Particlelike solutions to classical noncommutative gauge theory

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We construct perturbative static solutions to the classical field equations of noncommutative $U(1)$ gauge theory for the three cases: (a) space-time noncommutativity, (b) space-space noncommutativity, and (c) both (a) and (b). The solutions tend to the Coulomb solution at spatial infinity and are valid for intermediate values of the radial coordinate r. They yield a self-charge inside a sphere of radius r centered about the origin which increases with decreasing r for case (a), and decreases with decreasing r for case (b). For case (a) this may mean that the exact solution screens an infinite charge at the origin, while for case (b) it is plausible that the charge density is well behaved at the origin, as happens in Born-Infeld electrodynamics. For both cases (a) and (b) the self-energy in the intermediate region grows faster as r tends to the origin than that of the Coulomb solution. It then appears that the divergence of the classical self-energy is more severe in the noncommutative theory than it is in the corresponding commutative theory. We compute the lowest order effects of these solutions on the hydrogen atom spectrum and use them to put experimental bounds on the space-time and space-space noncommutative scales. For the former we get a significant improvement over previous bounds. We find that cases (a) and (b) have different experimental signatures.

I. INTRODUCTION

Recent quantum field theory investigations of noncommutative gauge theories have not been straightforward due to UV/IR mixing and related problems of renormalization. (For reviews, see [1,2].) On the other hand, important progress has been made in understanding the renormalization of other noncommutative field theories, in particular, the noncommutative ϕ^4 model [3]. There has also been a very recent attempt to apply similar methods to noncommutative gauge theories [4]. Concerning UV-IR mixing, it is absent from some theories upon using an approach based on the twisted action of the Poincare group [5,6], and it can be understood for noncommutative gauge theories in terms of an induced gravity action [7,8]. Nevertheless, the renormalization of noncommutative gauge theories is yet to be fully understood.

In light of the difficulties with the quantum field theory it may be useful to have a closer look at the classical field theory, and more specifically, for some behavior which may provide clues to these difficulties. Classical aspects of noncommutative gauge theories, and, in particular, their solutions, have been of recent interest. There has been much work done on the noncommutative analogues of vortex, monopole, and instanton solutions,[1] as well as the solutions of general relativity.[2]

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[1]For a small sample, see [9–14].
[2]Many different noncommutative deformations of general relativity have been found [15–18]. They have been used to compute corrections to black hole solutions. Approaches have ranged from smearing out the point singularity [19], to perturbatively solving noncommutative analogues of general relativity [20–22]. Corrections to conical singularities have also been computed [23].

Here we shall be concerned with classical aspects of noncommutative $U(1)$ gauge theory. One possible signal of regularization difficulties in the quantum theory could come from the classical self-energy of a charged particle, which may exhibit more singular behavior in the noncommutative gauge theory than appears in the corresponding commutative theory. We are, of course, referring to the infinite self-energy of the Coulomb solution. Thus there is motivation for studying properties of the noncommutative analogues of the Coulomb solution. The behavior of the fields and current density of the noncommutative solutions at the origin is of particular interest. As position eigenstates do not occur in theories with space-space noncommutativity, there can be no intrinsic notion of *points* for such theories. It has then been argued that the point charges of commutative gauge theories become smeared in noncommutative gauge theories [24], but this has not been demonstrated explicitly. It is noteworthy that such a desired behavior is seen in Born-Infeld electrodynamics [25]. The latter deformation of Maxwell theory has particlelike electrostatic solutions (or bions) which are characterized by a finite size. Bions are characterized by a finite self-energy and a smooth charge density at the origin (although a singularity in the electric field does remain at the origin). The Born-Infeld Lagrangian can be given explicitly and a simple expression results for the static solution. Unfortunately, this is not true for the case of noncommutative electrodynamics, or more precisely, for its equivalent description in terms of commutative gauge fields. Here we are referring to the known equivalence of commutative and noncommutative gauge theories expressed through the Seiberg-Witten map [26]. The Seiberg-Witten map has been given order-by-order in an expansion in the noncom-

mutativity parameter θ. This then leads to an order-by-order expansion for the effective Lagrangian expressed in terms of commutative fields. The zeroth order term (in the case of Abelian gauge fields) is the Maxwell action. Consequently, at best, one can obtain an order-by-order expansion for the field equations and any particlelike solutions. The expansion in θ corresponds to an expansion in one over the radial coordinate r for these solutions, and a Coulomb-like behavior is expected as $r \rightarrow \infty$. By going to higher orders in θ one can probe the intermediate region of the solutions. Although the nature of the solutions at the origin remains uncertain in this approach, the behavior of the fields along with the associated self-energy and charge density in intermediate region may provide clues to whether or not a singularity is present at $r = 0$.

In this article we shall construct static solutions to the classical field equations for noncommutative $U(1)$ gauge theory for three cases:

    (a) space-time noncommutativity,
    (b) space-space noncommutativity, and
    (c) both (a) and (b).

The solutions will tend to the Coulomb solution for $r \rightarrow \infty$ and are valid for intermediate values of r. Constant noncommutativity θ is assumed for all three cases. The operator algebra is realized on Minkowski space-time using the standard Groenewald-Moyal star product [27,28]. After applying the inverse Seiberg-Witten map [26], we obtain the commutative gauge fields associated with the solutions. We analyze the self-charge and energy distributions of these solutions for intermediate values of r, and compare the results with Born-Infeld, as well as Maxwell, electrodynamics. In order to see deviations from the self-charge and energy distributions of the Coulomb solution it is necessary to carry out the expansion of the Seiberg-Witten map up to second order in θ.[3] The results appear to confirm the above speculation that the classical self-energy singularity in noncommutative electrodynamics is more severe than that of the commutative theory. For case (b), results from the intermediate region also appear to indicate that the charge density is better behaved at the origin than it is in the commutative theory. On the other hand, the opposite appears to be true for case (a).

We shall also examine the effect on the hydrogen atom spectrum of replacing the Coulomb potential by a space-time noncommutative solution. Here effects are seen at first order in θ. (The first order effects on the spectrum of the space-space noncommutative solution were examined previously in [30] and were used to put experimental bounds on the noncommutative scale.) We shall treat the electron in the standard fashion, i.e., applying the standard (com-

mutative) Schrödinger equation. (It was shown in [31,32] that quantum mechanical spectra are unaffected by replacing the commutative Schrödinger equation with one associated with noncommuting time and space coordinates, provided that the spatial coordinates commute.) Here we can put limits on the space-time noncommutativity parameter. The bound is a significant improvement over previous results. We find that space-space and space-time noncommutative sources have distinct experimental signatures.

The outline of this article is the following: In Sec. II we review the Seiberg-Witten map and give expressions up to second order in θ. We apply it in Sec. III to obtain the second order corrections to commutative $U(1)$ gauge theory Lagrangian and Hamiltonian densities, while the static solutions are given in Sec. IV. In Sec. V we remark on a possible exact expression for the electrostatic Lagrangian for the case (a) of space-time noncommutativity. If the Lagrangian holds to all orders in the noncommutativity parameter, then we can show that the electrostatic fields of the particlelike solution must be singular at the origin. In Sec. VI we apply the lowest order results to the hydrogen atom spectrum. Concluding remarks are made in Sec. VII. We review the Born-Infeld solution in Appendix A, while we give expressions for the nonlinear field equations and the energy density of noncommutative electrodynamics in Appendix B.

II. SEIBERG-WITTEN MAP

As it will be essential for the analysis that follows, we here review the Seiberg-Witten map to second order in the noncommutativity parameter. We specialize to $U(1)$ gauge theory, and denote the commutative potentials by a_μ, with gauge variations $\delta a_\mu = \partial_\mu \lambda$, and field strengths $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. Seiberg and Witten [26] showed it can be mapped to noncommutative $U(1)$ gauge theory expressed in terms of noncommutative potentials A_μ,

$$(a, \lambda) \rightarrow (A = A(a), \Lambda = \Lambda(\lambda, a)), \quad (2.1)$$

with gauge variations

$$\delta A_\mu = \partial_\mu \Lambda - ie[A_\mu, \Lambda]_\star, \quad (2.2)$$

and field strengths

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]_\star, \quad (2.3)$$

where e is the coupling constant and $[,]_\star$ is here defined as the star commutator associated with the Groenewald-Moyal star product [27,28]

$$\star = \exp\left\{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right\}. \quad (2.4)$$

$\theta^{\mu\nu} = -\theta^{\nu\mu}$ are constant matrix elements and $\overleftarrow{\partial}_\mu$ and $\overrightarrow{\partial}_\mu$ are left and right derivatives, respectively, with respect to coordinates x^μ on some manifold M. Thus for two function

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[3]It was shown in [29] using a different approach that the Coulomb nature of the potential is preserved in noncommutative electrodynamics if one only examines first order effects.

f and g on M, $[f, g]_\star \equiv f \star g - g \star f$. The coordinates x^μ are associated with constant noncommutativity since

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}. \qquad (2.5)$$

The map (2.1) is required to satisfy

$$A_\mu(a + \partial\lambda) - A_\mu(a) = \partial_\mu \Lambda(\lambda, a)$$
$$- ie[A_\mu(a), \Lambda(\lambda, a)]_\star, \quad (2.6)$$

for infinitesimal transformation parameters Λ and λ. Solutions for the potentials, fields, and transformation parameters can be obtained in terms of expansions in $\theta^{\mu\nu}$ (or equivalently e)

$$A_\mu(a) = A_\mu^{(0)}(a) + eA_\mu^{(1)}(a) + e^2 A_\mu^{(2)}(a) + \cdots,$$
$$F_{\mu\nu}(a) = F_{\mu\nu}^{(0)}(a) + eF_{\mu\nu}^{(1)}(a) + e^2 F_{\mu\nu}^{(2)}(a) + \cdots, \quad (2.7)$$
$$\Lambda(\lambda, a) = \Lambda^{(0)}(\lambda, a) + e\Lambda^{(1)}(\lambda, a) + e^2 \Lambda^{(2)}(\lambda, a) + \cdots,$$

where the zeroth order corresponds to the commutative theory

$$A_\mu^{(0)}(a) = a_\mu, \qquad F_{\mu\nu}^{(0)}(a) = f_{\mu\nu}, \qquad \Lambda^{(0)}(\lambda, a) = \lambda. \qquad (2.8)$$

Explicit expressions up to the second order in $\theta^{\mu\nu}$ have been found by various authors [33–38]. Up to homogeneous terms, the first and second order solutions are given by

$$A_\mu^{(1)}(a) = \tfrac{1}{2}\theta^{\rho\sigma} a_\rho (f_{\mu\sigma} - \partial_\sigma a_\mu),$$
$$F_{\mu\nu}^{(1)}(a) = -\theta^{\rho\sigma}(a_\rho \partial_\sigma f_{\mu\nu} + f_{\mu\rho} f_{\sigma\nu}), \qquad (2.9)$$
$$\Lambda^{(1)}(\lambda, a) = \tfrac{1}{2}\theta^{\mu\nu} \partial_\mu \lambda a_\nu,$$

and

$$A_\mu^{(2)}(a) = \tfrac{1}{2}\theta^{\rho\sigma}\theta^{\eta\xi} a_\rho(a_\eta \partial_\xi f_{\sigma\mu} + \partial_\xi a_\mu \partial_\sigma a_\eta + f_{\sigma\eta} f_{\xi\mu}),$$
$$F_{\mu\nu}^{(2)}(a) = \tfrac{1}{2}\theta^{\rho\sigma}\theta^{\eta\xi}\{a_\rho(\partial_\sigma(a_\eta \partial_\xi f_{\mu\nu}) + \partial_\xi f_{\mu\nu} f_{\sigma\eta}$$
$$+ 2f_{\mu\xi}\partial_\sigma f_{\nu\eta} - 2f_{\nu\xi}\partial_\sigma f_{\mu\eta}) + 2f_{\mu\sigma}f_{\nu\xi}f_{\rho\eta}\},$$
$$\Lambda^{(2)}(\lambda, a) = \tfrac{1}{2}\theta^{\rho\sigma}\theta^{\eta\xi}\partial_\xi \lambda a_\rho \partial_\sigma a_\eta, \qquad (2.10)$$

respectively.

III. EFFECTIVE LAGRANGIAN

Here we give the effective Lagrangian and current density for noncommutative $U(1)$ gauge theory up to second order in θ. Although the first order results have been reported previously, the same is not true for the second order results which will be needed for discussions in Sec. IV. The details of the second order analysis appear in Appendix B. Here we also report on an alternative Lagrangian approach using auxiliary fields.

The considerations in the previous section do not involve dynamics. In general, (2.1) will not map solutions of a commutative gauge theory on a manifold M to solutions of the corresponding noncommutative gauge theory. Here we choose M to be four-dimensional Minkowski space-time. Solutions of the free commutative Maxwell equations $\partial^\mu f_{\mu\nu} = 0$ on M are not in general mapped to solutions of the free noncommutative field equations

$$\partial^\mu F_{\mu\nu} - ie[A^\mu, F_{\mu\nu}]_\star = 0, \qquad (3.1)$$

on M.

The converse is also true. Solutions to the free noncommutative field Eqs. (3.1) on M are not in general mapped to solutions to the free commutative Maxwell equations on M by the inverse Seiberg-Witten map

$$(A, \Lambda) \rightarrow (a = a(A), \lambda = \lambda(\Lambda, A)). \qquad (3.2)$$

The field Eqs. (3.1) are recovered from the Lagrangian density

$$L[A] = -\tfrac{1}{4}F_{\mu\nu} \star F^{\mu\nu}, \qquad (3.3)$$

which up to boundary terms[4] is $-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$. From (2.7) it can be expanded in $\theta^{\mu\nu}$ (or equivalently e) and expressed in terms of the commutative fields, giving the effective Lagrangian

$$L[A(a)] = \mathcal{L}[a]$$
$$= \mathcal{L}^{(0)}[a] + e\mathcal{L}^{(1)}[a] + e^2 \mathcal{L}^{(2)}[a] + \cdots. \quad (3.4)$$

We find that, up to second order, the effective $U(1)$ Lagrangian can be written as a function of only $f_{\mu\nu}$ (and $\theta^{\mu\nu}$), and not higher derivatives of the fields. At zeroth order we of course recover the free Maxwell Lagrangian

$$\mathcal{L}^{(0)}[a] = \tfrac{1}{4}\mathrm{Tr}f^2, \qquad (3.5)$$

while up to total derivatives, the first and second orders are given by

$$\mathcal{L}^{(1)}[a] = -\tfrac{1}{2}\mathrm{Tr}f^3\theta + \tfrac{1}{8}\mathrm{Tr}f^2\,\mathrm{Tr}f\theta, \qquad (3.6)$$

and

$$\mathcal{L}^{(2)}[a] = \tfrac{1}{4}\mathrm{Tr}(f^2\theta)^2 + \tfrac{1}{2}\mathrm{Tr}f^2(f\theta)^2 - \tfrac{1}{4}\mathrm{Tr}f^3\theta\,\mathrm{Tr}f\theta$$
$$+ \tfrac{1}{32}\mathrm{Tr}f^2(\mathrm{Tr}f\theta)^2 - \tfrac{1}{16}\mathrm{Tr}f^2\,\mathrm{Tr}(f\theta)^2. \qquad (3.7)$$

The first order correction $\mathcal{L}^{(1)}$ is well known [33,39,40]. The field equations resulting from variations of a in (3.4) state that there is a divergenceless field $B_{\mu\nu}$,

$$\partial^\mu B_{\mu\nu} = 0, \qquad (3.8)$$

which has an expansion in $\theta^{\mu\nu}$ (or equivalently e) and the field strengths $f_{\mu\nu}$:

$$B_{\mu\nu} = B_{\mu\nu}^{(0)} + eB_{\mu\nu}^{(1)} + e^2 B_{\mu\nu}^{(2)} + \cdots, \qquad (3.9)$$

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[4]In the next section we will examine solutions with a singularity at the origin. The boundary terms will affect the singularity. However, we shall only be concerned with the behavior of the fields away from the origin.

where the zeroth order is just $B^{(0)}_{\mu\nu} = f_{\mu\nu}$. The first and second orders are computed in Appendix B. Thus solutions of the noncommutative field Eqs. (3.1) are mapped under (3.2) to solutions of (3.8).

Alternatively, the field equations can be reexpressed in terms of Maxwell equations for f with an effective conserved current j_μ associated with the noncommutative self-interactions

$$\partial^\mu f_{\mu\nu} = j_\nu. \tag{3.10}$$

Some properties of these currents have been examined previously in [41,42]. After some work we find rather simple expressions for the current in an expansion in $\theta^{\mu\nu}$ (or equivalently e) up to the second order:

$$j_\nu = e j^{(1)}_\nu + e^2 j^{(2)}_\nu + \cdots, \qquad j^{(1)}_\nu = (f\theta)^{\rho\sigma} \partial_{(\rho} f_{\sigma)\nu},$$

$$j^{(2)}_\nu = -((f\theta)^2 + \tfrac{1}{2}\theta f^2\theta)^{\rho\sigma} \partial_{(\rho} f_{\sigma)\nu}, \tag{3.11}$$

where parenthesis indicate a symmetrization of indices.

The field Eqs. (3.8) and (3.10) can also be obtained from the Lagrangian

$$\mathcal{L}'(B, a) = \tfrac{1}{2}\text{Tr}Bf - \mathcal{L}''(B), \tag{3.12}$$

where here B as well as a are treated as independent variables and $\mathcal{L}''(B)$ only depends on B. Up to first order in $\theta^{\mu\nu}$ (or equivalently e), the latter is given by

$$\mathcal{L}''(B) = \frac{1}{4}\,\text{Tr}B^2 + \frac{e}{2}\,\text{Tr}B^3\theta - \frac{e}{8}\,\text{Tr}B^2\,\text{Tr}B\theta + \mathcal{O}(\theta^2). \tag{3.13}$$

The Hamiltonian density is obtained from (3.4) in the standard way

$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial(\partial_0 a_i)}\partial_0 a_i - \mathcal{L}$$

$$= \mathcal{H}^{(0)} + e\mathcal{H}^{(1)} + e^2\mathcal{H}^{(2)} + \cdots, \tag{3.14}$$

where as usual only the spatial components of the potential a_i are dynamical. The zeroth order term $\mathcal{H}^{(0)}$ is the Maxwell Hamiltonian, while the first two corrections are computed in Appendix B.

IV. STATIC SOLUTIONS

Here we look for static perturbative solutions to the field Eqs. (3.8) and (3.10) on four-dimensional Minkowski space M. (More precisely, M is defined with the spatial origin removed.) The perturbative expansion in $\theta^{\mu\nu}$ also corresponds to an expansion in one over the radial coordinate r for the solutions. The result can therefore be considered valid for intermediate values of r. Rather than solve the commutative field Eqs. (3.8) or (3.10) directly, it is simplest to first solve (3.1) for the noncommutative potentials A_μ and then apply the inverse Seiberg-Witten map (3.2) to get

the commutative potentials a_μ. This is the approach we follow below.

A. Space-time noncommutativity

Here we consider $\theta^{0i} \neq 0$ and $\theta^{ij} = 0$, where i is a space index and 0 the time index. We shall be concerned with electrostatic fields. The star commutator vanishes when acting between any static fields in this case, and so the noncommutative gauge field Eqs. (3.1) reduce to the commutative Maxwell equations. The Coulomb solution

$$A_0 = -\frac{e}{r}, \qquad A_i = 0, \tag{4.1}$$

is then exact in this case. Upon performing the inverse Seiberg-Witten map (3.2) of this solution one gets the following results for the commutative potentials and field strengths

$$a_0 = -\frac{e}{r} - \frac{e^3\theta^{0i}\hat{x}_i}{r^3} + \frac{e^5}{2r^5}((\theta^{0i})^2 - 5(\theta^{0i}\hat{x}_i)^2) + \cdots,$$

$$f_{i0} = \frac{e\hat{x}_i}{r^2}\left(1 + \frac{4e^2}{r^2}\theta^{0j}\hat{x}_i - \frac{5e^4}{2r^4}((\theta^{0i})^2 - 7(\theta^{0j}\hat{x}_j)^2) + \cdots\right)$$

$$- \frac{e^3\theta^{0i}}{r^4}\left(1 + \frac{5e^2}{r^2}\theta^{0j}\hat{x}_i + \cdots\right), \tag{4.2}$$

along with $a_i = f_{ij} = 0$. (The hat denotes a unit vector $\hat{x}_i = x_i/r$.) For this solution we can identify the current density in the Maxwell Eq. (3.10) with

$$j_0 = -\frac{4e^3}{r^5}\theta^{0j}\hat{x}_j + \frac{5e^5}{r^7}((\theta^{0i})^2 - 7(\theta^{0i}\hat{x}_i)^2) + \mathcal{O}(\theta^3),$$

$$j_i = 0, \qquad r > 0, \tag{4.3}$$

which breaks rotational invariance. From Gauss's law the resulting effective charge inside a sphere of radius r centered about the origin is

$$\frac{1}{4\pi}\int d\Omega r^2 \hat{x}_i f_{i0} = e\left(1 + \frac{5}{3}\frac{e^4}{r^4}(\theta^{0i})^2 + \mathcal{O}(\theta^3)\right), \tag{4.4}$$

which increases with decreasing r. Ω is the solid angle. This is in contrast to what happens for the electrostatic solution (bion) of Born-Infeld theory [see (A10)]. The bion charge inside a sphere of radius r goes smoothly to zero as $r \to 0$. In contrast, a singular source may be a general feature of gauge theories with space-time noncommutativity. Moreover, the singularity appears to be more severe than that which occurs in the commutative theory. The result (4.4) may indicate that the exact solution screens an infinite charge at the origin.

For the case of $\theta^{ij} = 0$ and electrostatic fields, the Lagrangian density (3.4) and energy density (3.14), simplify to

$$\mathcal{L}^{\theta^{ij}=0}_{es} = \tfrac{1}{2}(f_{i0})^2(1 - e(\theta^{0j}f_{j0}) + e^2(\theta^{0j}f_{j0})^2 + \cdots), \tag{4.5}$$

$$\mathcal{H}_{es}^{\theta^{ij}=0} = \tfrac{1}{2}(f_{i0})^2(1 - 2e(\theta^{0j}f_{j0}) + 3e^2(\theta^{0j}f_{j0})^2 + \cdots),$$
$$(4.6)$$

respectively. Substituting the solution (4.2) in the energy density (4.6) and averaging over a sphere of radius r gives

$$\frac{1}{4\pi}\int d\Omega\,\mathcal{H} = \frac{e^2}{2r^4}\left(1 + \frac{10}{3}\frac{e^4}{r^4}(\theta^{0i})^2 + \mathcal{O}(\theta^3)\right). \quad (4.7)$$

This is also in contrast to what happens for the bion [see (A11)], which has a finite self-energy. For the noncommutative solution (4.2), the self-energy in the intermediate region grows faster than that for a Coulomb point charge for decreasing r.

An alternative approach to obtaining solution (4.2) would be to solve the field Eqs. (3.8) directly. Since we are interested in electrostatic field configurations on $\mathbb{R}^3$ minus the origin we may try setting the divergenceless field $B_{\mu\nu}$ equal to the Coulomb solution. More generally, one can (actually, must) add homogeneous terms, in analogy to a multimoment expansion:

$$B_{i0} = \frac{e\hat{x}_i}{r^2} + c_1\frac{e^3}{r^4}(\theta^{0i} - 2\theta^{0j}\hat{x}_j\hat{x}_i)$$
$$+ c_2\frac{e^5}{r^6}((\theta^{0j})^2\hat{x}_i + 4\theta^{0j}\hat{x}_j\theta^{0i} - 7(\theta^{0j}\hat{x}_j)^2\hat{x}_i) + \cdots,$$
$$(4.8)$$

where c_a are arbitrary coefficients and $B_{ij} = 0$. Starting from the Lagrangian density (4.5) for electrostatic fields one obtains the following expansion up to second order for the divergenceless fields B_{i0} in terms of f_{i0}

$$B_{i0} = f_{i0}(1 - e(\theta^{0j}f_{j0}) + e^2(\theta^{0j}f_{j0})^2 + \cdots)$$
$$- e\theta^{0i}(f_{k0})^2(\tfrac{1}{2} - e(\theta^{0j}f_{j0}) + \cdots). \quad (4.9)$$

We can invert this expression to solve for the field strengths f_{i0} for the general solution (4.8). The result is

$$f_{i0} = \frac{e\hat{x}_i}{r^2} + \left(c_1 + \frac{1}{2}\right)\frac{e^3}{r^4}\theta^{0i} + (1 - 2c_1)\frac{e^3}{r^4}\theta^{0j}\hat{x}_j\hat{x}_i$$
$$+ \left(c_1 + c_2 + \frac{1}{2}\right)\frac{e^5}{r^6}(\theta^{0j})^2\hat{x}_i + (4c_2 + 1)\frac{e^5}{r^6}\theta^{0j}\hat{x}_j\theta^{0i}$$
$$+ (1 - 4c_1 - 7c_2)\frac{e^5}{r^6}(\theta^{0j}\hat{x}_j)^2\hat{x}_i + \cdots, \quad (4.10)$$

along with $f_{ij} = 0$. Lastly, if we impose the Bianchi identities, which here means $\partial_i f_{j0} = \partial_j f_{i0}$, we obtain the unique result up to second order that $c_1 = c_2 = -\frac{3}{2}$, and thus recover the solution (4.2).

B. Space-space noncommutativity

Here we consider $\theta^{0i} = 0$ and $\theta^{ij} \neq 0$. A static solution of the free noncommutative field Eqs. (3.1) on $\mathbb{R}^3$ (minus a point) $\times$ time was obtained perturbatively in [30]. It is given by

$$A_0 = -\frac{e}{r} + \frac{e^5}{4r^5}\left\{-(\theta^{ki}\hat{x}_i)^2 + \frac{1}{5}(\theta^{ij})^2\right\} + \mathcal{O}(\theta^3),$$
$$A_i = \frac{e^3\theta^{ij}\hat{x}_j}{4r^3} + \mathcal{O}(\theta^3). \quad (4.11)$$

Upon performing the inverse Seiberg-Witten map (3.2) of this solution to the commutative potentials and field strengths one gets

$$a_0(A) = -\frac{e}{r} + \frac{e^5(\theta^{ij})^2}{20r^5} + \mathcal{O}(\theta^3),$$
$$a_i(A) = \frac{e^3\theta^{ij}\hat{x}_j}{4r^3} + \mathcal{O}(\theta^3),$$
$$f_{i0} = \frac{e\hat{x}_i}{r^2} - \frac{e^5(\theta^{ij})^2\hat{x}_i}{4r^6} + \mathcal{O}(\theta^3),$$
$$f_{ij} = \frac{e^3}{r^4}\left(-\frac{1}{2}\theta^{ij} + \theta^{ik}\hat{x}_k\hat{x}_j - \theta^{jk}\hat{x}_k\hat{x}_i\right) + \mathcal{O}(\theta^3).$$
$$(4.12)$$

Unlike in the previous case, this is not an electrostatic solution of the field Eqs. (3.10) as magnetic fields are present (both in the commutative and noncommutative theory). Up to order θ^2, the charge density is rotationally invariant, while the current density is not

$$j_0 = \frac{e^5}{r^7}(\theta^{ij})^2 + \mathcal{O}(\theta^3), \qquad j_i = \frac{e^3}{r^5}\theta^{ij}\hat{x}_j + \mathcal{O}(\theta^3),$$
$$r > 0. \quad (4.13)$$

The resulting effective charge inside a sphere of radius r centered about the origin is

$$\frac{1}{4\pi}\int d\Omega\,r^2\hat{x}_i f_{i0} = e\left(1 - \frac{e^4(\theta^{ij})^2}{4r^4} + \mathcal{O}(\theta^3)\right), \quad (4.14)$$

which here decreases with decreasing r. This behavior is similar to that of the bion (A10). It is then a possibility that the charge distribution for the theory with space-space noncommutativity is smooth at the origin, and that one can associate the size of the distribution with $e\sqrt{|\theta^{ij}|}$, as has been done previously [24].

Substituting into the energy density computed in the Appendix (B5) and averaging over a sphere of radius r now gives

$$\frac{1}{4\pi}\int d\Omega\,\mathcal{H} = \frac{e^2}{2r^4}\left(1 + \frac{13}{24}\frac{e^4}{r^4}(\theta^{ij})^2 + \mathcal{O}(\theta^3)\right). \quad (4.15)$$

Once again, in contrast to the energy density of the Born-Infeld solution (A11), it grows faster than the self-energy of a Coulomb point charge as r tends to the origin.

C. General case

The solution (4.11) to the free noncommutative field Eqs. (3.1) applies in the general case where both θ^{0i} and θ^{ij} are nonzero. Upon performing the inverse Seiberg-Witten map (3.2) of (4.11) in this case, one gets

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$$a_0(A) = -\frac{e}{r} - \frac{e^3 \theta^{0i} \hat{x}_i}{r^3}$$
$$+ \frac{e^5}{2r^5} \left(\frac{1}{10} (\theta^{ij})^2 + (\theta^{0i})^2 - 5(\theta^{0i}\hat{x}_i)^2 \right) + \mathcal{O}(\theta^3),$$

$$a_i(A) = \frac{e^3 \theta^{ij} \hat{x}_j}{4r^3} + \frac{e^5}{8r^5} \theta^{0j} (3\theta^{ji} - 3\theta^{jk}\hat{x}_i \hat{x}_k + 8\theta^{ik}\hat{x}_j \hat{x}_k)$$
$$+ \mathcal{O}(\theta^3). \tag{4.16}$$

The charge density for the general case is simply the sum of (4.3) and (4.13) up to second order, while the current density contains mixed terms at second order; i.e. terms proportional to the product of both θ^{ij} and θ^{0k}. To obtain the effective charge inside a sphere of radius r up to second order in the general case, one adds the contributions (4.4) and (4.14). These second order contributions cancel when $(\theta^{0i})^2 = \frac{3}{20} (\theta^{ij})^2$. Further analysis of the general case is quite involved and does not appear to lead to novel results.

V. EXACT ELECTROSTATIC LAGRANGIAN FOR $\theta^{0i} \neq 0$, $\theta^{ij} = 0$?

This section is speculative in nature and offers a possible nonperturbative treatment for case (a), and if correct, gives the solution for all r. Having an exact Lagrangian could also lead to an investigation of noncommutative solutions which do not possess a commutative limit.

From the second order results for the electrostatic Lagrangian (4.5) and energy density (4.6) for the case $\theta^{0i} \neq 0$, $\theta^{ij} = 0$, it is tempting to surmise that the exact expressions for the electrostatic Lagrangian and energy density in this case are

$$\mathcal{L}_{es}^{\theta^{ij}=0} = \frac{\frac{1}{2}(f_{i0})^2}{1 + e\theta^{0j} f_{j0}} \tag{5.1}$$

and

$$\mathcal{H}_{es}^{\theta^{ij}=0} = \frac{\frac{1}{2}(f_{i0})^2}{(1 + e\theta^{0j} f_{j0})^2}, \tag{5.2}$$

respectively. The field equation following from (5.1) states that

$$B_{i0} = \frac{f_{i0}}{1 + e\theta^{0j} f_{j0}} - \frac{\frac{e}{2}\theta^{0i}(f_{k0})^2}{(1 + e\theta^{0j} f_{j0})^2} \tag{5.3}$$

has zero divergence (3.8). This can be inverted to obtain an expression for the field strength

$$f_{i0} = \frac{B_{i0} + \frac{\theta^{0i}}{e(\theta^{0n})^2}(1 - e\theta^{0m} B_{m0})}{\sqrt{(1 - e\theta^{0j} B_{j0})^2 - e^2(\theta^{0\ell})^2 (B_{k0})^2}} - \frac{\theta^{0i}}{e(\theta^{0\ell})^2}. \tag{5.4}$$

The field equations can then also be obtained from the Lagrangian

$$\mathcal{L}_{es}^{\theta^{ij}=0}(B, a) = \frac{1}{e^2(\theta^{0n})^2} \sqrt{(1 - e\theta^{0j} B_{j0})^2 - e^2(\theta^{0\ell})^2(B_{k0})^2}$$
$$+ \frac{\theta^{0i} B_{i0}}{e(\theta^{0n})^2} + B_{i0} f_{i0}, \tag{5.5}$$

where B and a are treated as independent variables as in (3.12). From (5.4) a well-defined solution should everywhere satisfy

$$e|\theta^{0\ell}||B_{k0}| + e\theta^{0j} B_{j0} \leq 1. \tag{5.6}$$

So, for example, we cannot simply set the divergenceless field B_{i0} proportional to the Coulomb term $\hat{x}^i/r^2$. The solution must include homogeneous terms as in (4.8). This is also needed for f_{i0} to satisfy the Bianchi identity.

Although we have not found a nontrivial exact solution to the electrostatic field equation $\partial_i B_{i0} = 0$ which goes to the Coulomb solution as $r \to \infty$, we can show that there are no regular solutions near the origin. The proof is by contradiction. Assume a power law behavior for the electrostatic potential near $r = 0$

$$a_0 \sim \gamma \frac{r(\theta^{0i}\hat{x}_i)}{e(\theta^{0k})^2} + \beta r^n (\theta^{0i}\hat{x}_i)^m, \quad \text{as } r \to 0, \tag{5.7}$$

where γ and β are constants and $n > 1$ and $m \geq 0$. Then to leading order in r, the field strength tends to a constant

$$f_{i0} \sim \frac{\gamma \theta^{0i}}{e(\theta^{0k})^2} + \beta r^{n-1}[(n-m)(\theta^{0j}\hat{x}_j)^m \hat{x}_i$$
$$+ m(\theta^{0j}\hat{x}_j)^{m-1}\theta^{0i}], \quad \text{as } r \to 0. \tag{5.8}$$

Upon substituting into (5.3) and computing the divergence to leading order in r, we get

$$\partial_i B_{i0} = \frac{\beta r^{n-2}(\theta^{0i}\hat{x}_i)^{m-2}}{1 + \gamma} \Big\{ (n-m)(n+m+1)(\theta^{0j}\hat{x}_j)^2$$
$$+ m(m-1)(\theta^{0j})^2 - \frac{(2+\gamma)\gamma}{(1+\gamma)^2(\theta^{0k})^2}((n-m)$$
$$\times (n-m-2)(\theta^{0j}\hat{x}_j)^4 + (n-m)(1+2m)$$
$$\times (\theta^{0j}\hat{x}_j)^2(\theta^{0\ell})^2 + m(m-1)(\theta^{0\ell})^4 \Big\}. \tag{5.9}$$

A regular asymptotic solution requires that all the coefficients of $(\theta^{0j}\hat{x}_j)^N$ in the braces vanish. From the vanishing of the $N = 4$ coefficient one gets (a) $n = m$ or (b) $n = m + 2$. From the vanishing of the $N = 0$ coefficient one gets $m = 0$ or 1, which for (a) is inconsistent with $n > 1$, and for (b) leads to γ not real (after demanding that the $N = 2$ coefficient vanishes). There are then no nontrivial asymptotic solutions of the form (5.7) near the origin.

VI. APPLICATION TO THE HYDROGEN ATOM

The effect of a space-space noncommutative source on the hydrogen atom at lowest order in θ^{ij} was considered in [30], and it was used to put experimental bounds on the

noncommutative scale. We first briefly review the argument below and then make analogous arguments for the case of space-time noncommutativity. We obtain a bound for the space-time noncommutativity parameter which is a substantial improvement over previous attempts.

We start with the standard nonrelativistic Hamiltonian for a charged spinning particle

$$H = \frac{1}{2m_e}(p_i - ea_i)^2 + ea_0 - \mu_B \epsilon^{ijk} f_{ij} S_k, \qquad (6.1)$$

where m_e is the electron mass, $\mu_B = \frac{e}{2m_e}$ denotes the Bohr magneton, and we use natural units $\hbar = c = 1$. For the case of space-space noncommutativity at leading order, a_0 is the Coulomb potential and a_i and f_{ij} are the first order expressions given in (4.12). Then

$$H = H_0 + H_1^{(ss)} + H_2^{(ss)}, \qquad (6.2)$$

where H_0 is the nonrelativistic hydrogen atom Hamiltonian

$$H_0 = \frac{1}{2m_e} p_i^2 - \frac{\alpha}{r}, \qquad (6.3)$$

and $H_1^{(ss)}$ and $H_2^{(ss)}$ are the leading perturbations

$$H_1^{(ss)} = \frac{\alpha^2}{4m_e} \frac{\vec{\theta} \cdot \vec{L}}{r^4},$$
$$\qquad\qquad (6.4)$$
$$H_2^{(ss)} = \frac{\alpha^2}{2m_e} \frac{2(\vec{S} \cdot \hat{x})(\vec{\theta} \cdot \hat{x}) - \vec{S} \cdot \vec{\theta}}{r^4},$$

where $\theta^{ij} = \epsilon^{ijk}\theta_k$ and α is the fine structure constant. $H_1^{(ss)}$ leads to corrections to the Lamb shifts of the $\ell \neq 0$ states, while $H_2^{(ss)}$ induces splittings in the $1s$ states. After choosing $\vec{\theta}$ in the third direction $\theta_i = \theta_{ss}\delta_{i3}$, one gets the following diagonal matrix elements

$$\langle H_1^{(ss)} \rangle_{2P_{1/2}^{\pm 1/2}} = \frac{\alpha^2 \theta_{ss}}{4m_e} \left\langle \frac{L_z}{r^4} \right\rangle_{2P_{1/2}^{\pm 1/2}} = \pm \frac{\alpha^6 m_e^3 \theta_{ss}}{144}, \qquad (6.5)$$

$$\langle H_2^{(ss)} \rangle_{1S_{1/2}^{\pm 1/2}} = -\frac{\alpha^2 \theta_{ss}}{6m_e} \left\langle \frac{S_z}{r^4} \right\rangle_{1S_{1/2}^{\pm 1/2}} = \mp \frac{1}{3} \alpha^5 m_e^2 \theta_{ss} \Lambda_{QCD},$$
$$\qquad\qquad (6.6)$$

using spectroscopic notation $n\ell_j^{m_j}$. To get a finite answer for (6.6) we inserted the Λ_{QCD} cutoff, taking into account the finite size of the nucleus. (This appears to be the best one can do without having a consistent treatment of noncommutative QCD.) According to [43] the current theoretical accuracy on the $2P$ Lamb shift is about 0.08 kHz. From the splitting (6.5), this then gives the bound

$$\theta_{ss} \lesssim (30 \text{ MeV})^{-2}. \qquad (6.7)$$

The current theoretical accuracy on the $1S$ shift is about 14 kHz [43]. From the splitting (6.6) and $\Lambda_{QCD} \sim 200$ MeV, this then gives the improved bound of

$$\theta_{ss} \lesssim (4 \text{ GeV})^{-2}. \qquad (6.8)$$

In the above treatment of the hydrogen atom we considered the gauge fields generated by the proton to be noncommutative, but we applied it to the standard Schrödinger equation, rather than its noncommutative counterpart. The relative coordinates of the Schrödinger equation were treated as commuting. In this regard, it has been noted that although the noncommutativity parameter is fixed in quantum field theory, this is not necessarily the case in noncommutative quantum mechanics where the different particle coordinates may be associated with different $\theta^{\mu\nu}$ [44,45]. For a multiparticle quantum system, the commutation relations for the different particles should, in principle, be derived starting from the noncommutative field theory. In [44] starting from a noncommutative version of QED the authors found that two Dirac particles of opposite charge have opposite noncommutativity, while the relative coordinates commute. As pointed out in [45], the correct approach for the hydrogen atom would have to include noncommutative QCD, which unfortunately is not well understood. A pragmatic approach would be to instead set separate bounds on the noncommutativity of the electron and nucleus.

In contrast to the above approach, the spatial coordinates do not commute in the space-space noncommutative version of the Schrödinger equation. The latter was considered previously for the hydrogen atom in [45], and it leads to the additional perturbation

$$H^{(e)} = -\frac{\alpha}{2} \frac{\vec{\theta} \cdot \vec{L}}{r^3} \qquad (6.9)$$

to H_0 and a further correction to the $2P$ Lamb shift. This gives yet another bound on the noncommutativity parameter $\theta_i = \theta_e \delta_{i3}$, here associated with the electron. The bound was correctly computed in [30] to be

$$\theta_e \lesssim (6 \text{ GeV})^{-2}. \qquad (6.10)$$

It remains to find the effect of a space-time noncommutative source on the hydrogen atom. Here we replace the standard Coulomb potential for the hydrogen atom with ea_0, with a_0 given by (4.2). We shall again only be interested in leading order effects. The hydrogen atom Hamiltonian is then

$$H = H_0 + H^{(st)}, \qquad H^{(st)} = -\frac{\alpha^2 \theta^{0i} \hat{x}_i}{r^3}. \qquad (6.11)$$

Unlike with the case of space-space noncommutativity, there are no diagonal matrix elements amongst the orbital angular momentum eigenstates for the perturbative term H^{st}. Choosing $\theta^{0i} = \theta_{st}\delta_{i3}$, the latter gives rise to nonvanishing matrix elements between states with $\Delta \ell = 1$ and $\Delta m = 0$. The matrix element between the degenerate 2s and 2p $m = 0$ states is

$\langle n = 2, \ell = 0, m_L = 0 | H^{(\text{st})} | n = 2, \ell = 1, m_L = 0 \rangle$

$$= -\frac{\alpha^5 m_e^3 \theta_{\text{st}}}{27}. \tag{6.12}$$

Upon including the electron spin (and the fine structure), this implies a mixing of the four degenerate $n = 2$, $j = \frac{1}{2}$ states (i.e., $2s_{1/2}$ and $2p_{1/2}$)

$$\langle 2s_{1/2}^{\pm 1/2} | H^{(\text{st})} | 2p_{1/2}^{\pm 1/2} \rangle = \pm \frac{\alpha^5 m_e^3 \theta_{\text{st}}}{27\sqrt{3}}. \tag{6.13}$$

Consequently, there is an energy splitting of the levels equal to

$$\Delta E = \frac{2\alpha^5 m_e^3 \theta_{\text{st}}}{27\sqrt{3}}, \tag{6.14}$$

and two sets of doubly degenerate levels result. Then from the above mentioned current theoretical accuracy on the $2P$ Lamb shift and (6.14), one gets the bound

$$\theta_{\text{st}} \lesssim (.6 \text{ GeV})^{-2}. \tag{6.15}$$

Previous bounds on the space-time noncommutativity parameter have been found using gravitational quantum well experiments [46–49]. Equation (6.15) is a significant improvement. Unlike with the case of space-space non-commutativity at lowest order, all of the $n = 2$, $j = \frac{1}{2}$ hydrogen atom states are shifted. Thus the experimental signature for space-time noncommutativity differs from that for space-space noncommutativity.

As before we considered the gauge fields generated by the proton to be noncommutative, but we applied the standard (commutative) Schrödinger equation for the electron. x_i does not commute with t in the space-time non-commutative version of the Schrödinger equation. However, it was shown in [31,32] that the quantum mechanical spectrum is unaffected by the replacement of the commutative Schrödinger equation with the one associated with noncommuting time and space coordinates (provided that the spatial coordinates commute).

VII. CONCLUSIONS

We have constructed perturbative static solutions to the classical field equations of noncommutative $U(1)$ gauge theory up to second order in θ for the three cases: (a) space-time noncommutativity, (b) space-space noncommutativity, (c) both (a) and (b). They tend to the Coulomb solution as $r \to \infty$. For case (a) the solution is electrostatic and the associated self-charge inside a sphere of radius r centered about the origin increases with decreasing r. This may signal that the exact solution screens an infinite charge at the origin. We proposed an exact expression for the Lagrangian and Hamiltonian in this case and, if it is correct, have shown that no nonsingular solutions exist at the origin. Magnetic as well as electric fields are present for the case (b) solution, and here the self-charge inside a

sphere of radius r centered about the origin decreases with decreasing r. It then becomes plausible that the charge density is well behaved at the origin, as happens in Born-Infeld electrodynamics. If so, the guess that charges become smeared in gauge theories with space-space non-commutativity would be valid. For both cases (a) and (b) the self-energy of the solutions in the intermediate region grows faster than that for a Coulomb point charge as r tends to the origin. It thus appears that the noncommutative solutions have infinite self-energy, contrary to the case of the Born-Infeld solution, and that the divergence of the classical self-energy in the noncommutative theory is more severe than its counterpart in the commutative theory.

We have also looked for the lowest order effects of these solutions on the hydrogen atom spectrum and used them to put experimental bounds on the space-time and space-space noncommutative scales. We found that the two different cases have different experimental signatures.

It is known that the star product realization of any given operator algebra on a noncommutative space is not unique. For example, for the case of constant noncommutativity one can also apply the Voros star product which is based on coherent states [50,51]. More generally, the star product belongs to a very large equivalence class of star products [52]. The different star products in the equivalence class are related by gauge transformations, and tools have been developed for writing down gauge theories based on these equivalence classes [53]. Since the analysis in this article relies on one particular choice of the star product; i.e., the Groenewald-Moyal star product, it is important to know how sensitive the above results are to this choice. It would be especially puzzling if the results obtained for the hydrogen atom spectrum depended on the choice of star product.

APPENDIX A: COMPARISON WITH THE BION

It is useful to compare the asymptotic behavior for the charge density and self-energy of the solutions found in Sec. IV with that of a known deformation of Maxwell theory, i.e., Born-Infeld theory. The Born-Infeld Lagrangian [25] on four-dimensional Minkowski space is constructed from the determinant of the matrix

$$h = \eta + \kappa f, \tag{A1}$$

where $f = [f_{\mu\nu}]$ is the field tensor, $\eta = [\eta_{\mu\nu}] = \text{diag}(-1, 1, 1, 1)$ is the flat metric tensor, and κ is a dimensionful constant. (In string theory the latter is identified with 2π times the string constant.) The Lagrangian density is

$$\mathcal{L}^{\,\mathrm{BI}} = \kappa^{-2}(1 - \sqrt{-\,\det h}),$$

$$\det h = -1 - \frac{\kappa^2}{2} f_{\mu\nu} f^{\mu\nu} + \frac{\kappa^4}{64} (\epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma})^2. \tag{A2}$$

One recovers the Maxwell action at lowest order in the expansion in κ. The field equations resulting from (A2) state that there is a divergenceless field $B_{\mu\nu}^{\mathrm{BI}}$, analogous to $B_{\mu\nu}$ in (3.8),

$$\partial^\mu B_{\mu\nu}^{\mathrm{BI}} = 0,$$

$$B_{\mu\nu}^{\mathrm{BI}} = \frac{1}{\sqrt{-\,\det h}} \left(f_{\mu\nu} - \frac{\kappa^2}{16} (\epsilon^{\alpha\beta\gamma\delta} f_{\alpha\beta} f_{\gamma\delta})(\epsilon_{\mu\nu\rho\sigma} f^{\rho\sigma}) \right). \tag{A3}$$

Alternatively, the field equations can be recast as Maxwell equations for the field strength $f_{\mu\nu}$ with an effective conserved current j_ν^{BI} as in (3.10)

$$\partial^\mu f_{\mu\nu} = j_\nu^{\mathrm{BI}}, \tag{A4}$$

where here $j_\nu^{\mathrm{BI}} = \partial^\mu(f_{\mu\nu} - B_{\mu\nu}^{\mathrm{BI}})$. The energy density is given by

$$\mathcal{H}^{\,\mathrm{BI}} = \frac{1}{\sqrt{-\,\det h}} \left(\frac{1}{2} (f_{ij})^2 + \frac{1}{\kappa^2} \right) - \frac{1}{\kappa^2}, \tag{A5}$$

and it is easy to check that the Maxwell energy density is recovered at zeroth order in a κ expansion.

For the case of electrostatics, (A2) and (A5) reduce to

$$\mathcal{L}_{es}^{\,\mathrm{BI}} = \kappa^{-2}(1 - \sqrt{1 - \kappa^2 (f_{i0})^2}),$$
$$\mathcal{H}_{es}^{\mathrm{BI}} = \kappa^{-2} \left(\frac{1}{\sqrt{1 - \kappa^2 (f_{i0})^2}} - 1 \right), \tag{A6}$$

respectively. Substituting the Coulomb solution for $B_{\mu\nu}^{\mathrm{BI}}$ on $\mathbb{R}^3$ (minus a point) $\times$ time

$$B_{i0}^{\mathrm{BI}} = e \frac{\hat{x}_i}{r^2}, \qquad B_{ij}^{\mathrm{BI}} = 0, \tag{A7}$$

gives the bion solution for $f_{\mu\nu}$

$$f_{i0} = \frac{e\hat{x}_i}{\sqrt{r^4 + \kappa^2 e^2}}, \qquad f_{ij} = 0, \tag{A8}$$

which satisfies the Bianchi identities $\partial_i f_{j0} = \partial_j f_{i0}$. The associated electrostatic potential can be expressed in terms of a hypergeometric function. [54] Using (A4) one then gets the following continuous charge distribution for the solution:

$$j_0^{\mathrm{BI}} = \frac{2\kappa^2 e^3}{r(r^4 + \kappa^2 e^2)^{3/2}}, \tag{A9}$$

and the following effective charge inside a sphere of radius r centered about the origin

$$\frac{1}{4\pi} \int d\Omega\, r^2 \hat{x}_i f_{i0} = \frac{e}{\sqrt{1 + (\frac{\kappa e}{r^2})^2}}$$

$$\rightarrow \begin{cases} e(1 - \frac{\kappa^2 e^2}{2r^4} + \cdots) & \text{as } r \rightarrow \infty \\ \frac{r^2}{\kappa}(1 - \frac{r^4}{2\kappa^2 e^2} + \cdots) & \text{as } r \rightarrow 0, \end{cases} \tag{A10}$$

which decreases monotonically to zero as r goes to zero. Substituting into (A5) gives the bion energy density

$$\mathcal{H}_{es}^{\mathrm{BI}} = \frac{1}{\kappa^2} \left(\sqrt{1 + (\frac{\kappa e}{r^2})^2} - 1 \right) \rightarrow \frac{e^2}{2r^4} \left(1 - \frac{\kappa^2 e^2}{4r^4} + \cdots \right)$$
$$\text{as } r \rightarrow \infty, \tag{A11}$$

whose integral is finite.

APPENDIX B: SOURCELESS FIELD AND HAMILTONIAN DENSITY

Here we give explicit expressions for (i) the sourceless field $B_{\mu\nu}$ in the noncommutative field theory, along with (ii) the Hamiltonian density in Sec. III, up to second order in $\theta^{\mu\nu}$.

(i) Varying the commutative field strengths $f_{\mu\nu}$ in the Lagrangian density (3.4) gives

$$\delta\mathcal{L} = \tfrac{1}{2} \mathrm{Tr} B \delta f, \tag{B1}$$

and the resulting field equations state that $B_{\mu\nu}$ satisfies (3.8). $B_{\mu\nu}$ is expanded in terms of $\theta^{\mu\nu}$ (or equivalently e) in (3.9). The zeroth order is just $B^{(0)} = f$, while

$$B^{(1)} = -(f^2\theta + f\theta f + \theta f^2) + \tfrac{1}{2} f \,\mathrm{Tr} f\theta + \tfrac{1}{4}\theta\,\mathrm{Tr} f^2,$$
$$B^{(2)} = \theta f^3 \theta + f\theta f^2\theta + \theta f^2\theta f + f(f\theta)^2 + (\theta f)^2 f$$
$$+ (f\theta)^2 f - \tfrac{1}{2}(f^2\theta + f\theta f + \theta f^2)\,\mathrm{Tr} f\theta - \tfrac{1}{4}\theta f\theta\,\mathrm{Tr} f^2$$
$$- \tfrac{1}{2}\theta\,\mathrm{Tr} f^3\theta + \tfrac{1}{8}\theta\,\mathrm{Tr} f\theta\,\mathrm{Tr} f^2 + \tfrac{1}{8} f(\mathrm{Tr} f\theta)^2$$
$$- \tfrac{1}{4} f\,\mathrm{Tr}(f\theta)^2. \tag{B2}$$

(ii) Up to the first class constraints, the Hamiltonian density in (3.14) can be reexpressed as

$$\mathcal{H} = 2\frac{\partial\mathcal{L}}{\partial f_{0i}} f_{0i} - \mathcal{L}. \tag{B3}$$

The coefficients $\mathcal{H}^{(n)}$ in the expansion given in (3.14) are then given by

$$\mathcal{H}^{(n)} = -(fB^{(n)})_{00} - \mathcal{L}^{(n)}, \tag{B4}$$

and so

$$\mathcal{H}^{(0)} = \tfrac{1}{2}(f_{i0})^2 + \tfrac{1}{4}(f_{ij})^2,$$

$$\mathcal{H}^{(1)} = (f^3\theta + f^2\theta f + f\theta f^2)_{00} + \tfrac{1}{2}\mathrm{Tr}f^3\theta + \tfrac{1}{2}\mathcal{H}^{(0)}\,\mathrm{Tr}f\theta$$
$$- \tfrac{1}{4}(f\theta)_{00}\,\mathrm{Tr}f^2,$$

$$\mathcal{H}^{(2)} = -((f^2\theta)^2 + f\theta f^2\theta f + f^2(f\theta)^2 + f^2(\theta f)^2$$
$$+ (f\theta)^2 f^2 + f\theta f^3\theta)_{00} - \tfrac{1}{4}\mathrm{Tr}(f^2\theta)^2 - \tfrac{1}{2}\mathrm{Tr}f^2(f\theta)^2$$
$$+ \tfrac{1}{2}(f\theta)_{00}\,\mathrm{Tr}f^3\theta + \tfrac{1}{4}[(f\theta)^2]_{00}\,\mathrm{Tr}f^2 + \tfrac{1}{2}\mathcal{H}^{(1)}\,\mathrm{Tr}f\theta$$
$$- \tfrac{1}{8}\mathcal{H}^{(0)}((\mathrm{Tr}f\theta)^2 + 2\,\mathrm{Tr}(f\theta)^2). \tag{B5}$$

The Hamiltonian function is properly expressed in terms of phase space variables, but as our purpose is only to use these expressions to examine the self-energy for the solutions given in Sec. IV, it is not necessary to reexpress $\mathcal{H}$ in terms of such variables.

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