \in \mathbb{S}^4$, it follows that

$$g = (R_{e_2} R_{\boldsymbol{b}})^{\sigma} f, \qquad \sigma \in \{0, 1\}, \tag{4.18}$$

where $\boldsymbol{b} \in \mathbb{S}^4$ and $f \in G_2$. If $\sigma = 0$, then we can choose an element of G_2 as a representative of Hg.

Let $\sigma = 1$. Since the product $R_{\bar{e}_4} R_{\boldsymbol{b}} \in G_2$, it follows that

$$g = g_1 f', \tag{4.19}$$

where $f' \in G_2$. Suppose

$$h = R_{e_5} R_{e_3} R_{e_4} R_{e_2}. \tag{4.20}$$

It follows from (1.10) that $1h = 1$. Hence, h belongs to the subgroup (4.15). Therefore,

$$Hg = Hhg_1 f' = Hf'', \tag{4.21}$$

where again $f'' \in G_2$. Thus, we can choose a representative of Hg in the subgroup G_2. The equality (4.6) is proven for $k = 1$.

B. The case $k = 2$

Obviously, the matrices $R_{ij} = \frac{1}{2}[R_{e_i}, R_{e_j}]$ are independent over $\mathbb{R}$. In addition, it follows from (1.19) that they satisfy the following commutation relations:

$$[R_{ij}, R_{kl}] = \delta_{ik} R_{jl} + \delta_{jl} R_{ik} - \delta_{il} R_{jk} - \delta_{jk} R_{il}. \tag{4.22}$$

Hence, the matrices R_{ij} form a basis of the algebra $A \simeq so(7)$. We consider the transformation

$$R_{e_i} \to J(KR_{e_i}K)J, \tag{4.23}$$

where the matrix J is defined in (3.20) and the matrix

$$K = \mathrm{diag}(1, -1, -1, 1, -1, 1, 1, -1). \tag{4.24}$$

Using the explicit form (3.18) of R_{e_i}, we prove that

$$\begin{aligned} J(KR_{e_i}K)J &= R_{e_i} \quad \text{for } i = 1, 2, 3, 4, \\ J(KR_{e_i}K)J &= -R_{e_i} \quad \text{for } i = 5, 6, 7. \end{aligned} \tag{4.25}$$

Obviously, the transformation (4.23) can be extend to an involutive automorphism of A. We consider the subalgebra $A_1 \subset A$ generated by the elements R_{ij}, where $i, j = 2, \ldots, 7$. It is obvious that $A_1 \simeq so(6)$. With respect to this automorphism the algebra A_1 can be decomposable into the direct sum (1.27) of the proper subspaces A_1^+ and A_1^-. It follows from (4.25) that $A_1^+ \simeq so(3) \oplus so(3)$. Since $so(6) \simeq su(4)$ and $so(3) \simeq su(2)$, it follows that the corresponding symmetric space is isomorphic to $SU(4)/H_2$.

We extend the involutive automorphism of A defined by (4.23) to an automorphism $\tilde{\sigma}$ of the corresponding simply connected Lie group Spin(7). We suppose that this group is embedded into the Clifford algebra $\mathrm{Cl}_{0,7}(\mathbb{R})$. Since the restriction of the homomorphism (1.20) to Spin(7) is injection, $\tilde{\sigma}$ induces an involutive automorphism σ of $G \in \mathrm{Aut}\mathbb{O}$. It is obvious that $G \simeq \mathrm{Spin}(7)$. On the other hand, for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{S}^6$ the product

$$R_{\boldsymbol{a}} R_{\boldsymbol{b}} = -R_{(\boldsymbol{a},\boldsymbol{b})} + \frac{1}{2}[R_{\boldsymbol{a}}, R_{\boldsymbol{b}}]. \tag{4.26}$$

Using (1.25), we prove that the automorphism σ of G is defined by (4.25). Obviously, the restrictions of σ to G_1 and G_2 can be also defined by (4.25).

Now, suppose H is a subgroup of G_1 invariant under the automorphism σ, and Hg is a right coset of G_1. As in the arguments above, we represent g in the form (4.18). If $\sigma = 0$, then we can choose an element of G_2 as a representative of Hg. If $\sigma = 1$, then g has the form (4.19). But it follows from (4.25) that the element (4.16) is invariant under the automorphism σ. Therefore, it belongs to H. Hence, we can choose a representative of Hg in the subgroup G_2. Since the groups H and H_2 are isomorphic, it follows that the equality (4.6) is proved for $k = 2$. This completes the proof of theorem 1.

VI. CLASSIFICATION OF BPS EQUATIONS

We have proven that to within equivalence there exists a unique system of BPS equations for every pair of values $d \leq 8$ and $\nu = \nu(d)$. In this section, we find all such systems of equations. However, we present first a general method allowing to obtain the systems of BPS equations.

Let V be a space of irreducible spinor representation of Spin(d) and $\Omega_1, \ldots, \Omega_{2^s}$: $V \rightarrow V$ be a finite set of linear operators satisfying the conditions

$$\sum_{\alpha=1}^{2^s} \Omega_\alpha = 1, \qquad \Omega_\alpha \Omega_\beta = \delta_{\alpha\beta} \Omega_\beta. \tag{5.1}$$

We say that the operators $\Omega_1, \ldots, \Omega_{2^s}$ make up a total orthogonal system of idempotent operators and the corresponding matrices make up a total orthogonal system of idempotent matrices. Obviously, every such operator is a projector onto a subspace in V. Moreover, with respect to this system of projectors the space V decomposes into the direct sum

$$V = V_1 \oplus \ldots \oplus V_{2^s} \tag{5.2}$$

of the subspaces $V_\alpha = \mathrm{Im}\Omega_\alpha$. The idempotent Ω_α is called primitive if it is not a sum of two nonzero mutually orthogonal idempotents. It is obvious that any projector is a sum of mutually orthogonal idempotents. Finally, if every idempotent in (5.1) is primitive, then we have a total orthogonal system of primitive idempotents.

Since irreducible spinor representations of Spin(d) are realized in the algebra $\mathrm{Cl}_{0,d-1}(\mathbb{R})$, we will find a total orthogonal system of primitive idempotents in this algebra. To this end, we choose a subset of monomials $E_1, \ldots, E_s$ in (1.4) such that

$$E_i^2 = 1, \qquad [E_i, E_j] = 0. \tag{5.3}$$

Further, we impose the condition (1.3) on the gamma matrices and define the 2^s matrices

$$\Omega[\alpha_1, \ldots, \alpha_s] = \frac{1}{2^s} \prod_{i=1}^{s} (1 + \alpha_i E_i), \tag{5.4}$$

where $\alpha_i = \pm 1$. It is easily shown that these matrices are Hermitian and satisfy the equalities (5.1). Since such notations of matrices are few inconveniently, we introduce new notations. To this end, we denote the matrices (5.4) by

$$\begin{aligned} \Omega_1 &= \Omega[1, \ldots, 1], \\ \Omega_2 &= \Omega[1, \ldots, -1], \quad \ldots, \\ \Omega_{2^s} &= \Omega[-1, \ldots, -1]. \end{aligned} \tag{5.5}$$

Notice that this way of ranking is used in the binary number system. Besides, we suppose that

$$\Omega_{\alpha_1 \ldots \alpha_r} = \sum_{i=1}^{r} \Omega_{\alpha_i}. \tag{5.6}$$

Further, with respect to the system of orthogonal idempotents (5.4) the algebra $\mathrm{Cl}_{0,d-1}(\mathbb{R})$ decomposes into the direct sum

$$\mathrm{Cl}_{0,d-1}(\mathbb{R}) = I_1 + \ldots + I_{2^s} \tag{5.7}$$

of left ideals $I_\alpha = \mathrm{Cl}_{0,d-1}(\mathbb{R})\Omega_\alpha$. And also, the idempotent Ω_α is primitive if and only if the left ideal I_α is minimal. It follows from Table I that all minimal left ideals of $\mathrm{Cl}_{0,d-1}(\mathbb{R})$ are isomorphic. Obviously, dimensions of minimal left ideals in $\mathrm{Cl}_{0,d-1}(\mathbb{R})$ and irreducible spinor representations of Spin(d) coincide. Let this dimension over $\mathbb{R}$ be 2^p. Then the quantity of mutually orthogonal primitive idempotents is 2^{d-p}. Hence, $\mathrm{Cl}_{0,d-1}(\mathbb{R})$ contains always $s = d - p$ monomials $E_1, \ldots, E_s$ satisfying the conditions (5.3).

After we find the primitive idempotent (5.4) [or monomials E_i] in $\mathrm{Cl}_{0,d-1}(\mathbb{R})$, we must find its isomorphic images in $\mathrm{Cl}_{d,0}(\mathbb{R})$. We can easily do it if we write the isomorphism

$$\mathrm{Cl}_{0,d-1}(\mathbb{R}) \rightarrow \mathrm{Cl}^0_{d,0}(\mathbb{R}) \tag{5.8}$$

in the explicit form

$$\Gamma_{a_1 \ldots a_k} \rightarrow \begin{cases} (\Gamma_{a_1 \ldots a_k})^\dagger & \text{for even } k, \\ (\Gamma_{a_1 \ldots a_k} \Gamma_d)^\dagger & \text{for odd } k. \end{cases} \tag{5.9}$$

Having the total orthogonal system of primitive idempotents in $\mathrm{Cl}^0_{d,0}(\mathbb{R})$, we easy find the BPS equations from (2.5). Note that the fraction ν of the unbroken supersymmetry can be found as

$$\nu = \frac{\dim I}{\dim \mathrm{Cl}_{0,d-1}(\mathbb{R})}, \tag{5.10}$$

where I is a left ideal of $\mathrm{Cl}_{0,d-1}(\mathbb{R})$ corresponding to the idempotent Ω. The dimension of I can be found in Table I. Now, we will construct BPS equations in the concrete dimensions.

A. The dimension $d \leq 3$

In these dimensions, the algebra $\mathrm{Cl}_{0,d-1}(\mathbb{R})$ is a division algebra. Therefore, any its left ideal is either trivial or coinciding with $Cl_{0,d-1}(\mathbb{R})$. It follows that the idempotent $\Omega = 0$ or 1. Thus, any system of BPS equations has only the trivial solution $F_{ab} = 0$.

B. Four dimensions

The algebra $Cl_{0,3}(\mathbb{R})$ decomposes into the direct sum of two minimal left ideals. Using the decomposition (5.7), we find $s = 1$. Further, we choose the monomial $E_1 = \Gamma_{123}$ in $Cl_{0,3}(\mathbb{R})$. Obviously, the square $E_1^2 = 1$. Using the mapping (5.9), we find the image of E_1 in $Cl^0_{4,0}(\mathbb{R})$ and next construct the total orthogonal system of primitive idempotents

$$\Omega_\alpha = \tfrac{1}{2}(1 \pm \Gamma_{1234}), \qquad (5.11)$$

where $\alpha = 1, 2$. Substituting Ω_1 in Eq. (2.5), we get the BPS equations

$$F_{ab} = \tfrac{1}{2}\varepsilon_{abcd}F_{cd}, \qquad (5.12)$$

where ε_{abcd} is the completely antisymmetric identity four tensor. Using (5.10), we find $\nu = 1/2$. Note that we consider the chiral representation.

C. Five dimensions

The algebra $Cl_{0,4}(\mathbb{R})$ also decomposes into the direct sum of two minimal left ideals. Hence, $s = 1$. We choose the monomial $E_1 = \Gamma_{1234}$ in $Cl_{0,5}(\mathbb{R})$, find its image in $Cl^0_{5,0}(\mathbb{R})$, and construct the total orthogonal system of primitive idempotents. Obviously, this system coincides with (5.11). Substituting Ω_1 in (2.5), we get the BPS equations

$$F_{ab} = \tfrac{1}{2}\varepsilon_{abcd}F_{cd}, \qquad F_{a5} = 0. \qquad (5.13)$$

It is obvious that the fraction of the unbroken supersymmetry $\nu = 1/2$.

D. Six dimensions

The algebra $Cl_{0,5}(\mathbb{R})$ decomposes into the direct sum of four minimal left ideals. In this case, $\nu = 1/4$ and $s = 2$. We choose the monomials $E_1 = \Gamma_{125}$ and $E_2 = \Gamma_{345}$ in this algebra. Obviously, they satisfy the conditions (5.3). Using (5.9), we find images of these monomials in $Cl^0_{6,0}(\mathbb{R})$ and construct the total orthogonal system of primitive idempotents

$$\Omega_\alpha = \tfrac{1}{4}(1 \pm \Gamma_{1234})(1 \pm \Gamma_{1256}). \qquad (5.14)$$

Substituting Ω_1 in Eq. (2.5), we get the following BPS equations:

$$\begin{gathered} F_{12} + F_{43} + F_{65} = 0, \qquad F_{13} + F_{24} = 0, \\ F_{16} + F_{52} = 0, \qquad F_{14} + F_{32} = 0, \qquad F_{35} + F_{64} = 0, \\ F_{15} + F_{26} = 0, \qquad F_{36} + F_{45} = 0. \end{gathered} \qquad (5.15)$$

Now, we consider the sum Ω_{12} of two primitive idempotents Ω_1 and Ω_2

$$\Omega_{12} = \tfrac{1}{2}(1 + \Gamma_{1234}). \qquad (5.16)$$

Obviously, the prototype of Ω_{12} in $Cl_{0,5}(\mathbb{R})$ is the identity of a left ideal I. Since $\dim I = 16$, it follows that $\nu = 2/4$. Substituting (5.16) in (2.5), we find the BPS equations

$$F_{ab} = \tfrac{1}{2}\varepsilon_{abcd}F_{cd}, \qquad F_{a5} = F_{a6} = 0. \qquad (5.17)$$

If we calculate the sum Ω_{123} of three primitive idempotents of the form (5.14) and substitute it to (2.5), then we get the system of BPS equations having only trivial solution. The alternative way to get this system is the following. We find systems of the form (5.15) for every Ω_α ($\alpha = 1, 2, 3$). Such systems are called primitive. Then the system corresponding to Ω_{123} is a system joining the systems for every Ω_α. It can be easily be checked that this joined system has only trivial solution.

E. Seven dimensions

The algebra $Cl_{0,6}(\mathbb{R})$ decomposes into the direct sum of eight minimal left ideals. In this case, $\nu = 1/8$ and $s = 3$. We choose the monomials $E_1 = \Gamma_{1234}$, $E_2 = \Gamma_{1256}$, and $E_3 = \Gamma_{164}$ in $Cl_{0,6}(\mathbb{R})$. Using (5.9), we find images of the monomials in $Cl^0_{7,0}(\mathbb{R})$ and construct the total orthogonal system of primitive idempotents

$$\Omega_\alpha = \tfrac{1}{8}(1 \pm \Gamma_{1234})(1 \pm \Gamma_{1256})(1 \pm \Gamma_{1476}). \qquad (5.18)$$

Substituting Ω_1 in (2.5), we get the following BPS equations

$$\begin{aligned} &F_{12} + F_{43} + F_{65} = 0, \qquad F_{13} + F_{24} + F_{75} = 0, \\ &F_{16} + F_{52} + F_{74} = 0, \qquad F_{14} + F_{32} + F_{67} = 0, \\ &F_{17} + F_{53} + F_{46} = 0, \qquad F_{15} + F_{26} + F_{37} = 0, \\ &F_{27} + F_{54} + F_{63} = 0. \end{aligned} \qquad (5.19)$$

Further, we consider the sum Ω_{12} of two primitive idempotents Ω_1 and Ω_2

$$\Omega_{12} = \tfrac{1}{4}(1 + \Gamma_{1234})(1 + \Gamma_{1256}). \qquad (5.20)$$

The dimension of left ideal corresponding to Ω_{12} is 16. Therefore, $\nu = 2/8$. The corresponding system of BPS equations has the form

$$\begin{aligned} &F_{12} + F_{43} + F_{65} = 0, \qquad F_{a7} = 0, \qquad F_{13} + F_{24} = 0, \\ &F_{16} + F_{52} = 0, \qquad F_{14} + F_{32} = 0, \qquad F_{35} + F_{64} = 0, \\ &F_{15} + F_{26} = 0, \qquad F_{36} + F_{45} = 0. \end{aligned} \qquad (5.21)$$

Now, we find the BPS equations corresponding to Ω_{123}. To this end, we write BPS equations for

$$\Omega_3 = \tfrac{1}{8}(1 + \Gamma_{1234})(1 - \Gamma_{1256})(1 + \Gamma_{1476}) \qquad (5.22)$$

and join them with the system (5.21). As result, we get the following BPS equations:

$$F_{ab} = \tfrac{1}{2}\varepsilon_{abcd}F_{cd}, \qquad F_{a5} = F_{a6} = F_{a7} = 0. \qquad (5.23)$$

Since the dimension of the corresponding left ideal is 24, it follows that $\nu = 3/8$. It can be easily be checked that the system of BPS equations constructed by means of four primitive idempotents has only trivial solution.

F. Eight dimensions

The algebra $Cl_{0,7}(\mathbb{R})$ decomposes into the direct sum of 16 minimal left ideals. Hence, $s = 4$. We choose in $Cl_{0,6}(\mathbb{R})$ the monomials $E_1 = \Gamma_{1234}$, $E_2 = \Gamma_{1256}$, $E_3 = \Gamma_{1476}$, and the monomial $E_4 = \Gamma_*$ defined in (2.14). We find its images in $Cl^0_{8,0}(\mathbb{R})$ and construct the total orthogonal system of primitive idempotents

$$\Omega_\alpha = \tfrac{1}{16}(1 \pm \Gamma_*)(1 \pm \Gamma_{1234})(1 \pm \Gamma_{1256})(1 \pm \Gamma_{1476}). \tag{5.24}$$

Obviously, we can find the BPS systems by the method that was used above. However, all these systems had been found in [24]. Therefore, we simply list them.

(1) $\nu = 1/16$, $\Omega = \Omega_1$

$$\begin{aligned} &F_{12} + F_{43} + F_{65} + F_{78} = 0,\\ &F_{13} + F_{24} + F_{75} + F_{86} = 0,\\ &F_{14} + F_{32} + F_{67} + F_{85} = 0,\\ &F_{15} + F_{26} + F_{37} + F_{48} = 0,\\ &F_{16} + F_{52} + F_{74} + F_{38} = 0,\\ &F_{17} + F_{53} + F_{46} + F_{82} = 0,\\ &F_{18} + F_{27} + F_{54} + F_{63} = 0. \end{aligned} \tag{5.25}$$

(2) $\nu = 2/16$, $\Omega = \Omega_{12}$

$$\begin{aligned} &F_{12} + F_{43} + F_{65} + F_{78} = 0,\\ &F_{13} + F_{24} = 0, \qquad F_{17} + F_{82} = 0,\\ &F_{37} + F_{48} = 0, \qquad F_{14} + F_{32} = 0,\\ &F_{18} + F_{27} = 0, \qquad F_{38} + F_{74} = 0,\\ &F_{15} + F_{26} = 0, \qquad F_{75} + F_{86} = 0,\\ &F_{46} + F_{53} = 0, \qquad F_{16} + F_{52} = 0,\\ &F_{67} + F_{85} = 0, \qquad F_{54} + F_{63} = 0. \end{aligned} \tag{5.26}$$

(3) $\nu = 3/16$, $\Omega = \Omega_{123}$

$$\begin{aligned} &F_{12} + F_{43} = 0, \qquad F_{13} + F_{24} = 0,\\ &F_{14} + F_{32} = 0, \qquad F_{56} + F_{87} = 0,\\ &F_{57} + F_{68} = 0, \qquad F_{58} + F_{76} = 0,\\ &F_{15} = F_{37} = F_{62} = F_{84},\\ &F_{16} = F_{25} = F_{38} = F_{47},\\ &F_{17} = F_{28} = F_{53} = F_{64},\\ &F_{18} = F_{45} = F_{63} = F_{72}. \end{aligned} \tag{5.27}$$

(4) $\nu = 4/16$, $\Omega = \Omega_{1234}$

$$\begin{aligned} &F_{12} + F_{43} = 0, \qquad F_{13} + F_{24} = 0,\\ &F_{14} + F_{32} = 0, \qquad F_{56} + F_{87} = 0,\\ &F_{57} + F_{68} = 0, \qquad F_{58} + F_{76} = 0,\\ &\quad F_{ab} = 0\\ &a \in \{1, 2, 3, 4\}, \qquad b \in \{5, 6, 7, 8\}. \end{aligned} \tag{5.28}$$

(5) $\nu = 5/16$, $\Omega = \Omega_{12\,345}$

$$\begin{aligned} &F_{12} = F_{34} = F_{56} = F_{78},\\ &F_{13} = F_{42} = F_{68} = F_{75},\\ &F_{14} = F_{23} = F_{76} = F_{85}, \qquad F_{ab} = 0\\ &a \in \{1, 2, 3, 4\}, \qquad b \in \{5, 6, 7, 8\}. \end{aligned} \tag{5.29}$$

(6) $\nu = 6/16$, $\Omega = \Omega_{123\,456}$

$$F_{12} = F_{34} = F_{56} = F_{78}, \tag{5.30}$$

and other components are zero. The system of BPS equations constructed by means of seven primitive idempotents of the form (5.24) has only trivial solution.

VII. DISCUSSIONS AND COMMENTS

In this paper, we systematically classified all possible BPS equations in Euclidean dimension $d \leq 8$ and presented a general method allowing to obtain the BPS equations in any dimension. In this section, we discuss symmetries of BPS equations and their connection with the self-dual Yang-Mills equations. Further, we find all BPS equations in the Minkowski space of dimension $d \leq 6$. In addition, we apply the obtained results to the supersymmetric Yang-Mills theories and to the low-energy effective theory of the heterotic string.

A. Symmetries of BPS equations

First, we consider a connection between BPS states and instantons in the Euclidean Yang-Mills theory. We note that the primitive system (5.25) can be rewritten in the form

$$F_{ab} = \tfrac{1}{2} f_{abcd} F_{cd}, \tag{6.1}$$

where f_{abcd} is a completely antisymmetric tensor with the following nonzero components:

$$\begin{aligned} f_{1234} &= f_{1256} = f_{1357} = f_{1476} = f_{2367} \\ &= f_{2457} = f_{3465} = 1, \\ f_{5678} &= f_{3476} = f_{2468} = f_{3258} = f_{1458} \\ &= f_{1368} = f_{1728} = 1. \end{aligned} \tag{6.2}$$

Let $d < 8$. Suppose that the components (6.2) with the indices $i > d$ equal to zero. Then we get the primitive system of BPS equations in dimension d. Obviously, this system has the form (6.1). Since any system of BPS equations is a system joining primitive systems, it also has the form (6.1). Thus, any BPS equation in Euclidean space of dimension $d \leq 8$ is equivalent to a self-dual Yang-Mills equation. It follows that any solution of BPS equations in the Euclidean super Yang-Mills theory in this dimension is an instanton solution.

We consider symmetries of the BPS equations. In Euclidean dimension $d \leq 8$, the group G of symmetries of BPS equations is a subgroup of $SO(8)$. On the other hand, the corresponding projection operator Ω is invariant under this subgroup. Using the canonical form of Ω, we easily find the group G. We list all such groups in Table II.

Note that these groups were first interpreted as groups of symmetries of the self-dual Yang-Mills equations in [1,2]. In the same place, an example of self-dual equations that differ from the BPS equations was found. These equations can be obtained if we deduce the equality of each term in each row of (5.25), i.e. $F_{12} = F_{43} = F_{65} = F_{78}$, etc., a set of 21 equations. It follows that a solution of the self-dual Yang-Mills equations is not necessarily a solution of BPS equations in the Euclidean super Yang-Mills theory.

Let us discuss a possibility of generalization of theorem 1. First note that we can consider arbitrary spinor representations of Spin(d). Then new systems of BPS equations appear in eight dimensions. We can easily construct such a system using the sum $\Omega_+ + \Omega_-$ of the idempotents

$$\Omega_\pm = \tfrac{1}{16}(1 \pm \Gamma_*)(1 + \Gamma_{1234})(1 + \Gamma_{1256})(1 + \Gamma_{1476}). \tag{6.3}$$

Obviously, it is a system joining the system (5.19) with the conditions $F_{a8} = 0$. Conversely, new BPS equations do not appear in dimension $d < 8$. This assertion is obvious for odd d, because any spinor representation of Spin(d) is irreducible in such a dimension. In order to prove this assertion for even $d < 8$, we use [24]. In this work, all BPS equations in even dimension $d \leq 8$ were found. And also in Euclidean dimension $d < 8$, arbitrary spinor representations of Spin(d) were considered. It was proven that only trivial BPS equations are in two dimensions. In four dimensions, the chiral BPS Eq. (5.12) and its antichiral analog were found. In six dimension, it was proven that BPS equations either have the form

$$\begin{aligned} F_{12} + \alpha_2 F_{34} + \alpha_1 F_{56} &= 0, & F_{13} + \alpha_2 F_{42} &= 0, \\ F_{14} + \alpha_2 F_{23} &= 0, & F_{15} + \alpha_1 F_{62} &= 0, \\ F_{16} + \alpha_1 F_{25} &= 0, & F_{35} + \alpha_1\alpha_2 F_{64} &= 0, \\ F_{36} + \alpha_1\alpha_2 F_{45} &= 0, \end{aligned} \tag{6.4}$$

where α_1, α_2 are two independent signs ± 1, or are a corollary of (6.4). The problem of equivalence is not being considered in this work. Nevertheless, we can prove that the four systems (6.4) defined by the choice of values of α_1 and α_2 are equivalent. Indeed, the permutation $(13) \times (24), (15)(26), (35)(46)$ of indices of F_{ab} leave invariant two systems and transpose the other two with each other. In turn, such transformations of BPS equations can be obtained by the transformations (2.8). Since the considered supersymmetry is global, it follows that these four systems of BPS equations are equivalent. It is obvious also that they are equivalent to the system (5.15). It is sufficient to put $\alpha_1 = \alpha_2 = 1$ in (6.4) and then use the permutation $(34) \times (56)$ of indices of F_{ab}. Thus, theorem 1 is true for any spinor representations of Spin(d) in dimension $d \leq 6$. Also, we prove that to within equivalence all BPS equations found in [24] are self-dual Yang-Mills equations.

B. BPS equations in the Minkowski spaces

The second possibility of generalization of theorem 1 is connected with an investigation of BPS equations in the Minkowski space. These equations also may be obtained by a dimensional reduction of the $D = 10$ $N = 1$ super Yang-Mills theory. Note that the method used above may be applied in this case. In particular, all constructions of Sec. III are remain true if we are restricted to the dimension $d < 8$. It is clear that we must correctly place the tensor indices in the text and also use the groups Spin$(d-1, 1)$, $SO(d-1, 1)$, and the anti-Hermitian matrices $i\Gamma_d$ instead of the groups Spin(d), $SO(d)$, and the Hermitian matrices Γ_d. The following weakened analog of theorem 1 is true.

Theorem 2. In the Minkowski space of dimension $d \leq 6$, there exists unique to within equivalence nontrivial system of BPS equations connected with constant chiral spinor.

Indeed, the matrix $\Omega_+ = 0$ or 1 in dimension $d = 1 + 1$. We consider dimension $d = 3 + 1$. Since the group Spin(3, 1) is isomorphic to $\mathrm{Sl}(2, \mathbb{C})$, Ω_+ is an Hermitian 2×2 matrix. It is obvious that this matrix can be reduced to the canonic form by conjugations of $\mathrm{Sl}(2, \mathbb{C})$. Now, we consider dimension $d = 5 + 1$. The group Spin(5, 1) is isomorphic to $SU^*(4)$. Hence, Ω_+ is an Hermitian 4×4 matrix. On the other hand, it was shown in Sec. V that this

TABLE II. Groups of symmetries of BPS equations.

$d = 4, 5$	$\nu = 1/2$			
	$SO(4)$			
$d = 6$	$\nu = 1/4$	$2/4$		
	$SU(3) \times U(1)/Z_3$	$SO(4) \times SO(2)$		
$d = 7$	$\nu = 1/8$	$2/8$	$3/8$	
	G_2	$SU(3) \times U(1)/Z_3$	$SO(4) \times SO(3)$	
$d = 8$	$\nu = 1/16$	$2/16, 6/16$	$3/16, 5/16$	$4/16$
	Spin(7)	$SU(4) \times U(1)/Z_4$	$Sp(2) \times SU(2)/Z_2$	$SO(4) \times SO(4)$

matrix can be reduced to the canonic form by conjugations of $Sp(2)$. Since $Sp(2) \subset SU^*(4)$, it follows that this reduction is possible in the considered case. Thus, there exists unique to within equivalence nontrivial system of BPS equations for any pair of values $d \leq 6$ and $\nu = \nu(d)$.

Now, we will construct these systems. It is obvious that in dimension $d = 1 + 1$, we have only the vacuum $F_{ab} = 0$. We consider dimension $d = 3 + 1$. If follows from (1.6) and Table I that

$$\mathrm{Cl}^0_{3,1}(\mathbb{R}) \simeq \mathrm{Cl}_{3,0}(\mathbb{R}) \simeq \mathbb{C}(2). \tag{6.5}$$

Therefore, the subalgebra $\mathrm{Cl}^0_{3,1}(\mathbb{R})$ decomposes into the direct sum of two minimal left ideals. We construct the total orthogonal system of primitive idempotents

$$\Omega_\alpha = \tfrac{1}{2}(1 \pm \Gamma_{14}). \tag{6.6}$$

It follows easily that the corresponding system of BPS equations has only the trivial solution $F_{ab} = 0$. We consider dimension $d = 5 + 1$. Since

$$\mathrm{Cl}^0_{5,1}(\mathbb{R}) \simeq \mathrm{Cl}_{5,0}(\mathbb{R}) \simeq \mathbb{H}(2) \oplus \mathbb{H}(2), \tag{6.7}$$

it follows that the subalgebra $\mathrm{Cl}^0_{5,1}(\mathbb{R})$ decomposes into the direct sum of four minimal left ideals. We construct the total orthogonal system of primitive idempotents

$$\Omega_\alpha = \tfrac{1}{4}(1 \pm \Gamma_{123456})(1 \pm \Gamma_{1234}). \tag{6.8}$$

Substituting Ω_1 in the Eq. (2.5), we get the BPS equations

$$F_{ab} = \tfrac{1}{2}\varepsilon_{abcd}F_{cd}, \qquad F_{a5} = -F_{a6}. \tag{6.9}$$

Conversely, the system of BPS equations constructed with the help of the idempotent

$$\Omega_{12} = \tfrac{1}{2}(1 + \Gamma_{123456}) \tag{6.10}$$

has only trivial solution. Hence, any nontrivial system of BPS equations in dimension $d = 5 + 1$ defined by the chiral representation of Spin(5, 1) is equivalent to the system (6.9). The theorem is proved.

C. BPS states in the supersymmetric Yang-Mills theories

Now we apply the obtained above results to the supersymmetric Yang-Mills theories. First, we note that for each choice of the infinitesimal supersymmetry parameter ε, there is a corresponding conserved supercharge Q. Out of this infinity of conserved supercharges, we wish to identify those that generate unbroken supersymmetries. An unbroken supersymmetry Q is simply a conserved supercharge that annihilates the vacuum state $|\Phi\rangle$. Saying that Q annihilates $|\Phi\rangle$ is equivalent to saying that for all operators U, $\langle\Phi|\{Q, U\}|\Phi\rangle = 0$. This will be so U is a bosonic operator, since then $\{Q, U\}$ is fermionic, so the real issue is whether $\langle\Phi|\{Q, U\}|\Phi\rangle$ vanishes when U is a fermionic operator. Now, when U is fermionic $\{Q, U\}$ is simply δU, the variation of U under the supersymmetry transformation generated by Q. Also, in the classical limit, δU and $\langle\Phi|\delta U|\Phi\rangle$ coincide. So finding an unbroken supersymmetry at tree level means finding a supersymmetry transformation such that $\delta U = 0$ for every fermionic field U. Also, in the classical limit, it is enough to check this for elementary fermion fields.

Further, the open superstring theory can be approximated at low energy by a supersymmetric Yang-Mills theory. Such theories are described by an action of the form

$$S = \int d^D x\Big(-\frac{1}{4}F^2 + \frac{i}{2}\bar{\psi}\Gamma \cdot D\psi\Big). \tag{6.11}$$

The supersymmetry transformations that leave (6.11) invariant are

$$\delta A_\mu = \frac{i}{2}\bar{\varepsilon}\Gamma_\mu\psi, \tag{6.12}$$

$$\delta\psi = -\tfrac{1}{4}F_{\mu\nu}\Gamma^{\mu\nu}\varepsilon, \tag{6.13}$$

where ε is a constant anticommuting spinor. It is well known that the supersymmetric Yang-Mills theory exists only in the $D = 3, 4, 6$, and 10. Using theorem 2, we prove that the condition $\delta\psi = 0$ in $D \leq 6$ is true only if either $F_{\mu\nu}$ is a solution of (6.9) or $F_{\mu\nu} = 0$.

We consider the dimension $D = 10$. The Majorana-Weyl spinor ψ in $D = 10$ has 16 real components. On shell these components must still satisfy the Dirac equation that relates eight of them to the other eight. Therefore, if the values of $F_{\mu\nu}$ are arbitrary, then it follows from (6.13) that only eight components of ε are independent. We choose an orthonormal basis in $\mathrm{Ker}(F_{\mu\nu}\Gamma^{\mu\nu}) \subset V$ and extend it to the spinor space V so that only eight components of ε are not zero. Then the condition $\delta\psi = 0$ requires that the projector Ω in (2.5) have the block diagonal form (2.16). It is obvious that it can be reduced to the canonic form by transformations from $SO(8) \subset SO(9,1)$. Hence, for every value of ν, there exists unique to within equivalence nontrivial system of BPS equations. In order to find these systems we construct the total orthogonal system of primitive idempotents

$$\Omega_\alpha = \tfrac{1}{32}(1 \pm \Gamma_*)(1 \pm \Gamma_{12\,345\,678})(1 \pm \Gamma_{1234}) \times (1 \pm \Gamma_{1256})(1 \pm \Gamma_{1476}) \tag{6.14}$$

of the Clifford algebra $\mathrm{Cl}^0_{9,1}(\mathbb{R})$. Substituting Ω_1 in Eq. (2.5), we get the following BPS equations

$$F_{ab} = \tfrac{1}{2} f_{abcd} F_{cd}, \qquad F_{a9} + F_{a10} = F_{910} = 0, \tag{6.15}$$

where f_{abcd} is a completely antisymmetric tensor with the components (6.2). Obviously, $\nu = 2/32$. The systems of BPS equations for other values of ν can be obtained by the method in Sec. VI. Thus, nontrivial state of unbroken supersymmetry in the supersymmetric Yang-Mills theory exist only for $D = 6$ and 10. Also, in dimension $D = 6$, such state is an instanton solutions of (6.9). In dimension $d = 10$, such states are either solutions of the system (6.15) or solutions of the BPS equations in eight dimensions adding the conditions $F_{a9} = F_{a10} = F_{910} = 0$.

D. Heterotic string solitons

In conclusion, we discuss the possibility of using the results obtained above to construct soliton solutions of the low-energy effective theory of the heterotic string. For the heterotic string, the low-energy effective action is identical to the $D = 10$, $N = 1$ supergravity and super Yang-Mills action. The bosonic part of this action reads

$$S = \frac{1}{2k^2}\int d^{10}x\sqrt{-g}e^{-2\phi} \times \left(R + 4(\nabla\phi)^2 - \frac{1}{3}H^2 - \frac{\alpha'}{30}\,\mathrm{Tr}F^2\right). \tag{6.16}$$

We are interested in solutions that preserve at least one supersymmetry. This requires that in 10 dimensions there exist at least one Majorana-Weyl spinor ε such that the supersymmetry variations of the fermionic fields vanish for such solutions

$$\delta\chi = F_{MN}\Gamma^{MN}\varepsilon, \tag{6.17}$$

$$\delta\lambda = \left(\Gamma^M \partial_M \phi - \frac{1}{6} H_{MNP}\Gamma^{MNP}\right)\varepsilon, \tag{6.18}$$

$$\delta\psi_M = \left(\partial_M + \frac{1}{4}\Omega_M^{AB}\Gamma_{AB}\right)\varepsilon. \tag{6.19}$$

Here, ϕ is the dilaton field, F_{MN} is the Yang-Mil1s field strength, and H is the gauge-invariant field strength of the antisymmetric tensor field B_{MN}. While we can arbitrarily specify the space-time metric and the dilaton field ϕ in trying to obey

$$\delta\chi = \delta\lambda = \delta\psi_M = 0, \tag{6.20}$$

we cannot arbitrarily specify F or H; they must obey certain Bianchi identities. In the string theory these identities have the form

$$dH = \alpha'(\mathrm{tr}R \wedge R - \tfrac{1}{30}\mathrm{Tr}F \wedge F). \tag{6.21}$$

Note that the connection Ω_M in (6.19) is a non-Riemannian. It is related to the usual spin connection ω by

$$\Omega_M^{AB} = \omega_M^{AB} - H_M^{AB}. \tag{6.22}$$

The analysis of (6.17), (6.18), and (6.19) is rather complicated in general, and so we simplify the discussion by assuming at the outset that the Majorana-Weyl spinor ε is constant. Further, we suppose that a subgroup G of $SO(9,1)$ is a group of symmetries of BPS equations, and we choose ε to be a G singlet of the Majorana-Weyl spinor. Then, for suitable G, there exists a completely antisymmetric tensor f_{abcd} such that the ansatz

$$g_{ab} = e^{\phi}\delta_{ab}, \qquad H_{abc} = \lambda f_{abcd}\partial^d\phi, \tag{6.23}$$

solves the supersymmetry equations with zero background fermi fields provided the Yang-Mills gauge fields satisfies the BPS equations. Such solutions were found in the works [14–23]. The obtained above classification of BPS equations in the Euclidean and Minkowski spaces permits to describe all such solutions at least with ansatz (6.23). It is interesting that at present, states of unbroken supersymmetry are very nearly the only examples known of compactified solutions of the equations; the other known examples are related in comparatively simple ways to states of unbroken supersymmetry.

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