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QCD-like theories on $R_3 \times S_1$: A smooth journey from small to large $r(S_1)$ with double-trace deformations

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We consider QCD-like theories with one massless fermion in various representations of the gauge group SU(N). The theories are formulated on $R_3 \times S_1$. In the decompactification limit of large $r(S_1)$ all these theories are characterized by confinement, mass gap, and spontaneous breaking of a (discrete) chiral symmetry (χSB). At small $r(S_1)$, in order to stabilize the vacua of these theories at a center-symmetric point, we suggest to perform a double-trace deformation. With this deformation, the theories at hand are at weak coupling at small $r(S_1)$ and yet exhibit basic features of the large $r(S_1)$ limit: confinement and χSB. We calculate the string tension, mass gap, bifermion condensates, and θ dependence. The double-trace deformation becomes dynamically irrelevant at large $r(S_1)$. Despite the fact that at small $r(S_1)$ confinement is Abelian, while it is expected to be non-Abelian at large $r(S_1)$, we argue that small and large $r(S_1)$ physics are continuously connected. If so, one can use small $r(S_1)$ laboratory to extract lessons about QCD and QCD-like theories on R_4.

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I. INTRODUCTION

Analyzing QCD and QCD-like theories on $R_3 \times S_1$ provides new insights in gauge dynamics at strong coupling and offers a new framework for discussing various ideas on confinement. The radius of the compact dimension $r(S_1)$ plays a role of an adjustable parameter, an obvious bonus and a welcome addition to a rather scarce theoretical toolkit available in strongly coupled gauge theories. As the circumference L of the circle S_1 varies, so does the dynamical pattern of the theory. For instance, at $L \ll \Lambda^{-1}$ in some instances the theory becomes weakly coupled. On the other hand, in the decompactification limit $L \gg \Lambda^{-1}$, we recover conventional four-dimensional QCD, with its most salient feature, non-Abelian confinement.

A qualitative picture of confinement in terms of the Polyakov line was suggested by Polyakov and Susskind long ago [1,2]. Assume that the compactified dimension is z. The Polyakov line (sometimes called the Polyakov loop) is defined as a path-ordered holonomy of the Wilson line in the compactified dimension

$$\mathcal{U} = P\exp\left\{i\int_0^L a_z dz\right\} \equiv VUV^{\dagger}, \tag{1}$$

where L is the size of the compact dimension while V is a matrix diagonalizing $\mathcal{U}$,

$$U = \mathrm{diag}\{v_1, v_2, \ldots, v_N\}. \tag{2}$$

According to Polyakov, non-Abelian confinement implies that the eigenvalues v_i are randomized: the phases of v_i wildly fluctuate over the entire interval $[0, 2\pi]$ so that

$$\langle \mathrm{Tr}U\rangle = 0. \tag{3}$$

The exact vanishing of $\langle \mathrm{Tr}U\rangle$ in pure Yang-Mills is the consequence of the unbroken Z_N center symmetry in the non-Abelian confinement regime. Introduction of dynamical fermions (quarks) generally speaking breaks the Z_N center symmetry at the Lagrangian level.[1] However, the picture of wild fluctuations of the phases of v_i's remains intact. Therefore, it is generally expected that $\langle \frac{1}{N}\mathrm{Tr}U\rangle$ is strongly suppressed even with the dynamical fermion fields that respect no center symmetry, $\langle \frac{1}{N}\mathrm{Tr}U\rangle \sim 0$. This expectation is supported by lattice simulations at finite temperatures [5] demonstrating that $\langle \mathrm{Tr}U\rangle$ is very close to zero at large L (low temperatures).

On the other hand, in QCD and QCD-like theories[2] at small L (high temperatures) the center-symmetric field configuration is dynamically disfavored. In many instances the vacuum is attained at $\langle \frac{1}{N}\mathrm{Tr}U\rangle = 1$. In this case, the effective low-energy theory is at strong coupling, and it is as hard to deal with it as with QCD on R_4. Typically, the small- and large-L domains are separated by a phase transition (or phase transitions). For instance, for S/AS with even N this is a Z_2 phase transition. Numerical studies show that for $N \geq 3$ there is a thermal phase transition between confinement and deconfinement phases. Similar

[1]It is still an emergent dynamical symmetry in the multicolor limit [3,4]; however, we limit ourselves to small N. In this paper parametrically N is of order one.

[2]By QCD-like theories we mean non-Abelian gauge theories without elementary scalars, e.g., Yang-Mills with fermions in the two-index symmetric or antisymmetric representation, to be referred to as S/AS; see below.

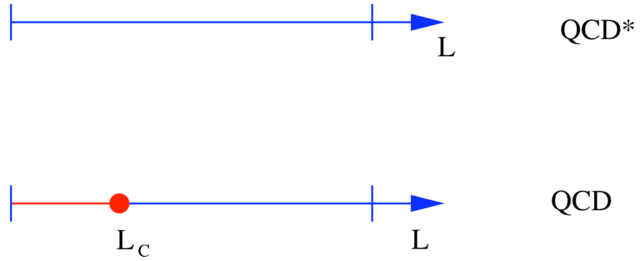

FIG. 1 (color online). Quantum chromodynamics as a function of compactified direction circumference before and after surgery (QCD and QCD*, respectively). L_c is the point of a phase transition.

numerical studies detect a temperature T_χ at which the broken chiral symmetry of $T = 0$ QCD gives place to restored chiral symmetry of high-T QCD. The phase transition at T_χ is that of the chiral symmetry restoration (the lower plot in Fig. 1).

In this case small-L physics says little, if anything, about large-L physics, our desired goal. We would like to create a different situation. We would like to design a theory which (i) in the decompactification large-L limit, tends to conventional QCD and its QCD-like sisters; (ii) at small L is analytically tractable and has both confinement and chiral symmetry breaking; and (iii) has as smooth transition between the small- and large-L domains as possible (the upper plot in Fig. 1). If this endeavor—rendering small- and large-L physics continuously connected—is successful, we could try to use small-L laboratory to extract lessons about QCD and QCD-like theories on R_4.

We will argue below that the goal can be achieved by performing a so-called double-trace deformation of QCD and QCD-like theories.[3] To this end we add a nonlocal operator

$$P[U(\mathbf{x})] = \frac{2}{\pi^2 L^4} \sum_{n=1}^{[N/2]} d_n |\mathrm{Tr} U^n(\mathbf{x})|^2 \quad \text{for SU}(N) \tag{4}$$

to the QCD action

$$\Delta S = \int_{R_3} d^3 x L P[U(\mathbf{x})], \tag{5}$$

where d_n are numerical parameters to be judiciously chosen. The theories obtained in this way will be labeled by asterisk. In minimizing $S + \Delta S$ the effect due to deformation (4) is two-fold. First, it tends to minimize $|\mathrm{Tr} U(\mathbf{x})|$. Second it tends to maximize the distance between the eigenvalues of U. It is necessary to have a polynomial of order $[N/2]$ to force the eigenvalues of the Polyakov line to be maximally apart from one another, i.e. to push the theory towards the center-symmetric point depicted in

[3]The double-trace deformations were previously discussed in the context of gauge/string theory dualities in [6–9], as well as in field theory [10–12].

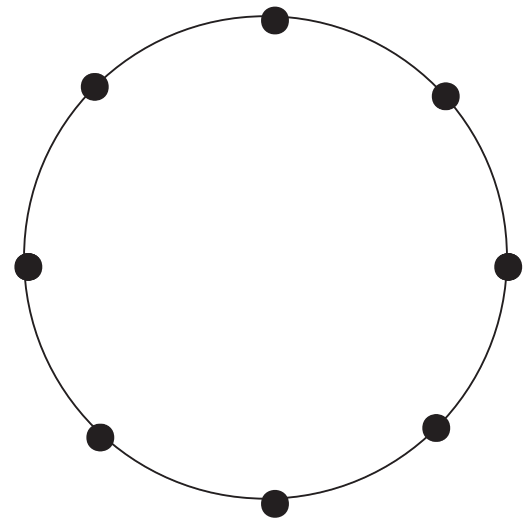

FIG. 2. Z_N symmetric vacuum fields v_k.

Fig. 2. Here $[x]$ stands for the integer part of x. To stabilize the vacuum sufficiently close to the center-symmetric configuration the coefficients d_n must be large enough, presumably, $d_n \sim 1$. Some technical details are discussed in the Appendix.

At large L, the deformation switches off and has no impact on the theory, i.e. QCD* $\approx$ QCD. However, at small L the impact is drastic. Given an appropriate choice of d_n's the deformation (5) forces the theory to pick up the following set[4] of the vacuum expectation values (VEVs):

$$v_k = e^{2\pi i k/N}, \qquad k = 1, \ldots, N, \tag{6}$$

(or permutations); see Fig. 2.

If we define

$$e^{iaL} \equiv U, \tag{7}$$

$$a = \sum_{\text{Cartan gen}} a_c T^c \equiv \mathrm{diag}\{a_1, a_2, \ldots, a_N\}, \qquad \sum_{k=1}^{N} a_k = 0, \tag{8}$$

it is obvious that Eq. (6) implies

$$\{La_i\} = \{-iL \ln v_i (\mathrm{mod}\ 2\pi)\} = \left\{-\frac{2\pi[N/2]}{N}, -\frac{2\pi([N/2]-1)}{N}, \ldots, \frac{2\pi[N/2]}{N}\right\}. \tag{9}$$

This means, in turn, that the theory is maximally Higgsed,

$$\mathrm{SU}(N) \to \mathrm{U}(1)^{N-1} \tag{10}$$

and weakly coupled at $L \ll \Lambda^{-1}$. The gauge bosons from the Cartan subalgebra (to be referred to as photons) remain

[4]More exactly, the set of VEVs will be very close to (6).

classically massless, while the off-diagonal gauge bosons (to be referred to as W-bosons) acquire large masses. The effective low-energy dynamics is that of compact QED. (See footnote [12], though.) It is not trivial. Dual photons acquire exponentially small masses nonperturbatively through the instanton-monopole mechanism [13,14]. The mass gap generation in the dual description amounts to linear Abelian confinement (at exponentially large distances). Chiral bifermion condensates are generated too [15,16]. Thus, the dynamical patterns in the small- and large-L domains do not seem to be that different from each other. Details are different (e.g. Abelian vs non-Abelian confinement), but gross features appear to be similar. It is not unreasonable to expect that there is no phase transition in L.

What is meant when we speak of Abelian/non-Abelian confinement [4,17]? In the former case the gauge group acting in the infrared (IR) and responsible for the flux tube formation is Abelian (i.e. $U(1) \times U(1) \ldots$). In the latter case we deal with a non-Abelian group in the infrared.

The best-known example exhibiting both regimes is the Seiberg-Witten solution [18] of a deformed $\mathcal{N} = 2$ super-Yang-Mills theory. If the deformation parameter μ is small,

$$|\mu| \ll \Lambda,$$

the SU(N) gauge group is spontaneously broken down to $U(1)^{N-1}$, and the confining string is a generalization of the Abrikosov vortex [19]. In the opposite limit

$$|\mu| \gg \Lambda,$$

the breaking of SU(N) down to $U(1)^{N-1}$ does not occur. The infrared dynamics is determined by SU(N); the corresponding flux tubes should be non-Abelian. Since the theory is holomorphic in μ, the Abelian and non-Abelian confinement regimes are expected to be smoothly connected.

Another example which should be mentioned (and which is close in formulation to what will be presented below) where it is believed that no phase transition in L takes place is $\mathcal{N} = 1$ supersymmetric Yang-Mills (SYM) theory on $R_3 \times S_1$ [15,16,20–22].

We expect that QCD* and QCD*-like theories are of this type—there is no phase transition between the Abelian confinement small-L and non-Abelian confinement large-L domains.

Conjecture: The deformed *one*-flavor QCD-like theories interpolate from small $r(S_1)$ to large $r(S_1)$ without phase transitions.

Since the theories under consideration are nonsupersymmetric we cannot back up this statement by holomorphy. Thus, the smoothness conjecture is on a somewhat weaker basis than in the Seiberg-Witten problem. However, arguments to be presented below can be viewed as at least some evidence in favor of the absence of the phase transition in L. More evidence can (and should) be provided by lattice studies.

In QCD-like theories with more than one flavor, chiral symmetry breaking (χSB) occurring on R_4 at strong coupling produces $N_f^2 - 1$ Goldstone mesons. Needless to say, it is impossible to get such Goldstones at weak coupling at small L. However, if one considers theories with *one* fermion flavor in the center-symmetric regime, there are no obvious reasons for a chiral phase transition. The chiral symmetry in such theories is discrete, and its spontaneous breaking results in domain walls rather than Goldstones. This phenomenon can show up both at strong and weak couplings. In this paper we will limit ourselves to QCD-like theories with a single flavor.

To be more exact, we will discuss in some detail SU(N) Yang-Mills theory with one fermion in the fundamental and two-index AS representations. Analysis of the two-index S fermion essentially runs parallel to that of the AS case. We will also dwell on SU(N) $\times$ SU(N) Yang-Mills with the bifundamental fermion. The number of colors N is *not* assumed to be large. The large-N limit and the case of fermions in the adjoint representation were treated elsewhere [4,14].

Among other results, we will, in particular, argue that many dynamical features of SU(N) $\times$ SU(N) orbifold QCD are remarkably close to those of SYM theory. The pattern of the chiral symmetry breaking, the mass gap, the nonperturbative spectrum, the k-string tensions—all of the above are demonstrated to coincide in these two theories.

The paper is organized as follows. In Sec. II we outline our formulation of the problem and briefly review general aspects of one-flavor QCD-like theories on R_4 and $R_3 \times S_1$. We also review dual description of three-dimensional Yang-Mills (the Georgi-Glashow model), and Polyakov's confinement. In Sec. III we consider the case of one fermion in the fundamental representation and solve the theory at small $r(S_1)$. In Sec. IV we carry out the same analysis in the SU(N) $\times$ SU(N) theory with one bifundamental fermion (orbifold theory). In Sec. V we consider Yang-Mills theory with one fermion in the two-index antisymmetric representation of SU(N). Section VI is devoted to θ dependence. In Sec. VII we discuss how our results are related to planar equivalence. Finally, Sec. VIII summarizes our results and outlines some problems for future investigation.

II. QCD AND QCD-LIKE THEORIES ON R_4 AND $R_3 \times S_1$: GENERAL ASPECTS

We will consider one-flavor QCD-like theories with the SU(N) gauge group and fermions in the following representations:

$$\mathcal{R} = \{\mathrm{F, AS, S, Adj, BF}\}, \tag{11}$$

where F stands for fundamental, AS and S are two-index antisymmetric and symmetric representations, Adj stands for adjoint, while BF for bifundamental. In all cases except Adj we deal with the Dirac fermion field, while in the adjoint case with the Majorana (Weyl) spinor. This is nothing but supersymmetric Yang-Mills (SYM) theory. In the BF case the gauge group is $SU(N) \times SU(N)$, with the fermion field being fundamental with respect to the first $SU(N)$ and antifundamental with respect to the second $SU(N)$. For the adjoint fermions we will use the following nomenclature. The theory with one Majorana flavor will be referred to as SYM, while in the case of two or more flavors we will speak of QCD(Adj).

The boundary conditions for fermions can be either periodic ($\mathcal{S}^+$) or antiperiodic ($\mathcal{S}^-$) in the compactified dimension. Yang-Mills theories with two-index fermions received much attention lately in connection with planar equivalence between such theories and SYM theory (see [23] and references therein). At $N = 3$ the AS theory is equivalent to F.

Theoretically the most informative is $\mathcal{N} = 1$ SYM theory. For periodic spin connection $\mathcal{S}^+$ this theory has unbroken center symmetry and broken discrete chiral symmetry for any $r(S_1)$. In fact, the chiral condensate $\langle \mathrm{Tr}\lambda\lambda \rangle$ was exactly calculated long ago [15,24], both on R_4 and $R_3 \times S_1$, and was shown to be totally independent of the value of $r(S_1)$. More recently, this theory was demonstrated [16] to possess Abelian confinement at small L. Therefore, there is no *obvious* obstruction for the L evolution to be smooth. We know that at L larger than the strong scale Λ^{-1}, the neutral sector observables in $\mathcal{N} = 1$ SYM theory and QCD(AS/S/BF) are remarkably close and only differ by mild $O(1/N)$ effects. However, the complex representation fermions break center symmetry at small $r(S^1)$ implying that these theories become drastically different from $\mathcal{N} = 1$ SYM theory. The double-trace deformation (5) is designed to maintain this similarity at small $r(S_1)$ too. One of the most intriguing findings of this paper is that the analytical tractability of $\mathcal{N} = 1$ SYM theory in the small-$r(S_1)$ limit is not necessarily a consequence of supersymmetry. The unbroken center symmetry is equally important.

Briefly summarizing our knowledge of other one-flavor QCD-like theories[5] on R_4 we can say the following. All these theories are expected to exhibit:

(i) Mass gap: there are no massless particles in the physical spectrum;

(ii) Non-Abelian confinement: the gauge group is not Higgsed, chromoelectric flux tubes are formed between quarks and antiquarks, these flux tubes are not stable, generally speaking, since the dynamical quark pair production can break them. No color-charged objects are present in the physical spectrum;

(iii) Discrete chiral symmetry breaking[6] for $\mathcal{R} = \{\mathrm{AS, S, BF, Adj}\}$: The one-flavor QCD-like theories on R_4 possess an axial U(1) symmetry at the classical level. Only a discrete subgroup of it, Z_{2h}, is the symmetry of the quantum theory,

$$Z_{2h} = \{Z_2, Z_{2N-4}, Z_{2N+4}, Z_{2N}, Z_{2N}\} \quad \text{for } \mathcal{R} = \{\mathrm{F, AS, S, BF, Adj}\}, \tag{12}$$

respectively. Here $2h$ is the number of the fermion zero modes in the instanton background. In all cases but F the axial Z_{2h} is spontaneously broken down to Z_2. Discrete symmetry breaking, unlike that of the continuous symmetries, does not lead to Goldstone bosons. Instead, the theory must possess h isolated vacua.

The above picture follows from multiple lattice calculations, and supersymmetry-based and large-N methods.

In this work the double-trace deformation of QCD($\mathcal{R}$) on $S_1 \times R_3$ with small $r(S_1)$ is used to stabilize the theories under consideration at (or, more exactly, very close to) a center-symmetric point. At small $r(S_1)$ the non-Abelian gauge group is Higgsed down to the maximal Abelian subgroup, but neither confinement nor the above chiral properties are lost. We will explicitly demonstrate confinement, the discrete chiral symmetry breaking, and mass gap generation.

On $S_1 \times R_3$ the Yang-Mills Lagrangian with one fermion flavor in the representation $\mathcal{R}$ takes the form

$$S = \int_{R_3 \times S_1} \frac{1}{g^2} \left[\frac{1}{2} \mathrm{Tr} F_{MN}^2(x) + i\bar{\Psi} \not{D} \Psi \right], \tag{13}$$

where Ψ is the four-dimensional Dirac spinor in the representation $\mathcal{R} = \{\mathrm{F, AS, S}\}$ of the gauge group $SU(N)$, F_{MN} is the non-Abelian gauge field strength,[7] and $\not{D} = \gamma_M D_M = \gamma_M(\partial_M + iA_M)$ is the covariant derivative acting on representation $\mathcal{R}$. For QCD(BF), the gauge group is $SU(N) \times SU(N)$ and gauge field part of the action must be replaced by

$$F_{MN}^2(x) \to F_{1,MN}^2(x) + F_{2,MN}^2(x).$$

In this theory the fermion is in the bifundamental representation. In terms of its Weyl components, the Dirac fermions are decomposed as

[5] A part of this knowledge is folklore.

[6] For F representation, the anomaly-free Z_2 is the fermion number and cannot be spontaneously broken. The theory has a unique vacuum.

[7] Throughout the paper we use the following notation: $M, N = 1, \ldots, 4$ are four-dimensional Lorentz indices while and $\mu, \nu = 1, 2, 3$ are three-dimensional indices. We normalize the Lie algebra generators as $\mathrm{Tr}\, t^A t^B = \frac{1}{2}\delta^{AB}$.

$$\Psi = \begin{pmatrix} \lambda \\ \bar{\psi} \end{pmatrix}, \tag{14}$$

where λ, ψ are two-component (complex) Weyl spinors. In three dimensions λ, ψ represent two Dirac spinors.

We must use the Kaluza-Klein (KK) mode decomposition for all fields in the Lagrangian. If we discard all modes other than zero we will arrive at a three-dimensional theory with a gauge field, a scalar field in the adjoint, and two three-dimensional spinors. The $S_1 \times R_3$ reduction of R_4 Yang-Mills does not quite lead to three-dimensional Yang-Mills, but at first, we will ignore this nuance, to be discussed in detail later, and will briefly review the phenomena that occur in three-dimensional Yang-Mills with a scalar field in the adjoint (discarding fermions for the time being).

Long ago Polyakov considered three-dimensional SU(2) Georgi-Glashow model (a Yang-Mills + adjoint Higgs system) in the Higgs regime [13]. In this regime SU(2) is broken down to U(1), so that at low energies the theory reduces to compact electrodynamics. The dual photon is a scalar field σ of the phase type (i.e. it is defined on the interval $[0, 2\pi]$)

$$F_{\mu\nu} = \frac{g_3^2}{4\pi}\varepsilon_{\mu\nu\rho}(\partial^\rho \sigma), \tag{15}$$

where g_3^2 is the three-dimensional gauge coupling with mass dimension $[g_3^2] = +1$. In perturbation theory the dual photon σ is massless. However, it acquires a mass due to instantons (technically, the latter are identical to the 't Hooft-Polyakov monopoles, after the substitution of one spatial dimension by imaginary time; that's why below we will refer to them as to the instanton-monopoles). In the vacuum of the theory, one deals with a gas of instantons interacting according to the Coulomb law. The dual photon mass is due to the Debye screening. In fact, the dual photon mass is determined by the one-instanton vertex

$$m_\sigma \sim m_W^{5/2} g_3^{-3} e^{-S_0/2}, \tag{16}$$

where S_0 is the one-instanton action,

$$S_0 = 4\pi \frac{m_W}{g_3^2}, \tag{17}$$

m_W is the lightest W-boson mass; see below. In terms of four-dimensional quantities $S_0 = 8\pi^2/(Ng^2)$. As a result, the low-energy theory is described by a three-dimensional sine-Gordon model,

$$\mathcal{L}_\sigma = \frac{g_3^2}{32\pi^2}(\partial_\mu \sigma)^2 + c_1 m_W^5 g_3^{-4} e^{-S_0}\cos\sigma, \tag{18}$$

where c_1 is an undetermined prefactor. The coefficient in front of $e^{-S_0}\cos\sigma$,

$$\mu \equiv c_1 m_W^5 g_3^{-4},$$

has mass dimension $[\mu] = +3$. The combination μe^{-S_0} is the monopole fugacity.

This model supports a domain line[8] (with σ field vortices at the endpoints) which in $1 + 2$ dimensions must be interpreted as a string. Since the σ field dualizes three-dimensional photon, the σ field vortices in fact represent electric probe charges in the original formulation, connected by the electric flux tubes which look like domain lines in the dual formulation.

Now, if we switch on massless adjoint fermions, as in [25], the mass gap generation does not occur in the Polyakov model *per se*. This is due to the fact that the instanton-monopoles acquire fermion zero modes which preclude the potential term as in Eq. (18). Correspondingly, the dual photons remain massless and the model no longer supports domain lines. The linear confinement is gone.

This situation changes, however, if three-dimensional Yang-Mills theory is obtained as a low-energy reduction of a four-dimensional gauge theory on $S_1 \times R_3$ with small $r(S_1)$. When the adjoint Higgs field is compact, as in Fig. 2, in addition to $N - 1$ 't Hooft-Polyakov monopole-instantons there is one extra monopole [whose existence is tied up to $\pi_1(S_1) \neq 0$]. It can be referred to as the Kaluza-Klein (KK) monopole-instanton.[9] Each of these monopoles carries fermion zero modes, hence they cannot contribute to the bosonic potential at the level e^{-S_0}. They can and do contribute at the level e^{-2S_0}.

Indeed, the bound state of the 't Hooft-Polyakov monopole-instanton with magnetic charge $\boldsymbol{\alpha}_i$ and antimonopole with charge $-\boldsymbol{\alpha}_{i+1}$ has no fermion zero modes: its topological charge coincides with that of the perturbative vacuum. Hence, such a bound state can contribute to the bosonic potential. Let

$$\Delta_{\text{aff}}^0 = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_N\} \tag{19}$$

denote the extended (affine) root system of SU(N) Lie algebra. If we normalize the magnetic and topological charges of the monopoles as

[8]Similar to the axion domain wall.

[9]The eigenvalues shown in Fig. 2 may be viewed as Euclidean D2-branes. N split branes support a spontaneously broken $U(1)^N$ gauge theory, whose U(1) center of mass decouples, and the resulting theory is $U(1)^{N-1}$. The $N - 1$ 't Hooft-Polyakov monopoles may be viewed as Euclidean D0 branes connecting the eigenvalues $(a_1 \to a_2)$, $(a_2 \to a_3), \ldots, (a_{N-1} \to a_N)$. Clearly, we can also have a monopole which connects $(a_N \to a_1)$ which owes its existence to the periodicity of the adjoint Higgs field, or equivalently, to the fact that the underlying theory is on $S_1 \times R_3$. Usually it is called the KK monopole. The Euclidean D0 branes with the opposite orientation, connecting $(a_j \leftarrow a_{j+1})$, $j = 1, \ldots, N$, are the antimonopoles. This viewpoint makes manifest the fact that the KK and 't Hooft-Polyakov monopoles are all on the same footing. The magnetic and topological charges of the monopoles connecting $(a_j \leftrightarrow a_{j+1})$ is $\pm((4\pi/g)\boldsymbol{\alpha}_j, \frac{1}{N})$ where the direction of the arrow is correlated with the sign of the charges.

$$\left(\int_{S^2} F, \int \frac{g^2}{32\pi^2} F^a_{MN}\tilde{F}^{MN,a}\right) = \left(\pm\frac{4\pi}{g}\boldsymbol{\alpha}_i, \pm\frac{1}{N}\right), \quad \text{for } \boldsymbol{\alpha}_i \in \pm\Delta^0_{\text{aff}}, \tag{20}$$

where $\boldsymbol{\alpha}_i$ stands for the simple roots of the affine Lie algebra then the following bound states are relevant:

$$\left[\frac{4\pi}{g}\boldsymbol{\alpha}_i, \frac{1}{N}\right] + \left[-\frac{4\pi}{g}\boldsymbol{\alpha}_{i+1}, -\frac{1}{N}\right] = \left[\frac{4\pi}{g}(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i+1}), 0\right]. \tag{21}$$

This pair is stable, as was shown in Ref. [14], where it is referred to as a magnetic bion. Thus, we can borrow Polyakov's discussion of magnetic monopoles and apply directly to these objects. The magnetic bions will induce a mass term for the dual photons via the Debye screening, the essence of Polyakov's mechanism.

The vacuum field (9) of the deformed SU(N) theory respects the (approximate) center symmetry Z_N. This field configuration breaks the gauge symmetry as indicated in (10). Because of the gauge symmetry breaking, electrically charged particles acquire masses. (By electric charges we mean charges with regards to $N-1$ "photons" of the low-energy theory.) The set of $N-1$ electric charges and masses of N lightest W-bosons are

$$\boldsymbol{q}_{W_\alpha} = g\boldsymbol{\alpha}, \qquad m_{W_\alpha} = \frac{2\pi}{NL}, \tag{22}$$

where $\boldsymbol{\alpha}_i$ ($i = 1, \ldots, N$) are the simple and affine roots of the SU(N) Lie algebra [see Eq. (27)]. Note that N lightest W-bosons are degenerate in the center-symmetric vacuum. The remaining $N^2 - N$ charged W-bosons can be viewed as composites of the above.

The stabilizing double-trace term (4) contributes to the self-interaction of the physical (neutral) Higgs fields. Assuming that all coefficients d are of order one, the masses of these fields are $\mathcal{O}(g/L)$. For instance, for SU (2) and SU(3) the physical Higgs masses are $(g\sqrt{d_1})/L$. These masses are much lighter than those of the W-bosons but much heavier than those of the fields in the effective low-energy Lagrangian [dual photons; see Eq. (24) below]. The stabilizing double-trace term (4) also contributes to corrections to the W-boson masses. They are expandable in g^2, i.e.

$$m_{W_\alpha} = \frac{2\pi}{NL}(1 + \mathcal{O}(g^2)).$$

In the SU(N) gauge theory with an adjoint fermion on $R_3 \times S_1$, which is Higgsed according to (10), the bosonic part of the effective low-energy Lagrangian is generated by the pairs (21), and hence the potential is proportional to e^{-2S_0}, rather than e^{-S_0} of the Polyakov problem. If we introduce an $(N-1)$-component vector $\boldsymbol{\sigma}$,

$$\boldsymbol{\sigma} \equiv (\sigma_1, \ldots, \sigma_{N-1}), \tag{23}$$

representing $N-1$ dual photons of the U(1)$^{N-1}$ theory, the bosonic part of the effective Lagrangian can be written as

$$\mathcal{L}(\sigma_1, \ldots, \sigma_{N-1}) = \frac{g_3^2}{32\pi^2}(\partial_\mu\boldsymbol{\sigma})^2 + c m_W^6 g_3^{-6} e^{-2S_0} \times \sum_{i=1}^{N} \cos(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i+1})\boldsymbol{\sigma}, \tag{24}$$

where c is an undetermined coefficient and g_3 is the three-dimensional coupling constant

$$g_3^2 = g^2 L^{-1}. \tag{25}$$

In terms of four-dimensional variables, the magnetic bion fugacity is

$$m_W^6 g_3^{-6} e^{-2S_0} \sim m_W^3 g^{-6} e^{-2S_0}. \tag{26}$$

We remind that $\boldsymbol{\alpha}_i$ ($i = 1, \ldots, N-1$) represent the magnetic charges of $(N-1)$ types of the 't Hooft-Polyakov monopoles while the affine root

$$\boldsymbol{\alpha}_N = -\sum_{i=1}^{N-1} \boldsymbol{\alpha}_i \tag{27}$$

is the magnetic charge of the KK monopole. Note that the bion configurations that contribute to the effective Lagrangian have magnetic charges $\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i+1}$ and vertices $e^{i(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i+1})\boldsymbol{\sigma}}$, corresponding to a product of a monopole vertex $e^{i\boldsymbol{\alpha}_i\boldsymbol{\sigma}}$ with charge $\boldsymbol{\alpha}_i$, and antimonopole vertex $e^{-i\boldsymbol{\alpha}_{i+1}\boldsymbol{\sigma}}$ with charge $-\boldsymbol{\alpha}_{i+1}$ (without the zero mode insertions). With the Z_N-symmetric vacuum field (9) all fugacities are equal.

Equation (24) implies that nonvanishing masses proportional to e^{-S_0} are generated for all σ's. They are much smaller than the masses in the Polyakov model in which they are $\sim e^{-S_0/2}$.

There are $N-1$ types of Abelian strings (domain lines). Their tensions are equal to each other and proportional to e^{-S_0}. Linear confinement develops at distances larger than e^{S_0}.

Needless to say, the physical spectrum in the Higgs/Abelian confinement regime is richer than that in the non-Abelian confinement regime. If in the latter case only color singlets act as asymptotic states, in the Abelian confinement regime all systems that have vanishing $N-1$ electric charges have finite mass and represent asymptotic states.

Note 1: For SU(2) and SU(3) Yang-Mills theories, the double-trace deformation is a particularly simple monomial

$$P[U(\mathbf{x})] = \frac{2}{\pi^2 L^4} d_1 |\mathrm{Tr} U(\mathbf{x})|^2 \quad \text{for SU(2), SU(3).} \tag{28}$$

Note 2: One can be concerned that the deformation potential is given in terms of multiwinding line operators, and looks nonlocal. In the $L\Lambda \ll 1$ region where the deformation is crucial, there is no harm in viewing the deforming operator as "almost local" since we are concerned with physics at scales much larger than the compactification scale. In the decompactification limit where the deformation is indeed nonlocal, it is not needed since its dynamical role is negligible. If one wants to be absolutely certain, one can insert a filter function as the coefficient of the double-trace operator which shuts it off exponentially $\sim e^{-L^2\Lambda^2}$ at large L in order not to deal with a nonlocal theory.

III. QCD WITH ONE FUNDAMENTAL FERMION

QCD(F) on R_4 possesses a $\mathrm{U}(1)_V \times \mathrm{U}(1)_A$ symmetry, at the classical level acting as

$$\Psi \to e^{i\alpha}\Psi, \qquad \Psi \to e^{i\beta\gamma_5}\Psi.$$

Because of nonperturbative effects, only the anomaly-free Z_2 subgroup of the $\mathrm{U}(1)_A$ is the genuine axial symmetry of the theory, the fermion number mod 2. This symmetry is already a part of the vector $\mathrm{U}(1)_V$ symmetry and, hence, cannot be spontaneously broken. However, a bifermion condensate (which does not break any chiral symmetry) is believed to exist on R_4 as well as on $S_1 \times R_3$ with sufficiently large $r(S_1)$.

The microscopic QCD Lagrangian also possesses the discrete symmetries C, P, T, and continuous three-dimensional Euclidean Lorentz symmetry SO(3). Thus, the symmetries of the original theory are

$$\mathrm{U}(1)_V \times C \times P \times T. \tag{29}$$

The double-trace deformation respects all these symmetries. (Otherwise this would explicitly contradict the claim made in Sec. I.) Below, we will construct a low-energy effective theory QCD(F)* assuming that the double-trace terms stabilize the theory in the center-symmetric vacuum. As usual, the set of all possible operators that can appear in the effective low-energy theory is restricted by the underlying symmetries (29).

Integrating out weakly coupled KK modes with nonvanishing frequencies

$$|\omega_n| \geq \frac{2\pi n}{L}, \qquad n \neq 0,$$

and adding the stabilizing deformation term (4) to the QCD (F) Lagrangian, we obtain the QCD(F)* theory. This is the Yang-Mills + compact adjoint Higgs system with fundamental fermions on R_3.

The action is[10]

$$S = \int_{R_3} \frac{L}{g^2}\bigg[\mathrm{Tr}\Big(\frac{1}{2}F_{\mu\nu}^2 + (D_\mu\Phi)^2 + g^2 V[\Phi]\Big) + i\bar{\lambda}(\sigma_\mu(\partial_\mu + iA_\mu) + i\sigma_4\Phi)\lambda + i\bar{\psi}(\sigma_\mu(\partial_\mu - iA_\mu) - i\sigma_4\Phi)\psi\bigg], \tag{30}$$

where ψ and λ are the two-component three-dimensional Dirac spinors which arise upon reduction of the four-dimensional Dirac spinor Ψ. Note that λ and ψ have opposite gauge charges, where λ and $\bar{\psi}$ are fundamental and $\bar{\lambda}$ and ψ are antifundamental. As usual, in Euclidean space, there is no relation between barred and unbarred variables, and they are not related to each other by conjugations.

The potential $V[\Phi]$ which is the sum of the one-loop potential and deformation potential has its minimum located at (6) [or (9)]. The fermion contribution to the effective one-loop potential involves terms such as $\mathrm{Tr}U + \mathrm{Tr}U^*$. These terms explicitly break the Z_N center symmetry and slightly shift the position of the eigenvalues of $\langle U\rangle$ from the minimum (6). However, this is a negligible $\mathcal{O}(g/d_n)$ effect suppressed by a judicious choice of the deformation parameters. Hence, we neglect this effect below.[11]

There are $N-1$ distinct U(1)'s in this model, corresponding to $N-1$ distinct electric charges. If we introduce a quark Ψ in the fundamental representation of SU(N) each component Ψ_i ($i = 1, \ldots, N$) will be characterized by a set of $N-1$ charges, which we will denote by $\boldsymbol{q}_{\Psi_i}$,

$$\boldsymbol{q}_{\Psi_i} = g\boldsymbol{H}_{ii}, \qquad i = 1, \ldots, N, \tag{31}$$

where $\boldsymbol{H}$ is the set of $N-1$ Cartan generators.

All fundamental fermions but two (one of each type ψ and λ) acquire masses due to gauge symmetry breaking. These masses are of order of $2\pi/L$ and depend on whether periodic or antiperiodic boundary conditions are imposed. The fermions that remain massless in perturbation theory are the ones corresponding to the vanishing (mod 2π) eigenvalue of the algebra-valued compact Higgs field Φ; see Eq. (9) [equivalently, $v = 1$; see Eq. (6)].

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[10]Our four-dimensional Dirac γ matrix conventions are

$$\gamma_M = \{\gamma_\mu, \gamma_4\}, \qquad \gamma_\mu = \sigma_1 \otimes \sigma_\mu, \qquad \gamma_4 = \sigma_2 \otimes I.$$

With this choice, the Dirac algebras in four and three dimensions are $\{\gamma_M, \gamma_N\} = 2\delta_{MN}$ and $\{\sigma_\mu, \sigma_\nu\} = 2\delta_{\mu\nu}$. It will be convenient to define $\bar{\sigma}_M = (\sigma_\mu, -iI) \equiv (\sigma_\mu, \sigma_4)$ and $\sigma_M = (\sigma_\mu, iI) \equiv (\sigma_\mu, -\sigma_4)$.

[11]If the eigenvalues are separated not equidistantly, yet the separations are nonvanishing for any pair, the gauge symmetry breaking $\mathrm{SU}(N) \to \mathrm{U}(1)^{N-1}$ still takes place. In the nonperturbative analysis below, this fact manifests itself as an unequal action (or fugacity) for different types of monopoles. The analysis in this latter case will not be qualitatively different.

Thus, the low-energy effective Lagrangian includes $N-1$ photons and two fermions. Their interactions (in particular, an induced mass gap) must arise due to nonperturbative effects.[12]

A. Nonperturbative effects and the low-energy Lagrangian

Nonperturbatively, there exist topologically stable, semiclassical field configurations—instanton-monopoles. If the adjoint Higgs field were noncompact, there would be $(N-1)$ types of fundamental monopoles. There is, however, an extra KK monopole which arises due to the fact that the underlying theory is formulated on a cylinder, $R_3 \times S_1$, or simply speaking, $\Phi(\mathbf{x})$ is compact. The magnetic and topological charges of the (anti)monopoles associated with root $\boldsymbol{\alpha}_i$ are given Eq. (20).

As follows from the explicit zero mode constructions of Jackiw and Rebbi [26] and the Callias index theorem [27], there are two fermion zero modes localized on one of the N constituent monopoles. van Baal *et al.* demonstrated [28–31] that as the boundary conditions of fermions vary in the background with nontrivial holonomy, the zero modes hop from a monopole to the next one. With fixed boundary conditions, they are localized, generally speaking, on a particular monopole.[13]

The above implies that one of the monopole-induced vertices has two fermion insertions (the one on which the fermion zero modes are localized) and other $N-1$ elementary monopoles have no fermion insertions (at the level e^{-S_0}). The set of the instanton-monopole-induced vertices can be summarized as follows:

$$\{e^{-S_0} e^{i\boldsymbol{\alpha}_1\boldsymbol{\sigma}}\lambda\psi,\, e^{-S_0} e^{i\boldsymbol{\alpha}_j\boldsymbol{\sigma}},\, j=2,\ldots,N\}, \tag{32}$$

plus complex conjugate for antimonopoles. Thus, the leading nonperturbatively induced interaction terms in the effective Lagrangian are

$$S^{\mathrm{QCD(F)}^*} = \int_{R_3}\bigg[\frac{g_3^2}{32\pi^2}(\partial_\mu\boldsymbol{\sigma})^2 + \frac{1}{g_3^2} i\bar{\Psi}\gamma^\mu(\partial_\mu + i\boldsymbol{q}_\Psi \boldsymbol{A}_\mu) + e^{-S_0}\Big(\tilde{\mu} e^{i\boldsymbol{\alpha}_1\boldsymbol{\sigma}}\lambda\psi + \mu \sum_{\boldsymbol{\alpha}_j \in (\Delta^0_{\mathrm{aff}} - \boldsymbol{\alpha}_1)} e^{i\boldsymbol{\alpha}_j\boldsymbol{\sigma}} + \mathrm{H.c.}\Big)\bigg], \tag{33}$$

where $\tilde{\mu}$ is dimensionless constant. Note the noncanonical normalization of the bosonic and fermionic terms. This choice for fermions will ease the derivations of certain physical quantities. It is clearly seen that in the infrared description of QCD(F)*, we must deal not only with the dual photons, but also with electrically charged fermions.

The three-dimensional effective Lagrangian respects the symmetries (29) of the microscopic (four-dimensional) theory. In particular, the fermion bilinears such as $\bar{\lambda}\lambda$ (allowed by $\mathrm{U}(1)_V$ and the Lorentz symmetry of the three-dimensional theory) are noninvariant under parity (see appendix in Ref. [25]) and, hence, cannot be generated. On the other hand, $\langle\lambda\psi\rangle \neq 0$ can and is generated. One can check that, up to order e^{-2S_0}, the Lagrangian (33) includes all possible operators allowed by the symmetries (29).

In the above Lagrangian, all operators are relevant in the renormalization-group sense. The fugacity has mass dimension $+3$. If the kinetic term for fermion is canonically normalized, the covariant photon-fermion interaction and instanton-monopole-induced term with the fermion insertion has dimension $+1$. Which operators will dominate the IR physics? The answer to this question requires a full renormalization-group analysis of all couplings. A preliminary investigation (along the lines of Ref. [33]) shows that quantum corrections in the running of the couplings are tame and do not alter the fact that the instanton-monopole vertex terms are the most relevant in the IR of QCD(F)*.

The $N-1$ linearly independent instanton-monopole vertices render all the $N-1$ dual photons massive, with masses proportional to $e^{-S_0/2}$. Thus, the dual scalars are pinned at the bottom of the potential

$$\mu e^{-S_0}\sum_{j=2}^{N}\cos\boldsymbol{\alpha}_j\boldsymbol{\sigma}. \tag{34}$$

As a result, the would-be massless fermions will also acquire a mass term of the type

$$\tilde{\mu} e^{-S_0}\lambda\psi. \tag{35}$$

The fermion mass is proportional to e^{-S_0}. Hence it is exponentially smaller than the dual photon mass $\sim e^{-S_0/2}$. Note that the fermion mass term is not associated with the spontaneous breaking of chiral symmetry. This circumstance, as well as the hierarchy of masses between the photons and fermions, is specific to one *fundamental* fermion and will change in the case of the two-index fermions.

[12]It is important to distinguish this theory from the case of the noncompact adjoint Higgs field, which is the Polyakov model with massless (complex-representation) fermions. Both theories have identical gauge symmetry breaking patterns: $\mathrm{SU}(N) \to \mathrm{U}(1)^{N-1}$. In perturbation theory, both theories reduce (by necessity) to compact QED_3 with fermions. However, it is possible to prove that the latter theory lacks confinement since photons remain massless nonperturbatively. This implies that if the symmetries at the cutoff scale are not specified, the question of confinement in compact QED_3 with massless fermions is ambiguous. The issue will be further discussed in a separate publication.

[13]More precisely, the Callias index applies to R_3. We need an index theorem for the Dirac operators in the background of monopoles on $R_3 \times S_1$. Such a generalization of the Callias index theorem was carried out in the work of Nye and Singer [32]. For a clear-cut lattice realization of the fermion zero modes explicitly showing on which monopole they are localized, see Ref. [28].

Since all $N-1$ dual photons become massive, a probe quark Q_i of every type ($i = 1, \ldots, N$) will be connected to its antiquark by a domain line/string with the tension[14]

$$T \sim g_3 \mu^{1/2} e^{-S_0/2}. \tag{36}$$

The string between Q_1 and $\bar{Q}_1$ is easily breakable due to pair production of λ's and ψ's. In other words, the external charge Q_1 will be screened by the dynamical fermions with charge $\boldsymbol{q}_{\Psi_1}$. The strings between Q_i and $\bar{Q}_i$ (with $i = 2, \ldots, N$) can break with an exponentially small probability due to pair creation of the KK modes of Ψ_i. This amounts, of course, to the conventional statement about large Wilson loops C,

$$\left\langle \frac{1}{N} \mathrm{Tr} W(C) \right\rangle \sim \frac{1}{N} \sum_{i=1}^{N} \left\langle e^{i \int_C \boldsymbol{H}_{ii} \boldsymbol{A}} \right\rangle = \frac{1}{N} e^{-\kappa P(C)} + \left(1 - \frac{1}{N}\right) e^{-T \mathrm{Area}(\Sigma)}, \tag{37}$$

where κ is the coefficient of the perimeter law, $P(C)$ is the perimeter of the loop C, the boundary of a surface Σ.

Remark: The product of the instanton-monopole-induced vertices is proportional to the Belyavin-Polyakov-Schwarz-Tyupkin (BPST) four-dimensional instanton vertex [34],

$$(e^{-S_0} e^{i \boldsymbol{\alpha}_1 \boldsymbol{\sigma}} \lambda \psi) \prod_{j=2}^{N} (e^{-S_0} e^{i \boldsymbol{\alpha}_j \boldsymbol{\sigma}}) \sim e^{-(8\pi^2/g^2)} \bar{\Psi}(1 + \gamma_5) \Psi \exp\left(i \sum_{i=1}^{N} \boldsymbol{\alpha}_i \boldsymbol{\sigma}\right) = e^{-(8\pi^2/g^2)} \bar{\Psi}(1 + \gamma_5) \Psi. \tag{38}$$

This is consistent with the fact that the instanton-monopoles can be viewed as the BPST instanton constituents. In Eq. (38) we used the fact that the sum of the N constituent instanton-monopole actions is in fact the BPST instanton action, and the sum of the magnetic and topological charges of the constituent monopoles gives the correct quantum numbers of the BPST R_4 instanton,

$$\sum_{i=1}^{N} \left(\int F, \int \frac{g^2}{32\pi^2} F^a_{MN} \tilde{F}^{MN,a} \right)_i = (0, 1), \tag{39}$$

see Eq. (20).

B. Bifermion condensate

As stated earlier, one-flavor QCD formulated on R_4 has no chiral symmetry whatsoever. The axial anomaly reduces the classical $\mathrm{U}(1)_A$ symmetry to Z_2. A bifermion condensate exists and breaks no chiral symmetry. We can evaluate the value of the chiral condensate in QCD(F)* in the small $r(S_1)$ regime. At large $r(S_1)$ (strong coupling) we know, from volume independence, that the condensate must get a value independent of the radius. Let b_0 denote the leading coefficient of the β function divided by N,

$$b_0 = \frac{1}{N}\left(\frac{11N}{3} - \frac{2N_f}{3}\right)\Bigg|_{N_f=1} = \frac{11}{3} - \frac{2}{3N}. \tag{40}$$

At weak coupling, $L\Lambda \ll 1$, the bifermion condensate in QCD(F)* receives its dominant contribution from the instanton-monopole with the fermion zero modes insertion, the first term in the second line in Eq. (33). The condensate is proportional to

$$\langle \lambda \psi \rangle \sim e^{-S_0} \sim e^{-(8\pi^2/g^2 N)}. \tag{41}$$

Above the scale $L\Lambda \sim 1$ we expect the bifermion condensate to be L-independent and saturate its value on R_4,

$$\langle \bar{\Psi}\Psi \rangle \sim \begin{cases} \Lambda^3 (\Lambda L)^{b_0 - 3} = \Lambda^3 (\Lambda L)^{(2/3)(1-N^{-1})}, & L\Lambda \ll 1, \\ \Lambda^3 (1 + \mathcal{O}(\frac{1}{\Lambda L})), & L\Lambda \gtrsim 1. \end{cases} \tag{42}$$

The above formula is testable on lattices.

It is natural to believe the saturation scale is associated with the transition from weak to strong coupling and restoration of the spontaneously broken gauge symmetry $\mathrm{U}(1)^{N-1} \to \mathrm{SU}(N)$. This is the regime where the theory passes from the Abelian to non-Abelian confinement. The effective theory (33) which is only valid at $L\Lambda \ll 1$ loses its validity when this parameter becomes of order one. Nonetheless, we do not expect phase transitions (or rapid crossovers) in the parameter $L\Lambda$. We expect physics of the two regimes to be continuously connected.

It would be immensely useful to study this passage on lattices. In the strong coupling regime, the volume dependent factors enter in observables only via subleading $\mathcal{O}(1/(L\Lambda))$ terms.

IV. QCD WITH ONE BIFUNDAMENTAL FERMION

Consider orbifold QCD, a gauge theory with the $\mathrm{SU}(N)_1 \times \mathrm{SU}(N)_2$ gauge group, and one bifundamental Dirac fermion, defined on $R_3 \times S_1$,

$$S^{\mathrm{QCD(BF)}} = \int_{R_3 \times S_1} \frac{1}{g^2} \mathrm{Tr}\left[\frac{1}{2} F^2_{1,MN}(x) + \frac{1}{2} F^2_{2,MN}(x) + i \bar{\Psi} \not{D} \Psi\right], \tag{43}$$

where

$$D_M \Psi = \partial_M \Psi + i A_{1,M} \Psi - i \Psi A_{2,M}.$$

The theory possesses a $\mathrm{U}(1)_V \times (Z_{2N})_A \times (Z_2)_I$ symmetry which acts on the elementary fields as

[14] This is also similar to the axion domain wall.

$$\begin{aligned} &\mathrm{U}(1)_V\colon\ \lambda \to e^{i\alpha}\lambda, \qquad \psi \to e^{-i\alpha}\psi, \\ &(Z_{2N})_A\colon\ \lambda \to e^{i(2\pi/2N)}\lambda, \qquad \psi \to e^{i(2\pi/2N)}\psi, \\ &(Z_2)_I\colon\ \lambda \leftrightarrow \psi, \qquad A_{\mu,1} \leftrightarrow A_{\mu,2}. \end{aligned} \tag{44}$$

The $(Z_{2N})_A$ symmetry is the anomaly-free subgroup of the axial $\mathrm{U}(1)_A$. It is a folklore statement that with sufficiently large $r(S_1)$, the chiral symmetry is broken down to Z_2 by the formation of the bifermion condensate,

$$\langle \bar{\Psi}\Psi \rangle = 4N\Lambda^3 \cos\left(\frac{2\pi k}{N}\right), \qquad k = 0, 1, \ldots, N-1, \tag{45}$$

marking N isolated vacua in the same manner as in $\mathcal{N} = 1$ SYM theory.

QCD(BF) on R_4 is believed to confine in the same way as $\mathcal{N} = 1$ SYM theory, possesses a mass gap, and N isolated vacua. We would like to shed some light on these issues by studying QCD(BF)* with small $r(S_1)$.

A. Deformed orbifold QCD

On $S_1 \times R_3$ we can deform original QCD(BF)

$$\begin{aligned} S = \int_{R_3} \frac{L}{g^2} \mathrm{Tr}\Big[&\frac{1}{2}F^2_{1,\mu\nu} + \frac{1}{2}F^2_{2,\mu\nu} + (D_\mu\Phi_1)^2 + (D_\mu\Phi_2)^2 \\ &+ g^2 V[\Phi_1, \Phi_2] + i\bar{\lambda}(\sigma_\mu(\partial_\mu\lambda + iA_{1,\mu}\lambda - i\lambda A_{2,\mu}) \\ &+ i\sigma_4(\Phi_1\lambda - \lambda\Phi_2)) \\ &+ i\bar{\psi}(\sigma_\mu(\partial_\mu\psi - iA_{1,\mu}\psi + i\psi A_{2,\mu}) \\ &- i\sigma_4(\Phi_1\psi - \psi\Phi_2))\Big], \end{aligned} \tag{46}$$

by adding double-trace terms (4) in such a way that the center symmetry is not broken in the vacuum. The center symmetry stability at weak coupling implies that the vacuum of the theory is located at

$$L\langle\Phi_1\rangle = L\langle\Phi_2\rangle = \mathrm{diag}\left(-\frac{2\pi[N/2]}{N}, -\frac{2\pi([N/2]-1)}{N}, \ldots, \frac{2\pi[N/2]}{N}\right), \qquad (\mathrm{mod}\ 2\pi); \tag{47}$$

cf. Equation (9). Consequently, in the weak coupling regime, the gauge symmetry is broken,

$$[\mathrm{SU}(N)]_1 \times [\mathrm{SU}(N)]_2 \to [\mathrm{U}(1)^{N-1}]_1 \times [\mathrm{U}(1)^{N-1}]_2. \tag{48}$$

In perturbation theory $2(N-1)$ photons remain massless while all off-diagonal gauge fields acquire masses in the range $[\frac{2\pi}{NL}, \frac{2\pi}{L}]$. The three-dimensional mass terms of the bifundamental fermions are determined by

$$\sum_{i,k=1}^{N} (a^1_i - a^2_k)\bar{\Psi}^k_i \gamma_4 \Psi^k_i,$$

where a^1_k, a^2_k are the eigenvalues of Φ_1 and Φ_2; see Eq. (47). The diagonal components of the bifundamental fermions

$$(\lambda^i_k, \psi^k_i)_{i=k}$$

remain massless to all orders in perturbation theory; we will refer to them as λ_i, ψ_i ($i = 1, \ldots, N$). Other components get masses $\sim 2\pi(i-k)/L$, and decouple in the low-energy limit, and so do the W-bosons.

The bifundamental fermions are electrically charged under the unbroken $[\mathrm{U}(1)^{N-1}]_1 \times [\mathrm{U}(1)^{N-1}]_2$ in a correlated fashion. If in Sec. III the electric charges of each fermion were characterized by an $(N-1)$-dimensional vector $\boldsymbol{q}_{\Psi_i}$, now they are characterized by concatenation of two such $N-1$ dimensional electric charge vectors

$$\boldsymbol{q}_{\lambda_i} = g(+\boldsymbol{H}_{ii}, -\boldsymbol{H}_{ii}), \qquad \boldsymbol{q}_{\psi_i} = g(-\boldsymbol{H}_{ii}, +\boldsymbol{H}_{ii}), \qquad i = 1, \ldots, N. \tag{49}$$

Thus, the low-energy effective Lagrangian in perturbation theory is

$$\begin{aligned} S^{\text{pert th}} = \int_{R_3} \frac{1}{g_3^2} \Big[&\sum_{a=1}^{N-1} \left(\frac{1}{4}F^{a,2}_{1,\mu\nu} + \frac{1}{4}F^{a,2}_{2,\mu\nu}\right) \\ &+ \sum_{i=1}^{N} i\bar{\Psi}_i\gamma_\mu(\partial_\mu + i\boldsymbol{H}_{ii}\boldsymbol{A}^1_\mu - i\boldsymbol{H}_{ii}\boldsymbol{A}^2_\mu)\Psi_i\Big]. \end{aligned} \tag{50}$$

The mass gap must arise due to nonperturbative effects, as in Sec. III. We will identify and classify nonperturbative effects induced by topologically nontrivial field configurations momentarily.

B. Nonperturbative low-energy effective Lagrangian

Nonperturbatively, the gauge symmetry breaking pattern (47) implies the existence of N types of instanton-monopoles associated with each gauge group. The magnetic and topological charges of these objects are

$$\left(\int_1 F, \int_1 \frac{g^2}{32\pi^2}F^a\tilde{F}^a; \int_2 F, \int_2 \frac{g^2}{32\pi^2}F^a\tilde{F}^a\right) = \begin{cases} (\pm\frac{4\pi}{g}\boldsymbol{\alpha}_i, \pm\frac{1}{N}, 0, 0) & , \\ (0, 0, \pm\frac{4\pi}{g}\boldsymbol{\alpha}_i, \pm\frac{1}{N}) & . \end{cases} \tag{51}$$

Consequently, each monopole generates two fermion zero modes, and the instanton-monopole vertices are

$$\begin{aligned}
\mathcal{M}_i^1&: \left(+\frac{4\pi}{g}\boldsymbol{\alpha}_i, +\frac{1}{N}, 0, 0\right): e^{+i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_1}(\lambda_i\psi_i + \lambda_{i+1}\psi_{i+1}),\\
\bar{\mathcal{M}}_i^1&: \left(-\frac{4\pi}{g}\boldsymbol{\alpha}_i, -\frac{1}{N}, 0, 0\right): e^{-i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_1}(\bar{\lambda}_i\bar{\psi}_i + \bar{\lambda}_{i+1}\bar{\psi}_{i+1}),\\
\mathcal{M}_i^2&: \left(0, 0, +\frac{4\pi}{g}\boldsymbol{\alpha}_i, +\frac{1}{N}\right): e^{+i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_2}(\lambda_i\psi_i + \lambda_{i+1}\psi_{i+1}),\\
\bar{\mathcal{M}}_i^2&: \left(0, 0, -\frac{4\pi}{g}\boldsymbol{\alpha}_i, -\frac{1}{N}\right): e^{-i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_2}(\bar{\lambda}_i\bar{\psi}_i + \bar{\lambda}_{i+1}\bar{\psi}_{i+1}),
\end{aligned} \tag{52}$$

where $\boldsymbol{\sigma}_1$ is the set of dual photons for $[\mathrm{U}(1)^{N-1}]_1$ while $\boldsymbol{\sigma}_2$ is the set of dual photons for $[\mathrm{U}(1)^{N-1}]_2$. In full analogy with the SYM theory, the $2N$ fermion zero modes of the BPST instanton split into N pairs: each instanton-monopole supports two fermion zero modes. This is a natural consequence of the Callias index theorem. (The same conclusion was also reached by Tong [35]).

As a result, the instanton-monopole contributions give rise to the following terms in the effective Lagrangian:

$$\Delta L^{\mathrm{QCD(BF)}*} = \mathrm{const} \times g^{-6}e^{-S_0}\sum_{\alpha_i\in\Delta^0_{\mathrm{aff}}}((e^{i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_1} + e^{i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_2}) \times (\lambda_i\psi_i + \lambda_{i+1}\psi_{i+1}) + \mathrm{H.c.}). \tag{53}$$

At the level e^{-S_0} the instanton-monopole effects in QCD(BF)* cannot provide mass terms for the dual photons. This situation is completely analogous to that in QCD(Adj)* where all instanton-monopoles have fermion zero modes and, hence, are unable to contribute to the bosonic potential for the dual photons $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$.

The situation drastically changes at order e^{-2S_0}. There are nontrivial effects which render the long-distance three-dimensional fields massive, implying confinement. An easy way to see that this is the case is to examine the symmetries of the theory.

Since $\mathrm{U}(1)_V \times (Z_{2N})_A \times (Z_2)_I$ is the symmetry of the microscopic theory, it must be manifest in the low-energy effective theory in three dimensions. The invariance of the instanton-monopole vertex under $\mathrm{U}(1)_V$ and $(Z_2)_I$ is manifest. At the same time, the $(Z_{2N})_A$ invariance requires combining the axial chiral symmetry with the discrete shift symmetry of the dual photon

$$(Z_{2N})_A\colon\ \lambda\psi \to e^{i(2\pi/N)}\lambda\psi, \qquad \boldsymbol{\sigma}_{1,2} \to \boldsymbol{\sigma}_{1,2} - \frac{2\pi}{N}\boldsymbol{\rho}, \tag{54}$$

where $\boldsymbol{\rho}$ is the Weyl vector defined by

$$\boldsymbol{\rho} = \sum_{j=1}^{N-1}\boldsymbol{\mu}_k, \tag{55}$$

and $\boldsymbol{\mu}_k$ stand for the $N-1$ fundamental weights of the associated Lie algebra, defined through the reciprocity relation

$$\frac{2\boldsymbol{\alpha}_i\boldsymbol{\mu}_j}{\boldsymbol{\alpha}_i^2} = \boldsymbol{\alpha}_i\boldsymbol{\mu}_j = \delta_{ij}. \tag{56}$$

Using the identities

$$\boldsymbol{\alpha}_N\boldsymbol{\rho} = -(N-1), \qquad \boldsymbol{\alpha}_i\boldsymbol{\rho} = 1, \qquad i = 1, \ldots, N-1, \tag{57}$$

the vertex operator

$$e^{i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_{1,2}} \to e^{i\boldsymbol{\alpha}_i(\boldsymbol{\sigma}_{1,2}-(2\pi/N)\boldsymbol{\rho})} = e^{-i(2\pi/N)}e^{i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_{1,2}}, \quad i = 1, \ldots, N, \tag{58}$$

rotates in the opposite direction compared with the fermion bilinear, by the same amount. Hence, the instanton-monopole-induced vertex

$$(e^{i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_1} + e^{i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_2})(\lambda_i\psi_i + \lambda_{i+1}\psi_{i+1})$$

is invariant under the discrete chiral symmetry.

The discrete shift symmetry (54), as opposed to the continuous shift symmetry, cannot prohibit mass term for the dual photons. At best, it can postpone its appearance in the e^{-S_0} expansion. Hence, such a mass term must be, and is, generated.

As in SYM theory, at level e^{-2S_0} there exist magnetically charged bound monopole-antimonopole pairs with no fermion zero modes. These stable pairs were referred to as magnetic bions in [16]. In QCD(BF)*, the bions come in a wider variety than in SYM theory. The analogs of the magnetic bions that appear in SYM theory are the pairs of the type $\mathcal{M}_i^1$ and $\bar{\mathcal{M}}_{i\pm1}^1$ (and $1 \leftrightarrow 2$). Despite the repulsive Coulomb interactions between these two monopoles they form bound states due to the fermion exchange between them, with the combined effect

$$\sim \frac{1}{r} + \log r.$$

The corresponding bound state is stable.

Since the fermion zero modes in QCD(BF)* communicate with the monopoles in both gauge groups, the fermion zero mode exchange also generates logarithmic attractive interactions between the monopoles $\mathcal{M}_i^1$ in the first gauge group and the antimonopoles $\bar{\mathcal{M}}_{i\pm1}^2$ in the second. Note that there is no Coulomb interaction between these two since the first instanton-monopole is charged under the $[\mathrm{U}(1)^{N-1}]_1$ gauge subgroup of $[\mathrm{U}(1)^{N-1}]_1 \times [\mathrm{U}(1)^{N-1}]_2$ while the second is charged under $[\mathrm{U}(1)^{N-1}]_2$. Thus, the stable magnetic bions in QCD(BF)*, their magnetic and topological charges, and the vertices they generate are

$$
\begin{aligned}
&\mathcal{B}_i^1: \left(\frac{4\pi}{g}(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1}), 0, 0, 0\right): c_1 e^{-2S_0} e^{i(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1})\boldsymbol{\sigma}_1} \\
&\mathcal{B}_i^2: \left(0, 0, \frac{4\pi}{g}(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1}), 0\right): c_1 e^{-2S_0} e^{i(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1})\boldsymbol{\sigma}_2} \\
&\mathcal{B}_{i,i}^{12}: \left(\frac{4\pi}{g}\boldsymbol{\alpha}_i, \frac{1}{N}, -\frac{4\pi}{g}\boldsymbol{\alpha}_i, -\frac{1}{N}\right): c_2 e^{-2S_0} e^{i(\boldsymbol{\alpha}_i\boldsymbol{\sigma}_1 - \boldsymbol{\alpha}_i\boldsymbol{\sigma}_2)} \\
&\mathcal{B}_{i,i-1}^{12}: \left(\frac{4\pi}{g}\boldsymbol{\alpha}_i, \frac{1}{N}, -\frac{4\pi}{g}\boldsymbol{\alpha}_{i-1}, -\frac{1}{N}\right): c_2 e^{-2S_0} e^{i(\boldsymbol{\alpha}_i\boldsymbol{\sigma}_1 - \boldsymbol{\alpha}_{i-1}\boldsymbol{\sigma}_2)} \\
&\mathcal{B}_{i,i+1}^{12}: \left(\frac{4\pi}{g}\boldsymbol{\alpha}_i, \frac{1}{N}, -\frac{4\pi}{g}\boldsymbol{\alpha}_{i+1}, -\frac{1}{N}\right): c_2 e^{-2S_0} e^{i(\boldsymbol{\alpha}_i\boldsymbol{\sigma}_1 - \boldsymbol{\alpha}_{i+1}\boldsymbol{\sigma}_2)}.
\end{aligned}
\tag{59}
$$

The vertices for antibions (such as $\bar{\mathcal{B}}_i^1$) are the complex conjugates of the ones given above. The above bions are stable due to the attractive fermion pair exchange between their constituents. Note that the constituents of the bions $\mathcal{B}_i^1$ and $\mathcal{B}_i^2$, unlike the ones of $\mathcal{B}_{i,i}^{12}$, $\mathcal{B}_{i,i+1}^{12}$, $\mathcal{B}_{i,i-1}^{12}$ need to compete with the Coulomb repulsion for stability. Thus, in principle, there are no (symmetry or microscopic) reasons for the prefactor of the first two to be the equal to the ones of the latter. Therefore, we assume they are not.

As a result, we obtain the bion-induced bosonic potential in QCD(BF)* in the form

$$
\begin{aligned}
V_{\rm bion}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = m_W^3 g^{-6} e^{-2S_0} \sum_{i=1}^{N} [&c_1(e^{i(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1})\boldsymbol{\sigma}_1} \\
&+ e^{i(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1})\boldsymbol{\sigma}_2}) + c_2(2e^{i(\boldsymbol{\alpha}_i\boldsymbol{\sigma}_1 - \boldsymbol{\alpha}_i\boldsymbol{\sigma}_2)} \\
&+ e^{i(\boldsymbol{\alpha}_i\boldsymbol{\sigma}_1 - \boldsymbol{\alpha}_{i-1}\boldsymbol{\sigma}_2)} + e^{i(\boldsymbol{\alpha}_i\boldsymbol{\sigma}_1 - \boldsymbol{\alpha}_{i+1}\boldsymbol{\sigma}_2)})] + \text{H.c.}
\end{aligned}
\tag{60}
$$

In full analogy with the superpotential in SYM* theory, it is convenient to define a prepotential in QCD(BF)*. To this end we introduce the function

$$
\mathcal{W}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = m_W g^{-4} e^{-S_0} \sum_{\boldsymbol{\alpha}_i \in \Delta_0^{\rm aff}} (e^{i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_1} + e^{i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_2}), \tag{61}
$$

to be referred to as prepotential. Note that the prepotential, as well as its derivatives, transform homogeneously under the Z_{2N} shift symmetry (54),

$$
Z_{2N}: \mathcal{W}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) \to e^{-i(2\pi/N)} \mathcal{W}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2).
$$

Now, it is easy to express the bion-induced potential in terms of the prepotential in the form which is manifestly invariant under the Z_{2N} shift and $(Z_2)_I$ interchange symmetries,

$$
\begin{aligned}
V(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = g_3^2 \sum_{a=1}^{N-1} \bigg(&c_+ \left| \frac{\partial \mathcal{W}}{\partial \sigma_{1,a}} + \frac{\partial \mathcal{W}}{\partial \sigma_{2,a}} \right|^2 \\
&+ c_- \left| \frac{\partial \mathcal{W}}{\partial \sigma_{1,a}} - \frac{\partial \mathcal{W}}{\partial \sigma_{2,a}} \right|^2 \bigg).
\end{aligned}
\tag{62}
$$

We are finally ready to present the low-energy effective theory for QCD(BF)*,

$$
\begin{aligned}
L^{\rm QCD(BF)*} = &\frac{g_3^2}{32\pi^2} [(\partial\boldsymbol{\sigma}_1)^2 + (\partial\boldsymbol{\sigma}_2)^2] + V_{\rm bion}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) \\
&+ \frac{1}{g_3^2} \sum_{i=1}^{N} i\bar{\Psi}_i \gamma_\mu (\partial_\mu + i\boldsymbol{H}_{ii}\boldsymbol{A}_\mu^1 - i\boldsymbol{H}_{ii}\boldsymbol{A}_\mu^2)\Psi_i \\
&+ c g^{-6} e^{-S_0} \sum_{\alpha_i \in \Delta_{\rm aff}^0} ((e^{+i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_1} + e^{+i\boldsymbol{\alpha}_i\boldsymbol{\sigma}_2}) \\
&\times (\lambda_i\psi_i + \lambda_{i+1}\psi_{i+1}) + \text{H.c.}).
\end{aligned}
\tag{63}
$$

Like in other QCD-like theories with complex-representation fermions [such as QCD(F/AS/S)*], but unlike the ones with real-representation fermions [such as SYM theory or QCD(Adj)], we have both the electric and magnetic couplings. The Lagrangian (63) includes all relevant terms allowed by symmetries up to $\mathcal{O}(e^{-3S_0})$.

The important question at this stage is which operators in our effective Lagrangian (33) are most important at large distances in the renormalization-group sense. Apparently, the fugacity (the coefficient in front of the bion vertices) has dimension +3 and is dominant in the IR. The quantum-mechanical corrections are negligible. This suggests that in the IR the effects produced by magnetically charged bions are most relevant.

C. Vacuum structure and chiral symmetry realization

The low-energy effective theory respects all symmetries of the underlying gauge theory $U(1)_V \times (Z_{2N})_A \times (Z_2)_I$ and C, P, T. These symmetries may be spontaneously broken. By studying dynamics of the effective theory we demonstrate that the breaking pattern is

$$
U(1)_V \times (Z_{2N})_A \times (Z_2)_I \to U(1)_V \times (Z_2)_A \times (Z_2)_I, \tag{64}
$$

leading to the occurrence of N isolated vacua.

In Eq. (63) the Z_{2N} chiral symmetry is entangled with the shift symmetry of the dual photon (54), just like in SYM theory. There are N isolated vacua in the $(Z_2)_I$ invariant subspace related to each other by the action of the Z_N shift symmetry. These vacua are located at

$$
\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2 = \left\{0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, \frac{2(N-1)\pi}{N}\right\}\boldsymbol{\rho} \tag{65}
$$

in the field space. The choice of a given vacuum spontaneously breaks the Z_N shift symmetry and, hence, the chiral symmetry.

Let $|\Omega_k\rangle$ denote one of the N vacuum states ($k = 1, \ldots, N$). Following the techniques of [15,22], we observe that the chiral condensate is proportional to the monopole-induced term e^{-S_0}. The renormalization-group β function of QCD(BF)* is identical to that of SYM theory up to $\mathcal{O}(1/N^2)$ corrections. The first coefficients are just identical. Thus,

$$e^{-S_0} \equiv e^{-(8\pi^2/g^2N)} = \Lambda^3(\Lambda L)^{b_0-3}, \tag{66}$$

where b_0 denotes the leading coefficient of the β function divided by N. At one-loop order in QCD(BF)*

$$b_0 = 3.$$

Thus, the chiral condensate in QCD(BF)* is

$$\langle\Omega_k|\mathrm{Tr}\bar{\Psi}\Psi|\Omega_k\rangle = 2N\Lambda^3 e^{i(2\pi k/N)} + \mathrm{H.c.} \tag{67}$$

There is no L dependence in the condensate in QCD(BF)* at one-loop level, just like in SYM theory.

D. Mass gap and confinement

The small fluctuation analysis around any of the N minima is sufficient to see that there are no massless modes in the infrared description of the QCD(BF)*. The choice of the vacuum breaks the discrete chiral symmetry rendering all fermions massive. The bion-induced potential makes all $2(N-1)$ photons massive. This shows that every particle-like excitation must have a finite mass $m \sim e^{-S_0}$. There are no physical states in the mass range $[0, m)$ in the physical Hilbert space of the theory. Since the global Z_N center group symmetry and $(Z_2)_I$ interchange symmetry are unbroken, the physical states can be expressed as the mutual eigenstates of these symmetries. The Fourier transform

$$\sigma_{\pm,k} = (\sigma_{1,k} \pm \sigma_{2,k}) \equiv \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{i(2\pi jk/N)}\boldsymbol{H}_{jj}(\boldsymbol{\sigma}_1 \pm \boldsymbol{\sigma}_2) \tag{68}$$

diagonalizes the mass matrix. The masses of the dual photons are proportional to $\exp(-S_0)$. More exactly,[15]

$$m_{\sigma_{\pm,k}} = \sqrt{c_\pm}\Lambda(\Lambda L)^2\left(2\sin\frac{\pi k}{N}\right)^2, \qquad \Lambda L \ll 1. \tag{69}$$

Any probe charge one might consider is coupled to a number of σ fields. The thickness of the domain line (string) attached to the probe charge is determined by the inverse mass of the lightest σ field ($k = 1$). It is worth noting that the string has a substructure corresponding to the contribution of the next-to-lightest, next-to-next-to-lightest and so on σ's. The fermion masses are of the same order of magnitude in the same regime, as seen from Eq. (53),

$$m_{\Psi_i} = c\Lambda(\Lambda L)^2. \tag{70}$$

Now we are ready to discuss strings in QCD(BF)* at small L. Let us consider a heavy probe quark $Q^{i_1\ldots i_m}_{j_1\ldots j_n}$ and its antiquark $\overline{Q^{i_1\ldots i_m}_{j_1\ldots j_n}}$ in a color-singlet state at an exponentially large distance from each other. If $m \neq n$ the string (domain line) forming between these probe objects is unbreakable. Light dynamical fermions of the low-energy theory cannot screen the electric charges of the probe quarks. However, if $m = n$ some strings (i.e. those attached to the probes for which every index i is equal to some j) will break through pair creation of light dynamical fermions. Assume $|n - m| \equiv k \neq 0$. Then the tensions of these unbreakable k strings can be found by calculating the tensions of the domain lines supported by the theory (63). These tensions are of the order of $\Lambda^2(\Lambda L)$ in the $\Lambda L \ll 1$ Abelian confinement regime while at $\Lambda L \gtrsim 1$, in the non-Abelian confinement regime, they tend to Λ^2 times a numerical coefficient.

To the best of our knowledge, this is the first analytic demonstration of χSB, mass gap generation, and linear confinement in QCD(BF)*. This theory exhibits all expected nontrivial features of QCD(BF) on R_4.

V. QCD WITH ONE AS FERMION

Now we will discuss QCD with one antisymmetric Dirac fermion[16] on $R_3 \times S_1$. The theory possesses a $\mathrm{U}(1)_V \times Z_{2N-4}$ symmetry, Z_{2N-4} being the anomaly-free subgroup of the axial $\mathrm{U}(1)_A$. The action of the symmetry on the elementary fields is as follows:

$$\begin{aligned} &\mathrm{U}(1)_V\colon\ \lambda \to e^{i\alpha}\lambda, \qquad \psi \to e^{-i\alpha}\psi, \\ &(Z_{2N-4})_A\colon\ \lambda \to e^{i(2\pi/(2N-4))}\lambda, \qquad \psi \to e^{i(2\pi/(2N-4))}\psi. \end{aligned} \tag{71}$$

It is believed that for sufficiently large $r(S_1)$, the chiral symmetry is broken down to Z_2 by the bifermion condensate $\langle\psi\lambda\rangle \neq 0$

$$\langle\bar{\Psi}\Psi\rangle \sim N\Lambda^3 e^{i(2\pi k/(N-2))} + \mathrm{H.c.}$$

resulting in $N-2$ isolated vacua. The QCD(AS) theory on R_4 must confine the same way as $\mathcal{N} = 1$ SYM theory and possess a mass gap. Since the discussion is quite similar to the case of QCD(BF)*, we will be brief.

A. Deformed orientifold QCD

In the small $r(S_1)$ regime, the gauge symmetry is broken, $\mathrm{SU}(N) \to \mathrm{U}(1)^{N-1}$. Without loss of generality we can take $N = 2m + 1$. The case $N = 2m$ can be dealt with in a similar manner.

In perturbation theory the massless fields are $N-1$ diagonal photons and $N-2$ charged fermions. The $N^2 - N$ off-diagonal W-bosons and $N^2 - 2N + 2$ fermions acquire masses in the range $[\frac{2\pi}{LN}, \frac{2\pi}{L})$ and decouple from infrared physics.

The AS fermions Ψ_{ij} acquire three-dimensional mass terms given by

[15] Powers of g and numerical factors are omitted here and in similar expressions below.

[16] Discussion of QCD with the symmetric representation fermion is parallel.

$$\sum_{i,j=1}^{N}(a_i + a_j)\bar{\Psi}^{[ij]}\gamma_4\Psi_{[ij]},$$

where a_k's are given in Eq. (9). Hence,

$$m_{ij} = \frac{2\pi}{LN}([i+j] \bmod N).$$

Thus, the fermion components $\Psi_{i,N-i}$ remain massless to all orders in perturbation theory. Let us label

$$\Psi_{i,N-i} \equiv \Psi_i, \qquad i = 1, \ldots, N-1.$$

The electric charges of these degrees of freedom under the unbroken gauge group is

$$\boldsymbol{q}_{\Psi_i} = g(\boldsymbol{H}_{ii} + \boldsymbol{H}_{N-i,N-i}), \qquad i = 1, \ldots, N. \tag{72}$$

Since the fermion is antisymmetric in its indices, we may parametrize the set of the massless fermions as

$$\begin{aligned}\Psi &= \{\Psi_1, \ldots, \Psi_{m-1}, \Psi_m, \Psi_{m+1}, \Psi_{m+2}, \ldots, \Psi_{2m}\} \\ &= \{\Psi_1, \ldots, \Psi_{m-1}, \Psi_m, -\Psi_m, -\Psi_{m-1}, \ldots, -\Psi_1\}.\end{aligned} \tag{73}$$

The IR action in perturbation theory is

$$\begin{aligned}S = \int_{R_3} \frac{1}{g_3^2}\bigg[\frac{1}{4}\sum_{a=1}^{N-1}(F^a_{\mu\nu})^2 \\ + 2\sum_{i=1}^{m} i\bar{\Psi}_i\gamma_\mu(\partial_\mu + i(\boldsymbol{H}_{ii} + \boldsymbol{H}_{N-i,N-i})\boldsymbol{A}_\mu)\Psi_i\bigg].\end{aligned} \tag{74}$$

B. Nonperturbative effects

In QCD(AS)* on small $S_1 \times R_3$ there are N types of instanton-monopoles because of the pattern of the gauge symmetry breaking $SU(N) \to U(1)^{N-1}$ via a compact adjoint Higgs field. The $2N-4$ fermion zero modes of the BPST R_4 instanton split into $N-2$ pairs of the instanton-monopole zero modes in a slightly different way than that in SYM* theory and QCD(BF)*. The $N-2$ instanton-monopoles have two fermion zero modes each, while the remaining two monopoles have no zero modes. It is useful to present the monopole-instanton vertices in QCD(AS)* due to a nontrivial structure of their zero modes,

$$\begin{aligned}\mathcal{M}_1 &= e^{-S_0}e^{i\boldsymbol{\alpha}_1\boldsymbol{\sigma}}(\lambda_1\psi_1 + \lambda_2\psi_2),\\ \mathcal{M}_2 &= e^{-S_0}e^{i\boldsymbol{\alpha}_2\boldsymbol{\sigma}}(\lambda_2\psi_2 + \lambda_3\psi_3), \ldots,\\ \mathcal{M}_{m-1} &= e^{-S_0}e^{i\boldsymbol{\alpha}_{m-1}\boldsymbol{\sigma}}(\lambda_{m-1}\psi_{m-1} + \lambda_m\psi_m),\\ \mathcal{M}_m &= e^{-S_0}e^{i\boldsymbol{\alpha}_m\boldsymbol{\sigma}}(2\lambda_m\psi_m),\\ \mathcal{M}_{m+1} &= e^{-S_0}e^{i\boldsymbol{\alpha}_{m+1}\boldsymbol{\sigma}}(\lambda_m\psi_m + \lambda_{m-1}\psi_{m-1}), \ldots,\\ \mathcal{M}_{2m-2} &= e^{-S_0}e^{i\boldsymbol{\alpha}_{2m-2}\boldsymbol{\sigma}}(\lambda_3\psi_3 + \lambda_2\psi_2),\\ \mathcal{M}_{2m-1} &= e^{-S_0}e^{i\boldsymbol{\alpha}_{2m-1}\boldsymbol{\sigma}}(\lambda_2\psi_2 + \lambda_1\psi_1),\\ \mathcal{M}_{2m} &= e^{-S_0}e^{i\boldsymbol{\alpha}_{2m}\boldsymbol{\sigma}},\\ \mathcal{M}_{2m+1} &= e^{-S_0}e^{i\boldsymbol{\alpha}_{2m+1}\boldsymbol{\sigma}}.\end{aligned} \tag{75}$$

Consequently, the contribution to the QCD(AS)* Lagrangian induced by monopole-instantons takes the form

$$\Delta L \sim \sum_{i=1}^{2m+1}(\mathcal{M}_i + \bar{\mathcal{M}}_i). \tag{76}$$

Since the $N-2$ monopoles carry compulsory fermionic zero mode insertions, they cannot induce a mass term for all the dual photons if $N \geq 4$. As seen from Eq. (76), two of the monopole-instantons do contribute to the bosonic potential, but this is insufficient to render all photons massive for $N \geq 4$. (At $N = 3$, QCD(AS)* and QCD(F)* are the same theories.) Thus, in order to render all the photons massive, we need to incorporate effects of order e^{-2S_0}, and introduce the magnetic bions. Before doing so let us show that the underlying symmetries of QCD(AS)* allow mass terms for all dual photons to be generated.

Since $U(1)_V \times (Z_{2N-4})_A$ is the symmetry of the microscopic theory, it must be a symmetry of the long-distance theory. The invariance under $U(1)_V$ is manifest. The invariance under the $(Z_{2N-4})_A$ necessitates intertwining the axial chiral symmetry with a discrete shift symmetry of the dual photon

$$\begin{aligned}(Z_{2N-4})_A\colon\ \lambda\psi &\to e^{i(2\pi/(N-2))}\lambda\psi,\\ \boldsymbol{\sigma} &\to \boldsymbol{\sigma} - \frac{2\pi}{N-2}\boldsymbol{\rho}_{\mathrm{AS}},\end{aligned} \tag{77}$$

where

$$\boldsymbol{\rho}_{\mathrm{AS}} \equiv \sum_{j=1}^{N-2}\boldsymbol{\mu}_k \tag{78}$$

and $\boldsymbol{\mu}_k$ are the $N-1$ fundamental weights of the associated Lie algebra. Note that the parameter $\boldsymbol{\rho}_{\mathrm{AS}}$ is not exactly the Weyl vector, which appears in SYM* theory and QCD(BF)*. Rather, it can be represented as

$$\boldsymbol{\rho}_{\mathrm{AS}} = \boldsymbol{\rho} - \boldsymbol{\mu}_{N-1}. \tag{79}$$

Using the identities

$$\boldsymbol{\alpha}_{N-1}\boldsymbol{\rho}_{\rm AS} = 0, \qquad \boldsymbol{\alpha}_N\boldsymbol{\rho}_{\rm AS} = -(N-2)$$
$$\boldsymbol{\alpha}_i\boldsymbol{\rho}_{\rm AS} = 1, \qquad i = 1, \ldots N-2 \tag{80}$$

we observe that the vertex operators $e^{i\boldsymbol{\alpha}_i\boldsymbol{\sigma}}$ transform under the discrete shift symmetry

$$\boldsymbol{\sigma} \to \boldsymbol{\sigma} - \frac{2\pi}{N-2}\boldsymbol{\rho}_{\rm AS}$$

as

$$Z_{N-2}\colon\ e^{i\boldsymbol{\alpha}_{2m}\boldsymbol{\sigma}} \to e^{i\boldsymbol{\alpha}_{2m}\boldsymbol{\sigma}}, \qquad e^{i\boldsymbol{\alpha}_{2m+1}\boldsymbol{\sigma}} \to e^{i\boldsymbol{\alpha}_{2m+1}\boldsymbol{\sigma}},$$
$$e^{i\boldsymbol{\alpha}_i\boldsymbol{\sigma}} \to e^{-i(2\pi/(N-2))}e^{i\boldsymbol{\alpha}_i\boldsymbol{\sigma}}, \qquad i = 1, \ldots 2m-1. \tag{81}$$

Hence, the monopole-induced interactions (76) are invariant under $(Z_{2N-4})_A$ given in (77). The discrete shift symmetry allows mass terms for all dual photons at order e^{-2S_0}.

In QCD(AS)*, there are novel topological excitations as is the case in QCD(BF)*. The zero mode structure of monopole-instantons suggests that other than the magnetic bions common with SYM* theory, there are magnetic bions of a more exotic variety,

$$\mathcal{B}^1_i\colon \left(\frac{4\pi}{g}(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1}), 0\right)\colon\ c_1 e^{-2S_0}e^{i(\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_{i-1})\boldsymbol{\sigma}},$$
$$\mathcal{B}^{12}_{i,i}\colon \left(\frac{4\pi}{g}(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{2m-i}), 0\right)\colon\ c_2 e^{-2S_0}e^{i(\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_{2m-i})\boldsymbol{\sigma}},$$
$$\mathcal{B}^{12}_{i,i-1}\colon \left(\frac{4\pi}{g}(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{2m-i-1}), 0\right)\colon\ c_2 e^{-2S_0}e^{i(\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_{2m-i+1})\boldsymbol{\sigma}},$$
$$\mathcal{B}^{12}_{i,i+1}\colon \left(\frac{4\pi}{g}(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{2m-i-1}), 0\right)\colon\ c_2 e^{-2S_0}e^{i(\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_{2m-i-1})\boldsymbol{\sigma}}. \tag{82}$$

Here in the first line summation runs over $i = 1, \ldots, 2m-1$ while in the second, third and fourth lines over $i = 1, \ldots, m-1$. The pairing of the constituent monopoles follows from the structure of the fermion zero modes. The magnetic bion $\mathcal{B}^1_i$ is held together due to the attractive fermionic pair exchanges which overcomes the Coulomb repulsion between its constituents. The constituents of the latter bions $\mathcal{B}^{12}_{i,i}$ and $\mathcal{B}^{12}_{i,i\pm1}$ do not interact via the Coulomb law, rather they experience just the fermion pair exchange. Consequently, the combined effect of the magnetic bions (which is of order e^{-2S_0})

$$V_{\rm bion}(\boldsymbol{\sigma}) = m_W^3 g^{-6}\left[\sum_{i=1}^{2m-1}\mathcal{B}^1_i + \sum_{i=1}^{m-1}(\mathcal{B}^{12}_{i,i} + \mathcal{B}^{12}_{i,i+1} + \mathcal{B}^{12}_{i,i-1})\right] + \text{H.c.} \tag{83}$$

and two monopole-instantons $\mathcal{M}_{2m}$, $\mathcal{M}_{2m+1}$ gives rise to the bosonic potential which renders all $N-1$ dual photons massive which, in turn, leads to string (domain line) formation. Assembling perturbative and nonperturbative effects we get

$$L^{\rm QCD(AS)^*} = \frac{g_3^2}{32\pi^2}(\partial\boldsymbol{\sigma})^2 + V_{\rm bion}(\boldsymbol{\sigma}) + \sum_{i=2m}^{2m+1}(\mathcal{M}_i + \bar{\mathcal{M}}_i) + \frac{2}{g_3^2}\sum_{i=1}^{m}\bar{\Psi}_i\gamma_\mu(i\partial_\mu + (\boldsymbol{H}_{ii} + \boldsymbol{H}_{N-i,N-i})\boldsymbol{A}_\mu)\Psi_i + \sum_{i=1}^{2m-1}(\mathcal{M}_i + \bar{\mathcal{M}}_i). \tag{84}$$

In QCD(F/BF)* we had both electric couplings and monopole and bion-induced magnetic interactions. By the same token in QCD(AS)* interactions of the electric and magnetic type are present. (This is unlike what we have in SYM* theory.) The monopole and bion-induced effects are dominant.

In the effective low-energy theory (84), the $(Z_{2N-4})_A$ chiral symmetry is entangled with the shift symmetry of the dual photon. Examination of the bosonic potential in QCD(AS)* reveals $N-2$ gauge inequivalent isolated vacua located at

$$\boldsymbol{\sigma} = \left\{0, \frac{2\pi}{N-2}, \frac{4\pi}{N-2}, \ldots, \frac{2(N-3)\pi}{N-2}\right\}\boldsymbol{\rho}_{\rm AS}. \tag{85}$$

As usual, we label these $N-2$ vacuum states by $|\Omega_k\rangle$, ($k = 1, \ldots, N-2$). Choosing a vacuum we spontaneously break the Z_{N-2} symmetry.

The chiral condensate in the vacuum $|\Omega_k\rangle$ can be calculated along the same lines as in QCD(BF)*,

$$\langle\Omega_k|{\rm Tr}\bar{\Psi}\Psi|\Omega_k\rangle = 2(N-2)\begin{Bmatrix}\Lambda^3(\Lambda L)^{4/3N}, & \Lambda L \ll 1,\\ \Lambda^3, & \Lambda L \gtrsim 1,\end{Bmatrix} \times \cos\left(\frac{2\pi k}{N-2}\right), \tag{86}$$

where there is a weak L dependence at small L. This follows from the $\mathcal{O}(1/N)$ difference in b_0, the first β-function coefficient of QCD(AS), and SYM theories divided by N. In QCD(AS)

$$b_0 = 3 + \frac{4}{3N}.$$

Remark on the Callias and Atiyah-Singer index theorems: On R_4, the global aspect of the chiral anomaly is expressed by the Atiyah-Singer index theorem. BPST instanton is associated with $2h$ fermionic zero modes, where $2h = \{2, 2N, 2N, 2N-4, 2N+4\}$ for QCD(F/adj/BF/AS/S), respectively. In QCD($\mathcal{R}$)* at small $r(S_1)$, due to the gauge symmetry breaking, the four-dimensional instanton splits into N monopoles. In the small-$r(S_1)$ (weak coupling) regime, the instanton should be viewed as a composite object, with the magnetic and topological charges as in Eq. (39), built of N types of elementary monopoles with charges $\frac{4\pi}{g}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_N)$. The $2h$ fermion zero modes split into groups which associate themselves with the above N monopoles as follows:

$$\begin{aligned} \text{QCD(F)}&: 2 \to \{2, 0, \ldots, 0, 0, 0\}, \\ \text{SYM}&: 2N \to \{2, 2, \ldots, 2, 2, 2\}, \\ \text{QCD(BF)}&: 2N \to \{2, 2, \ldots, 2, 2, 2\}, \\ \text{QCD(AS)}&: 2N-4 \to \{2, 2, \ldots, 2, 0, 0\}, \\ \text{QCD(S)}&: 2N+4 \to \{2, 2, \ldots, 2, 4, 4\}. \end{aligned} \tag{87}$$

The numbers on the right-hand side are the Callias indices for the corresponding monopoles. Strictly speaking, the Callias index theorem is formulated for the Yang-Mills + noncompact adjoint Higgs system on $\mathbb{R}^3$ [27]. Its generalization to $\mathbb{R}^3 \times S^1$ is carried out by Nye and Singer [32]. To study the index theorems we need to find the kernels of the Dirac operators $\not{D}$ and $\not{D}^\dagger$ in the background of the appropriate topological excitation. The kernel is the set of zero eigenstates of the Dirac operator. The difference of the dimensions of the kernels gives the number of zero mode attached to a given topological excitation. Thus, we observe the following relation between the Atiyah-Singer index $\mathcal{I}_{\text{inst}}$ and the Callias index $\mathcal{I}_{\boldsymbol{\alpha}_i}$,

$$\mathcal{I}_{\text{inst}} = \sum_{\boldsymbol{\alpha}_i \in \Delta^0_{\text{aff}}} \mathcal{I}_{\boldsymbol{\alpha}_i}, \tag{88}$$

or

$$\begin{aligned} &\dim \ker \not{D}_{\text{inst}} - \dim \ker \not{D}^\dagger_{\text{inst}} \\ &= \sum_{\boldsymbol{\alpha}_i \in \Delta^0_{\text{aff}}} (\dim \ker \not{D}_{\boldsymbol{\alpha}_i} - \dim \ker \not{D}^\dagger_{\text{inst}}). \end{aligned} \tag{89}$$

VI. θ DEPENDENCE

There is one more interesting aspect of the theory which has not yet been discussed, namely, θ dependence. It is well-known that in pure Yang-Mills theory on R_4 physical quantities, e.g. string tensions, do depend on θ, and physical periodicity in θ is 2π. Introduction of one massless quark in representation $\mathcal{R}$ eliminates θ dependence of physical quantities since one can eliminate the θ term through an appropriate chiral rotation of the fermion field, as a result of the chiral anomaly. This does not mean that various order parameters, e.g. the bifermion condensate, are θ independent. If a small fermion mass term is added, physical quantities acquire θ dependence; all θ-dependent effects are proportional to the fermion mass m.

Let us ask ourselves what happens on $R_3 \times S_1$, in deformed theories. At first, let us consider pure Yang-Mills, assuming that $\theta \neq 0$. Then the instanton-monopole-induced vertices at level e^{-S_0} are

$$\mathcal{L} = e^{-S_0} \sum_{j=1}^{N} \mu_j e^{i\boldsymbol{\alpha}_j \boldsymbol{\sigma} + i\theta/N} + \text{H.c.} \tag{90}$$

By globally shifting

$$\boldsymbol{\sigma} \to \boldsymbol{\sigma} - \frac{\theta}{N}\boldsymbol{\rho}, \tag{91}$$

where $\boldsymbol{\rho}$ is the Weyl vector, and using the identities (57), we can rewrite the instanton-monopole vertices in the form

$$\mathcal{L} = e^{-S_0} \sum_{j=1}^{N-1} \mu_j e^{i\boldsymbol{\alpha}_j \boldsymbol{\sigma}} + \mu_N e^{-S_0} e^{i\boldsymbol{\alpha}_N \boldsymbol{\sigma} + i\theta} + \text{H.c.}, \tag{92}$$

where the 2π periodicity is more transparent. In both Eqs. (90) and (92) the vacuum angle dependence is explicit.

Introducing one fundamental fermion, and localizing the fermionic zero mode on the monopole with charge α_N without loss of generality, we get, instead of (90) and (92)

$$\begin{aligned} \mathcal{L} &= \tilde{\mu}_N e^{-S_0} e^{i\boldsymbol{\alpha}_N \boldsymbol{\sigma} + i\theta/N} \lambda\psi + e^{-S_0} \sum_{j=1}^{N-1} \mu_j e^{i\boldsymbol{\alpha}_j \boldsymbol{\sigma} + i\theta/N} + \text{H.c.} \\ &= \tilde{\mu}_N e^{-S_0} e^{i\boldsymbol{\alpha}_N \boldsymbol{\sigma} + i\theta} \lambda\psi + e^{-S_0} \sum_{j=1}^{N-1} \mu_j e^{i\boldsymbol{\alpha}_j \boldsymbol{\sigma}} + \text{H.c.}, \end{aligned} \tag{93}$$

where we used (91) in passing to the second step. It is clear in the latter form that the θ dependence can be completely absorbed in the fermion fields,

$$\{\psi, \lambda\} \to \{\psi e^{-i\theta/2}, \lambda e^{-i\theta/2}\}. \tag{94}$$

If the fermion mass term $m\psi\lambda$ is added, the θ dependence can no longer be absorbed in the definition of the fermion field. Performing (94) we change the phase of the mass parameter. Correspondingly, one can expect physical θ dependent effects proportional to m, such as the vacuum energy density

$$\mathcal{E}(\theta) \sim m\langle\bar{\Psi}\Psi\rangle \cos\theta, \tag{95}$$

in parallel with the behavior of the undeformed theory on R_4.

Analysis of the θ dependence in QCD(BF)* is even easier technically. The magnetic bion vertices have no θ dependence because each of them represents the product of a monopole and antimonopole vertex in which the θ dependence cancels. Moreover, the monopole-induced vertices are

$$\begin{aligned} \Delta L^{\text{QCD(BF)}^*} = e^{-S_0} \sum_{\boldsymbol{\alpha}_i \in \Delta^0_{\text{aff}}} &((e^{i\boldsymbol{\alpha}_i \boldsymbol{\sigma}_1 + i\theta/N} + e^{i\boldsymbol{\alpha}_i \boldsymbol{\sigma}_2 + i\theta/N}) \\ &\times (\lambda_i \psi_i + \lambda_{i+1}\psi_{i+1}) + \text{H.c.}). \end{aligned} \tag{96}$$

The θ dependence can be readily absorbed in the fermion fields with the following redefinition:

$$\{\psi_i, \lambda_i\} \to e^{-i\theta/(2N)}\{\psi_i, \lambda_i\}. \tag{97}$$

If we introduce very small mass terms for the fermion fields, $m \ll \Lambda(\Lambda L)$, then it is obvious that the θ dependence reappears in the vacuum energy density,

$$\mathcal{E}(\theta) = \min_k \mathcal{E}_k(\theta) \equiv \min_k \left[m\Lambda^3 \cos\left(\frac{\theta}{N} + \frac{2\pi k}{N}\right)\right], \quad k = 1, \ldots, N. \tag{98}$$

Turning on a nonvanishing mass term lifts the N-fold degeneracy of the vacua $|\Omega_k\rangle$. The vacuum labeled by the integer k turns into a state with energy $\mathcal{E}_k(\theta)$. Each one of the N branches is $2\pi N$ periodic in θ. Consequently, the vacuum energy density is physically 2π periodic,

$$\mathcal{E}_{\text{vac}}(\theta + 2\pi) = \mathcal{E}_{\text{vac}}(\theta).$$

This is precisely the expected behavior of undeformed QCD(BF) on R_4.

In the case of QCD(AS)* the overall picture emerging from our analysis is quite similar (albeit there are some minor differences subleading in $1/N$) and also matches the known θ dependence of QCD(AS) on R_4.

VII. REMARKS ON PLANAR EQUIVALENCE

Similarity of the dynamical aspects of QCD(BF/AS/S)* (with fermions in the two-index representation) and $\mathcal{N} = 1$ SYM* theory is evident. Given that they are quantum theories with distinct matter content and distinct microscopic symmetries, this similarity is remarkable. We explicitly showed that in the small-$r(S_1)$ regime, QCD(BF/AS/S)* confine through the magnetic bion mechanism in the same way as $\mathcal{N} = 1$ SYM* theory. Moreover, spontaneous breaking of the discrete chiral symmetries is similar in these two cases too. The bifermion condensate is saturated by a monopole-instanton with appropriate fermion zero mode structure. The calculated mass gaps are quite alike in both cases. Clearly, our analysis makes it manifest that solvability of $\mathcal{N} = 1$ SYM* theory at weak coupling is due to the unbroken center symmetry. Supersymmetry is secondary in this regime.

In fact, an intimate relation between SYM theory and its orientifold-orbifold daughters exists not only at small $r(S_1)$ but also in the decompactification limit of large $r(S_1)$. If the number of colors $N \to \infty$, there is a well-defined equivalence between $\mathcal{N} = 1$ SYM and QCD(BF/AS/S) which goes under the name of planar equivalence [36–40]. The necessary conditions for planar equivalence to be valid nonperturbatively are (i) interchange $(Z_2)_I$ symmetry is unbroken in QCD(BF), (ii) C conjugation symmetry is unbroken in QCD(AS/S). It is generally believed that these conditions are met [3].

The large N equivalence is a useful tool to translate nonperturbative data of SYM theory to its daughters (and vice versa) on R_4. Planar equivalence is valid also on $R_3 \times S_1$. The equivalence establishes an isomorphism on a subspace of the Hilbert space of these theories. Let us grade the Hilbert space of SYM theory with respect to $(-1)^F$ where F is the fermion number, as

$$\mathcal{H}^{\text{SYM}} = \mathcal{H}^{\text{SYM}+} \oplus \mathcal{H}^{\text{SYM}-}. \tag{99}$$

Similarly, the Hilbert spaces of QCD(BF) and QCD(AS/S) can be graded respect to the $1 \leftrightarrow 2$ interchange symmetry in the first case and charge conjugation in the second. Planar equivalence is an isomorphism between the even subspaces of the Hilbert spaces

$$\mathcal{H}^{\text{SYM}+} \equiv \mathcal{H}^{\text{QCD(BF)}+} \equiv \mathcal{H}^{\text{QCD(AS)}+}. \tag{100}$$

(The full Hilbert spaces are by no means isomorphic.)

If one performs periodic compactifications[17] of QCD (BF/AS/S) on $R_3 \times S_1$, with small $r(S_1)$, the $1 \leftrightarrow 2$ interchange symmetry of QCD(BF)* and C invariance of QCD(AS/S)* do break spontaneously, along with the spatial center symmetry [35,41]. (For related lattice studies showing the breaking and restoration of C, see [42,43].)

Certain order parameters which probe the interchange symmetry and C invariance are topologically nontrivial [44], e.g.

$$\text{Tr}(U_1^k) - \text{Tr}(U_2^k), \qquad \text{QCD(BF)}^*$$
$$\text{and} \quad \text{Tr}(U^k) - \text{Tr}(U^{*k}) \qquad \text{QCD(AS)}^*. \tag{101}$$

These operators are charged under the center symmetry and odd under $(Z_2)_I$ and C. In QCD(BF/AS/S)* stabilization of the center symmetry automatically implies vanishing of the expectation values of the order parameters (101).

There are also order parameters which are neutral under the center symmetry, yet charged under $(Z_2)_I$ and C. For example, the odd combination of the Wilson loops $W_1(C) - W_2(C)$ or $\text{Tr}F_1^2 - \text{Tr}F_2^2$ in QCD(BF)* and $W(C) - W^*(C)$ in QCD(AS)* are of this type. The unbroken center symmetry does not restrict the expectation value of such operators. Our dynamical analysis in Secs. IV and V shows that spontaneous breaking of $(Z_2)_I$ and C symmetry definitely does not take place at small $r(S_1)$. Arguments why this must be the case also on R_4 are summarized in Ref. [3].

VIII. CONCLUSIONS AND PROSPECTS: ABELIAN VS NON-ABELIAN CONFINEMENT

The aspects of QCD* theories that we studied are valid in the limit $L\Lambda \ll 1$, where the weak coupling regime sets in. We presented arguments that one-flavor QCD($\mathcal{R}$)* theories are continuously connected to one-flavor QCD($\mathcal{R}$) on R_4. We demonstrated, through an explicit calculation at small $r(S_1)$, existence of the mass gap, linear confinement, and discrete χSB. These are indeed the most salient features of QCD-like theories on R_4.

In the small-$r(S_1)$ domain, the QCD* theories are characterized by the fact that the gauge symmetry is Higgsed

[17] In thermal compactification, only the center symmetry breaks spontaneously; the interchange symmetry and C invariance remain unbroken [41]. Thus, planar equivalence for orbifold and orientifold daughters remains valid in the high temperature deconfined phase.

down to a maximal Abelian subgroup $U(1)^{N-1}$. Thus, at small $r(S_1)$ we deal with Abelian confinement, while it is expected to give place to non-Abelian confinement in the decompactification limit.

What happens as we increase $L\Lambda$ gradually, all the way to $L \to \infty$? At a scale of the order $L\Lambda \sim 1$, we lose the separation of scales between the W-bosons and the nonperturbatively gapped photons. Thus, our effective low-energy description (which includes only light bosonic and fermionic degrees of freedom) ceases to be valid. At and above $\Lambda \sim 1/L$ the theory is strongly coupled in the IR, and the full non-Abelian gauge group is operative. Thus, the confinement mechanism in this regime must be non-Abelian.

This situation is completely analogous to the Seiberg-Witten solution [18] of four-dimensional $\mathcal{N} = 2$ SYM theory exhibiting mass gap and linear confinement upon a μ deformation breaking $\mathcal{N} = 2$ down to $\mathcal{N} = 1$. If $\mu/\Lambda \ll 1$, the Seiberg-Witten theory in the IR is in the regime of broken gauge symmetry, i.e. $SU(N) \to U(1)^{N-1}$, where it is solvable. For $\mu/\Lambda \gtrsim 1$, one loses the separation of scales between the W-bosons and nonperturbatively gapped photons. The full gauge symmetry is restored. In this regime, the low-energy theory approaches pure $\mathcal{N} = 1$ SYM theory. The confining strings must be non-Abelian. Currently no controllable analytical approaches allowing one to continue the Seiberg-Witten solution to the domain $\mu/\Lambda \gg 1$ are known, and yet there are good reasons to believe that this continuation is smooth.

Conceptually the relation between μ-deformed $\mathcal{N} = 2$ and $\mathcal{N} = 1$ SYM theories on R_4 is parallel to that between one-flavor QCD* on $R_3 \times S_1$ and QCD on R_4. Both theories realize confinement via the following pattern

$$\mathrm{SU}(N) \overset{\text{Higgsing}}{\longrightarrow} [\mathrm{U}(1)]^{N-1} \overset{\text{nonperturbative}}{\longrightarrow} \text{no massless modes.} \tag{102}$$

Existence of an intermediate Abelian gauge theory in the IR is the key to analytical calculability in both cases.

In both cases by tuning the relevant parameter, μ/Λ or $L\Lambda$, respectively, from small to large values, we can remove the intermediate step of "Abelianization." In this paper we presented a number of arguments in favor of no phase transitions separating the Abelian and non-Abelian confinement regimes. It is desirable to develop a special technique allowing one to perform "integrating in" of the W-bosons (and their partners) gradually. If this task can be achieved this could provide a direct route to QCD and QCD-like theories on R_4.

If we are right and the transition from QCD* to QCD-like theories is smooth, this smoothness could explain a long-standing puzzle. The point is that a rather vaguely defined method which goes under the name of the maximal Abelian projection seems to give sensible results in the lattice calculations. The reason might be the proximity of the Abelian confinement regime we discussed in the body of this paper.

The status of QCD-like theories with massless or very light fermions with exact or approximate chiral symmetry significantly improved in the recent years [45,46]. It is highly desirable to implement QCD* theories on lattices, and then carry out an in-depth study of the transition from Abelian to non-Abelian confinement.

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APPENDIX: CENTER STABILIZATION

Let $U(\mathbf{x})$ be the path-ordered holonomy of the Wilson line wrapping S_1 at the point $\mathbf{x} \in R_3$. It is known that for complex representation fermions (F/AS/S/BF), the center symmetry is broken down at sufficiently small $r(S^1)$ regardless of the spin connections of fermions. For adjoint fermions with periodic spin connection, the spatial center symmetry is not broken at small $r(S_1)$, whereas for antiperiodic (thermal) boundary conditions the temporal center symmetry is broken at sufficiently high temperatures.

An easy way to see this is to evaluate the one-loop Coleman-Weinberg effective potential induced by quantum fluctuations by using the background field method (e.g. [41,47]). The minimum of the *classical* action is achieved at the vanishing value of the gauge field strength, and constant but arbitrary values of the $U(\mathbf{x})$. Quantum corrections lift the degeneracy.

One can evaluate the one-loop-potentials for one-flavor QCD-like theories. In the gauge in which the Polyakov line is represented by a constant and diagonal matrix one obtains[18]

$$V_{\mathrm{eff}}[U] = \frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} T_n, \tag{A1}$$

where

$$\begin{aligned} T_n &= -|\mathrm{Tr}U^n|^2 + a_n(\mathrm{Tr}U^n + \mathrm{Tr}U^{*n}), \qquad (\mathrm{F}), \\ T_n &= (-1 + a_n)|\mathrm{Tr}U^n|^2, \qquad (\mathrm{adj}), \end{aligned} \tag{A2}$$

[18]In the multiflavor generalization (with N_f fermions) one must replace $a_n \to a_n N_f$.

$$T_n = \frac{1}{2}(-1 + a_n)|\mathrm{Tr}U_1^n + \mathrm{Tr}U_2^n|^2 + \frac{1}{2}(-1 - a_n)|\mathrm{Tr}U_1^n - \mathrm{Tr}U_2^n|^2, \qquad (\mathrm{BF}), \quad (A3)$$

$$T_n = \frac{1}{4}(-1 + a_n)|\mathrm{Tr}U^n + \mathrm{Tr}U^{*n}|^2 + \frac{1}{4}(-1 - a_n)|\mathrm{Tr}U^n - \mathrm{Tr}U^{*n}|^2 \mp \frac{1}{2}a_n(\mathrm{Tr}U^{2n} + \mathrm{Tr}U^{*2n}), \qquad (\mathrm{AS/S}). \quad (A4)$$

Here a_n are prefactors which depend on the fermion boundary conditions

$$a_n = \begin{cases} (-1)^n & \text{for } \mathcal{S}^-, \\ 1 & \text{for } \mathcal{S}^+. \end{cases} \quad (A5)$$

Note that

$$\mathcal{C}(\mathrm{Tr}U^n \pm \mathrm{Tr}(U^*)^n) = \pm(\mathrm{Tr}U^n \pm \mathrm{Tr}(U^*)^n), \quad \mathcal{I}(\mathrm{Tr}U_1^n \pm \mathrm{Tr}(U_2)^n) = \pm(\mathrm{Tr}U_1^n \pm \mathrm{Tr}(U_2)^n). \quad (A6)$$

The minimum of the effective potential presented above is located at

$$U \sim \mathrm{Diag}(1, 1, \ldots, 1) \text{ all } \mathcal{R} \text{ with } \mathcal{S}^- \text{ and F/BF/AS/S with } \mathcal{S}^+, \quad (A7)$$
$$U = \mathrm{Diag}(1, e^{i(2\pi/N)}, \ldots, e^{i(2\pi(N-1)/N)}) \text{ adj with } \mathcal{S}^+.$$

Thus, the (spatial or temporal) center symmetry is broken in all theories, except QCD(Adj) with the periodic spin connection $\mathcal{S}^+$. In the cases of broken center symmetry the small and large radius physics on $S_1 \times R_3$ are separated by a phase transition. In all these cases the fermions essentially decouple from infrared physics, and the theory at small $r(S_1)$ has not much in common with the theory at large $r(S_1)$.

The center symmetry breaking is induced by destabilizing double-trace operators such as e.g. $-|\mathrm{Tr}U|^2$ and their multiwinding counterparts. One can stabilize the center symmetry while respecting the underlying symmetries of the theories at hand by adding a stabilizing polynomial in the appropriate variable up to the winding number $[N/2]$ with judiciously chosen coefficients. This will overwhelm the one-loop effect, and make the center-symmetric point a stable vacuum in the small-$r(S_1)$ regime.

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