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Energy-momentum distribution in static and nonstatic cosmic string space-times

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We elaborate the problem of energy-momentum in general relativity by energy-momentum prescriptions theory. In this regard, we calculate Møller, Landau-Lifshitz, Papapetrou, Einstein, Bergmann, Tolman, and Weinberg's energy-momentum complexes in static and nonstatic cosmic string space-times. We obtain strong coincidences between the results. These coincidences can be considered an extension of Virbhadra's viewpoint that different energy-momentum prescriptions may provide some basis to define a unique quantity. In addition, our results disagree with Lessner's belief about Møller's prescription and support the Virbhadra's conclusion about the power of Einstein's prescription.

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I. INTRODUCTION

In classical mechanics and even in special relativity, we can always introduce a two-indice, symmetric tensorial quantity, i.e. T_a^b, which is called the energy-momentum tensor and represents the energy and momentum of matter and nongravitational fields sources. Besides the mentioned properties (being tensorial and symmetric), it has an important special characteristic: it is localized. This means that in every point of the manifold the quantity of energy-momentum is conserved. In other words, the energy-momentum tensor is a divergenceless quantity. In fact, in any local point of manifold no contribution of this quantity produces and none eliminates. We have

$$T^b_{a,b} = 0. \tag{1}$$

Equation (1) is the definition of energy-momentum conservation—known as conservation laws. Since the energy and momentum are two important, conserved quantities in physics, people are interested to keep them (as usual form of conservation laws) unchanged in all fields of physics, especially in the theory of general relativity (GR). But in GR, ordinary derivatives transform to covariant derivatives. So we have [1]

$$T^b_{a;b} = \frac{1}{\sqrt{-g}}(\sqrt{-g}T^b_a)_{,b} - \Gamma^b_{ac}T^c_b = 0. \tag{2}$$

It is obvious from Eq. (2) that T_a^b no longer satisfies $T^b_{a,b} = 0$, but as noted before, we are interested in having a similar equation in GR. We add an additional term to T_a^b, e.g. t_a^b, so that the summation of these two terms remains divergenceless. In reality, the quantity that is actually conserved in the sense of Eq. (1) is some effective quantity which is given (in one variant) by Eq. (20.18) of MTW [2] as ${}_{\mathrm{eff}}T_a^b = (T_a^b + t_a^b)$. In other variants, we obtain

$${}_{\mathrm{eff}}T_a^b = (-g)^{n/2}(T_a^b + t_a^b), \tag{3}$$

where $g = \det(g_{ab})$ and n is a positive integer that indicates the weight. For each of these ${}_{\mathrm{eff}}T_a^b$, Eq. (2) can be rewritten as

$${}_{\mathrm{eff}}T^b_{a,b} = 0. \tag{4}$$

Conserved quantity ${}_{\mathrm{eff}}T_a^b$ refers to the flux and density of energy and momentum of gravitational systems. In fact, coming from special relativity to GR, we add a contribution of gravitational fields, t_a^b, to the contribution of matter and all nongravitational fields, T_a^b. Einstein, himself, proposed the first prescription for ${}_{\mathrm{eff}}T_a^b$ just after GR's formulation in 1916. Then, many other persons such as Møller [3], Landau-Lifshitz [4], Papapetrou [5], Bergmann [6], Tolman [7], and Weinberg [1] gave different prescriptions. All proposed expressions are called energy-momentum complexes, because they can be expressed as a combination of a tensor, T_a^b, and a pseudotensor, t_a^b. However, by using this (adding t_a^b to T_a^b) we could solve the problem of having nonzero divergence of an energy-momentum tensor, but some serious problems arise. Actually, it can be shown that t_a^b does not obey tensor transformations. This nontensorial property of ${}_{\mathrm{eff}}T_a^b$ has caused these complexes not to satisfy the required covariance and be coordinate dependent; it is the main problem of using energy-momentum complexes. Some authors tried to introduce new coordinate independent prescriptions. In fact, with the exception of a few prescriptions, including Penrose [8], Møller [3], and Komar's [9] prescriptions, for other energy-momentum complexes one gets physically meaningful results only in the Cartesian coordinate system. The next problem is that it is not necessary for ${}_{\mathrm{eff}}T_a^b$ to be symmetric in all prescriptions. We can define conserved

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TABLE I. Comparison of the differences between energy-momentum prescriptions in symmetry and suitable coordinate systems needed.

Prescription	Coordinate System	Symmetry
Møller	any coordinate system	antisymmetric
Landau-Lifshitz	Cartesian	symmetric
Papapetrou	Cartesian	symmetric
Einstein	Cartesian	antisymmetric
Bergmann	Cartesian	nonsymmetric
Tolman	Cartesian	nonsymmetric
Weinberg	Cartesian	symmetric

angular momentum quantity only for symmetric prescriptions [1]. In this regard, the antisymmetric characteristic of Einstein's prescription was the main motivation for Landau and Lifshitz to look for an alternative prescription for energy-momentum which is symmetric. We have listed some prevalent and well-known prescriptions (that we have used in the next sections) and their properties in Table I.

For making the subject clearer, it should be noted that ${}_{\rm eff}T_a^b$ can be written as the divergence of some *superpotential* $H_a^{[bc]}$ that is antisymmetric in its two upper indices [10] as

$$ {}_{\rm eff}T_a^b = H_{a,c}^{[bc]}. \tag{5}$$

In addition, a new function like U_a^{bc} can also play the role of $H_a^{[bc]}$ if it has the following conditions:

$$U_a^{bc} = H_a^{[bc]} + \Psi_a^{bc} \quad (\Psi_{a,c}^{bc} \equiv 0, \ \text{or} \ \Psi_{a,cb}^{bc}). \tag{6}$$

Then, the quantity Θ_a^b which is defined by this new superpotential remains conserved locally:

$$\Theta_a^b = U_{a,c}^{bc} \Rightarrow \Theta_{a,b}^b = 0. \tag{7}$$

Using this freedom on the choice of the superpotential, authors like Einstein and Tolman arrived through different methods at the following superpotentials [11]:

$$H_a^{[bc]} = \frac{1}{2\kappa}\tilde{g}_{ae}(\tilde{g}^{eb}\tilde{g}^{dc} - \tilde{g}^{ec}\tilde{g}^{db})_{,d} \qquad \text{(Einstein)}, \tag{8}$$

$$\tau_a^{bc} = H_a^{[bc]} + \frac{1}{2\kappa}(\delta_a^c\tilde{g}^{db} - \delta_a^d\tilde{g}^{cb})_{,d} \qquad \text{(Tolman)}, \tag{9}$$

where $\tilde{g}^{ab} = \sqrt{-g}g^{ab}$.

Considering the above discussion, there are many prescriptions for new energy-momentum density (${}_{\rm eff}T_a^b$) [1,3–9] in which their differences are in a curl term. Each of them has its own advantages and disadvantages and there has not been proven any preferences between them. However, Palmer [12] and Virbhadra [13] discussed the importance of Einstein's energy-momentum prescription and Lessner [14] believed that Møller's prescription is a powerful tool for calculating the energy-momentum in GR.

The problems associated with the concept of energy-momentum complexes resulted in some researchers even doubting the concept of energy-momentum localization. Misner *et al.* [2] argued that to look for a local energy-momentum is looking for the right answer to the wrong question. He showed that the energy can be localized only in systems which have spherical symmetry. Cooperstock and Sarracino [15] proved that if energy is localizable for spherical systems, then it can be localized in any system. In 1990, Bondi [16] argued that a nonlocalizable form of energy is not allowed in GR. Recently, besides energy-momentum prescriptions theories, it was suggested that another solution for the energy problem in GR is in agreement with energy-momentum prescriptions theoriers about the localization of energy, i.e. *tele-parallel gravity* (for example see [17]). On the other hand, some people do not believe in localization of energy and momentum in GR. In addition, some physicists propose a new concept in this regard: *quasilocalization* (for example see [18]). Unlike energy-momentum prescriptions theories, quasilocalization theory does not restrict one to use a particular coordinate system, but this theory also has its drawbacks [19,20]. In general, there has been no generally accepted definition for energy and momentum in GR until now. Chang *et al.* in Ref. [21] showed that every energy-momentum complex can be associated with a distinct boundary term which gives the quasilocal energy-momentum. By this way, he dispels doubts expressed about the physical meaning of energy-momentum complexes.

For a long time, there has been an uncertainty that different energy-momentum complexes would give different results for a given space-time. Many researchers considered different prescriptions and obtained interesting results. Virbhadra *et al.* [13,22–25] investigated several examples of the space-times and showed that different prescriptions could provide exactly the same results for a given space-time. Aguirregabiria *et al.* [26] proved the consistency of the results obtained by using the different energy-momentum complexes for any Kerr-Schild class metric and revived the energy-momentum prescriptions theory after a long period of time.

In this paper we extend the previous works by calculating the energy of static and nonstatic cosmic string space-times in a specific region by seven well-known energy-momentum prescriptions: Møller, Landau-Lifshitz, Papapetrou, Einstein, Bergmann, Tolman, and Weinberg. We obtain encouraging results which show interesting coincidences among the results calculated by different prescriptions. Our results about Møller's prescription disagree with Lessner's viewpoint but support Virbhadra's conclusion that Einstein's prescription is the best available method for computing energy-momentum in a given space-time. The rest of the paper is organized as follows. In Sec. II we introduce energy-momentum complexes which we use in the following sections. Section III contains an introduction to static and nonstatic cosmic string space-times. The method and results of calculations are written in Sec. IV. In Sec. V we summarize and conclude with some remarks and discussions.

The conventions we use are geometrized units in which the speed of light in vacuum c is taken to be equal to 1 and the metric has signature $(+ - - -)$. Latin indices take values $0 \ldots 3$.

II. ENERGY-MOMENTUM PRESCRIPTIONS

Among many different forms proposed for energy-momentum pseudotensors, in this article we shall use Møller, Landau-Lifshitz, Papapetrou, Einstein, Bergmann, Tolman, and Weinberg's prescriptions. As in the previous works in the literature, here we try to show the compatibilities and to find out any existing discrepancies between predictions of these prescriptions when applying to static and nonstatic cosmic string space-times. Specific forms of each energy-momentum pseudotensor, conservation laws, and energy-momentum 4-vectors are listed in Table II briefly. Interested readers can refer to the mentioned references for details. In the last column of Table II, Gauss's theorem is used. In the surface integrals n_a represents the components of a normal one form over an infinitesimal surface element ds. The results of calculations according to each of the individual forms will be shown in the following sections.

III. COSMIC STRING SPACE-TIMES

The structure of the very early universe is one of the most interesting subjects of theoretical physics, as it remains a mystery. Cosmologists have generally assumed that at very early stages of its evolution, the Universe went through a number of phase transitions. One of the immediate consequences of these phase transitions is the formation of defects or mismatches in the orientation of the Higgs field in causally disconnected regions [27]. Cosmic strings are one of remarkable topological defects that have received particular attention because of their cosmological implications. The double quasar problem can be explained by strings and galaxy formation might also be generated by density fluctuation in the early universe due to strings [28].

Suppose an infinitely long, thin, straight, static string lying along the z axis with the following stress-energy tensor

$$T_a^b = \mu \delta(x)\delta(y)\mathrm{diag}(1, 0, 0, 1), \tag{10}$$

where μ is the mass per unit length of the string in the z direction. Considering space-time symmetries, Einstein's field equations lead to a well-known solution for case $\Lambda = 0$ in a polar cylindrical coordinate system (t, ρ, ϕ, z) [29–31]:

$$ds^2 = dt^2 - dz^2 - d\rho^2 - (1 - 4G\mu)^2\rho^2 d\phi^2. \tag{11}$$

For $\Lambda \neq 0$ Einstein's field equations lead to general form

TABLE II. Energy-momentum prescriptions.

Prescription	Energy-momentum pseudotensor	Conservation laws	Energy-momentum 4-vector
Møller [3]	$M_i^k = \frac{1}{8\pi}\chi_{i,l}^{kl}$ $\chi_i^{kl} = \sqrt{-g}(\frac{\partial g_{ip}}{\partial x^q} - \frac{\partial g_{iq}}{\partial x^p})g^{kq}g^{lp}$	$\frac{\partial M_i^k}{\partial x^k} = 0$	$P_i = \iiint M_i^0 dx^1 dx^2 dx^3$ $= \frac{1}{8\pi}\iint \chi_i^{0a} n_a ds$
Landau-Lifshitz [4]	$L^{ik} = \frac{1}{16\pi}\lambda_{,lm}^{iklm}$ $\lambda^{iklm} = -g(g^{ik}g^{lm} - g^{il}g^{km})_{,m}$	$\frac{\partial L^{ik}}{\partial x^k} = 0$	$P^i = \iiint L^{i0} dx^1 dx^2 dx^3$ $= \frac{1}{16\pi}\iint \lambda_{,m}^{i0\alpha m} n_a ds$
Papapetrou [5]	$\Sigma^{ik} = \frac{1}{16\pi}N_{,lm}^{iklm}$ $N^{iklm} = \sqrt{-g}(g^{ik}\eta^{lm} - g^{il}\eta^{km} + g^{lm}\eta^{ik} - g^{lk}\eta^{im})$ $\eta^{ik} = \mathrm{diag}(1, -1, -1, -1)$	$\frac{\partial \Sigma^{ik}}{\partial x^k} = 0$	$P^i = \iiint \Sigma^{i0} dx^1 dx^2 dx^3$ $= \frac{1}{16\pi}\iint N_{,m}^{i0\alpha m} n_a ds$
Einstein [3]	$\Theta_i^k = \frac{1}{16\pi}H_{i,l}^{kl}$ $H_i^{kl} = -H_i^{lk} = \frac{g_{in}}{\sqrt{-g}}[-g(g^{kn}g^{lm} - g^{ln}g^{km})]_{,m}$	$\frac{\partial \Theta_i^k}{\partial x^k} = 0$	$P_i = \iiint \Theta_i^0 dx^1 dx^2 dx^3$ $= \frac{1}{16\pi}\iint H_i^{0a} n_a ds$
Bergman [6]	$B^{ik} = \frac{1}{16\pi}\beta_{,m}^{ikm}$ $\beta^{ikm} = g^{ir}\nu_r^{km}$ $\nu_i^{kl} = -\nu_i^{lk} = \frac{g_{in}}{\sqrt{-g}}[-g(g^{kn}g^{lm} - g^{ln}g^{km})]_{,m}$	$\frac{\partial B^{ik}}{\partial x^k} = 0$	$P^i = \iiint B^{i0} dx^1 dx^2 dx^3$ $= \frac{1}{16\pi}\iint \beta_{,m}^{i0\alpha m} n_a ds$
Tolman [7]	$T_i^k = \frac{1}{8\pi}U_{i,l}^{kl}$ $U_i^{kl} = \sqrt{-g}(-g^{pk}V_{ip}^l + \frac{1}{2}g_i^k g^{pm}V_{pm}^l)$ $V_{jk}^i = -\Gamma_{jk}^i + \frac{1}{2}g_j^i\Gamma_{mk}^m + \frac{1}{2}g_k^i\Gamma_{mj}^m$	$\frac{\partial T_i^k}{\partial x^k} = 0$	$P_i = \iiint T_i^0 dx^1 dx^2 dx^3$ $= \frac{1}{8\pi}\iint U_i^{0a} n_a ds$
Weinberg [1]	$W^{ik} = \frac{1}{16\pi}D_{,l}^{ikl}$ $D^{ijk} = \frac{\partial h_a^a}{\partial x_i}\eta^{jk} - \frac{\partial h_a^a}{\partial x_j}\eta^{ik} - \frac{\partial h^{ai}}{\partial x^a}\eta^{jk} + \frac{\partial h^{aj}}{\partial x^a}\eta^{ik} + \frac{\mathrm{partial}h^{ik}}{\partial x_j} - \frac{\partial h^{jk}}{\partial x_i}$ $h_{ik} = g_{ik} - \eta_{ik}$	$\frac{\partial W^{ik}}{\partial x^k} = 0$	$P^i = \iiint W^{i0} dx^1 dx^2 dx^3$ $= \frac{1}{16\pi}\iint D^{i0a} n_a ds$

of static cosmic string space-time with the following line element in a polar cylindrical coordinate system (t, ρ, ϕ, z) [32]:

$$ds^2 = \cos^{4/3}\left(\frac{\sqrt{3\Lambda}}{2}\rho\right)(dt^2 - dz^2) - d\rho^2 - \frac{4(1-4G\mu)}{3\Lambda}\cos^{4/3}\left(\frac{\sqrt{3\Lambda}}{2}\rho\right)\tan^2\left(\frac{\sqrt{3\Lambda}}{2}\rho\right)d\phi^2, \tag{12}$$

where $\Lambda \to 0$ reduces to the previous metric, Eq. (11). Investigating the nonstatic solution of the cosmic strings, Einstein's field equations lead to nonstatic cosmic string space-time with the following line element in a polar cylindrical coordinate system (t, ρ, ϕ, z) [32]:

$$ds^2 = dt^2 - e^{2\sqrt{(\Lambda/3)}t}[d\rho^2 + (1-4G\mu)^2\rho^2 d\phi^2 + dz^2], \tag{13}$$

where $\Lambda \to 0$ or $t = 0$ reduces to Eq. (11). In the next section we calculate the energy of these space-times by different energy-momentum prescriptions.

IV. CALCULATIONS

A. Method

As mentioned in Sec. II, for calculating the energy of a given space-time in a specific region by energy-momentum prescriptions we should integrate energy-momentum superpotentials over a suitable surface in space-time. So, we should calculate the superpotential components and then indicate the normal vector over the infinitesimal surface element. In the two next subsections integrations are over a cylindrical surface surrounding the length L from the string symmetrically with radius ρ.

It should be noted that in the Cartesian coordinate system [considering $\phi = \arctan(\frac{y}{x})$, and $\rho = \sqrt{x^2+y^2}$] Eqs. (12) and (13) transform to the following line elements [Eq. (14) and (15)], respectively:

$$\begin{aligned} ds^2 = {} & \cos^{4/3}\alpha dt^2 \\ & - \frac{1}{3}\frac{3\Lambda x^2(x^2+y^2)\cos^{2/3}\alpha + 4a^2y^2\sin^2\alpha}{\Lambda(x^2+y^2)\cos^{2/3}\alpha}dx^2 \\ & - \frac{2}{3}\frac{3\Lambda(x^2+y^2)\cos^{2/3}\alpha - 4a^2\sin^2\alpha}{\Lambda(x^2+y^2)^2\cos^{2/3}\alpha}xydxdy \\ & - \frac{1}{3}\frac{3\Lambda x^2(x^2+y^2)\cos^{2/3}\alpha + 4a^2x^2\sin^2\alpha}{\Lambda(x^2+y^2)\cos^{2/3}\alpha}dy^2 \\ & - \cos^{4/3}\alpha dz^2, \end{aligned} \tag{14}$$

where $a = (1-4G\mu)$ and $\alpha = \frac{\sqrt{3\Lambda}}{2}\rho$, and

$$\begin{aligned} ds^2 = {} & dt^2 - e^{\alpha t}\frac{x^2+a^2y^2}{x^2+y^2}dx^2 + 2e^{\alpha t}\frac{a^2-1}{x^2+y^2}xydxdy \\ & - e^{\alpha t}\frac{y^2+a^2x^2}{x^2+y^2}dy^2 - e^{\alpha t}dz^2, \end{aligned} \tag{15}$$

where $a = (1-4G\mu)$ and $\alpha = 2\sqrt{\frac{\Lambda}{3}}$ [remember that α in Eq. (14) is different from that defined in Eq. (15)].

Everywhere we use $ds = \rho d\phi dz$ as the infinitesimal surface element. In a polar cylindrical coordinate system (allowed in Møller's prescription) we have $n_a = (0, 1, 0, 0)$, and in the Cartesian coordinate system (all prescriptions) we have $n_a = (0, \frac{x}{\rho}, \frac{y}{\rho}, 0)$. Summarizing all of the above, for calculating the energy, after extracting the needed superpotential components, we must calculate surface integrals over a cylindrical surface with a suitable normal vector n_a that depends on the used coordinate system. Following this method, in the next subsections we bring needed nonzero components of superpotentials and final energy results (i.e. P^0 or P_0) which are calculated by different energy-momentum prescriptions. Exact expressions of energy (except for the Møller prescription) are very well defined but long and complicated. So, we restricted ourselves to study the manner of energy around $\Lambda = 0$. Calculations for static ($\Lambda \neq 0$) and nonstatic cosmic string space-times are classified in two separate subsections. Meanwhile, it should be noted that we have done similar calculations by using static ($\Lambda = 0$) cosmic string space-time [line element Eq. (11)] and that these results are presented in Table III directly.

B. Static cosmic string ($\Lambda \neq 0$)

Defining $\alpha = \frac{\sqrt{3\Lambda}}{2}\rho$ and $a = 1 - 4G\mu$ energy can be calculated by different energy-momentum prescriptions as follows:

1. Møller prescription

Using Table II and Eq. (12) we find that nonzero needed components of χ_i^{kl} are

$$\chi_t^{tx} = -\frac{4}{3}\frac{a\sin^2\alpha}{x^2+y^2}x, \tag{16}$$

$$\chi_t^{ty} = -\frac{4}{3}\frac{a\sin^2\alpha}{x^2+y^2}y. \tag{17}$$

Using the surface integral (Table II) energy can be obtained as

$$_M E = -\tfrac{1}{3}aL\sin^2\alpha. \tag{18}$$

After Taylor expansion around $\Lambda = 0$ we have

$$_M E = -\tfrac{1}{4}a\rho^2 L\Lambda + \tfrac{1}{16}a\rho^4 L\Lambda^2 - \tfrac{1}{160}a\rho^6 L\Lambda^3 + \ldots, \tag{19}$$

TABLE III. Energy of three different cosmic string space-times in a cylinder with length L and radius ρ surrounding symmetrically the string ($\alpha = 2\sqrt{\frac{\Lambda}{3}}$ and $a = 1 - 4G\mu$).

Prescriptions	Static ($\Lambda = 0$)	Static ($\Lambda \neq 0$)	Nonstatic ($\Lambda \neq 0$)
Møller	0	$-\frac{1}{4}a\rho^2L\Lambda + \frac{1}{16}a\rho^4L\Lambda^2 - \frac{1}{160}a\rho^6L\Lambda^3 + \ldots$	0
Landau-Lifshitz	$\frac{L(1-a^2)}{8}$	$\frac{(1-a^2)}{8}L + \frac{(3a^2-1)}{16}L\rho^2\Lambda - \frac{(7a^2-1)}{128}L\rho^4\Lambda^2 + \ldots$	$\frac{(1-a^2)}{8}e^{2\alpha t}L$
Papapetrou	$\frac{L(1-a^2)}{8a}$	$\frac{(1-a^2)}{8a}L + \frac{(3a^2-1)}{16a}L\rho^2\Lambda - \frac{(17a^2-2)}{320a}L\rho^4\Lambda^2 + \ldots$	$\frac{(1-a^2)}{8a}e^{(1/2)\alpha t}L$
Einstein	$\frac{L(1-a^2)}{8a}$	$\frac{(1-a^2)}{8a}L + \frac{(3a^2-1)}{16a}L\rho^2\Lambda - \frac{(17a^2-2)}{320a}L\rho^4\Lambda^2 + \ldots$	$\frac{(1-a^2)}{8a}e^{(1/2)\alpha t}L$
Bergmann	$\frac{L(1-a^2)}{8a}$	$\frac{(1-a^2)}{8a}L + \frac{a}{8}L\rho^2\Lambda + \frac{1}{640}\frac{(11a^2-1)}{a}L\rho^4\Lambda^2 + \ldots$	$\frac{(1-a^2)}{8a}e^{(1/2)\alpha t}L$
Tolman	$\frac{L(1-a^2)}{8a}$	$\frac{(1-a^2)}{8a}L + \frac{(3a^2-1)}{16a}L\rho^2\Lambda - \frac{(17a^2-2)}{320a}L\rho^4\Lambda^2 + \ldots$	$\frac{(1-a^2)}{8a}e^{(1/2)\alpha t}L$
Weinberg	$\frac{L(1-a^2)}{4a^2}$	$\frac{(1-a^2)}{4a^2}L + \frac{1}{4}L\rho^2\Lambda + \frac{(34a^4-7a^2-2)}{160a^4}L\rho^4\Lambda^2 + \ldots$	$\frac{(1-a^2)}{4a^2}(-e^{\alpha t}a^2 + 1 + a^2)e^{-2\alpha t}L$

that for $\Lambda = 0$ vanishes immediately. In addition, as we expect, the same energy expression [Eq. (18)] is obtained by using the Møller energy-momentum prescription with this metric in a polar cylindrical coordinate system instead of a Cartesian coordinate system.

2. *Landau-Lifshitz prescription*

Using Table II and Eq. (14) we find that nonzero needed components of λ^{iklm} are

$$\lambda^{ttxx} = -\frac{1}{3}\cos^{2/3}\alpha\frac{3y^2\Lambda\cos^{2/3}\alpha(x^2+y^2) + 4a^2x^2\sin^2\alpha}{\Lambda(x^2+y^2)^2}, \tag{20}$$

$$\lambda^{ttyy} = -\frac{1}{3}\cos^{2/3}\alpha\frac{3x^2\Lambda\cos^{2/3}\alpha(x^2+y^2) + 4a^2y^2\sin^2\alpha}{\Lambda(x^2+y^2)^2}, \tag{21}$$

$$\lambda^{ttzz} = -\frac{4}{3}\frac{a^2\sin^2\alpha}{\Lambda(x^2+y^2)\cos^{2/3}\alpha}, \tag{22}$$

$$\lambda^{ttxy} = \lambda^{ttyx} = \frac{1}{3}\cos^{2/3}\alpha\frac{3\Lambda\cos^{2/3}\alpha(x^2+y^2) - 4a^2\sin^2\alpha}{\Lambda(x^2+y^2)^2}. \tag{23}$$

After surface integration (Table II) and Taylor expansion around $\Lambda = 0$, we obtain

$$_{LL}E = \frac{(1-a^2)}{8}L + \frac{(3a^2-1)}{16}L\rho^2\Lambda - \frac{(7a^2-1)}{128}L\rho^4\Lambda^2 + \ldots \tag{24}$$

3. *Papapetrou prescription*

Using Table II and Eq. (14) we find that nonzero needed components of N^{iklm} are

$$N^{ttxx} = -\frac{1}{6}\sqrt{\frac{3}{\Lambda}}\frac{1}{a\cos\alpha\sin\alpha(x^2+y^2)^{3/2}}[\cos^{8/3}(x^2+y^2) \times(3\Lambda y^2 - 4a^2) + 4a^2x^2\sin^2\alpha\cos^2\alpha + 4a^2\cos^{2/3}\alpha(x^2+y^2)], \tag{25}$$

$$N^{ttyy} = -\frac{1}{6}\sqrt{\frac{3}{\Lambda}}\frac{1}{a\cos\alpha\sin\alpha(x^2+y^2)^{3/2}}[\cos^{8/3}(x^2+y^2) \times(3\Lambda x^2 - 4a^2) + 4a^2y^2\sin^2\alpha\cos^2\alpha + 4a^2\cos^{2/3}\alpha(x^2+y^2)], \tag{26}$$

$$N^{ttzz} = -\frac{4\sqrt{3}}{3}\frac{a\sin\alpha}{\sqrt{\Lambda(x^2+y^2)}\cos^{1/3}\alpha}, \tag{27}$$

$$N^{ttxy} = N^{ttyx} = -\frac{\sqrt{3}}{6}\frac{\cos\alpha}{\sqrt{\Lambda(x^2+y^2)}} \times\frac{3\Lambda\cos^{2/3}\alpha(x^2+y^2) - 4a^2\sin^2\alpha}{a(x^2+y^2)\sin\alpha}. \tag{28}$$

After surface integration (Table II) and Taylor expansion around $\Lambda = 0$ we obtain

$$_{P}E = \frac{(1-a^2)}{8a}L + \frac{(3a^2-1)}{16a}L\rho^2\Lambda - \frac{(17a^2-2)}{320a}L\rho^4\Lambda^2 + \ldots \tag{29}$$

4. *Einstein prescription*

Using Table II and Eq. (14), we find that complicated quantities of H_t^{tx} and H_t^{ty} are only nonzero needed components of superpotential. After surface integration (Table II) and Taylor expansion around $\Lambda = 0$ we obtain

$$_{E}E = \frac{(1-a^2)}{8a}L + \frac{(3a^2-1)}{16a}L\rho^2\Lambda - \frac{(17a^2-2)}{320a}L\rho^4\Lambda^2 + \ldots \quad (30)$$

5. Bergmann prescription

Using Table II, and Eq. (14), we find that complicated quantities of B^{ttx} and B^{tty} are only nonzero needed components of the superpotential. After surface integration (Table II) and Taylor expansion around $\Lambda = 0$ we obtain

$$_{B}E = \frac{(1-a^2)}{8a}L + \frac{1}{8}L\rho^2 a\Lambda + \frac{(11a^2-1)}{640a}L\rho^4\Lambda^2 + \ldots \quad (31)$$

6. Tolman prescription

Using Table II and Eq. (14), we find that complicated quantities of U_t^{tx} and U_t^{ty} are only nonzero needed components of the superpotential. After surface integration (Table II) and Taylor expansion around $\Lambda = 0$ we obtain

$$_{T}E = \frac{(1-a^2)}{8a}L + \frac{(3a^2-1)}{16a}L\rho^2\Lambda - \frac{(17a^2-2)}{320a}L\rho^4\Lambda^2 + \ldots \quad (32)$$

7. Weinberg prescription

Using Table II and Eq. (14), we find that complicated quantities of D^{xtt} and D^{ytt} are only nonzero needed components of the superpotential. After surface integration (Table II) and Taylor expansion around $\Lambda = 0$ we obtain

$$_{W}E = \frac{(1-a^2)}{4a^2}L + \frac{1}{4}L\rho^2\Lambda + \frac{(34a^4-7a^2-2)}{160a^4}L\rho^4\Lambda^2 + \ldots \quad (33)$$

C. Nonstatic cosmic string

With $\alpha = 2\sqrt{\frac{\Lambda}{3}}$ and $a = 1 - 4G\mu$ different energy-momentum prescriptions can be evaluated as follows.

1. Møller prescription

Using Table II and Eq. (15) we find that nonzero components of χ_i^{kl} are

$$\chi_x^{xy} = -\chi_x^{yx} = \frac{(a-1)e^{(1/2)\alpha t}}{a(x^2+y^2)}y, \quad (34)$$

$$\chi_y^{yx} = -\chi_y^{xy} = \frac{(a-1)e^{(1/2)\alpha t}}{a(x^2+y^2)}x. \quad (35)$$

After surface integration, we find that the integral of energy vanishes. As we expect, in a polar cylindrical coordinate system we obtain the same result for the energy integral as we have obtained in the Cartesian coordinate system i.e. $_{\text{Møller}}E = 0$.

2. Landau-Lifshitz prescription

Using Table II and Eq. (15) we find that nonzero needed components of λ^{iklm} are

$$\lambda^{ttxx} = -e^{2\alpha t}\frac{y^2+a^2x^2}{x^2+y^2}, \quad (36)$$

$$\lambda^{ttyy} = -e^{2\alpha t}\frac{x^2+a^2y^2}{x^2+y^2}, \quad (37)$$

$$\lambda^{ttzz} = -e^{2\alpha t}a^2, \quad (38)$$

$$\lambda^{ttxy} = \lambda^{ttyx} = e^{2\alpha t}\frac{(1-a^2)}{x^2+y^2}xy. \quad (39)$$

After surface integration (Table II) we obtain

$$_{LL}E = \frac{(1-a^2)}{8}e^{2\alpha t}L. \quad (40)$$

3. Papapetrou prescription

Using Table II and Eq. (15) we find that nonzero needed components of N^{iklm} are

$$N^{ttxx} = -\frac{(y^2+a^2x^2) + e^{\alpha t}a^2(x^2+y^2)}{a(x^2+y^2)}e^{(1/2)\alpha t}, \quad (41)$$

$$N^{ttyy} = -\frac{(y^2+a^2y^2) + e^{\alpha t}a^2(x^2+y^2)}{a(x^2+y^2)}e^{(1/2)\alpha t}, \quad (42)$$

$$N^{ttzz} = -(1+e^{\alpha t})ae^{(1/2)\alpha t}, \quad (43)$$

$$\frac{(1-a^2)}{a(x^2+y^2)}xye^{(1/2)\alpha t}. \quad (44)$$

After surface integration (Table II) we obtain

$$_{P}E = \frac{(1-a^2)}{8a}e^{(1/2)\alpha t}L. \quad (45)$$

4. Einstein prescription

Using Table II and Eq. (15) we obtain nonzero needed components of H_i^{kl}

$$H_t^{tx} = \frac{(1-a^2)e^{1/2}}{a(x^2+y^2)}x, \quad (46)$$

$$H_t^{ty} = \frac{(1-a^2)e^{1/2}}{a(x^2+y^2)} y. \qquad (47)$$

After surface integration (Table II) we obtain

$$_E E = \frac{(1-a^2)}{8a} e^{(1/2)\alpha t} L. \qquad (48)$$

5. Bergmann prescription

Using Table II and Eqs. (15) and (32) we find nonzero needed components of B^{ikl} as

$$B^{ttx} = \frac{(1-a^2)e^{1/2}}{a(x^2+y^2)} x, \qquad (49)$$

$$B^{tty} = \frac{(1-a^2)e^{1/2}}{a(x^2+y^2)} y. \qquad (50)$$

After surface integration (Table II) we obtain

$$_B E = \frac{(1-a^2)}{8a} e^{(1/2)\alpha t} L. \qquad (51)$$

6. Tolman prescription

Using Table II and Eq. (15) we find that nonzero needed components of U_i^{kl} are

$$U_t^{tx} = \frac{(1-a^2)e^{1/2}}{2a(x^2+y^2)} x, \qquad (52)$$

$$U_t^{ty} = \frac{(1-a^2)e^{1/2}}{2a(x^2+y^2)} y. \qquad (53)$$

After surface integration (Table II) we obtain

$$_T E = \frac{(1-a^2)}{8a} e^{(1/2)\alpha t} L. \qquad (54)$$

7. Weinberg prescription

Using Table II and Eq. (15) we obtain that nonzero needed components of D^{ikl} are

$$D^{xtt} = \frac{(1-a^2)(e^{\alpha t}a^2 - 1 - a^2)}{a^4(x^2+y^2)} e^{-2\alpha t} x, \qquad (55)$$

$$D^{ytt} = \frac{(1-a^2)(e^{\alpha t}a^2 - 1 - a^2)}{a^4(x^2+y^2)} e^{-2\alpha t} y. \qquad (56)$$

After surface integration (Table II) we obtain

$$_W E = \frac{(1-a^2)(-e^{\alpha t}a^2 + 1 + a^2)}{4a^2} e^{-2\alpha t} L. \qquad (57)$$

V. CONCLUSIONS AND REMARKS

In the previous section we calculated the energy of static and nonstatic cosmic string space-times in a cylinder with length L and radius ρ surrounding the string symmetrically. We have summarized all obtained results in Table III.

Regarding the contents of Table III:

(i) It is concluded that the energy turns out to be finite and well defined in all prescriptions for these space-times.

(ii) Substituting $a = 1$ in the first column, all prescriptions give energy equal to zero that is completely consistent. Because, if $a = 1$, Eq. (11) reduces to the Minkowski line element of which its energy is equal to zero in any arbitrary region.

(iii) For static ($\Lambda \neq 0$) cosmic string space-time Einstein, Tolman, and Papapetrou's prescriptions lead to the same results. In addition, when $\Lambda \to 0$, the Bergmann prescription is added to this list. For the nonstatic case Einstein, Papapetrou, Tolman, and Bergmann prescriptions have the same result. This coincidence supports and extends Virbhadra's viewpoint [26] that different energy-momentum prescriptions may provide some basis to defining a unique quantity. However, the remaining prescriptions give different energy densities (because of a noncovariant property of pseudotensors).

(iv) As we expect, for $\Lambda \to 0$ energy expressions in the second and third columns reduce to their corresponding expressions in the first column. It should be noted that we calculated the components of the second column separately, by using line element Eq. (11) in energy-momentum prescriptions.

(v) Unlike other prescriptions, Møller's prescription leads to a zero quantity for energy. This shortcoming is in contradiction with Lessner's viewpoint and supports Virbhadra's conclusion. Lessner [14] believed that Møller's prescription is a powerful tool for calculating the energy-momentum pseudotensors in GR, and Virbhadra [13] concluded that Einstein's energy-momentum prescription is still the best available method for computing energy-momentum in a given space-time.

(vi) Reviewing the results shows that adding a factor a in denominators of Landau-Lifshitz's results causes this prescription to also give equivalent results (in comparison with Einstein, Tolman, and Papapetrou's prescriptions). In other words Landau-Lifshetz's results are different with other similar results (Einstein, Tolman, and Papapetrou) just in a factor a in the denominator. This dilemma is due to the fact that the conserved quantity in the Landau-Lifshitz prescription is $_{\rm eff}T_a^b = (-g)(T_a^b + t_a^b)$ (weight + 2) instead of $_{\rm eff}T_a^b = \sqrt{-g}(T_a^b + t_a^b)$ (see Eq. (3) and Ref. [4]) in which its weight is +1. So, we should be careful about using this expression with weight +2 in our integration (see [33], chapter 7). Calculating energy by using a correction to Landau-Lifshitz prescription i.e. $\lambda^{iklm} = \sqrt{-g}(g^{ik}g^{lm} - g^{il}g^{km})_{,m}$ instead of $\lambda^{iklm} = (-g)(g^{ik}g^{lm} - g^{il}g^{km})_{,m}$ (Table II) leads to consistent results.

(vii) In the final remark we would like to raise some points on the validity of the metric (12). Equation (12) with $\mu \to 0$ faces with some problems to represent de Sitter space-time (dss). It has intrinsic singularities at $\rho = \frac{n\pi}{\sqrt{3\Lambda}}$, $n =$ odd [34], while dss is free of them. The standard form of dss is

$$ds^2 = dt^2 - \frac{3}{\Lambda}\cosh^2\left(\sqrt{\frac{3}{\Lambda}}t\right)(d\chi^2 + \sin^2\chi \times (d\theta^2 + \sin^2\theta d\phi^2)), \tag{58}$$

where $-\infty < t < +\infty$, $0 \le \chi \le \pi$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. By the following transformations,

$$\hat{t} = \sqrt{\frac{3}{\Lambda}}\log\left(\sinh\left(\sqrt{\frac{3}{\Lambda}}t\right) + \cosh\left(\sqrt{\frac{3}{\Lambda}}t\right) + \cos\chi\right), \tag{59}$$

$$\hat{x} = \frac{\cosh(\sqrt{\frac{3}{\Lambda}}t)\sin\chi\cos\theta}{\sinh(\sqrt{\frac{3}{\Lambda}}t) + \cosh(\sqrt{\frac{3}{\Lambda}}t) + \cos\chi}, \tag{60}$$

$$\hat{y} = \frac{\cosh(\sqrt{\frac{3}{\Lambda}}t)\sin\theta\cos\phi}{\sinh(\sqrt{\frac{3}{\Lambda}}t) + \cosh(\sqrt{\frac{3}{\Lambda}}t) + \cos\chi}, \tag{61}$$

$$\hat{z} = \frac{\cosh(\sqrt{\frac{3}{\Lambda}}t)\sin\theta\sin\phi}{\sinh(\sqrt{\frac{3}{\Lambda}}t) + \cosh(\sqrt{\frac{3}{\Lambda}}t) + \cos\chi}. \tag{62}$$

The metric (58) transforms to the steady state form or the so-called half de Sitter metric:

$$ds^2 = d\hat{t}^2 - \exp\left(2\sqrt{\frac{3}{\Lambda}}t\right)(d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2). \tag{63}$$

In a polar coordinate system, it takes the form

$$ds^2 = d\hat{t}^2 - \exp\left(2\sqrt{\frac{3}{\Lambda}}t\right)(d\hat{r}^2 + \hat{r}^2 \times (d\theta^2 + \sin^2\theta d\phi^2)). \tag{64}$$

Then by transformation,

$$r = \exp\left(\sqrt{\frac{3}{\Lambda}}t\right)\hat{r}, \tag{65}$$

$$t = \hat{t} - \frac{1}{2}\sqrt{\frac{3}{\Lambda}}\ln\left(1 - \frac{\Lambda}{3}r^2\right). \tag{66}$$

The metric (64) can be transformed to the static form

$$ds^2 = \left(1 - \frac{\Lambda}{3}r^2\right)dt^2 - \frac{1}{(1 - \frac{\Lambda}{3}r^2)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{67}$$

Writing the metric (67) in a cylindrical coordinate system (t, ρ, φ, z), we have

$$ds^2 = \left(1 - \frac{\Lambda}{3}r^2\right)dt^2 - \frac{1 - \frac{\Lambda}{3}z^2}{1 - \frac{\Lambda}{3}r^2}d\rho^2 - 2\frac{\frac{\Lambda}{3}\rho z}{1 - \frac{\Lambda}{3}r^2}d\rho dz - \frac{1 - \frac{\Lambda}{3}\rho^2}{1 - \frac{\Lambda}{3}r^2}dz^2 - \rho^2 d\varphi^2. \tag{68}$$

Now the difference between Eqs. (12) and (68) when $\mu \to 0$ is quite evident. This means although the static metric (12) is an exact solution for Einstein equations, there is some doubt as to whether it actually fulfills precisely the requirements of the space-time associated with a cylindrical cosmic string located in a cosmological constant background.

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[1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of General Theory of Relativity* (Wiley and Sons, New York, 1972).
[2] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973), p. 467, 603.
[3] C. Møller, Ann. Phys. (N.Y.) **4**, 347 (1958).
[4] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Press, Reading, Mass., 1987), p. 280.
[5] A. Papapetrou, Proceedings of the Royal Irish Academy Section B Biological, geological and chemical science **52**, 11 (1948).
[6] P. G. Bergmann and R. Thompson, Phys. Rev. **89**, 400 (1953).
[7] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, New York, 1934), p. 227.
[8] R. Penrose, Proc. R. Soc. A **381**, 53 (1982).
[9] A. Komar, Phys. Rev. **113**, 934 (1959).

[10] Marc Favata, Phys. Rev. D **63** 064013 (2001).
[11] N. Pinto Neto and P. I. Trajtenberg, Braz. J. Phys. **30**, no. 1 (2000).
[12] T. Palmer, Gen. Relativ. Gravit. **12**, 149 (1980).
[13] K. S. Virbhadra, Phys. Rev. D **60**, 104041 (1999).
[14] G. Lessner, Gen. Relativ. Gravit. **28**, 527 (1996).
[15] F. I. Cooperstock and R. S. Sarracino, J. Phys. A **11**, 877 (1978).
[16] H. Bondi, Proc. R. Soc. A **427**, 249 (1990).
[17] T. Vargas, Gen. Relativ. Gravit. **36**, 1255 (2004).
[18] S. A. Hayward, Phys. Rev. D **49**, 831 (1994).
[19] D. Christodoulou and S. T. Yau, *Mathematics and General Relativity*, edited by J. Isenberg (AMS, Providence Rhode Island, 1988).
[20] G. Berqvist, Classical Quantum Gravity **9**, 831 (1994).
[21] C. C. Chang, J. M. Nester, and C. M. Chen, Phys. Rev. Lett. **83**, 1897 (1999).
[22] K. S. Virbhadra, Phys. Rev. D **42**, 2919 (1990).
[23] N. Rosen and K. S. Virbhadra, Gen. Relativ. Gravit. **25**, 429 (1993).
[24] K. S. Virbhadra and J. C. Parikh, Phys. Lett. B **317**, 312 (1993).
[25] K. S. Virbhadra and J. C. Parikh, Phys. Lett. B **331**, 302 (1994).
[26] J. M. Aguirregabiria, A. Chamorro, and K. S. Virbhadra, Gen. Relativ. Gravit. **28**, 1393 (1996).
[27] T. W. B. Kibble, J. Phys. A **9**, 1387 (1976).
[28] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, England, 1994).
[29] A. Vilenkin, Phys. Rep. **121**, 263 (1985).
[30] J. R. Gott, Astrophys. J. **288**, 422 (1985).
[31] P. J. E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, Princeton, New Jersey, 1993).
[32] Amir H. Abbassi, Amir M. Abbassi, and H. Razmi, Phys. Rev. D **67**, 103504 (2003).
[33] R. A. D'Inverno, *Introducing Einstein's Relativety* (Oxford University Press, New York, 1992).
[34] S. Bhattacharya and A. Lahiri, arXiv:0807.0543.