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Note on quantum Minkowski space

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In this work, some interesting details about quantum Minkowski space and quantum Lorentz group structures are revealed. The task is accomplished by generalizing an approach adopted in a previous work where quantum rotation group and quantum Euclidean space structures have been investigated. The generalized method is based on a mapping relating the q-spinors (precisely the tensor product of dotted and undotted fondamental q-spinors) to Minkowski q-vectors. As a result of this mapping, the quantum analog of Minkowski space is constructed (with a definite metric). Also, the matrix representation of the quantum Lorentz group is determined together with its corresponding q-deformed orthogonality relation.

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I. INTRODUCTION

It is not without reason to say that the most successful model of theoretical physics in the twentieth century is in a sense related to some kind of deformation: Special relativity and quantum mechanics, on which is built the notorious standard model of particle physics, can be viewed as deformed versions of Galilei's relativity and classical mechanics, respectively [1], where the deformation parameters are the velocity of light c for the former and the Planck constant h for the latter.

In any case, many authors think that the deformation of a theory is not unusual in physics and may give answers to insoluble questions. For example, at present considerable efforts are made in studying the $\star$-deformed Minkowski space-time as well as the topics which rely upon its structure [2].[1] Likewise, we are interested in Minkowski space-time deformed structure but within Manin's meaning of deformed spaces [3]. The author, in [3], by evoking the notion of quantum planes that are spaces on which quantum groups act or coact (see also Ref. [4]), has stimulated us to investigate in Ref. [5] how one can apply Manin's view in the peculiar case of quantum Euclidean space and the quantum rotation group. This enabled us to construct the quantum analog of Euclidean space with a definite metric on one hand, and on the other hand we determined the matrix representation of the $SO_q(3)$ quantum group together with its corresponding orthogonality relation. In this work, we intended to generalize the analysis developed in [5] to the quantum Lorentz group and quantum Minkowski space. There has been a proposal regarding this problem in [6]. See also [7], where the question of probing the structure of the quantum Lorentz group has been carried out via an algebraic formulation. That is to say the quantum Lorentz group has been studied through its corresponding quantum Lie algebra which is equivalent to the direct product of two quantum $SU(2)$ Lie subalgebras. However, as we worked out an approach [5] which is rather geometric in treating the quantum Euclidean space,[2] we decided to pursue our analysis to recover the four-dimensional quantum Minkowski space case.

The aim of this paper is to reveal more details about quantum Minkowski space structure by using its link with the dotted and undotted q-spinors. So, by generalizing the prescription presented in [5], some interesting aspects on quantum Minkowski space and quantum Lorentz group structures are determined.

The paper is organized as follows. In Sec. II, we briefly review some algebraical aspects concerning both dotted and undotted q-spinors, wherein our notations are implemented. Then the link between q-spinors and Minkowski q-vectors is established. Thereby, the quantum Minkowski metric is determined. In Sec. III, the general matrix representation of the $SO_q(3, 1)$ quantum group and the corresponding q-orthogonality relation are derived. Section IV is devoted to some concluding remarks.

II. THE LINK BETWEEN *q*-SPINORS AND MINKOWSKI *q*-VECTORS

Let ψ^α be an element of the $SL_q(2)$ quantum plane, so that

$$\psi^\alpha = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}.$$

Such elements will be called undotted q-spinors. We assume q is real. Moreover, let $SL_q(2)$ coactions on contravariant q-spinor components be defined by

$$\Delta_L(\psi^\alpha) = T^\alpha{}_\beta \otimes \psi^\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \qquad (1)$$

leaving invariant the bilinear form $(\psi^1\phi^2 - q\psi^2\phi^1)$. ψ

[1]The cited references are just for illustration.

[2]The term *geometric* means that we have employed geometrical tools as the metric, the quantum distance, and the orthogonality relation.

and ϕ stand for any two undotted q-spinors and a, b, c, d the T-matrix elements, satisfying the quantum Yang-Baxter equation

$$R^{\alpha\beta}{}_{\gamma\delta}T^{\gamma}{}_{\sigma}T^{\delta}{}_{\tau} = T^{\alpha}{}_{\gamma}T^{\beta}{}_{\delta}R^{\gamma\delta}{}_{\sigma\tau}, \tag{2}$$

where R is a 4-by-4 matrix whose elements are given by

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{3}$$

Similarly, the left coaction of the $SL_q(2)$ quantum group on covariant q-spinor components reads [5]

$$\Delta_L(\psi_\alpha) = S^{-1}(T^{\beta}{}_{\alpha}) \otimes \psi_\beta = \begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix} \otimes \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{4}$$

leaving invariant the bilinear form $(\psi_1\phi_2 - q^{-1}\psi_2\phi_1)$. We note that the covariant and contravariant q-spinor components are related by means of the quantum metrics $\epsilon_{\alpha\beta}$ and $\epsilon^{\beta\alpha}$ such that

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \tag{5}$$

$$\psi^\alpha = \epsilon^{\beta\alpha}\psi_\beta. \tag{6}$$

Their entrees are

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}, \tag{7}$$

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix}. \tag{8}$$

Moreover, one can verify that the q-spinor components obey the known noncommuting rule of the Manin quantum plane [3], namely,

$$\psi^1\psi^2 = q\psi^2\psi^1, \tag{9}$$

or

$$\psi_1\psi_2 = q^{-1}\psi_2\psi_1. \tag{10}$$

As a generalization of (9), for any two q-spinors ψ and ϕ, we have (see [6])

$$\begin{aligned} &\psi^1\phi^1 = \phi^1\psi^1, \qquad \psi^1\phi^2 = q^{-1}[\phi^2\psi^1 + \lambda\phi^1\psi^2], \\ &\psi^2\phi^1 = q^{-1}\phi^1\psi^2, \qquad \psi^2\phi^2 = \phi^2\psi^2, \end{aligned} \tag{11}$$

or

$$\begin{aligned} &\psi_1\phi_1 = \phi_1\psi_1, \qquad \psi_2\phi_1 = q^{-1}[\phi_1\psi_2 + \lambda\phi_2\psi_1], \\ &\psi_1\phi_2 = q^{-1}\phi_2\psi_1, \qquad \psi_2\phi_2 = \phi_2\psi_2. \end{aligned} \tag{12}$$

Similarly, let us denote by $\psi^{\dot\alpha}$ ($\psi_{\dot\alpha}$) the contravariant (covariant) dotted q-spinor components. They transform as follows:

$$\Delta_L(\psi^{\dot\alpha}) = S^{-1}(\bar{T}^{\dot\alpha}{}_{\dot\beta}) \otimes \psi^{\dot\beta}, \tag{13}$$

$$\Delta_L(\psi_{\dot\alpha}) = \bar{T}^{\dot\beta}{}_{\dot\alpha} \otimes \psi_{\dot\beta}, \tag{14}$$

leaving invariant the bilinear forms $\phi^{\dot2}\psi^{\dot1} - q^{-1}\phi^{\dot1}\psi^{\dot2}$ and $\phi_{\dot2}\psi_{\dot1} - q\phi_{\dot1}\psi_{\dot2}$, respectively. Noting that $\bar{T}$ is the Hermitian conjugate matrix of the matrix T satisfying the quantum Yang-Baxter equation,

$$R^{\dot\gamma\dot\delta}{}_{\dot\alpha\dot\beta}\bar{T}^{\dot\tau}{}_{\dot\delta}\bar{T}^{\dot\sigma}{}_{\dot\gamma} = \bar{T}^{\dot\delta}{}_{\dot\beta}\bar{T}^{\dot\gamma}{}_{\dot\alpha}R^{\dot\sigma\dot\tau}{}_{\dot\gamma\dot\delta}, \tag{15}$$

and the dotted q-spinors are defined such that

$$\psi^{\dot\alpha} \sim (\psi_\alpha)^\star, \qquad \psi_{\dot\alpha} \sim (\psi^\alpha)^\star.$$

Here, "$\sim$" means "transforms as." At this point, we note that the dotted and undotted R matrices are the same. Now, to lower or raise the dotted indices, one can use the quantum metrics

$$\epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix}, \tag{16}$$

$$\epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}. \tag{17}$$

Furthermore, one can verify that the q-spinor components $\psi^{\dot\alpha}$ obey the following noncommuting rule:

$$\psi^{\dot1}\psi^{\dot2} = q\psi^{\dot2}\psi^{\dot1} \tag{18}$$

or

$$\psi_{\dot1}\psi_{\dot2} = q^{-1}\psi_{\dot2}\psi_{\dot1}. \tag{19}$$

More generally, we have

$$\begin{aligned} &\psi^{\dot1}\phi^{\dot1} = \phi^{\dot1}\psi^{\dot1}, \qquad \psi^{\dot2}\phi^{\dot1} = q[\phi^{\dot1}\psi^{\dot2} - \lambda\phi^{\dot2}\psi^{\dot1}], \\ &\psi^{\dot1}\phi^{\dot2} = q\phi^{\dot2}\psi^{\dot1}, \qquad \psi^{\dot2}\phi^{\dot2} = \phi^{\dot2}\psi^{\dot2}, \end{aligned} \tag{20}$$

or

$$\begin{aligned} &\psi_{\dot1}\phi_{\dot1} = \phi_{\dot1}\psi_{\dot1}, \qquad \psi_{\dot1}\phi_{\dot2} = q[\phi_{\dot2}\psi_{\dot1} - \lambda\phi_{\dot1}\psi_{\dot2}], \\ &\psi_{\dot2}\phi_{\dot1} = q\phi_{\dot1}\psi_{\dot2}, \qquad \psi_{\dot2}\phi_{\dot2} = \phi_{\dot2}\psi_{\dot2}. \end{aligned} \tag{21}$$

Using the dotted and undotted R matrices, the relations in (11), (12), (20), and (21) can be put under these forms:

$$\psi^\alpha\phi^\beta = \frac{1}{q}R^{\alpha\beta}{}_{\gamma\delta}\phi^\gamma\psi^\delta, \tag{22}$$

$$\psi^{\dot\alpha}\phi^{\dot\beta} = q(R^{-1})^{\dot\alpha\dot\beta}{}_{\dot\gamma\dot\delta}\phi^{\dot\gamma}\psi^{\dot\delta}, \tag{23}$$

$$\psi_\alpha\phi_\beta = \frac{1}{q}R^{\gamma\delta}{}_{\beta\alpha}\phi_\delta\psi_\gamma, \tag{24}$$

$$\psi_{\dot\alpha}\phi_{\dot\beta} = q(R^{-1})^{\dot\gamma\dot\delta}{}_{\dot\beta\dot\alpha}\phi_{\dot\delta}\psi_{\dot\gamma}. \quad (25)$$

Where R^{-1} is the inverse matrix of R, numerically

$$R^{-1} = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}. \quad (26)$$

We point out that in our approach the representation relevant for the quantum Lorentz group $SO_q(3,1)$ will be derived by requiring the following nontrivial commutation relations:

$$\psi^1\phi^{\dot 1} = \phi^{\dot 1}\psi^1, \qquad \psi^1\phi^{\dot 2} = q^{-1}[\phi^{\dot 2}\psi^1 + \lambda\phi^{\dot 1}\psi^2],$$
$$\psi^2\phi^{\dot 1} = q^{-1}\phi^{\dot 1}\psi^2, \qquad \psi^2\phi^{\dot 2} = \phi^{\dot 2}\psi^2,$$

or equivalently

$$\psi^\alpha\phi^{\dot\beta} = \frac{1}{q}R^{\alpha\dot\beta}{}_{\dot\gamma\delta}\phi^{\dot\gamma}\psi^\delta. \quad (27)$$

In view of (1), (13), and (27), we also have a nontrivial commutation relation between the matrices T and $S^{-1}(\bar T)$ as well as between the transposed matrices $\tilde T$ and $S^{-1}(\tilde{\bar T})$ (see [5]), namely,

$$R^{\alpha\dot\beta}{}_{\dot\gamma\delta}S^{-1}(\bar T^{\dot\gamma}{}_{\dot\sigma})T^\delta{}_\rho = T^\alpha{}_\delta S^{-1}(\bar T^{\dot\beta}{}_{\dot\gamma})R^{\delta\dot\gamma}{}_{\dot\sigma\rho}, \quad (28)$$

$$R^{\alpha\dot\beta}{}_{\dot\gamma\delta}S^{-1}(\tilde{\bar T}^{\dot\gamma}{}_{\dot\sigma})\tilde T^\delta{}_\rho = \tilde T^\alpha{}_\delta S^{-1}(\tilde{\bar T}^{\dot\beta}{}_{\dot\gamma})R^{\delta\dot\gamma}{}_{\dot\sigma\rho}. \quad (29)$$

Now let us introduce the big q-spinor Ψ with four components $(\psi^1, \psi^2, \psi^{\dot 1}, \psi^{\dot 2})$; then the mixed q-spinor $\Psi^{\alpha\dot\beta}$ will read[3]

$$\Psi^{\alpha\dot\beta} = \begin{pmatrix} \psi^1\phi^{\dot 1} \\ \psi^1\phi^{\dot 2} \\ \psi^2\phi^{\dot 1} \\ \psi^2\phi^{\dot 2} \end{pmatrix}. \quad (30)$$

By construction, one can verify that

$$\Psi^{\alpha\dot\beta}\Psi_{\dot\beta\alpha} = \text{Inv} \equiv \text{Invariant}. \quad (31)$$

Using the quantum metrics (7) and (17), we obtain instead of (31)

$$q^{-1}\Psi^{1\dot 1}\Psi^{2\dot 2} - \Psi^{1\dot 2}\Psi^{\dot 1 2} - \Psi^{2\dot 1}\Psi^{\dot 2 1} + \Psi^{2\dot 2}\Psi^{\dot 1 1} = \text{Inv}. \quad (32)$$

But as we also have [thanks to (27)]

$$\Psi^{1\dot 1} = \Psi^{\dot 1 1}, \qquad \Psi^{2\dot 2} = \Psi^{\dot 2 2},$$
$$\Psi^{1\dot 2} = \frac{1}{q}[\Psi^{\dot 2 1} + \lambda\Psi^{\dot 1 2}], \qquad \Psi^{2\dot 1} = q^{-1}\Psi^{\dot 1 2}, \quad (33)$$

(32) becomes

$$q^{-1}\Psi^{1\dot 1}\Psi^{2\dot 2} - q\Psi^{1\dot 2}\Psi^{2\dot 1} - \Psi^{2\dot 1}[q\Psi^{1\dot 2} - \lambda q\Psi^{2\dot 1}] + q\Psi^{2\dot 2}\Psi^{1\dot 1} = \text{Inv}. \quad (34)$$

Once more the quantum metrics are used, this time $\epsilon^{\dot\alpha\dot\beta}$, in order to transform (34) into

$$-[q^{-1}\Psi^1{}_{\dot 2}\Psi^2{}_{\dot 1} - q\Psi^1{}_{\dot 1}\Psi^2{}_{\dot 2} - q\Psi^2{}_{\dot 2}\Psi^1{}_{\dot 1} - \lambda\Psi^2{}_{\dot 2}\Psi^2{}_{\dot 2} + q\Psi^2{}_{\dot 1}\Psi^1{}_{\dot 2}] = \text{Inv}. \quad (35)$$

At this level, we can propose a mapping between the q-spinors and Minkowski q-vectors:

$$\Psi^1{}_{\dot 1} = q^{-1}A^0 + A^3, \qquad \Psi^2{}_{\dot 2} = qA^0 - A^3,$$
$$\Psi^1{}_{\dot 2} = A^-, \qquad \Psi^2{}_{\dot 1} = A^+. \quad (36)$$

In view of (36), and the expression between brackets in (35) becomes

$$q^{-1}A^-A^+ + qA^+A^- + (q+q^{-1})A^3A^3 - q^2(q+q^{-1})A^0A^0 = \text{Inv}. \quad (37)$$

It should be noticed that the usual Minkowski space distance is recovered when $q \to 1$. One can also see appearing in (37) the q-deformed metric of quantum Minkowski space; indeed

$$\text{Inv} = G_{\mu\nu}A^\mu A^\nu. \quad (38)$$

The latter formula is nothing but the invariant quantum Minkowski "distance." So it becomes possible to obtain from (37) the quantum metric G:

$$G = \begin{pmatrix} -q^2(q+q^{-1}) & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & (q+q^{-1}) \end{pmatrix}. \quad (39)$$

Note that the rows and columns of $G_{\mu\nu}$ are labeled by 0, +, −, 3. In the restrictive case where $(\psi^\alpha)^\star \sim \psi_\alpha$, A^0 will vanish and the quantum metric reduces to

$$g = \begin{pmatrix} 0 & q & 0 \\ q^{-1} & 0 & 0 \\ 0 & 0 & (q+q^{-1}) \end{pmatrix}, \quad (40)$$

as was found in [5].

III. THE MATRIX REPRESENTATION OF THE QUANTUM LORENTZ GROUP

Now let us determine the general matrix representation of the quantum Lorentz group $SO_q(3,1)$. First of all, let us see how the mixed q-spinor $\Psi^\alpha{}_{\dot\beta}$ can be transformed under

[3] It should be noticed that $\psi^{\dot\alpha}\phi^\beta \nsim \phi^\beta\psi^{\dot\alpha}$ as far as we have a nontrivial commutation relation between the T and $\bar T$ matrices [see (28)]; then $\psi^{\dot\alpha}\phi^\beta$ cannot be added to define the mixed q-spinor $\Psi^{\alpha\dot\beta}$.

$SL_q(2)$ coactions. By definition we have

$$\Delta_L(\Psi^{\alpha}{}_{\dot{\beta}}) = T^{\alpha}{}_{\gamma}\bar{T}^{\dot{\delta}}{}_{\dot{\beta}} \otimes \Psi^{\gamma}{}_{\dot{\delta}}. \tag{41}$$

Explicitly,

$$\begin{aligned}
\Delta_L(\Psi^1{}_{\dot{1}}) &= aa^{\star} \otimes \Psi^1{}_{\dot{1}} + ab^{\star} \otimes \Psi^1{}_{\dot{2}} + ba^{\star} \otimes \Psi^2{}_{\dot{1}} + bb^{\star} \otimes \Psi^2{}_{\dot{2}},\\
\Delta_L(\Psi^1{}_{\dot{2}}) &= ac^{\star} \otimes \Psi^1{}_{\dot{1}} + ad^{\star} \otimes \Psi^1{}_{\dot{2}} + bc^{\star} \otimes \Psi^2{}_{\dot{1}} + bd^{\star} \otimes \Psi^2{}_{\dot{2}},\\
\Delta_L(\Psi^2{}_{\dot{1}}) &= ca^{\star} \otimes \Psi^1{}_{\dot{1}} + cb^{\star} \otimes \Psi^1{}_{\dot{2}} + da^{\star} \otimes \Psi^2{}_{\dot{1}} + db^{\star} \otimes \Psi^2{}_{\dot{2}},\\
\Delta_L(\Psi^2{}_{\dot{2}}) &= cc^{\star} \otimes \Psi^1{}_{\dot{1}} + cd^{\star} \otimes \Psi^1{}_{\dot{2}} + dc^{\star} \otimes \Psi^2{}_{\dot{1}} + dd^{\star} \otimes \Psi^2{}_{\dot{2}}.
\end{aligned}$$

In view of (36), one can transform (41) into

$$\Delta_L(A^{\mu}) = M^{\mu}{}_{\nu} \otimes A^{\nu}, \tag{42}$$

where $M^{\mu}{}_{\nu}$ is a matrix with the following entrees:

$$\begin{pmatrix}
\frac{q^{-1}(aa^{\star}+cc^{\star})+q(bb^{\star}+dd^{\star})}{Q} & \frac{ba^{\star}+dc^{\star}}{Q} & \frac{ab^{\star}+cd^{\star}}{Q} & \frac{aa^{\star}+cc^{\star}-bb^{\star}-dd^{\star}}{Q}\\
q^{-1}ca^{\star}+qdb^{\star} & da^{\star} & cb^{\star} & ca^{\star}-db^{\star}\\
q^{-1}ac^{\star}+qbd^{\star} & bc^{\star} & ad^{\star} & ac^{\star}-bd^{\star}\\
\frac{aa^{\star}-q^{-2}cc^{\star}+q^2bb^{\star}-dd^{\star}}{Q} & \frac{qba^{\star}-q^{-1}dc^{\star}}{Q} & \frac{qab^{\star}-q^{-1}cd^{\star}}{Q} & \frac{q(aa^{\star}-bb^{\star})-q^{-1}(cc^{\star}-dd^{\star})}{Q}
\end{pmatrix}.$$

The rows and columns are labeled by 0, +, −, 3, and $Q = q + q^{-1}$. Furthermore, by using (2), (15), (28), and (29) one can verify that

$$\tilde{M}GM = G. \tag{43}$$

The latter equation is nothing but the q-deformed orthogonality relation of the $SO_q(3,1)$ quantum group, where G is the quantum Minkowski metric given by (39) and $\tilde{M}$ the transposed matrix of M. Also, it should be mentioned that nontrivial commutation relations between quantum Minkowski vector components have been found, namely,

$$\begin{aligned}
&A^0A^3 = A^3A^0, \qquad A^0A^+ = q^{-2}A^+A^0,\\
&A^0A^- = q^2A^-A^0, \qquad A^3A^+ = q^2A^+A^3,\\
&A^3A^- = q^{-2}A^-A^3,\\
&A^-A^+ = q^2A^+A^-\\
&\qquad - q\lambda(q^2A^0A^0 - qA^0A^3 - qA^3A^0 + A^3A^3).
\end{aligned} \tag{44}$$

To arrive at these nontrivial commutation relations, we took into account the proposed mapping (36) and Eqs. (22), (23), and (27). It should be stressed that from the construction given above, the quantum rotation group $SO_q(3)$ appears as a substructure; i.e.,

$$SO_q(3) \subset SO_q(3,1). \tag{45}$$

For seeing that, it suffices to go to the restrictive case where $(\psi^{\alpha})^{\star} \sim \psi_{\alpha}$; then $SL_q(2) \to SU_q(2)$. That is,

$$T^{\alpha}{}_{\beta} = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \to \begin{pmatrix} a & b\\ -qb^{\star} & a^{\star}\end{pmatrix}$$

and $SO_q(3,1) \to SO_q(3)$. That is to say,

$$M^{\mu}{}_{\nu} \to M_3 = \begin{pmatrix} a^{\star}a^{\star} & -qb^{\star}b^{\star} & -(1+q^2)a^{\star}b^{\star}\\ -qbb & aa & -(1+q^2)ba\\ ba^{\star} & ab^{\star} & aa^{\star}-bb^{\star}\end{pmatrix},$$

as was found in [5] but by carrying out the following changes:

$$b \to (-qb^{\star}), \qquad (-qb^{\star}) \to b.$$

This is because in [5] we used the representation

$$T = \begin{pmatrix} a & -qb^{\star}\\ b & a^{\star}\end{pmatrix}$$

for $SU_q(2)$ quantum group transformations.

IV. CONCLUDING REMARKS

In this paper we constructed the four-dimensional quantum Minkowski space, which is the representation space of the $SO_q(3,1)$ quantum group, with a quantum Minkowski metric (39) and a quantum Minkowski "length" (37). In the limit $q \to 1$ this space reduces to classical Minkowski space. We also derived the general matrix representation of the $SO_q(3,1)$ quantum Lorentz group together with its corresponding q-deformed orthogonality relation. This construction seems to be complete. Indeed, quantum Euclidean space and the quantum rotation group appeared within this study as substructures of quantum Minkowski space and the quantum Lorentz group, respectively. On the other hand, we know that the $SL(2,C)$ group has been considered, for a long time, as the most interesting track to build the gauge theory of Einstein's general relativity [8]. Thus, one wonders whether the $SL_q(2)$ quantum group as realized in this paper can add something new to the theory of gravitation, once its corresponding gauge theory of gravitation is established.[4] On the other hand, in comparing the present material with [6], we conclude the following: First of all the two approaches are geometrical in their formulation. The main difference is that in [6] the time component is a central element in the algebra of the coordinates (by construction). This property makes possible the retrieval of the quantum $SO(3)$ group structure just by posing the time component equal to zero. In our case, the time component appears as an ordinary component with nontrivial commutation relations [Eq. (44)]. So when we want to retrieve the quantum $SO(3)$ group structure, it just suffices to make identical the dotted spinor component with its corresponding undotted one $[(\psi^{\alpha})^{\star} \sim \psi_{\dot{\alpha}} = \psi_{\alpha}]$ as in the classical case ($q = 1$) [10]. Moreover, it seems to us constructive to mention that in the present prescription we used a notation in which classical coordinates and invariants appear in their deformed form. For example, the time component of the deformed four-vector, for $q = 1$, becomes simply the probability of presence of the fermion (spinor) once $SL(2) \rightarrow SU(2)$. Furthermore, in view of the relations (44), we believe that the nontriviality does appear in the case $q \neq 1$ only by complexifying the real coordinates, in the present situation $(A_x, A_y) \rightarrow (A^+ = A_x + iA_y, A^- = A_x - iA_y)$. These coordinates (A^+, A^-) seem to be at the origin of the observed nontriviality in (44). Classically A^+ and A^- are objects similar to charged particle fields. Therefore, it seems to us that only "charged" components can exhibit nontrivial commutation relations.

[4] At this point, we should mention that the work [9] will be of great interest in formulating the quantum gauge theory of gravitation as it contains a detailed differential calculus over the $SL_q(2)$ quantum group.

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