

Separability of the massive Dirac equation in 5-dimensional Myers-Perry black hole geometry and its relation to a rank-three Killing-Yano tensor

Shuang-Qing Wu (吴双清)

College of Physical Science and Technology, Central China Normal University, Wuhan, Hubei 430079, People's Republic of China
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The Dirac equation for the electron around a five-dimensional rotating black hole with two different angular momenta is separated into purely radial and purely angular equations. The general solution is expressed as a superposition of solutions derived from these two decoupled ordinary differential equations. By separating variables for the massive Klein-Gordon equation in the same spacetime background, I derive a simple and elegant form for the Stäckel-Killing tensor, which can be easily written as the square of a rank-three Killing-Yano tensor. I have also explicitly constructed a symmetry operator that commutes with the scalar Laplacian by using the Stäckel-Killing tensor, and the one with the Dirac operator by the Killing-Yano tensor admitted by the five-dimensional Myers-Perry metric, respectively.

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I. INTRODUCTION

It is well known that the four-dimensional Kerr geometry [1] possesses a lot of miraculous properties that not only can the geodesic Hamilton-Jacobi equation [2] and the Klein-Gordon scalar field equation [2] be separated and decoupled into purely radial and purely angular parts, but also the massless nonzero-spin field equations [3] as well as the equilibrium equation for a stationary cosmic string [4]. These separability properties are shown to be closely connected with the existence of an additional integral of motion associated with the second order symmetric Stäckel-Killing tensor discovered in the Kerr metric by Carter [2]. The separation of the variables of Dirac's equation for massive fields in the Kerr geometry using the Newman-Penrose formalism [5], however, had only succeeded since Chandrasekhar's remarkable work [6]. Shortly after that, this result was extended by Page and other people [7] to the four-dimensional rotating charged Kerr-Newman black hole background.

As has been remarked by Chandrasekhar [8], the most striking feature of the Kerr metric is the separability of all the standard wave equations in it. For some of these equations, their separability has been understood as a consequence of the existence of certain tensor fields, which have been found to be associated with a Killing spinor. Walker and Penrose [9] demonstrated that the Carter's fourth constant can be constructed out of the Weyl spinor. Subsequently, the separability of Dirac's equation has been explained by Carter and McLennaghan [10] in terms of the existence of a Killing-Yano tensor, whose spinorial image is a two-index Killing spinor. Physically, Killing-Yano tensors and operators constructed from them have been associated with angular momentum. It has also been shown by a lot of people [11] that Killing-Yano tensors and the Killing spinor play a crucial role in separation of variables for the Maxwell's equation ($s = 1$), Rarita-Schwinger's

equation ($s = 3/2$), and the gravitational perturbation equation in the Kerr geometry. The separation of various equations can be understood in terms of different order differential operators that characterized the separation constants that appeared in the separable solutions. The differential operators characterizing separation constants [12] are also symmetry operators of the various field equations in question. The essential property that allows the construction of such operators is the existence of a Killing-Yano tensor in the Kerr spacetime. These results have been shown to hold for more general classes of type-D vacuum metrics; see Ref. [13] for a comprehensive review.

In recent years, higher-dimensional generalizations of the Kerr black hole and their properties have attracted considerable attention [14], in particular, in the context of string theory, with the discovery of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, and with the advent of brane-world theories [15], raising the possibility of direct observation of Hawking radiation and of as probes of large spatial extra dimensions in future high energy colliders [16]. In the brane-world scenarios, our physical world is represented by a four-dimensional brane embedded in the higher-dimensional bulk spacetime. Brane-world models of spacetimes with large extra dimensions allow for the existence of higher-dimensional black holes whose geometry can be approximately described by the classical solutions of vacuum Einstein equation, thus predicting the possibility of mini-black hole production in a high energy factory. The metrics describing the isolated rotating black holes in higher dimensions were first constructed by Myers and Perry [17] as the asymptotically flat generalizations of the well-known four-dimensional Kerr vacuum solution. By introducing a nonzero cosmological constant, Hawking, *et al.* [18] obtained the asymptotically nonflat generalizations in five dimensions with two independent angular momenta and in higher dimensions with just one nonzero angular momentum parameter. Further

vacuum generalizations to all dimensions have been made recently in [19]. Quite recently, an exact charged generalization of the Kerr-Newman solution in five dimensions was obtained in [20] within the framework of minimally gauged supergravity theory. Other rotating charged black hole solutions in five-dimensional gauged and ungauged supergravity were also obtained in [21–24].

It is generally accepted that symmetries play a key role in the study of physical effects in the gravitational fields of black holes. Initiated by the work of Frolov and his collaborators (see [25] for a review and references therein), recently there has been a resurgence of interest [26–37] in the study of “hidden” symmetry and separation of variables properties of the Klein-Gordon scalar equation, the Hamilton-Jacobi equation, and stationary strings [38] in higher dimensions [39]. Remarkably, it was shown that the five-dimensional Myers-Perry [17] metric possesses a number of miraculous properties similar to the Kerr metric. Namely, it allows the separation of variables in the geodesic Hamilton-Jacobi equation and the separability of the massless Klein-Gordon scalar field equation [26]. These properties are also intimately connected with the existence of the second order Stäckel-Killing tensor [26] admitted by the five-dimensional Myers-Perry black hole geometry. It was further demonstrated that this rank-two Stäckel-Killing tensor can be constructed from its “square root,” a rank-three Killing-Yano tensor [27]. Following the procedure of Carter’s construction [40] in four dimensions, Frolov *et al.* [27] started from a potential 1-form to generate a rank-two conformal Killing-Yano tensor [41], whose Hodge dual is just the expected Killing-Yano tensor. Subsequently, these results have further been extended [27,35–37] to general higher-dimensional rotating black hole solutions with Newman-Tamburino-Unti (NUT) charges [19].

However, less is known about the separability of Dirac’s equation and other higher-spin fields and its relation to the Killing-Yano tensor in higher dimensions [42,43]. Therefore it is important to investigate the separability of the fermion field equation and its relation to the Killing-Yano tensor in higher-dimensional rotating black holes. In this paper, I will report my past unpublished work (done in the October of 2004) on the separation of variables for a massive Dirac equation in five-dimensional rotating Myers-Perry black holes with two unequal angular momenta [17]. I will also present my recent construction of an explicit symmetry operator that commutes with the standard Dirac operator, making use of the rank-three Killing-Yano tensor which can be viewed as the square root of a rank-two symmetric Stäckel-Killing tensor. In addition, the separated parts of a massive Klein-Gordon equation in the five-dimensional Myers-Perry background are used to construct a simple and elegant expression of the Stäckel-Killing tensor. Note that these symmetry operators are directly constructed from the separated solutions of the

Klein-Gordon equation and Dirac’s equation in the background geometry considered here.

The outline of this paper goes as follows. In Sec. II, the action of the five-dimensional gravity and fermions is given and the fünfbein form of Dirac’s equation is formulated. Using Clifford algebra and the spinor representation of $SO(4,1)$, I construct the spinor-connection 1-form which is necessary for the fermion field equation in curved space-time. Sec. III is devoted to dealing with the separation of variables of Dirac’s equation in a five-dimensional Myers-Perry black hole geometry. This section consists of three subsections. In Sec. III A, a new form of the five-dimensional Myers-Perry metric is expressed in the Boyer-Lindquist coordinate which admits an explicit construction of local orthonormal coframe 1-forms (pentad). A brief review of the relevant symmetry properties of the Myers-Perry metric is also presented. In Sec. III B, the spinor connection is obtained by making use of the homomorphism between the $SO(4,1)$ group and its spinor representation which is derived from the Clifford algebra defined by the anticommutation relations of the gamma matrices. In Sec. III C, the massive Dirac equation in five-dimensional Myers-Perry black hole is separated into purely radial and purely angular equations. Sec. IV is also divided into three parts. In this section, the separated solutions of a massive Klein-Gordon equation is used to construct a concise expression for the Stäckel-Killing tensor. From the separated part of Dirac’s equation, I also explicitly construct a first order symmetry operator that commutes with the Dirac operator by using the rank-three Killing-Yano tensor. The last section V is a brief summary of this paper and the related work under preparation. Possible applications of this work to further research are given here. In Appendix A, the affine spin-connection 1-forms are calculated by the first Cartan structure equation from the exterior differential of the pentad. Appendix B displays the five-dimensional Myers-Perry metric in a manner similar to the Plebanski solution [44] in four dimensions.

II. FÜNFBEIN FORMALISM OF DIRAC FIELD EQUATION IN 5-DIMENSIONAL CURVED SPACE

It is well known that there exist two different but equivalent formalisms for the four-dimensional gravity, namely, the orthonormal tetrad formalism [45] and the null-tetrad (Newman-Penrose) formalism [5]. Dirac’s equation in four dimensions was reformulated within the Newman-Penrose formalism first by Chandrasekhar [6] and then extended to the charged case by Page [7]. To my knowledge, a higher-dimensional generalization of the Newman-Penrose formalism was established in [46,47], but no similar work was given for the Dirac equation, subject to the purpose here. In absence of a similar Newman-Penrose formalism in five dimensions, in this paper I will work out the Dirac equation within the orthonormal pentad formalism [48]. In a forth-

coming paper [49], a seminull pentad formalism of the Dirac equation was constructed in the five-dimensional relativity similar to the famous work of Chandrasekhar's [6]. The Dirac equation has been shown to be decoupled into purely radial and purely angular parts which agree with the results presented here.

In curved background spacetime, the action of the five-dimension gravity and fermions is given by

$$S = \int d^5x \sqrt{-g} \left[\frac{-\mathcal{R}}{16\pi} + i\bar{\psi} \gamma^A e_A{}^\mu (\partial_\mu + \Gamma_\mu) \psi + i\mu_e \bar{\psi} \psi \right], \quad (1)$$

where \mathcal{R} is the five-dimensional curvature scalar of the metric $g_{\mu\nu}$, ψ is a four-component Dirac spinor, μ_e is the mass of the electron, Γ_μ is the spinor-connection, $e_A{}^\mu$ is the fünfbein (pentad), and γ^A 's are the five-dimensional gamma matrices. My conventions are as follows: Latin letters A, B denote local orthonormal (Lorentz) frame indices $\{0, 1, 2, 3, 5\}$, while Greek letters μ, ν run over five-dimensional spacetime coordinate indices $\{t, r, \theta, \phi, \psi\}$. Units are used as $G = \hbar = c = 1$ throughout this paper.

The Dirac equation can be deduced from the action (1) by variation with respect to the spinor field as

$$(\mathbb{H}_D + \mu_e)\Psi = [\gamma^A e_A{}^\mu (\partial_\mu + \Gamma_\mu) + \mu_e]\Psi = 0, \quad (2)$$

where the fünfbein $e_A{}^\mu$ and its inverse $e^A{}_\mu$ are defined by the spacetime metric $g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu$ with $\eta_{AB} = \text{diag}(-1, 1, 1, 1, 1)$ being the flat (Lorentz) metric tensor. For my purpose in this paper, I choose gamma matrices γ^A obeying the anticommutation relations (Clifford algebra)

$$\{\gamma^A, \gamma^B\} \equiv \gamma^A \gamma^B + \gamma^B \gamma^A = 2\eta^{AB}, \quad (3)$$

and take an explicit representation of the Clifford algebra as follows:

$$\begin{aligned} \gamma^0 &= i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, & \gamma^1 &= i \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}, \\ \gamma^2 &= i \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, & \gamma^3 &= i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \\ \gamma^5 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = -i\gamma^0\gamma^1\gamma^2\gamma^3, \end{aligned} \quad (4)$$

where σ^i 's are the Pauli matrices, and I is a 2×2 identity matrix, respectively.

In order to derive the spinor-connection 1-form $\Gamma = \Gamma_\mu dx^\mu \equiv \Gamma_A e^A$, I first compute the spin-connection 1-form $\omega_{AB} = \omega_{AB\mu} dx^\mu \equiv f_{ABC} e^C$ in the orthonormal frame, i.e., the 1-form (pentad) $e^A = e^A{}_\mu dx^\mu$ satisfying the torsion-free condition

$$de^A + \omega^A{}_B \wedge e^B = 0, \quad \omega_{AB} = \eta_{AC} \omega^C{}_B = -\omega_{BA}. \quad (5)$$

To obtain the spinor-connection 1-form Γ from ω_{AB} , I can make use of the homomorphism between the $SO(4,1)$ group and its spinor representation which is derived from the Clifford algebra (3). The $SO(4,1)$ Lie algebra is defined by the ten antisymmetric generators $\Sigma^{AB} = [\gamma^A, \gamma^B]/(2i)$ which gives the spinor representation, and the spinor-connection Γ can be regarded as a $SO(4,1)$ Lie-algebra-valued 1-form. Using the isomorphism between the $SO(4,1)$ Lie algebra and its spinor representation, i.e., $\Gamma_\mu = (i/4)\Sigma^{AB}\omega_{AB\mu} = (1/4)\gamma^A\gamma^B\omega_{AB\mu}$, I can immediately construct the spinor-connection 1-form

$$\Gamma = \frac{1}{8}[\gamma^A, \gamma^B]\omega_{AB} = \frac{1}{4}\gamma^A\gamma^B\omega_{AB} = \frac{1}{4}\gamma^A\gamma^B f_{ABC} e^C. \quad (6)$$

Now in terms of the local differential operator $\partial_A = e_A{}^\mu \partial_\mu$, the Dirac equation (2) can be rewritten in the local Lorentz frame as

$$[\gamma^A (\partial_A + \Gamma_A) + \mu_e]\Psi = 0, \quad (7)$$

where $\Gamma_A = e_A{}^\mu \Gamma_\mu = (1/4)\gamma^B\gamma^C f_{BCA}$ is the component of the spinor connection in the local Lorentz frame. Note that the five-dimensional Clifford algebra has two different but reducible representations (they can differ by the multiplier of a γ^5 matrix). It is usually assumed that fermion fields are in a reducible representation of the Clifford algebra. In other words, one can work with the Dirac equation in a four-component spinor formalism like in the four-dimensional case, and just needs to take the γ^5 matrix as the fifth basis vector component.

III. DIRAC FIELD EQUATION IN 5-DIMENSIONAL MYERS-PERRY BLACK HOLE

In this section, I will present a new form for the five-dimensional Myers-Perry metric in the Boyer-Lindquist coordinates. One major advantage of these coordinates is that it allows us to construct a local orthonormal pentad with which the Dirac equation can be decoupled into purely radial and purely angular parts.

A. Metric of a 5-dimensional Myers-Perry black hole

The metric of a five-dimensional rotating black hole with two independent angular momenta was first obtained by Myers and Perry [17] in 1986. The solution with a negative cosmological constant was given by Hawking *et al.* [18] in 1999. The line element of the Myers-Perry metric can be recast into an elegant form in the Boyer-Lindquist coordinates as

$$\begin{aligned}
 ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = \eta_{AB} e^A \otimes e^B \\
 &= -\frac{\Delta_r}{\Sigma} (dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi)^2 + \frac{\Sigma}{\Delta_r} dr^2 \\
 &\quad + \Sigma d\theta^2 + \frac{\sin^2 \theta \cos^2 \theta}{p^2 \Sigma} [(b^2 - a^2) dt + (r^2 + a^2) a d\phi \\
 &\quad - (r^2 + b^2) b d\psi]^2 + \frac{1}{r^2 p^2} [-abdt + (r^2 + a^2) \\
 &\quad \times b \sin^2 \theta d\phi + (r^2 + b^2) a \cos^2 \theta d\psi]^2, \quad (8)
 \end{aligned}$$

where

$$\begin{aligned}
 \Delta_r &= (r^2 + a^2)(r^2 + b^2)/r^2 - 2M, \\
 \Sigma &= r^2 + p^2, \\
 p &= \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}.
 \end{aligned}$$

The metric determinant for this spacetime is $\sqrt{-g} = r\Sigma \sin\theta \cos\theta$, and the contra-invariant metric tensor can be read accordingly from

$$\begin{aligned}
 l^\mu \partial_\mu &= \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \Delta_r} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi + \frac{b}{r^2 + b^2} \partial_\psi \right) + \partial_r, \\
 n^\mu \partial_\mu &= \frac{(r^2 + a^2)(r^2 + b^2)}{2r^2 \Sigma} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi + \frac{b}{r^2 + b^2} \partial_\psi \right) - \frac{\Delta_r}{2\Sigma} \partial_r, \\
 m^\mu \partial_\mu &= \frac{1}{\sqrt{2}(r + ip)} \left\{ \partial_\theta + i \frac{\sin\theta \cos\theta}{p} \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2 \theta} \partial_\phi - \frac{b}{\cos^2 \theta} \partial_\psi \right] \right\}, \\
 \bar{m}^\mu \partial_\mu &= \frac{1}{\sqrt{2}(r - ip)} \left\{ \partial_\theta - i \frac{\sin\theta \cos\theta}{p} \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2 \theta} \partial_\phi - \frac{b}{\cos^2 \theta} \partial_\psi \right] \right\}, \\
 k^\mu \partial_\mu &= \frac{1}{rp} (ab \partial_t + b \partial_\phi + a \partial_\psi).
 \end{aligned} \quad (10)$$

These vectors are geodesic and satisfy the following orthogonal relations

$$l^\mu n_\mu = -1, \quad m^\mu \bar{m}_\mu = 1, \quad k^\mu k_\mu = 1, \quad (11)$$

and all others are zero.

Here I briefly sketch the construction of a seminull pentad formalism in five dimensions, analogous to the four-dimensional Newman-Penrose null-tetrad formalism. For the Myers-Perry black hole (8) which has a topology of S^3 sphere, a most convenient seminull pentad should endow it with a pair of real principal null vectors, a pair of

$$\begin{aligned}
 g^{\mu\nu} \partial_\mu \partial_\nu &= \eta^{AB} \partial_A \otimes \partial_B \\
 &= -\frac{(r^2 + a^2)^2 (r^2 + b^2)^2}{r^4 \Delta_r \Sigma} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi \right. \\
 &\quad \left. + \frac{b}{r^2 + b^2} \partial_\psi \right)^2 + \frac{\Delta_r}{\Sigma} \partial_r^2 + \frac{1}{\Sigma} \partial_\theta^2 + \frac{\sin^2 \theta \cos^2 \theta}{p^2 \Sigma} \\
 &\quad \times \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2 \theta} \partial_\phi - \frac{b}{\cos^2 \theta} \partial_\psi \right]^2 \\
 &\quad + \frac{1}{r^2 p^2} (ab \partial_t + b \partial_\phi + a \partial_\psi)^2. \quad (9)
 \end{aligned}$$

The Myers-Perry metric (8) possesses three Killing vectors (∂_t , ∂_ϕ , and ∂_ψ). In addition, it also admits a rank-two symmetric Stäckel-Killing tensor [26], which can be written as the square of a rank-three Killing-Yano tensor [27]. The existence of such tensors ensures the separation of variables in the geodesic Hamilton-Jacobi equation and the separability of the massless Klein-Gordon scalar field equation [26]. In this paper, it will be shown that the separability of Dirac's equation in this spacetime background is also closely associated with the existence of the rank-three Killing-Yano tensor.

The spacetime metric (8) is of Petrov type-D [47,50]. It possesses a pair of real principal null vectors $\{l, n\}$, a pair of complex principal null vectors $\{m, \bar{m}\}$, and one real, spatial-like unit vector k . Similar to the four-dimensional Kerr black hole case, they can be constructed to be of Kinnersley-type as follows:

complex principal null vectors, and a real unit vector, which obey the above orthogonal relations (11). In terms of these vectors, the metric can be written as

$$ds^2 = -l \otimes n - n \otimes l + m \otimes \bar{m} + \bar{m} \otimes m + k \otimes k. \quad (12)$$

I shall refer to this seminull pentad formalism as the $2\bar{2}1$ formalism. In a forthcoming paper [49], the Dirac equation has been reformulated within this seminull pentad formalism and can be decoupled into purely radial and purely

angular parts in the five-dimensional Myers-Perry black hole geometry.

On the other hand, for black ring solutions [51] whose horizon topology is $S^2 \times S^1$, the most suitable seminull pentad formalism should possess a real, timelike unit vector \mathbf{k} and two pairs of complex principal null vectors $\{\mathbf{m}_1, \bar{\mathbf{m}}_1\}$ and $\{\mathbf{m}_2, \bar{\mathbf{m}}_2\}$, satisfying the orthonormal relations: $\mathbf{k}^\mu \mathbf{k}_\mu = \mathbf{m}_1^\mu \bar{\mathbf{m}}_{1\mu} = \mathbf{m}_2^\mu \bar{\mathbf{m}}_{2\mu} = 1$. Working within such a $1\bar{2}\bar{2}$ formalism, the metric tensor can be written as $g_{\mu\nu} = -\mathbf{k}_\mu \mathbf{k}_\nu + \mathbf{m}_{1\mu} \bar{\mathbf{m}}_{1\nu} + \bar{\mathbf{m}}_{1\mu} \mathbf{m}_{1\nu} + \mathbf{m}_{2\mu} \bar{\mathbf{m}}_{2\nu} + \bar{\mathbf{m}}_{2\mu} \mathbf{m}_{2\nu}$.

B. Construction of covariant spinor differential operator

In the local Lorentz form of Dirac's equation, we need to find the local differential operator $\partial_A = e_A^\mu \partial_\mu$ and the spinor connection $\Gamma_A = e_A^\mu \Gamma_\mu$ subject to the Myers-Perry

metric (8). The orthonormal basis 1-vectors ∂_A dual to the pentad e^A constructed in the Appendix Eq. (A1) are

$$\begin{aligned}\partial_0 &= \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \sqrt{\Delta_r \Sigma}} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi + \frac{b}{r^2 + b^2} \partial_\psi \right), \\ \partial_1 &= \sqrt{\frac{\Delta_r}{\Sigma}} \partial_r, \quad \partial_2 = \frac{1}{\sqrt{\Sigma}} \partial_\theta, \\ \partial_3 &= \frac{\sin\theta \cos\theta}{p \sqrt{\Sigma}} \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2\theta} \partial_\phi - \frac{b}{\cos^2\theta} \partial_\psi \right], \\ \partial_5 &= \frac{1}{rp} (ab \partial_t + b \partial_\phi + a \partial_\psi).\end{aligned}\quad (13)$$

Taking use of the local Lorentz frame component Γ_A and the gamma matrices with relation $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, I get the composite expression

$$\begin{aligned}\gamma^A \Gamma_A &= \frac{1}{4} \gamma^A \gamma^B \gamma^C f_{BCA} \\ &= \gamma^1 \sqrt{\frac{\Delta_r}{\Sigma}} \left(\frac{\Delta'_r}{4\Delta_r} + \frac{1}{2r} + \frac{r}{2\Sigma} \right) + \gamma^2 \frac{1}{\sqrt{\Sigma}} \left[\frac{1}{2} \cot\theta - \frac{1}{2} \tan\theta - \frac{(a^2 - b^2) \sin\theta \cos\theta}{2\Sigma} \right] - \frac{(a^2 - b^2)r \sin\theta \cos\theta}{2\Sigma^{3/2} p} \gamma^0 \gamma^1 \gamma^3 \\ &\quad - \frac{ab}{2r^2 p} \gamma^0 \gamma^1 \gamma^5 + \frac{p\sqrt{\Delta_r}}{2\Sigma^{3/2}} \gamma^0 \gamma^2 \gamma^3 + \frac{ab}{2rp^2} \gamma^2 \gamma^3 \gamma^5 \\ &= \gamma^1 \sqrt{\frac{\Delta_r}{\Sigma}} \left(\frac{\Delta'_r}{4\Delta_r} + \frac{1}{2r} + \frac{r - ip\gamma^5}{2\Sigma} \right) + \gamma^2 \frac{1}{\sqrt{\Sigma}} \left[\frac{1}{2} \cot\theta - \frac{1}{2} \tan\theta - \frac{(a^2 - b^2) \sin\theta \cos\theta}{2\Sigma p} (p + ir\gamma^5) \right] \\ &\quad + \frac{iab}{2r^2 p^2} \gamma^0 \gamma^1 (r + ip\gamma^5),\end{aligned}\quad (14)$$

where a prime denotes the partial differential with respect to the coordinates r and θ .

Combining this formula with the spinor differential operator

$$\begin{aligned}\gamma^A \partial_A &= \gamma^0 \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \sqrt{\Delta_r \Sigma}} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi + \frac{b}{r^2 + b^2} \partial_\psi \right) + \gamma^1 \sqrt{\frac{\Delta_r}{\Sigma}} \partial_r + \gamma^2 \frac{1}{\sqrt{\Sigma}} \partial_\theta \\ &\quad + \gamma^3 \frac{\sin\theta \cos\theta}{p \sqrt{\Sigma}} \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2\theta} \partial_\phi - \frac{b}{\cos^2\theta} \partial_\psi \right] + \gamma^5 \frac{1}{rp} (ab \partial_t + b \partial_\phi + a \partial_\psi),\end{aligned}\quad (15)$$

I find that the covariant Dirac differential operator in the local Lorentz frame is

$$\begin{aligned}\gamma^A (\partial_A + \Gamma_A) &= \gamma^0 \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \sqrt{\Delta_r \Sigma}} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi + \frac{b}{r^2 + b^2} \partial_\psi \right) + \gamma^1 \sqrt{\frac{\Delta_r}{\Sigma}} \left(\partial_r + \frac{\Delta'_r}{4\Delta_r} + \frac{1}{2r} + \frac{r - ip\gamma^5}{2\Sigma} \right) \\ &\quad + \gamma^2 \frac{1}{\sqrt{\Sigma}} \left[\partial_\theta + \frac{1}{2} \cot\theta - \frac{1}{2} \tan\theta - \frac{(a^2 - b^2) \sin\theta \cos\theta}{2\Sigma p} (p + ir\gamma^5) \right] \\ &\quad + \gamma^3 \frac{\sin\theta \cos\theta}{p \sqrt{\Sigma}} \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2\theta} \partial_\phi - \frac{b}{\cos^2\theta} \partial_\psi \right] + \gamma^5 \frac{1}{rp} (ab \partial_t + b \partial_\phi + a \partial_\psi) \\ &\quad + \frac{iab}{2r^2 p^2} \gamma^0 \gamma^1 (r + ip\gamma^5).\end{aligned}\quad (16)$$

C. Separation of variables in Dirac equation

With the above preparation in hand, I am now ready to decouple the Dirac equation. Substituting the above spinor differential operator into Eq. (7), the Dirac equation in the five-dimensional Myers-Perry metric reads

$$\begin{aligned} & \left\{ \gamma^0 \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \sqrt{\Delta_r \Sigma}} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi + \frac{b}{r^2 + b^2} \partial_\psi \right) + \gamma^1 \sqrt{\frac{\Delta_r}{\Sigma}} \left(\partial_r + \frac{\Delta_r'}{4\Delta_r} + \frac{1}{2r} + \frac{r - ip\gamma^5}{2\Sigma} \right) \right. \\ & + \gamma^2 \frac{1}{\sqrt{\Sigma}} \left[\partial_\theta + \frac{1}{2} \cot\theta - \frac{1}{2} \tan\theta - \frac{(a^2 - b^2) \sin\theta \cos\theta}{2\Sigma p} i\gamma^5 (r - ip\gamma^5) \right] + \gamma^3 \frac{\sin\theta \cos\theta}{p\sqrt{\Sigma}} \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2\theta} \partial_\phi - \frac{b}{\cos^2\theta} \partial_\psi \right] \\ & \left. + \gamma^5 \frac{1}{rp} (ab\partial_t + b\partial_\phi + a\partial_\psi) + \frac{iab}{2r^2 p^2} \gamma^0 \gamma^1 (r + ip\gamma^5) + \mu_e \right\} \Psi = 0. \quad (17) \end{aligned}$$

Multiplying $(r - ip\gamma^5)\sqrt{r + ip\gamma^5} = \sqrt{\Sigma}(r - ip\gamma^5)$ by the left to the above equation, and after some lengthy algebra manipulations I finally obtain

$$\begin{aligned} & \left\{ \gamma^0 \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \sqrt{\Delta_r}} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi + \frac{b}{r^2 + b^2} \partial_\psi \right) + \gamma^1 \sqrt{\Delta_r} \left(\partial_r + \frac{\Delta_r'}{4\Delta_r} + \frac{1}{2r} \right) + \gamma^2 \left(\partial_\theta + \frac{1}{2} \cot\theta - \frac{1}{2} \tan\theta \right) \right. \\ & + \gamma^3 \frac{\sin\theta \cos\theta}{p} \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2\theta} \partial_\phi - \frac{b}{\cos^2\theta} \partial_\psi \right] + \left(\frac{\gamma^5}{p} - \frac{i}{r} \right) (ab\partial_t + b\partial_\phi + a\partial_\psi) \\ & \left. + \frac{iab}{2} \left(\frac{1}{p^2} + \frac{1}{r^2} \right) \gamma^0 \gamma^1 + \mu_e (r - ip\gamma^5) \right\} (\sqrt{r + ip\gamma^5} \Psi) = 0. \quad (18) \end{aligned}$$

At this stage, I assume that the spin-1/2 fermion fields are in a reducible representation of the Clifford algebra, which can be taken as a four-component Dirac spinor. Applying the explicit representation (4) for the gamma matrices and adopting the following ansatz for the separation of variables

$$\sqrt{r + ip\gamma^5} \Psi = e^{i(m\phi + k\psi - \omega t)} \begin{pmatrix} R_2(r) S_1(\theta) \\ R_1(r) S_2(\theta) \\ R_1(r) S_1(\theta) \\ R_2(r) S_2(\theta) \end{pmatrix}, \quad (19)$$

I find that the Dirac equation in the five-dimensional Myers-Perry metric can be decoupled into the purely radial parts

$$\begin{aligned} & \left[\sqrt{\Delta_r} D_r - i \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \sqrt{\Delta_r}} \left(\omega - \frac{ma}{r^2 + a^2} - \frac{kb}{r^2 + b^2} \right) \right] R_1 \\ & = \left[\lambda + i\mu_e r - \frac{ab}{2r^2} - \frac{i}{r} (ab\omega - mb - ka) \right] R_2, \quad (20) \end{aligned}$$

$$\begin{aligned} & \left[\sqrt{\Delta_r} D_r + i \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \sqrt{\Delta_r}} \left(\omega - \frac{ma}{r^2 + a^2} - \frac{kb}{r^2 + b^2} \right) \right] R_2 \\ & = \left[\lambda - i\mu_e r - \frac{ab}{2r^2} + \frac{i}{r} (ab\omega - mb - ka) \right] R_1, \quad (21) \end{aligned}$$

and the purely angular parts

$$\begin{aligned} & \left\{ L_\theta + \frac{\sin\theta \cos\theta}{p} \left[(a^2 - b^2)\omega - \frac{ma}{\sin^2\theta} + \frac{kb}{\cos^2\theta} \right] \right\} S_1 \\ & = \left[\lambda + \mu_e p + \frac{ab}{2p^2} + \frac{1}{p} (ab\omega - mb - ka) \right] S_2, \quad (22) \end{aligned}$$

$$\begin{aligned} & \left\{ L_\theta - \frac{\sin\theta \cos\theta}{p} \left[(a^2 - b^2)\omega - \frac{ma}{\sin^2\theta} + \frac{kb}{\cos^2\theta} \right] \right\} S_2 \\ & = \left[-\lambda + \mu_e p - \frac{ab}{2p^2} + \frac{1}{p} (ab\omega - mb - ka) \right] S_1, \quad (23) \end{aligned}$$

in which I have introduced two operators

$$D_r = \partial_r + \frac{\Delta_r'}{4\Delta_r} + \frac{1}{2r}, \quad L_\theta = \partial_\theta + \frac{1}{2} \cot\theta - \frac{1}{2} \tan\theta.$$

Now the separated radial equation and the angular equation can be reduced into a master equation containing only one component. For the radial part, I write them explicitly as

$$\begin{aligned}
& \frac{1}{r} \sqrt{\Delta_r} D_r (r \sqrt{\Delta_r} D_r R_1) + \left\{ \frac{(r^2 + a^2)^2 (r^2 + b^2)^2}{r^4 \Delta_r} \left(\omega - \frac{ma}{r^2 + a^2} - \frac{kb}{r^2 + b^2} \right)^2 - \frac{(ab\omega - mb - ka)^2}{r^2} \right. \\
& + 2\mu_e (ab\omega - mb - ka) - \mu_e^2 r^2 - \lambda^2 + \lambda \frac{ab}{r^2} - \frac{a^2 b^2}{4r^4} - \frac{\lambda + 2i\mu_e r + ab/(2r^2)}{\lambda r + i\mu_e r^2 - ab/(2r) - i(ab\omega - mb - ka)} \Delta_r D_r \\
& + \left[\frac{2i}{r} + i \frac{\Delta'_r}{2\Delta_r} - \frac{\mu_e r - iab/r^2 + (ab\omega - mb - ka)/r}{\lambda r + i\mu_e r^2 - ab/(2r) - i(ab\omega - mb - ka)} \right] \frac{(r^2 + a^2)(r^2 + b^2)}{r^2} \left(\omega - \frac{ma}{r^2 + a^2} - \frac{kb}{r^2 + b^2} \right) \\
& \left. - \frac{2i}{r} [(2r^2 + a^2 + b^2)\omega - ma - kb] \right\} R_1 = 0, \tag{24}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{r} \sqrt{\Delta_r} D_r (r \sqrt{\Delta_r} D_r R_2) + \left\{ \frac{(r^2 + a^2)^2 (r^2 + b^2)^2}{r^4 \Delta_r} \left(\omega - \frac{ma}{r^2 + a^2} - \frac{kb}{r^2 + b^2} \right)^2 - \frac{(ab\omega - mb - ka)^2}{r^2} \right. \\
& + 2\mu_e (ab\omega - mb - ka) - \mu_e^2 r^2 - \lambda^2 + \lambda \frac{ab}{r^2} - \frac{a^2 b^2}{4r^4} - \frac{\lambda - 2i\mu_e r + ab/(2r^2)}{\lambda r - i\mu_e r^2 - ab/(2r) + i(ab\omega - mb - ka)} \Delta_r D_r \\
& - \left[\frac{2i}{r} + i \frac{\Delta'_r}{2\Delta_r} + \frac{\mu_e r + iab/r^2 + (ab\omega - mb - ka)/r}{\lambda r - i\mu_e r^2 - ab/(2r) + i(ab\omega - mb - ka)} \right] \frac{(r^2 + a^2)(r^2 + b^2)}{r^2} \left(\omega - \frac{ma}{r^2 + a^2} - \frac{kb}{r^2 + b^2} \right) \\
& \left. + \frac{2i}{r} [(2r^2 + a^2 + b^2)\omega - ma - kb] \right\} R_2 = 0. \tag{25}
\end{aligned}$$

From the above decoupled master equations, it is easy to see that they are more complicated than the four-dimensional case derived by Chandrasekhar [2]. As for the exact solution to these equations, I expect they can be recast into the confluent form of Heun equation [52].

The case occurs similarly for the angular parts if I adopt $p = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}$ rather than θ itself as a variable. Moreover, the angular part can be transformed into the radial part if I make the replacement $p = ir$ in the case $M = 0$.

IV. CONSTRUCTION OF SYMMETRY OPERATORS IN TERMS OF STÄCKEL-KILLING AND KILLING-YANO TENSORS

In the last section, I have explicitly shown that Dirac's equation is separable in the five-dimensional Myers-Perry black hole spacetime. In this section, I will demonstrate that this separability is intimately related to the very existence of a rank-three Killing-Yano tensor admitted by the Myers-Perry metric. Specifically speaking, I will construct a symmetry operator that commutes with the scalar Laplacian by using the Stäckel-Killing tensor, and another one that commutes with the Dirac operator by the Killing-Yano tensor. These symmetry operators are directly constructed from the separated solutions of the Klein-Gordon equation and Dirac's equation.

A. Stäckel-Killing tensor from the separated solution of the Klein-Gordon equation

In this subsection, I will present a simple and elegant form for the Stäckel-Killing tensor, which can be easily written as the square of a rank-three Killing-Yano tensor.

This symmetric tensor is constructed from the separated solution of the Klein-Gordon scalar field equation in the five-dimensional Myers-Perry metric.

To begin with, let us consider a massive Klein-Gordon scalar field equation

$$(\square - \mu_0^2)\Phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) - \mu_0^2 \Phi = 0, \tag{26}$$

together with the ansatz $\Phi = R(r)S(\theta)e^{i(m\phi + k\psi - \omega t)}$. In the background spacetime metric (8), the massive scalar field equation reads

$$\begin{aligned}
& \left\{ - \frac{(r^2 + a^2)^2 (r^2 + b^2)^2}{r^4 \Delta_r \Sigma} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi + \frac{b}{r^2 + b^2} \partial_\psi \right)^2 \right. \\
& + \frac{1}{r \Sigma} \partial_r (r \Delta_r \partial_r) + \frac{1}{\Sigma \sin \theta \cos \theta} \partial_\theta (\sin \theta \cos \theta \partial_\theta) \\
& + \frac{\sin^2 \theta \cos^2 \theta}{p^2 \Sigma} \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2 \theta} \partial_\phi - \frac{b}{\cos^2 \theta} \partial_\psi \right]^2 \\
& \left. + \frac{1}{r^2 p^2} (ab \partial_t + b \partial_\phi + a \partial_\psi)^2 - \mu_0^2 \right\} \Phi = 0. \tag{27}
\end{aligned}$$

Apparently, it can be separated into a radial part and an angular part,

$$\begin{aligned}
& \frac{1}{r} \partial_r (r \Delta_r \partial_r R) + \left\{ \frac{(r^2 + a^2)^2 (r^2 + b^2)^2}{r^4 \Delta_r} \right. \\
& \times \left(\omega - \frac{ma}{r^2 + a^2} - \frac{kb}{r^2 + b^2} \right)^2 - \frac{1}{r^2} (ab\omega - mb - ka)^2 \\
& \left. - \mu_0^2 r^2 - \lambda^2 \right\} R(r) = 0, \tag{28}
\end{aligned}$$

$$\begin{aligned} & \frac{1}{\sin\theta \cos\theta} \partial_\theta (\sin\theta \cos\theta \partial_\theta S) - \left\{ \frac{\sin^2\theta \cos^2\theta}{p^2} \left[(a^2 - b^2)\omega \right. \right. \\ & \quad \left. \left. - \frac{ma}{\sin^2\theta} + \frac{kb}{\cos^2\theta} \right]^2 + \frac{1}{p^2} (ab\omega - mb - ka)^2 \right. \\ & \quad \left. + \mu_0^2 p^2 - \lambda^2 \right\} S(\theta) = 0, \end{aligned} \quad (29)$$

which can be transformed into the confluent form of the Heun equation [28,52].

Now from the separated Eqs. (28) and (29), I can construct a new dual field equation as follows:

$$\begin{aligned} & \left\{ -p^2 \frac{(r^2 + a^2)^2 (r^2 + b^2)^2}{r^4 \Delta_r \Sigma} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi \right. \right. \\ & \quad \left. \left. + \frac{b}{r^2 + b^2} \partial_\psi \right)^2 + \frac{p^2}{r \Sigma} \partial_r (r \Delta_r \partial_r) \right. \\ & \quad \left. - \frac{r^2}{\Sigma \sin\theta \cos\theta} \partial_\theta (\sin\theta \cos\theta \partial_\theta) - \frac{r^2 \sin^2\theta \cos^2\theta}{p^2 \Sigma} \right. \\ & \quad \times \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2\theta} \partial_\phi - \frac{b}{\cos^2\theta} \partial_\psi \right]^2 \\ & \quad \left. + \frac{p^2 - r^2}{r^2 p^2} (ab \partial_t + b \partial_\phi + a \partial_\psi)^2 - \lambda^2 \right\} \Phi = 0, \end{aligned} \quad (30)$$

from which I can extract a second order symmetric tensor—the so-called Stäckel-Killing tensor

$$\begin{aligned} K^{\mu\nu} \partial_\mu \partial_\nu &= -p^2 \frac{(r^2 + a^2)^2 (r^2 + b^2)^2}{r^4 \Delta_r \Sigma} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi \right. \\ & \quad \left. + \frac{b}{r^2 + b^2} \partial_\psi \right)^2 + p^2 \frac{\Delta_r}{\Sigma} \partial_r^2 - r^2 \frac{1}{\Sigma} \partial_\theta^2 \\ & \quad - r^2 \frac{\sin^2\theta \cos^2\theta}{p^2 \Sigma} \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2\theta} \partial_\phi \right. \\ & \quad \left. - \frac{b}{\cos^2\theta} \partial_\psi \right]^2 + \frac{p^2 - r^2}{r^2 p^2} (ab \partial_t + b \partial_\phi + a \partial_\psi)^2. \end{aligned} \quad (31)$$

This symmetric tensor $K_{\mu\nu} = K_{\nu\mu}$ obeys the equation [9]

$$K_{\mu\nu;\rho} + K_{\nu\rho;\mu} + K_{\rho\mu;\nu} = 0, \quad (32)$$

and is equivalent to the one found in [26,27], up to an ignorable constant.

In the local Lorentz coframe (A1), it has a simple, diagonal form $K_{AB} = \text{diag}(-p^2, p^2, -r^2, -r^2, p^2 - r^2)$. Using the Stäckel-Killing tensor, the above dual equation can be written in a coordinate-independent form

$$(\mathbb{K} - \lambda^2)\Phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} K^{\mu\nu} \partial_\nu \Phi) - \lambda^2 \Phi = 0. \quad (33)$$

It is obvious that the operator \mathbb{K} commutes with the scalar Laplacian \square . Working out the commutator $[\mathbb{K}, \square] = 0$ yields the Killing equation (32) and the integrability con-

dition for the Stäckel-Killing tensor. These two operators have a classical analogue. In classical mechanics, the scalar Laplacian \square corresponds to the Hamiltonian $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$, while the operator \mathbb{K} to the Carter's constant $K_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$. They are two integrals of motion in addition to three constants from the Killing vector fields ∂_t , ∂_ϕ , and ∂_ψ .

B. Killing-Yano potential, (conformal) Killing-Yano tensor, and Stäckel-Killing tensor

Before constructing a first-order symmetry operator that commutes with the Dirac operator, I first give a brief review on the recent work [27,35,36] about the construction of the Stäckel-Killing tensor from the (conformal) Killing-Yano tensor.

Penrose and Floyd [53] discovered that the Stäckel-Killing tensor for the four-dimensional Kerr metric can be written in the form $K_{\mu\nu} = f_{\mu\rho} f_{\nu}{}^\rho$, where the skew-symmetric tensor $f_{\mu\nu} = -f_{\nu\mu}$ is the Killing-Yano tensor [54–56] obeying the equation $f_{\mu\nu;\rho} + f_{\mu\rho;\nu} = 0$. Using this object, Carter and McLenaghan [10] constructed a first-order symmetry operator that commutes with the massive Dirac operator. In the case of a four-dimensional Kerr black hole ($D = 4$), the Killing-Yano tensor f is of the rank two, its Hodge dual $k = -*f$ is a rank-two, antisymmetric, conformal Killing-Yano tensor [41] obeying the equation

$$\begin{aligned} k_{\alpha\beta;\gamma} + k_{\alpha\gamma;\beta} &= \frac{1}{D-1} (g_{\alpha\beta} k^\mu{}_{\gamma;\mu} + g_{\gamma\alpha} k^\mu{}_{\beta;\mu} \\ & \quad - 2g_{\beta\gamma} k^\mu{}_{\alpha;\mu}). \end{aligned} \quad (34)$$

This equation can be equivalently rewritten in the form proposed by Penrose [57]

$$\mathcal{P}_{\alpha\beta\gamma} = k_{\alpha\beta;\gamma} + \frac{1}{D-1} (g_{\beta\gamma} k^\mu{}_{\alpha;\mu} - g_{\gamma\alpha} k^\mu{}_{\beta;\mu}) = 0. \quad (35)$$

A conformal Killing-Yano tensor k is dual to the Killing-Yano tensor if and only if it is closed $dk = 0$. This fact implies that there exists a potential 1-form \hat{b} so that $k = d\hat{b}$. Carter [40] is the first one who found this potential to generate the Killing-Yano tensor for the Kerr-Newman black hole.

Recently, these results have further been extended [27,35–37] to general higher-dimensional rotating black hole solutions. In the case of $D = 5$ dimensions, it was demonstrated [27] that the rank-two Stäckel-Killing tensor can be constructed from its “square root”, a rank-three, totally antisymmetric Killing-Yano tensor. Following Carter's procedure [40], Frolov *et al.* [27] found a potential 1-form to generate a rank-two conformal Killing-Yano tensor [41], whose Hodge dual $f = *k$ is a rank-three Killing-Yano tensor.

Now restricting ourselves to the five-dimensional Myers-Perry metric, it is easy to check that the following object constructed from the rank-three Killing-Yano tensor

$$K_{\mu\nu} = -\frac{1}{2}f_{\mu\alpha\beta}f_{\nu}{}^{\alpha\beta}, \quad (36)$$

is just the rank-two, Stäckel-Killing tensor given in Eq. (31). The rank-three Killing-Yano tensor f obeying the equation

$$f_{\alpha\beta\mu;\nu} + f_{\alpha\beta\nu;\mu} = 0, \quad (37)$$

can be taken as the Hodge dual $f = *k$ of the 2-form $k = d\hat{b}$ via the following definition:

$$f_{\alpha\beta\gamma} = (*k)_{\alpha\beta\gamma} = \frac{1}{2}\sqrt{-g}\varepsilon_{\alpha\beta\gamma\mu\nu}k^{\mu\nu}. \quad (38)$$

The Killing-Yano potential found for the five-dimensional Myers-Perry metric is [27]

$$2\hat{b} = (-r^2 + a^2\cos^2\theta + b^2\sin^2\theta)dt + (r^2 + a^2)\sin^2\theta d\phi + (r^2 + b^2)b\cos^2\theta d\psi, \quad (39)$$

from which a conformal Killing-Yano tensor can be constructed

$$k = d\hat{b} = re^0 \wedge e^1 + pe^2 \wedge e^3. \quad (40)$$

Adopting the convention $\varepsilon^{01235} = 1 = -\varepsilon_{01235}$ for the totally antisymmetric tensor density ε_{ABCDE} , I find that

the Killing-Yano tensor is given by

$$f = *k = (-pe^0 \wedge e^1 + re^2 \wedge e^3) \wedge e^5. \quad (41)$$

In what follows, I shall show that this rank-three Killing-Yano tensor and its exterior differential

$$\begin{aligned} W = df &= -4\frac{ab}{rp}e^0 \wedge e^1 \wedge e^2 \wedge e^3 + 4\frac{(a^2 - b^2)\sin\theta\cos\theta}{p\sqrt{\Sigma}} \\ &\times e^0 \wedge e^1 \wedge e^2 \wedge e^5 + 4\sqrt{\frac{\Delta_r}{\Sigma}}e^1 \wedge e^2 \wedge e^3 \wedge e^5 \end{aligned} \quad (42)$$

play a central role in constructing a first order symmetry operator that commutes with the Dirac operator.

C. Killing-Yano tensor from the separated solution of the Dirac equation

The last task is to construct a first order symmetry operator that commutes with the Dirac operator, parallel to the work done by Carter and McLenaghan [10] in the case of a four-dimensional Kerr black hole. I proceed to construct such an operator from the separated solutions (20)–(23) of the Dirac equation. After some tedious algebra manipulations, I find that the following equation:

$$\begin{aligned} &\left\{ \gamma^0 p \sqrt{\Delta_r} \left(\partial_r + \frac{\Delta'_r}{4\Delta_r} + \frac{1}{2r} \right) + \gamma^1 p \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \sqrt{\Delta_r}} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi + \frac{b}{r^2 + b^2} \partial_\psi \right) \right. \\ &\quad + \gamma^2 (-r) \frac{\sin\theta \cos\theta}{p} \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2\theta} \partial_\phi - \frac{b}{\cos^2\theta} \partial_\psi \right] + \gamma^3 r \left(\partial_\theta + \frac{1}{2} \cot\theta - \frac{1}{2} \tan\theta \right) \\ &\quad \left. - i\gamma^0 \gamma^1 \frac{\Sigma}{rp} (ab\partial_t + b\partial_\phi + a\partial_\psi) + \frac{ab}{2} \left(\frac{ip}{r^2} + \frac{\gamma^5 r}{p^2} \right) + \lambda(\gamma^5 r - ip) \right\} (\sqrt{r + ip\gamma^5} \Psi) = 0, \end{aligned} \quad (43)$$

is a dual one to the Dirac equation (18). Expanding it, I get

$$\begin{aligned} &\left\{ \gamma^0 p \sqrt{\frac{\Delta_r}{\Sigma}} \left(\partial_r + \frac{\Delta'_r}{4\Delta_r} + \frac{1}{2r} + \frac{r - ip\gamma^5}{2\Sigma} \right) + \gamma^1 p \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \sqrt{\Delta_r \Sigma}} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi + \frac{b}{r^2 + b^2} \partial_\psi \right) \right. \\ &\quad + \gamma^2 (-r) \frac{\sin\theta \cos\theta}{p\sqrt{\Sigma}} \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2\theta} \partial_\phi - \frac{b}{\cos^2\theta} \partial_\psi \right] + \gamma^3 r \frac{1}{\sqrt{\Sigma}} \left[\partial_\theta + \frac{1}{2} \cot\theta - \frac{1}{2} \tan\theta \right. \\ &\quad \left. - \frac{(a^2 - b^2)\sin\theta \cos\theta}{2\Sigma p} i\gamma^5 (r - ip\gamma^5) \right] - i\gamma^0 \gamma^1 (r + i\gamma^5 p) \frac{1}{rp} (ab\partial_t + b\partial_\phi + a\partial_\psi) + \frac{iab}{2rp} + \gamma^5 \left(\lambda - \frac{ab}{2r^2} + \frac{ab}{2p^2} \right) \right\} \Psi = 0. \end{aligned} \quad (44)$$

I do not hope $\gamma^5 \lambda$ appears in the above equation, therefore I can multiply it the γ^5 matrix by the left so as to rewrite it as

$$\begin{aligned}
& \left\{ \gamma^5 \gamma^0 p \sqrt{\frac{\Delta_r}{\Sigma}} \left(\partial_r + \frac{\Delta_r'}{4\Delta_r} + \frac{1}{2r} + \frac{r - ip\gamma^5}{2\Sigma} \right) + \gamma^5 \gamma^1 p \frac{(r^2 + a^2)(r^2 + b^2)}{r^2 \sqrt{\Delta_r \Sigma}} \left(\partial_t + \frac{a}{r^2 + a^2} \partial_\phi + \frac{b}{r^2 + b^2} \partial_\psi \right) + \gamma^5 \gamma^2 (-r) \right. \\
& \quad \times \frac{\sin\theta \cos\theta}{p\sqrt{\Sigma}} \left[(a^2 - b^2) \partial_t + \frac{a}{\sin^2\theta} \partial_\phi - \frac{b}{\cos^2\theta} \partial_\psi \right] + \gamma^5 \gamma^3 r \frac{1}{\sqrt{\Sigma}} \left[\partial_\theta + \frac{1}{2} \cot\theta - \frac{1}{2} \tan\theta - \frac{(a^2 - b^2) \sin\theta \cos\theta}{2\Sigma p} \right. \\
& \quad \left. \left. \times i\gamma^5 (r - ip\gamma^5) \right] + (p\gamma^0 \gamma^1 - r\gamma^2 \gamma^3) \frac{1}{rp} (ab\partial_t + b\partial_\phi + a\partial_\psi) + \frac{iab}{2rp} \gamma^5 + \lambda - \frac{ab}{2r^2} + \frac{ab}{2p^2} \right\} \Psi = 0, \quad (45)
\end{aligned}$$

which can be put into an operator form

$$(\mathbb{H}_f + \lambda)\Psi = 0. \quad (46)$$

My final aim is to find the explicit expression for this symmetry operator \mathbb{H}_f . The process to construct such an operator is more involved than the one to treat with the Dirac operator $\mathbb{H}_D = \gamma^\mu \nabla_\mu = \gamma^\mu (\partial_\mu + \Gamma_\mu)$. However, the final result is extremely simple,

$$\mathbb{H}_f = -\frac{1}{2} \gamma^\mu \gamma^\nu f_{\mu\nu}{}^\rho (\partial_\rho + \Gamma_\rho) - \frac{1}{64} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma W_{\mu\nu\rho\sigma}. \quad (47)$$

Using the definition $W_{\mu\nu\rho\sigma} = -f_{\mu\nu\rho;\sigma} + f_{\nu\rho\sigma;\mu} - f_{\rho\sigma\mu;\nu} + f_{\sigma\mu\nu;\rho}$ and the property of gamma matrices as well as $f^\rho{}_{\mu\nu;\rho} = 0$, I can also write the above operator in another form:

$$\mathbb{H}_f = -\frac{1}{2} \gamma^\mu \gamma^\nu f_{\mu\nu}{}^\rho \nabla_\rho + \frac{1}{16} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma f_{\mu\nu\rho;\sigma}. \quad (48)$$

The symmetry operator \mathbb{H}_f constructed here has a lot of correspondences in different contexts. It is the five-dimensional analogue to the nonstandard Dirac operator discovered by Carter and McLenaghan [10] for the four-dimensional Kerr metric, which generates generalized angular momentum quantum number. This operator corresponds to the nongeneric supersymmetric generator in pseudoclassical mechanics [58]. Moreover, the 2-form field $L_{\mu\nu} = f_{\mu\nu\rho} \dot{x}^\rho$ is parallel-propagated along the geodesic with a cotangent vector \dot{x}^μ , whose square is just the Carter's constant $-(1/2)L_{\mu\nu}L^{\mu\nu} = K_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$.

The existence of a rank-three Killing-Yano tensor is enough to explain the separability of the Dirac equation in the five-dimensional Myers-Perry vacuum background. The operator \mathbb{H}_f commutes with the standard Dirac operator \mathbb{H}_D . Expanding the commutation relation $[\mathbb{H}_D, \mathbb{H}_f] = 0$ yields the Killing-Yano equation (37) and the integrability condition for the rank-three Killing-Yano tensor f .

V. CONCLUDING REMARKS

In this paper, I have investigated the separability of the Dirac equation in the five-dimensional Myers-Perry metric and its relation to a rank-three Killing-Yano tensor. First, the field equation for Dirac fermions in five-dimensional relativity is formulated in a fünfbein formalism. The

spinor-connection is constructed by the method of the Clifford algebra and its derived Lie algebra $SO(4,1)$. Second, an orthonormal pentad has been established for the Myers-Perry metric so that one can easily deal with the Dirac equation in this background geometry. It is obviously shown that Dirac's equation in the Myers-Perry metric can be separated into purely radial and purely angular parts. Finally, from the separated solutions of the massive Klein-Gordon equation and Dirac's equation, I have constructed two symmetry operators that commute with the scalar Laplacian and the Dirac operator, respectively. A simple form for the Stäckel-Killing tensor was given so that it can be easily understood as the square of a rank-three Killing-Yano tensor.

A comparison with the previously published work [42,43] is made here. First, my work covers the partial work done in [42] about the separation of variables of Dirac's equation in the Myers-Perry black hole with two equal-magnitude angular momenta. In [43], Dirac's equation was separated in general higher-dimensional rotating Kerr-AdS-NUT black hole background. However, because the role of angular momenta becomes less obvious, it seems difficult to directly apply that work to study various properties of the Dirac field. What is more, symmetry operators that (anti-)commute with the Dirac operator have not been found there. Although my work can serve as a special case of that work and is announced later than it, the results presented in this paper can be directly applied to study various aspects [59] of fermion fields, for example, Hawking radiation [60], emission rates [61], quasinormal modes, instability [62], etc. On the other hand, a non-standard Dirac operator has been explicitly constructed here. In addition, the representation of gamma matrices adopted in this paper is different from that used in [43].

In a subsequent work [49], I have constructed a seminull pentad formalism of the Dirac equation in the five-dimensional relativity similar to the widely used null-tetrad formalism [5]. The Dirac equation can be shown to be decoupled into purely radial and purely angular parts which agree with the equations obtained here. The agreement assures that the Clifford-algebra formalism is equivalent to my seminull pentad formalism. A paper based upon my previously unpublished notes is being written.

Finally, the present work can be directly extended to the case of five-dimensional rotating black holes with a non-zero cosmological constant [18]. In another forthcoming

paper, the present work has been generalized to the charged case of five-dimensional rotating black holes in minimal gauged and ungauged supergravity [20] with the inclusion of a Chern-Simons term. It is found there that the usual Dirac equation cannot be separated by variables. To ensure the separability of fermion fields in this Einstein-Maxwell-Chern-Simons background geometry, one must include an additional term in the action of spin-1/2 fields. A paper on this aspect is in preparation.

It is also an interesting question to investigate the separability of higher-spin field equations (for example, Maxwell's equation and Rarita-Schwinger's equation) in the five-dimensional Myers-Perry metric and its relation to a rank-three Killing-Yano tensor.

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APPENDIX A: PENTAD AND CONNECTION 1-FORMS

The new form of the five-dimensional Myers-Perry metric (8) admits the following local Lorentz basis of 1-forms (pentad) defined as $e^A = e^A{}_{\mu} dx^{\mu}$ orthonormal with respect to η_{AB} ,

$$\begin{aligned} e^0 &= \sqrt{\frac{\Delta_r}{\Sigma}}(dt - a\sin^2\theta d\phi - b\cos^2\theta d\psi), & e^1 &= \sqrt{\frac{\Sigma}{\Delta_r}}dr, & e^2 &= \sqrt{\Sigma}d\theta, \\ e^3 &= \frac{\sin\theta \cos\theta}{p\sqrt{\Sigma}}[(b^2 - a^2)dt + (r^2 + a^2)ad\phi - (r^2 + b^2)bd\psi], \\ e^5 &= \frac{1}{rp}[-abd\psi + (r^2 + a^2)b\sin^2\theta d\phi + (r^2 + b^2)a\cos^2\theta d\psi]. \end{aligned} \quad (A1)$$

These coframe 1-forms are different from those used in [63,64].

After some algebraic computations, I obtain the exterior differential of the coframe 1-forms as

$$\begin{aligned} de^0 &= -\left(\sqrt{\frac{\Delta_r}{\Sigma}}\right)_{,r} e^0 \wedge e^1 + \frac{(a^2 - b^2)\sin\theta \cos\theta}{\Sigma^{3/2}} e^0 \wedge e^2 - \frac{2p\sqrt{\Delta_r}}{\Sigma^{3/2}} e^2 \wedge e^3, & de^1 &= \frac{(a^2 - b^2)\sin\theta \cos\theta}{\Sigma^{3/2}} e^1 \wedge e^2, \\ de^2 &= \frac{r\sqrt{\Delta_r}}{\Sigma^{3/2}} e^1 \wedge e^2, & de^3 &= -\frac{2(a^2 - b^2)r\sin\theta \cos\theta}{\Sigma^{3/2}p} e^0 \wedge e^1 + \frac{r\sqrt{\Delta_r}}{\Sigma^{3/2}} e^1 \wedge e^3 + \frac{p}{\sin\theta \cos\theta} \left(\frac{\sin\theta \cos\theta}{p\sqrt{\Sigma}}\right)_{,\theta} e^2 \wedge e^3, \\ de^5 &= -\frac{2ab}{r^2p} e^0 \wedge e^1 + \frac{1}{r}\sqrt{\frac{\Delta_r}{\Sigma}} e^1 \wedge e^5 + \frac{2ab}{rp^2} e^2 \wedge e^3 - \frac{(a^2 - b^2)\sin\theta \cos\theta}{p^2\sqrt{\Sigma}} e^2 \wedge e^5. \end{aligned} \quad (A2)$$

The spin-connection 1-form $\omega^A{}_B = \omega^A{}_{B\mu} dx^{\mu} = f^A{}_{BC} e^C$ can be found from the Cartan's first structure equation (5) as follows:

$$\begin{aligned} \omega^0{}_1 &= \left(\sqrt{\frac{\Delta_r}{\Sigma}}\right)_{,r} e^0 - \frac{(a^2 - b^2)r\sin\theta \cos\theta}{\Sigma^{3/2}p} e^3 - \frac{ab}{r^2p} e^5, & \omega^0{}_2 &= -\frac{(a^2 - b^2)\sin\theta \cos\theta}{\Sigma^{3/2}} e^0 - \frac{p\sqrt{\Delta_r}}{\Sigma^{3/2}} e^3, \\ \omega^0{}_3 &= -\frac{(a^2 - b^2)r\sin\theta \cos\theta}{\Sigma^{3/2}p} e^1 + \frac{p\sqrt{\Delta_r}}{\Sigma^{3/2}} e^2, & \omega^0{}_5 &= -\frac{ab}{r^2p} e^1, & \omega^1{}_2 &= -\frac{(a^2 - b^2)\sin\theta \cos\theta}{\Sigma^{3/2}} e^1 - \frac{r\sqrt{\Delta_r}}{\Sigma^{3/2}} e^2, \\ \omega^1{}_3 &= -\frac{(a^2 - b^2)r\sin\theta \cos\theta}{\Sigma^{3/2}p} e^0 - \frac{r\sqrt{\Delta_r}}{\Sigma^{3/2}} e^3, & \omega^1{}_5 &= -\frac{ab}{r^2p} e^0 - \frac{1}{r}\sqrt{\frac{\Delta_r}{\Sigma}} e^5, \\ \omega^2{}_3 &= -\frac{p\sqrt{\Delta_r}}{\Sigma^{3/2}} e^0 - \frac{p}{\sin\theta \cos\theta} \left(\frac{\sin\theta \cos\theta}{p\sqrt{\Sigma}}\right)_{,\theta} e^3 - \frac{ab}{rp^2} e^5, & \omega^2{}_5 &= -\frac{ab}{rp^2} e^3 + \frac{(a^2 - b^2)\sin\theta \cos\theta}{p^2\sqrt{\Sigma}} e^5, & \omega^3{}_5 &= \frac{ab}{rp^2} e^2. \end{aligned} \quad (A3)$$

The local Lorentz frame component Γ_A can be easily read from the spinor-connection 1-form $\Gamma \equiv \Gamma_A e^A = (1/4)\gamma^A \gamma^B \omega_{AB}$ as

$$\begin{aligned}
\Gamma_0 &= \frac{1}{2} \left[- \left(\sqrt{\frac{\Delta_r}{\Sigma}} \right)_r \gamma^0 \gamma^1 + \frac{(a^2 - b^2) \sin\theta \cos\theta}{\Sigma^{3/2}} \gamma^0 \gamma^2 - \frac{(a^2 - b^2) r \sin\theta \cos\theta}{\Sigma^{3/2} p} \gamma^1 \gamma^3 - \frac{ab}{r^2 p} \gamma^1 \gamma^5 - \frac{p \sqrt{\Delta_r}}{\Sigma^{3/2}} \gamma^2 \gamma^3 \right], \\
\Gamma_1 &= \frac{1}{2} \left[\frac{(a^2 - b^2) r \sin\theta \cos\theta}{\Sigma^{3/2} p} \gamma^0 \gamma^3 + \frac{ab}{r^2 p} \gamma^0 \gamma^5 - \frac{(a^2 - b^2) \sin\theta \cos\theta}{\Sigma^{3/2}} \gamma^1 \gamma^2 \right], \\
\Gamma_2 &= \frac{1}{2} \left[- \frac{p \sqrt{\Delta_r}}{\Sigma^{3/2}} \gamma^0 \gamma^3 - \frac{r \sqrt{\Delta_r}}{\Sigma^{3/2}} \gamma^1 \gamma^2 + \frac{ab}{r p^2} \gamma^3 \gamma^5 \right], \\
\Gamma_3 &= \frac{1}{2} \left[\frac{(a^2 - b^2) r \sin\theta \cos\theta}{\Sigma^{3/2} p} \gamma^0 \gamma^1 + \frac{p \sqrt{\Delta_r}}{\Sigma^{3/2}} \gamma^0 \gamma^2 - \frac{r \sqrt{\Delta_r}}{\Sigma^{3/2}} \gamma^1 \gamma^3 - \frac{p}{\sin\theta \cos\theta} \left(\frac{\sin\theta \cos\theta}{p \sqrt{\Sigma}} \right)_{,\theta} \gamma^2 \gamma^3 - \frac{ab}{r p^2} \gamma^2 \gamma^5 \right], \\
\Gamma_5 &= \frac{1}{2} \left[\frac{ab}{r^2 p} \gamma^0 \gamma^1 - \frac{1}{r} \sqrt{\frac{\Delta_r}{\Sigma}} \gamma^1 \gamma^5 - \frac{ab}{r p^2} \gamma^2 \gamma^3 + \frac{(a^2 - b^2) \sin\theta \cos\theta}{p^2 \sqrt{\Sigma}} \gamma^2 \gamma^5 \right].
\end{aligned} \tag{A4}$$

APPENDIX B: PLEBANSKI-LIKE FORM OF THE $D = 5$ MYERS-PERRY METRIC

In some cases, it is more convenient to use $p = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}$ rather than θ as the appropriate angle coordinate. In doing so, the five-dimensional Myers-Perry metric can be put in a symmetric manner as follows

$$\begin{aligned}
ds^2 &= - \frac{\Delta_r}{\Sigma} \left[dt - \frac{(p^2 - a^2)a}{b^2 - a^2} d\phi - \frac{(p^2 - b^2)b}{a^2 - b^2} d\psi \right]^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_p} dp^2 + \frac{\Delta_p}{\Sigma} \left[dt + \frac{(r^2 + a^2)a}{b^2 - a^2} d\phi \right. \\
&\quad \left. + \frac{(r^2 + b^2)b}{a^2 - b^2} d\psi \right]^2 + \left(\frac{ab}{rp} \right)^2 \left[dt - \frac{(r^2 + a^2)(p^2 - a^2)}{(b^2 - a^2)a} d\phi - \frac{(r^2 + b^2)(p^2 - b^2)}{(a^2 - b^2)b} d\psi \right]^2,
\end{aligned} \tag{B1}$$

where

$$\begin{aligned}
\Delta_r &= (r^2 + a^2)(r^2 + b^2)/r^2 - 2M, \\
\Delta_p &= -(p^2 - a^2)(p^2 - b^2)/p^2, \quad \Sigma = r^2 + p^2.
\end{aligned}$$

The following coordinate transformations:

$$\begin{aligned}
t &= \tau + (a^2 + b^2)u + a^2 b^2 v, \\
\phi &= a(u + b^2 v), \quad \psi = b(u + a^2 v),
\end{aligned} \tag{B2}$$

send the metric to a Plebanski-like form [44]

$$\begin{aligned}
ds^2 &= - \frac{\Delta_r}{\Sigma} (d\tau + p^2 du)^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_p} dp^2 \\
&\quad + \frac{\Delta_p}{\Sigma} (d\tau - r^2 du)^2 \\
&\quad + \left(\frac{ab}{rp} \right)^2 [d\tau + (p^2 - r^2)du - r^2 p^2 dv]^2,
\end{aligned} \tag{B3}$$

in which the role of angular momenta becomes less clear.

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