PHYSICAL REVIEW D **78,** 064051 (2008)

Measurement analysis and quantum gravity

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(Received 18 February 2008; published 18 September 2008)

We consider the question of whether consistency arguments based on measurement theory show that the gravitational field must be quantized. Motivated by the argument of Eppley and Hannah, we apply a DeWitt-type measurement analysis to a coupled system that consists of a gravitational wave interacting with a mass cube. We also review the arguments of Eppley and Hannah and of DeWitt, and investigate a second model in which a gravitational wave interacts with a quantized scalar field. We argue that one cannot conclude from the existing gedanken experiments that gravity has to be quantized. Despite the many physical arguments which speak in favor of a quantum theory of gravity, it appears that the justification for such a theory must be based on empirical tests and does not follow from logical arguments alone.

DOI: 10.1103/PhysRevD.78.064051 PACS numbers: 04.60.−m, 03.65.Ta

I. INTRODUCTION

The theoretical analysis of the measurement process has played a crucial role in the development of quantum theory. A famous example is the discussion between Einstein and Bohr on the role of the uncertainty relations, which took place during the Solvay conferences in Brussels at the end of the 1920s. Another example is the analysis of quantum electrodynamical field quantities by Bohr and Rosenfeld in 1933 [1]. We are interested in the question of whether an analysis along these lines could also be of help in the search for a quantum theory of gravity.

Quantum gravity does not yet exist in a final form, but many promising, although competing approaches, exist [2]. Among them are string theory, quantum geometrodynamics, loop quantum gravity, path-integral quantization, and others. It would be of interest to see which variables can be accessible in a quantum measurement and whether they can be measured with arbitrary accuracy or not. In the canonical approaches to quantum gravity, candidates for such variables are the three-dimensional metric and the second fundamental form, or holonomies and fluxes of the densitized triad; in string theory; the candidates for the most fundamental variables are less clear.

The present paper can be seen as preparation for such an analysis of quantum-gravitational variables. It addresses the question of whether a measurement analysis could disclose that the gravitational field *must* be quantized for consistency. There exist various papers in the literature that claim that this question must be answered in the affirmative, see for example [3,4]. In our paper, we show that such a conclusion cannot be drawn. Many physical arguments speak in favor of quantum gravity (such as the existence of the singularity theorems in the classical theory) [2], but the final justification can only come from an empirical test and not from logical arguments alone.

Our paper is organized as follows. In Sec. II, we address the gedanken experiment discussed in [4]. We disclose gaps in the chain of argument which invalidate the conclusion drawn that gravity must be quantized. In Sec. III, we apply the formalism of quantum measurement analysis developed by Bryce DeWitt to such a gedanken experiment. This serves the purpose of putting the heuristic discussion of [4] on a quantitative level, but is also of interest in its own right—as a study of the relationship between classical and quantum theory. We show that (and how) the uncertainties present in one system entail uncertainties in the system to which it is coupled, but that this does not enforce the quantization of the coupled system. In Sec. IV, we present an explicit counterexample to the claim that a system coupled to a quantum system must necessarily also be of quantum nature: we discuss a hybrid model with a consistent coupling between classical gravity and a quantized scalar field. In Sec. V, we consider the argument made in [3] that the quantum theory must be extended to all physical systems and show that this conclusion is not justified. Sec. VI gives a brief summary and an outlook.

II. CRITIQUE OF THE EPPLEY AND HANNAH GEDANKEN EXPERIMENT

In 1977, Eppley and Hannah proposed a gedanken experiment which was meant to demonstrate that the gravitational field must be quantized [4]. They considered the interaction of a *classical* gravitational wave with a *quantum* system, and argued that this would lead to a violation of momentum conservation, to a violation of the uncertainty principle, or to the transmission of signals faster than light. Since nothing special about gravity seems to enter their line of thought (one is, in particular, at the level of a linearized wave), their arguments should hold for any classical wave, in particular, for an electromagnetic wave. One thus seems to be led to the conclusion that any system that is coupled to a quantum system must also be a quantum system.

A gedanken experiment should, of course, be realizable at least in principle. It was recently emphasized by Mattingly that this is not the case for the Eppley-Hannah (EH) model; for example, their detector must be so massive as to be within its own Schwarzschild radius [5]. Furthermore, it was argued by Huggett and Callender that the violations of physical principles are only present in the Copenhagen interpretation of quantum mechanics and are thus, at least partially, resolvable within alternative interpretations [6]. Here we show that even without the question of realizability or interpretation, it is not correct to say that the EH gedanken experiment entails the necessity of quantizing the gravitational field.

Eppley and Hannah consider the interaction of a classical gravitational wave of small momentum with a quantum particle described by a wave function ψ. They restrict to the case in which the wavelength, λ, of the wave is much smaller than the position uncertainty, Δx, of the particle in order for the interaction to lead to a measurement of the position of the quantum particle. For the generation and detection of the gravitational wave, they make the following assumption: the uncertainties of the quantum system which is used to prepare and detect the wave will result in a negligible perturbation of the classical wave. To discuss this assumption, we shall present a DeWitt-type measurement analysis in the following section. Here we take their assumption for granted. When the quantum particle is probed by the classical wave, it is furthermore assumed that no direct perturbation of the particle occurs because of the small wave momentum. This constitutes an "ideal measurement" in the sense of John von Neumann, a condition that is now also called "quantum nondemolition measurement," cf. [7]. (We postpone a discussion of this assumption until the end of this section; here, we also take it for granted.)

Eppley and Hannah now distinguish between two possibilities: either the gravitational wave leads to a collapse of the wave function of the particle, or it does not. If it does, they argue that this would entail either momentum nonconservation or a violation of the uncertainty relation. If it does not collapse, they argue that there will be a transmission of information with superluminal speed.

Let us consider the first case of an assumed collapse. Such a collapse is not part of the standard linear quantum theory because it would violate the superposition principle. One can, of course, modify quantum mechanics in order to accommodate such a collapse, cf. chapter 8 of [7] or [8]. In fact, one popular class of such models are models of gravity-induced collapse of the wave function. Such collapse models typically introduce new constants of nature. They often have problems with conservation laws, so some effort is required in order to construct a model that is in accordance with such laws. It is thus not surprising that EH arrive at problems with momentum nonconservation, but this by itself should not be taken as a logical argument to quantize gravity. Therefore, before any definite statement about a possible violation of momentum conservation or the uncertainty relations can be made, the interaction of the gravitational wave with the quantum particle must be discussed quantitatively within a definite collapse model. The arguments of [4] are therefore inconclusive for this case.

Let us assume, then, that the gravitational wave does not collapse the wave function ψ of the particle. Eppley and Hannah then consider an EPR-type situation where one particle decays into two other particles which together are in a singlet state (e.g., a π^0 decaying into two γ): if an observer measures photon 1 in horizontal polarization, photon 2 will be found in vertical polarization, and vice versa. Eppley and Hannah then argue that a classical gravitational wave scattered off from particle 2 can distinguish between particle 2 having a definite polarization or particle 2 being in a superposition of both polarizations. The gravitational wave could thus instantaneously "see" whether a measurement at particle 1 was done—the corresponding information would then have propagated with superluminal speed. This conclusion is, however, not correct. Before the measurement of particle 1, the total state of photons and detector is in the entangled state

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2)|\Phi_0\rangle, \tag{1}$$

where $|\uparrow\rangle_1$ ($|\downarrow\rangle_1$) denotes horizontal (vertical) polarization of photon 1 (and similarly for photon 2), and $|\Phi_0\rangle$ denotes the initial (switched off) state of the detector which will measure photon 1. After the detector has measured photon 1, the initial state $|\Psi_0\rangle$ will have evolved into the new entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2|\Phi_\uparrow\rangle - |\downarrow\rangle_1|\uparrow\rangle_2|\Phi_\downarrow\rangle), \tag{2}$$

where $|\Phi_\uparrow\rangle$ ($|\Phi_\downarrow\rangle$) denotes the state of the detector after it has measured the polarization of photon 1 to be horizontal (vertical). The important point is that photon 2, by itself, is not in a pure state, neither before nor after the measurement of photon 1. It finds itself in a mixed state which is obtained from the total pure state of system plus detector by tracing out the states of particle 1 and the detector. This leads for *both* (1) and (2) to the same density operator for photon 2,

$$\hat{\rho} = \frac{1}{2}(|\uparrow\rangle_2\langle\uparrow|_2 + |\downarrow\rangle_2\langle\downarrow|_2). \tag{3}$$

In both cases, photon 2, by itself, is in a mixed state of horizontal and vertical polarization with equal probability one half. The gravitational wave "sees" the same mixed state for photon 2, independent of whether a measurement of photon 1 has been performed or not; thus, no superluminal communication is possible. The difference between the total states (1) and (2) can only be seen after both photons are brought together: in case (1) they will interfere, while in case (2) they will not; in the latter case the information about the original superposition has been delocalized into correlations of the detector state with its natural environment, that is, decoherence has occurred [7].

The above argument assumes that the linear structure of quantum theory remains untouched. This, of course, corresponds to an Everett interpretation. If, on the other hand, one assumes that a measurement on photon 1 collapses its wave function into $|\uparrow\rangle_1$ or $|\downarrow\rangle_1$, a superluminal communication might in principle be possible, cf. [9]. This can happen, for example, in the context of semiclassical gravity (see, e.g., [2]) where the source of the gravitational field is taken to be the quantum expectation value of the energy-momentum tensor. Semiclassical gravity introduces a nonlinearity into quantum theory, which in principle can be experimentally tested [9]. The possibility of such an example does not prove, however, that a mixed classical-quantum coupling without superluminal communication is impossible. In fact, Sec. IV presents a candidate for such a theory. In Appendix B we shall discuss a simple gedanken experiment of a classical particle interacting with a quantum system; we shall show there that the mixed classical-quantum coupling of Sec. IV can successfully deal with such a situation. This is possible because the classical and quantum sectors still exhibit some entanglement.

One of the assumptions that we have taken for granted here, namely, that the momentum of the wave is sufficiently small for the particle not to be perturbed, leads to difficulties if the scattering of the gravitational wave is going to be used to measure the position of the particle. Let us assume, as Eppley and Hannah do, that the interaction is such that incoming plane gravitational waves are scattered by the quantum system with the result that spherical waves are emitted. To achieve this, a certain amount of energy needs to be transferred from the incoming gravitational wave to the quantum system, because the system will not emit gravitational waves unless its quadrupole or higher multipole modes are excited. Given that a quantum system typically has quadrupole modes that are quantized, one would expect this energy to be of the order of $h\nu$, where ν is some characteristic frequency associated with the transition to the quadrupole mode. But if we assume that the incoming gravitational wave carries a negligible amount of momentum, then it will not be able to transfer a sufficient amount of energy to the quantum system to excite its higher modes and no scattering can take place. And if we allow the gravitational wave to carry sufficient energy for scattering to occur, then we cannot rule out a transfer of energy and momentum that may be sufficient for the quantum system to be left in a state that does not violate the Heisenberg uncertainty relation, independent of any particular model that might be used to describe collapse of the wave function.

A more detailed description of the interaction term is required before we can make more conclusive statements regarding the outcome of the EH gedanken experiment. Presumably, the gravitational wave will interact with the quantized system through gravitational effects only. If so, then the form of the interaction will be determined by the way in which gravity couples to matter. In the next sections, we present examples where these considerations are taken into account and particular interaction terms are examined.

III. DEWITT-TYPE MEASUREMENT ANALYSIS

A. General analysis

Motivated, in particular, by the EH gedanken experiment discussed in the last section, we present a general measurement analysis of an interaction between a gravitational wave and a nongravitational system. For this purpose, we apply the general formalism introduced by Bryce DeWitt, cf. [3,10–12]. We consider this type of analysis for two reasons. First, DeWitt's approach provides a descriptive and straightforward method of modeling a measurement (or any other influence of one system on another). Second, it provides a convenient way of developing a concrete realization of the EH gedanken experiment, see Sec. III B below. Before describing our model, we review the general framework.

The starting point of the measurement analysis developed by DeWitt is the observation that a measurement of a physical observable requires an interaction between the system and the apparatus. A measurement is then interpreted as recording the resulting change in the state of the apparatus, that is, the difference between the configurations with and without the interaction taking place. For example, for a voltage measurement this would be 2 V if the initial value increases (due to the coupling between apparatus and system) from 5 to 7 V.

There are four basic ingredients that enter into the analysis and which need to be modeled theoretically: the physical system, the measuring apparatus, the choice of system observable that is being measured, and the coupling term that describes the interaction between the system and the apparatus. At this level of description, the focus is on possible theoretical limitations rather than on the practical limitations that are unavoidable when carrying out measurements in a laboratory. Therefore, other complications that are encountered in real experiments are not considered here. Once these four ingredients are defined, one can follow DeWitt's method and calculate the change in the configuration of the apparatus induced by the coupling to the system. Two assumptions are needed to make the equations tractable. First, the coupling is assumed to be weak. This is an essential assumption that allows a perturbative approach using first- and second-order terms only. Moreover, this assumption corresponds to the idea of a careful measurement which imparts only a small disturbance and is therefore justified when investigating "…uncontrollable uncertainties which remain in spite of all precautions" [10]. Second, it is assumed that the second-order terms which involve the disturbance of the apparatus can be neglected. This approximation is valid because of the usual $1/m^2$ dependence of the Green functions of the given systems together with the condition that the apparatus be much more massive than the system. Such an assumption helps simplify the equations, but one may also take such terms into consideration if necessary.

The first step in this perturbative analysis is to solve the equations for the system and apparatus while neglecting the interaction term, that is, the equations

$$S[\phi]_{,i} = 0, \qquad \Sigma[\theta]_{,I} = 0,$$

where $S[\phi]$ denotes the action functional of the uncoupled system, $\Sigma[\theta]$ the action functional of the uncoupled apparatus, ϕ are system variables and θ are apparatus variables. The comma denotes functional differentiation and lowercase (capital) latin letters are used to indicate derivatives with respect to system (apparatus) variables. We make use of DeWitt's condensed notation, where a summation over discrete indices indicates an integration over the argument of the field (e.g., given $A_i(x, t)$ and $B_i(x, t)$, A_iB^i should be read as $\sum_i \int \mathrm{d}x\mathrm{d}t A_iB^i$). The solutions of the uncoupled equations will be called ϕ_0 and θ_0, respectively.

Once the uncoupled field configurations are calculated, one introduces the interaction term $\Omega[\phi, \theta]$. The interaction is assumed to last for a finite time only, which means that all terms describing the perturbations that arise due to the interaction (i.e., $\delta\phi$, $\delta\theta$, etc.) must satisfy *retarded* boundary conditions,

$$\lim_{t\to-\infty} \delta\phi(t) = 0, \qquad \lim_{t\to-\infty} \delta\theta(t) = 0. \tag{4}$$

If we take into consideration the interaction term, the total action functional takes the form

$$S[\phi] + \Sigma[\theta] \to S[\phi] + \Sigma[\theta] + g\Omega[\phi, \theta], \tag{5}$$

with a small dimensionless coupling constant g.

We now introduce the assumption that the coupling is weak and expand the equations derived from (5) around the solutions ϕ_0 and θ_0 of the uncoupled equations,

$$S_{,i}[\phi_0 + \delta\phi] + g\Omega_{,i}[\phi_0 + \delta\phi, \theta_0 + \delta\theta] = 0,$$
$$\Sigma_{,I}[\theta_0 + \delta\theta] + g\Omega_{,I}[\phi_0 + \delta\phi, \theta_0 + \delta\theta] = 0,$$

where due to the small coupling the stationary points are expected to lie close to ϕ_0 and θ_0. This leads to the functional Taylor series

$$S_{,i}[\phi_0] + S_{,ij}[\phi_0]\delta\phi^j + \frac{1}{2}S_{,ijk}[\phi_0]\delta\phi^j\delta\phi^k + g\Omega_{,i}[\phi_0, \theta_0] + g\Omega_{,ij}[\phi_0, \theta_0]\delta\phi^j + g\Omega_{,iI}[\phi_0, \theta_0]\delta\theta^I + \cdots = 0 \tag{6}$$

and

$$\Sigma_{,I}[\theta_0] + \Sigma_{,IJ}[\theta_0]\delta\theta^J + \frac{1}{2}\Sigma_{,IJK}[\theta_0]\delta\theta^J\delta\theta^K + g\Omega_{,I}[\phi_0, \theta_0] + g\Omega_{,IJ}[\phi_0, \theta_0]\delta\theta^J + g\Omega_{,Ii}[\phi_0, \theta_0]\delta\phi^i + \cdots = 0. \tag{7}$$

In principle, Eqs. (6) and (7) can be used to derive solutions to any order. Within DeWitt's scheme one has to calculate from (6) the change in the configuration of the system only to first order, that is, to solve

$$S_{,ij}[\phi_0]\delta\phi^j = -g\Omega_{,i}[\phi_0, \theta_0]. \tag{8}$$

Backreaction of the apparatus change onto the system (which would correspond to the term $g\Omega_{,iI}\delta\theta^I$) is thus not considered. Note that terms involving the first functional derivative of the action vanish because ϕ_0 and θ_0 are solutions of the uncoupled equations of motion. To get an expression for the apparatus which takes into account the system changes, it is necessary to solve Eq. (7) to second order. The calculation simplifies if we now make use of the second assumption and neglect all second-order terms involving $\delta\theta$ and any $\delta\theta$ terms that appear together with $\delta\phi$ terms of the same order. When this is done, we are led from (7) to the equations

$$\Sigma_{,IJ}[\theta_0]\delta\theta^J = -g\Omega_{,I}[\phi_0, \theta_0] - g\Omega_{,Ii}[\phi_0, \theta_0]\delta\phi^i. \tag{9}$$

The last term describes the backreaction of the system on the apparatus. It is also possible to have the special case where the interaction between system and apparatus causes no system disturbance, that is, with $\delta\phi^i = 0$. In this case, one needs to use (7) without the above approximation of neglecting certain terms in $\delta\theta$.

Finally, one needs to choose an appropriate apparatus observable, A, to calculate the change, δA, which arises due to the disturbance, $\delta\theta$, within the apparatus configuration,

$$\delta A = A_{,I}\delta\theta^I. \tag{10}$$

While the brief description given here does not present DeWitt's original procedure in full (which makes use of Peierls brackets and Green's functions, see Sec. IV C), it is sufficient for the analysis presented in the next subsection, where we apply the formalism to a concrete model. Note that so far the analysis is entirely classical.

B. Mass-cube interacting with a gravitational wave packet

The motivation for this model arose from the critique of [4] presented in Sec. II. While the actual results and interpretations at the end of this subsection are heuristic, it will be shown that they support our critique mentioned before.

We consider a cube with homogeneous mass density interacting with a linear gravitational wave packet. The cube is characterized by its edge length, a, and its mass, M. Its speed is assumed to be small compared to the speed of light (nonrelativistic approximation). It can be described by the action

$$S = \int \mathrm{d}t \frac{1}{2} M \dot{\mathbf{r}}^2 = \frac{1}{2} \int \mathrm{d}^4 x \rho \dot{\mathbf{r}}^2. \tag{11}$$

The mass density of the cube is given by

$$\begin{aligned}\rho = \frac{M}{a^3} \Theta\Big(x_1 - x_{1_{\rm cm}}(t) + \frac{a}{2}\Big)\Big[1 - \Theta\Big(x_1 - x_{1_{\rm cm}}(t) - \frac{a}{2}\Big)\Big] \\ \times \Theta\Big(x_2 - x_{2_{\rm cm}}(t) + \frac{a}{2}\Big)\Big[1 - \Theta\Big(x_2 - x_{2_{\rm cm}}(t) - \frac{a}{2}\Big)\Big] \\ \times \Theta\Big(x_3 - x_{3_{\rm cm}}(t) + \frac{a}{2}\Big)\Big[1 - \Theta\Big(x_3 - x_{3_{\rm cm}}(t) - \frac{a}{2}\Big)\Big];\end{aligned} \tag{12}$$

the subscript "cm" is an abbreviation for "center of mass," and the $x_{i_{\rm cm}}$ are the time-dependent coordinates of the center of mass of the cube, that is, they correspond to the aforementioned dynamical variable of the system ϕ. A possible solution of the free equation of motion, that is, the solution corresponding to the ϕ_0 above, is

$$\mathbf{r}_{\rm cm}(t) = (0, vt, 0), \tag{13}$$

which is unaccelerated motion in x_2-direction with velocity v.

If the cube consists of noninteracting particles, the components of its stress-energy tensor read

$$T_{00} = \rho c^2 \qquad T_{ik} = \rho u_i u_k = \rho \dot{x}_{i_{\rm cm}}(t) \dot{x}_{k_{\rm cm}}(t),$$

where ρ is given by (12). (We have neglected here possible internal stresses of the cube, so the cube is in first approximation interpreted as a set of noninteracting particles.) The linear gravitational wave packet is, as usual, a superposition of plane waves, which are solutions of the linear Einstein equations. In the spirit of the EH-model, the gravitational wave plays here the role of the "apparatus." The Einstein equations follow from the variation of the Einstein-Hilbert action

$$\Sigma[g_{\mu\nu}] = \int d^4x \frac{\sqrt{-g} R}{2\kappa},$$

where $\kappa = 8\pi G/c^4$. For the limit of linearized gravity one gets

$$\Box \Psi_{\mu\nu} = 0. \tag{14}$$

Here, $g_{\mu\nu} \approx \eta_{\mu\nu} + 2h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$, $\Psi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\lambda_\lambda$, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, and we use the harmonic gauge which corresponds to $\Psi_{\mu\nu,}{}^{\nu} = 0$. We are interested in solutions of (14) that represent gravitational waves propagating into a direction of maximal coupling (perpendicular to the path of the cube), for example, the x_1-direction:

$$\Psi^{\omega_0}_{\mu\nu}(x) = \mathrm{Re}(A_+ e^+_{\mu\nu} e^{-i\omega_0(t-(x_1/c))}), \tag{15}$$

where $e^+_{\mu\nu}$ denotes a polarization tensor which reads

$$e^+_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

this describes the $+$-polarization, see, for example, [13], chapter 35. A superposition of such solutions with Gaussians $A_0 \exp(-b(\omega_0 - \omega)^2/2)$ according to

$$\begin{aligned}\Psi_{\mu\nu}(x_1, t) &= \mathrm{Re}\Big[e^+_{\mu\nu} A_+ \\ &\quad \times \int d\omega_0 A_0 e^{-(b/s)(\omega_0-\omega)^2 - i\omega_0(t-(x_1/c))}\Big] \\ &\equiv \mathrm{Re}\Big[e^+_{\mu\nu} A \sqrt{\frac{2\pi}{b}} e^{-(1/2b)(t-(x_1/c))^2 - i\omega(t-(x_1/c))}\Big],\end{aligned} \tag{16}$$

where $A = A_+ A_0$, yields the requested linear gravitational wave packet. We note that this solution $\Psi_{\mu\nu}$ corresponds to θ_0 in Sec. III A.

The coupling between wave and cube is modeled via a linear interaction of the wave and the stress-energy tensor of the mass contribution. This constitutes a nonlinear coupling between the gravitational field and the particles, and thus goes beyond the usual limit in which the particles are only considered as test particles in an external gravitational wave. The interaction is switched on at $t = T$ in order to follow DeWitt's procedure that all disturbances have to obey Eq. (4); hence we assume

$$\Omega[\mathbf{r}(t), \Psi_{\mu\nu}(x_1, t)] = \int d^4x T_{\mu\nu} \Psi^{\mu\nu} \theta(t - T). \tag{17}$$

The contraction of $T_{\mu\nu}$ with $\Psi^{\mu\nu}$ in this form of interaction defines which states of motion will yield nonvanishing contributions. A brief look at $e^+_{\mu\nu}$ shows that only the T_{22} and T_{33} components will couple to the scattering gravitational wave. Moreover, they are going to contribute with opposite sign. This indicates that certain systems, for example, an isotropic scalar field, would not couple to such a gravitational wave because the interaction would be proportional to $k_2^2 - k_3^2 = k_2^2 - k_2^2 = 0$.

For this model, the interaction causes disturbances $\delta\mathbf{r}_{\rm cm}$ and $\delta\Psi$ of the initially free solutions (13) and (15) which, if assumed small, are given by the general expressions (8) and (9). Following the procedure described above (i.e., to take the changes within the system into account in order to calculate those of the apparatus), we first compute $\delta\mathbf{r}_{\rm cm}$.

We shall use (8) with x_j for the ϕ^j and $\psi_{\mu\nu}$ for the θ^J; we set $g = 1$. Then,

$$S_{,ij}\delta x^j = -M\delta\ddot{x}^i(t), \tag{18}$$

and, since $T_{\mu\nu}\psi^{\mu\nu} = \rho\dot{x}_2^2\psi^{22}$,

$$-\frac{\delta\Omega}{\delta x^i(t)} = -\int d^3x\left[\frac{\partial\rho}{\partial x^i}\dot{x}_2^2\psi^{22} - \delta_2^i\frac{d}{dt}(2\rho\dot{x}_2\psi^{22})\right] \times \theta(t-T). \tag{19}$$

We shall consider first the disturbance in $x_2(t)$. Equation (8) reads for this case

$$-M\delta\ddot{x}_2(t) = -\frac{\delta\Omega}{\delta x^2(t)}. \tag{20}$$

For the right-hand side we find that only the second term in (19) contributes. Setting $T = 0$, we then get

$$\begin{aligned}-\frac{\delta\Omega}{\delta x^2(t)} &= \int d^3x\left[\frac{d}{dt}(2\rho\dot{x}_2\psi^{22})\right]\theta(t)\\ &= -\frac{2Mvc}{a}\theta(t)(\psi_+^{22} - \psi_-^{22})\\ &\quad -\frac{2Mv}{a}\delta(t)\int_{-a/2}^{a/2}dx_1\psi^{22}(x_1, t),\end{aligned} \tag{21}$$

where

$$\psi_+^{22} = A\sqrt{\frac{2\pi}{b}}e^{-(t_+^2/2b)-i\omega t_+},$$

$$\psi_-^{22} = A\sqrt{\frac{2\pi}{b}}e^{-(t_-^2/2b)-i\omega t_-},$$

and $t_+ = t + a/2c$ and $t_- = t - a/2c$. In the following we do not consider the last term occurring in (21), since it is proportional to $\delta(t)$ and thus only effective at the initial time $t = 0$. We then get from (20)

$$\delta\ddot{x}_2(t) = \frac{2vcA}{a}\sqrt{\frac{2\pi}{b}}\theta(t)\,\mathrm{Re}(e^{-(t_-^2/2b)-i\omega t_-} - e^{-(t_+^2/2b)-i\omega t_+}). \tag{22}$$

In a similar way one finds for the second time derivatives of the other disturbances,

$$\begin{aligned}-M\delta\ddot{x}_1(t) &= -\frac{\delta\Omega}{\delta x^1(t)} = -\frac{Mv^2}{a}\theta(t)(\psi_+^{22} - \psi_-^{22})\\ &= \frac{v}{2c}\delta\ddot{x}_2(t),\\ -M\delta\ddot{x}_3(t) &= -\frac{\delta\Omega}{\delta x^3(t)} = 0.\end{aligned} \tag{23}$$

One recognizes from these equations that $\delta x_1(t)$ is smaller than $\delta x_2(t)$ by a factor $v/2c$; since we work in the nonrelativistic approximation $v/c \ll 1$, we only have to consider $\delta x_2(t)$ in the following. Therefore, integrating (22) twice, one arrives at the result for $\delta x_2(t)$,

$$\begin{aligned}\delta x_2(t) = \frac{2vcA\pi}{a}\sqrt{2b}e^{-(b\omega^2/2)}\theta(t)\times\mathrm{Re}\Bigg(&\left[\frac{t}{\sqrt{2b}} + \sqrt{\frac{b}{2}}\left(-\frac{a}{2cb} + i\omega\right)\right]\mathrm{erf}\left[\frac{t}{\sqrt{2b}} + \sqrt{\frac{b}{2}}\left(-\frac{a}{2cb} + i\omega\right)\right]\\ &- \left[\frac{t}{\sqrt{2b}} + \sqrt{\frac{b}{2}}\left(\frac{a}{2cb} + i\omega\right)\right]\mathrm{erf}\left[\frac{t}{\sqrt{2b}} + \sqrt{\frac{b}{2}}\left(\frac{a}{2cb} + i\omega\right)\right]\\ &+ \frac{1}{\sqrt{\pi}}(e^{-((t/\sqrt{2b})+\sqrt{b/2}[-(a/2cb)+i\omega])^2} - e^{-((t/\sqrt{2b})+\sqrt{b/2}[(a/2cb)+i\omega])^2})\Bigg) + C_1t + C_2,\end{aligned} \tag{24}$$

where the constants C_1 and C_2 depend on the initial conditions.

We are interested in the limit where t can be assumed large, that is, after the wave has passed through and the measurement is completed. Using the asymptotic property of the error function,

$$x\mathrm{erf}(x) + \frac{e^{-x^2}}{\sqrt{\pi}} \sim x,$$

one gets

$$\delta x_2(t) \sim -2vA\pi e^{-(b\omega^2/2)} + C_1t + C_2 \sim C_1t. \tag{25}$$

It is not surprising that asymptotically $\delta x_2(t)$ increases linearly with time, since the dust particles comprising the cube do not interact.

We want to impose initial conditions such that $\delta x_2(0) = 0$ and $\delta\dot{x}_2(0) = 0$. After a straightforward calculation we find that $C_2 = 0$ and

$$C_1 = -\frac{2vcA}{a[(\frac{a}{2bc})^2 + \omega^2]}\sqrt{\frac{2\pi}{b}}e^{-a^2/8bc^2} \times\left[\frac{a}{bc}\cos\left(\frac{\omega a}{2c}\right) - 2\omega\sin\left(\frac{\omega a}{2c}\right)\right]. \tag{26}$$

According to (25), the asymptotic behavior of $\delta x_2(t)$ is determined by this constant C_1.

The next step in DeWitt's procedure is the application of (9) in order to calculate the backreaction on the gravitational wave, $\delta\psi_{\mu\nu}$. In this equation, only the functional derivative of Ω (Eq. (17)) with respect to ψ^{22} is nonzero, since $T_{\mu\nu}\psi^{\mu\nu} = \rho\dot{x}_2^2\psi^{22}$; a derivative with respect to I and J in (9) is thus a derivative with respect to ψ^{22}. The left-hand side of (9) yields

$$\frac{c^4}{4\pi G}\Box\delta\psi_{\mu\nu}(x).$$

On the right-hand side, the first term gives

$$-\frac{\delta\Omega}{\delta\psi_{\mu\nu}} = -\rho\dot{x}_2^2\theta(t)\delta_2^\mu\delta_2^\nu. \quad (27)$$

The second term also yields a contribution only for $\mu = 2$, $\nu = 2$, and reads

$$-\sum_i \frac{\delta\Omega}{\delta\psi_{22}\delta x^i}\delta x^i = -v^2\left(2\rho D_1 + v\frac{\partial\rho}{\partial x_2}D_0 + v\frac{\partial\rho}{\partial x_2}D_1 t\right) \times \theta(t), \quad (28)$$

where $D_0 = -2A\pi e^{-b\omega^2/2}$ and $D_1 = C_1/v$. Taking all together, (9) reads

$$\Box\delta\psi_{22} = \frac{4\pi G}{c^4}v^2\left(2\rho D_1 + v\frac{\partial\rho}{\partial x_2}D_0 + v\frac{\partial\rho}{\partial x_2}D_1 t\right)\theta(t). \quad (29)$$

We solve this equation with the help of the usual retarded Green function,

$$D_r(t-t', \mathbf{r}-\mathbf{r}') = \frac{\theta(t-t')}{4\pi|\mathbf{r}-\mathbf{r}'|}\delta\left(t - t' - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right),$$

under the assumption that $|\mathbf{r}-\mathbf{r}'| \approx |\mathbf{r}| \equiv r$; we also introduce the retarded time

$$t_r \equiv t - \frac{|\mathbf{r}-\mathbf{r}'|}{c} \approx t - \frac{r}{c}.$$

After some straightforward calculations, we get the result

$$\delta\psi_{22}(\mathbf{r}, t) = \frac{GMv^2}{c^4 r}\left(1 - \frac{2C_1}{v}\right)\theta(t_r). \quad (30)$$

The first term on the right-hand side corresponds to the first term on the right-hand side of (9); the form of the resulting term is well known from the generation of gravitational waves by, for example, the circular motion of two masses (here it is a consequence of the interaction (17)). The second term originates from the second term in (9) and describes the backreaction coming from δx^i. We expect it to be much smaller than the first term. Let us perform some numerical estimates.

We take, for example, the values $\omega = 1$ Hz and $a = 1$ m (but the precise values are not important). We then have

$$\frac{\omega a}{2c} \sim 10^{-9} \ll 1,$$

and we can therefore approximate $\cos\frac{\omega a}{2c} \approx 1$ and $\sin\frac{\omega a}{2c} \approx \frac{\omega a}{2c}$ in (26). The ratio of the second term to the first term in the parentheses of (26) is then given by $\omega^2 b$, which is much bigger than 1 if we assume a narrow packet for the gravitational wave (16) (as we shall do). We then also find that the first term in the denominator of (26) is much smaller than the second term. We thus get

$$\frac{2C_1}{v} \approx 4A\sqrt{\frac{2\pi}{b}}e^{-a^2/8bc^2}. \quad (31)$$

The size of the backreaction in (30) is thus proportional to the amplitude of the gravitational wave as expected. If we take $A/\sqrt{b} \sim 10^{-20}$ (which should be a realistic value for an astrophysical gravitational wave), we see that the backreaction term is tiny. Since the exponential in (31) is ≈ 1 for our values of parameters, we get from (30)

$$\delta\psi_{22}(\mathbf{r}, t) \approx \frac{GMv^2}{c^4 r}\left(1 - 4A\sqrt{\frac{2\pi}{b}}\right). \quad (32)$$

We have obtained this result from the application of the general expression (9). This formula follows after neglecting certain terms in the more general formula (7). But can this be justified? After all, a gravitational wave is not a massive apparatus in the original sense of DeWitt's analysis. If we compare (9) with (7), there are only two terms that are being neglected, the third and fifth terms of (7). One then needs to justify neglecting these two terms. In our case, the fifth term is identically zero because the interaction term (17) is linear in the field $\psi^{\mu\nu}$, and therefore its second functional derivative is zero. To justify neglecting the third term, notice that this term is proportional to the square of $\delta\psi_{22}$. But, according to the approximate calculation that has been carried out, see (30), $\delta\psi_{22}$ is of order v^2/c^4. One can therefore assume that the third term must be of order v^4/c^8, and therefore it is safe to neglect it in the nonrelativistic approximation. The approximation (9) is thus a good one because of the form of the interaction and the nonrelativistic approximation.

We now need to consider a suitable "apparatus quantity" of the gravitational wave to be measured, cf. (10). We choose the energy density of the gravitational wave,

$$A(\Psi_{\mu\nu}) \equiv T_{00}^{\rm GW} = \frac{c^2}{8\pi G}\dot{\Psi}_{\mu\nu}\dot{\Psi}^{\mu\nu}. \quad (33)$$

The change of $T_{00}^{\rm GW}$ is given by

$$\delta T_{00}^{\rm GW} = \frac{c^2}{8\pi G}\int d^3x'dt'\frac{\delta T_{00}^{\rm GW}}{\delta\Psi^{\mu\nu}}\delta\Psi^{\mu\nu} = -\frac{c^2}{4\pi G}\ddot{\psi}^{22}\delta\psi_{22}, \quad (34)$$

since $\delta\dot{\psi}_{22} = 0$ (neglecting, again, a delta-function contribution). If we use again $b^{-1} \ll \omega^2$ and set $t \approx x_1/c$ (because only then do we get a noticeable contribution from a packet peaked around $t - x_1/c$), we get $\ddot{\psi}^{22} \approx -A\omega^2\sqrt{2\pi/b}$, and therefore

$$\delta T_{00}^{\mathrm{GW}} \approx \frac{A\omega^2 M v^2}{4\pi r c^2}\sqrt{\frac{2\pi}{b}}\left(1 - 4A\sqrt{\frac{2\pi}{b}}\right). \tag{35}$$

(Note that the dependence on a has dropped out in this limit.) The measurement of $\delta T_{00}^{\mathrm{GW}}$ thus yields directly information about the kinetic energy of the cube (and also about a if higher-order corrections are taken into account).

These results, although derived for a classical mass cube, are also valid, at least approximately, for a quantized system. This follows from the fact that the equations of motion for the center of mass of the free quantum system are also solved by (13). The solutions of the uncoupled equations are therefore the same; however, in the case of a quantum system, the interaction term may differ from (17) by terms of order $\hbar$, leading to corresponding quantum corrections to the disturbances. These corrections should be negligible if the quantum system is large enough, in which case our solutions are also valid for a gravitational wave packet interacting with a quantum mass cube. The effect of the quantum uncertainties of the system on the disturbance of the apparatus observable is estimated below.

Regarding the EH gedanken experiment, it is of interest to consider the limit of $\delta T_{00}^{\mathrm{GW}}$ in which the amplitude A tends to zero. Taking $A \to 0$ leads to a vanishing disturbance of the apparatus observable:

$$\lim_{A\to 0} \delta T_{00}^{\mathrm{GW}} = 0,$$

which suggests that obviously no measurement can be achieved within this limit. Another way of coming into contact with the EH gedanken experiment, at least on a heuristical level, is to introduce quantum uncertainties for the mass cube. The particles comprising the cube obey the uncertainty principle. These uncertainties limit the accuracy for the measurement of the edge length a and introduce an uncertainty that depends on the particles' position uncertainty Δx. A simultaneous momentum measurement will be restricted by the quantum-mechanical momentum uncertainty $\Delta p \gtrsim \hbar/\Delta x$. It is only at this stage that the quantum theory comes into play (and only in the weak form of the uncertainty relation). The resulting classical uncertainty for T_{00}^{GW} then becomes

$$\Delta T_{00}^{\mathrm{GW}} = \sqrt{\left(\frac{\partial \delta T_{00}^{\mathrm{GW}}}{\partial a}\Delta a\right)^2 + \left(\frac{\partial \delta T_{00}^{\mathrm{GW}}}{\partial v}\Delta v\right)^2}$$
$$\geq \sqrt{\left(\frac{\partial \delta T_{00}^{\mathrm{GW}}}{\partial a}\Delta x\right)^2 + \left(\frac{\partial \delta T_{00}^{\mathrm{GW}}}{\partial v}\frac{\hbar}{2m\Delta x}\right)^2}.$$

The minimum uncertainty is found for

$$\Delta x_{\min} = \sqrt{\frac{\hbar}{2m}\frac{\delta T_{00\ ,v}^{\mathrm{GW}}}{\delta T_{00\ ,a}^{\mathrm{GW}}}},$$

and therefore

$$\Delta T_{00}^{\mathrm{GW}} \geq \sqrt{\frac{\hbar}{m}\delta T_{00\ ,v}^{\mathrm{GW}}\delta T_{00\ ,a}^{\mathrm{GW}}}.$$

Thus, a coupling between a weak gravitational wave and a test body obeying the uncertainty principle in the manner described above unavoidably transfers a minimum amount of uncertainty to the classical system, as can be seen here for the averaged energy density of the scattered gravitational wave (for more arguments on the transfer of uncertainties, see Sec. IV). But this disagrees with the requirement in [4] that the gravitational wave, considered as a classical system, should be free of all uncertainties during the whole measurement. This indicates another weakness of the arguments presented in the EH gedanken experiment. We thus conclude from this discussion that the corresponding assumption made in [4] is unrealistic. Whether the derived minimum uncertainty is big enough to strictly invalidate the EH-arguments is a different question and beyond the scope of our discussion.

IV. INTERACTING CLASSICAL-QUANTUM SYSTEMS

In this section, we construct an explicit model to describe the interaction of a classical gravitational wave with a quantized field. This provides a counterexample to the claims made in [3,4] that a system coupled to a quantum system must, for consistency reasons, also be of quantum nature. A number of different methods have been proposed to model "mixed" classical-quantum systems; here, we use the formalism of ensembles in configuration space [14]. This formalism is applicable to both classical and quantum systems and allows a general and consistent description of interactions between them. In particular, the correct equations of motion for the classical and quantum sectors are recovered in the limit of no interaction, conservation of probability and energy are satisfied, uncertainty relations hold for conjugate quantum variables, and the formalism allows a backreaction of the quantum system on the classical system.

We consider one of the simplest models: a two-dimensional version of the scalar theory of gravity of Nordström [15,16] with a quantized massive scalar field. Although not a viable theory of gravity [2,17], the theory of Nordström appears to be the simplest one that has all the ingredients that are necessary to model a gedanken experiment of the Eppley and Hannah type. There are gravitational waves in the theory, the coupling to the scalar field is uniquely determined, and the calculations simplify some-what because the whole theory can be formulated in Minkowski space-time (although the reformulation due to Einstein and Fokker [18] shows that one may relax the requirement of a flat space-time and provide a geometric interpretation of the theory). The simple model discussed in this section already shows the main features expected of a more realistic description while presenting fewer technical complications, the main advantage being that the interaction term is simpler than the one derived from general relativity. This will allow us to get explicit expressions for the state of the coupled classical-quantum system that are valid to first order in the coupling parameter for particular classes of fields.

The scalar theory of gravity of Nordström is based on the Lagrangian density [2]

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_{\mathrm{matter}} = -\frac{1}{2}\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - gT\phi + \mathcal{L}_{\mathrm{matter}},$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1)$ is the two-dimensional Minkowski metric, the commas indicate partial derivatives, g is the coupling constant, $T = \eta^{\mu\nu}T_{\mu\nu}$ is the trace of the energy-momentum tensor $T_{\mu\nu}$ and $\mathcal{L}_{\mathrm{matter}}$ is the Lagrangian density of matter. It will be convenient to consider the Hamiltonian formulation of the theory, which in two dimensions and for the particular case where the matter consists of a massive scalar field ψ takes the form

$$H = \frac{1}{2}\int dx(\pi_\phi^2 + \phi'^2) + \frac{1}{2}\int dx(\pi_\psi^2 + \psi'^2 + m^2\psi^2) + gm^2\int dx\phi\psi^2. \quad (36)$$

Here, the prime indicates a derivative with respect to the spatial coordinate.

The description of "mixed" classical-quantum systems of Ref. [14] is based on a canonical formalism for describing statistical ensembles on configuration space (which is here the space spanned by the fields ϕ and ψ). The state of the system is described in terms of a probability P together with its canonically conjugate variable S, and the equations of motion are derived from an ensemble Hamiltonian. For the model that we are considering in this section, the equations for P and S are of the form

$$\dot{S} + \int dx\Big\{\frac{1}{2}\Big[\Big(\frac{\delta S}{\delta\phi}\Big)^2 + \Big(\frac{\delta S}{\delta\psi}\Big)^2\Big] + \frac{\hbar^2}{8}\Big[\frac{1}{P^2}\Big(\frac{\delta P}{\delta\psi}\Big)^2 - \frac{2}{P}\frac{\delta^2 P}{\delta\psi^2}\Big] - \frac{1}{2}\phi\phi'' + \frac{1}{2}\psi[-\psi'' + (1 + 2g\phi)m^2\psi]\Big\} = 0 \quad (37)$$

and

$$\dot{P} + \int dx\Big[\frac{\delta}{\delta\phi}\Big(P\frac{\delta S}{\delta\phi}\Big) + \frac{\delta}{\delta\psi}\Big(P\frac{\delta S}{\delta\psi}\Big)\Big] = 0, \quad (38)$$

where the overdot indicates a derivative with respect to the time coordinate.

Equation (37) has the form of a modified Hamilton-Jacobi equation, and in the limit where $\hbar$ goes to zero it becomes identical to the Hamilton-Jacobi equation that corresponds to Eq. (36). Equation (38) can be interpreted as a continuity equation for the probability P. There are some subtle issues concerning the physical interpretation of S within the formalism of ensembles in configuration space which we can discuss only briefly here. To maintain full generality, S should not be regarded as a field momentum density potential. In particular, for an ensemble of classical fields with uncertainty described by probability P, it will not be assumed that the field momentum density of a member of the ensemble is a well-defined quantity proportional to the functional derivative of S, as it is done in the usual deterministic interpretation of the Hamilton-Jacobi functional equation. This avoids forcing a similar deterministic interpretation in the quantum and quantum-classical cases. A deterministic picture can be recovered for classical ensembles precisely in those cases in which trajectories are operationally defined [14].

A. Solution for the noninteracting case

We first derive solutions for the noninteracting case (i.e., we set $g = 0$). We assume

$$S[\phi, \psi, t) = S^c[\phi, t) - E^q t, \quad (39)$$

$$P[\phi, \psi, t) = P^c[\phi, t)P^q[\psi], \quad (40)$$

where E^q is a constant. With this ansatz, (37) and (38) take the simpler form

$$-\dot{S}^c + E^q = \frac{1}{2}\int dx\Big\{\Big(\frac{\delta S^c}{\delta\phi_x}\Big)^2 - \frac{\hbar^2}{A^q}\frac{\delta^2 A^q}{\delta\psi_x^2} - \phi_x\phi_x'' - \psi_x[\psi_x'' - m^2\psi_x]\Big\}, \quad (41)$$

$$\dot{P}^c = -\int dx\Big[\frac{\delta}{\delta\phi_x}\Big(P^c\frac{\delta S^c}{\delta\phi_x}\Big)\Big] = 0, \quad (42)$$

where we have introduced $A^q \equiv \sqrt{P^q}$. To solve these equations, we will use standard techniques developed for the Schrödinger functional representation of quantum field theory [19,20]. Here, however, we work in a representation where the pair of canonically conjugate functionals P and S are taken as fundamental variables. Although it is possible to introduce a wave functional for the total system by means of a transformation of the form $\Psi = \sqrt{P}e^{iS/\hbar}$, there is no clear advantage in doing this as (41) and (42) do not become linear in this case.

We consider solutions which are of the form

$$S^c[\phi, t] = \frac{1}{2}\iint dydz\phi_y F_{yz}(t)\phi_z,$$

$$P^c[\phi, t] = N^c(t)e^{-(1/2)\iint dadb(\phi_a - \beta_a(t))K_{ab}(t)(\phi_b - \beta_a(t))},$$

$$P^q[\psi] = N^q\Big(\frac{1}{\hbar}\iint dydz\gamma_y G_{yz}\psi_z\Big)^2 e^{-(1/\hbar)\iint dadb\psi_a G_{ab}\psi_b}.$$

Equation (41) leads to

$$E^q - \frac{\hbar}{2}\int dxG_{xx} - \frac{\iiint dxdydz\gamma_y G_{yx}G_{xz}\psi_z}{\iint dydz\gamma_y G_{yz}\psi_z} = 0, \quad (43)$$

$$\dot{F}_{yz} + \int dxF_{yx}F_{xz} - \partial_z^2\delta(y-z) = 0, \quad (44)$$

$$-\int dxG_{yx}G_{xz} - [\partial_z^2 - m^2]\delta(y-z) = 0, \quad (45)$$

while Eq. (42) leads to

$$\frac{\dot{N}^c}{N^c} - \frac{1}{2}\iint dydx(\beta_y K_{yx}\beta_x)^{\cdot} + \int dxF_{xx} = 0, \quad (46)$$

$$\int dy(\beta_y K_{yz})^{\cdot} + \iint dydx\beta_y K_{yx}F_{xz} = 0, \quad (47)$$

$$-\frac{1}{2}\dot{K}_{yz} - \int dxK_{yx}F_{xz} = 0. \quad (48)$$

To solve these equations, we introduce (real) basis functions $f_x^{(k)}$ that satisfy the eigenvalue equation

$$-\partial_x^2 f_x^{(k)} = k^2 f_x^{(k)}$$

as well as orthonormality and completeness relations of the form $\int dxf_x^{(k)}f_x^{(m)} = \delta_{km}$ and $\sum_k f_x^{(k)}f_y^{(k)} = \delta(x-y)$. We have assumed discrete eigenvalues to simplify the notation, but the extension to continuous ones is straightforward. We now expand all quantities in terms of these basis functions and use these expressions in (43)–(48). Equations (44), (45), and (48) lead to representations for the kernels of the form

$$F_{xy} = -\sum_k k\tan(kt)f_x^{(k)}f_y^{(k)}, \quad (49)$$

$$G_{xy} = \sum_k \sqrt{k^2 + m^2}\, f_x^{(k)}f_y^{(k)}, \quad (50)$$

$$K_{yx} = \sum_k \frac{\tau_k}{\cos^2(kt)} f_x^{(k)}f_y^{(k)}, \quad (51)$$

where the τ_k are arbitrary constants. Equations (43) and (47) lead to representations for the functions γ_x and β_x and also determine the value of E^q,

$$\gamma_x = f_x^{(a)}, \quad (52)$$

$$E^q = \frac{\hbar}{2}\int dxG_{xx} + \sqrt{a^2 + m^2}, \quad (53)$$

$$\beta_x = \sum_k w_k \cos(kt)f_x^{(k)}, \quad (54)$$

where the w_k and a are arbitrary constants. The term on the right-hand side of (53) diverges and needs to be renormalized. Fortunately, the divergence can be easily isolated and removed in this formalism: it is present in the first term of the right-hand side. This prescription for dealing with the energy renormalization is essentially the same one that is used when dealing with the Schrödinger wave functional representation of quantum field theory [20]. In our case, the issue of the renormalization prescription is not a crucial one, and we will not discuss it further. Finally, N^q is proportional to a constant while N^c is determined by Eq. (46) and is formally given by

$$N^c \sim \frac{1}{\Pi_k \cos(kt)} \quad (55)$$

in the generic case where none of the τ_k vanishes.

The physical interpretation of these solutions can be summarized briefly as follows. The solution $\{S^c[\phi, t), P[\phi, \psi, t)\}$ provides the field-theoretic generalization of a solution which describes a particular ensemble in configuration space for a one-dimensional classical harmonic oscillator (i.e., a one-dimensional oscillator state which is prepared with zero momentum and no momentum uncertainty, but localized in space—see Appendix A for details); $A^q[\psi]$ is the Schrödinger wave functional for a one-particle state specified by the eigenfunction $\gamma_x = f_x^{(a)}$ and of energy E^q [20].

B. Solution for the interacting case ($g \ll 1$)

We now want to turn on the interaction and consider the case where $g \ll 1$. A general solution to (37) and (38) valid to first order in g seems quite complicated, although in principle possible. We derive here a solution that is correct to first order in g for a particular class of states. In this solution, the expression for S, given by (39), remains of the same form, but the expression for P, given by (40), is modified to $P[\phi, \psi, t) = P^c[\phi, t)P^{cq}[\phi, \psi]$ with

$$P^{cq}[\phi, \psi] = N^q[\phi]\Big(\frac{1}{\hbar}\iint dydz\gamma_y G_{yz}[\phi]\psi_z\Big)^2 \times e^{-(1/\hbar)\iint dadb\psi_a G_{ab}[\phi]\psi_b}.$$

That is, we now allow the kernel G_{xy} and the normalization factor N^q to be functionals of ϕ. With this new ansatz, (37) and (38) lead to a set of equations similar to the set (43)–(48) that we derived previously. Note that we do not distinguish explicitly between quantities that are evaluated with $g = 0$ and with $g \neq 0$ in the rest of this section in order not to clutter the notation.

We first consider the three equations derived from (37): Equations (43) and (44) remain as before, but (45) is replaced by

$$-\int dx G_{yx}G_{xz} - [\partial_z^2 - (1+2g\phi_z)m^2]\delta(y-z) = 0. \tag{56}$$

A solution of (56) that is correct to first order in g is given by

$$G_{xy}[\phi] = \sum_k \sqrt{k^2 + m^2(1+2gI^k)}\, g_x^{(k)} g_y^{(k)} \simeq \sum_k \sqrt{k^2+m^2}\left(1 + \frac{gI^k}{k^2/m^2+1}\right) g_x^{(k)} g_y^{(k)},$$

where the dependence of $G_{xy}[\phi]$ on ϕ_x comes in through

$$I^k = \int dx f_x^{(k)} \phi_x f_x^{(k)}.$$

The $g_x^{(k)}$ are modified eigenfunctions determined by first-order perturbation theory,

$$g_x^{(k)} = f_x^{(k)} + \sum_{n\neq k} \epsilon^{(kn)} f_x^{(n)}$$

with

$$\epsilon^{(kn)} = 2gm^2 \sum_{n\neq k}\left[\frac{\int dx f_x^{(k)} \phi_x f_x^{(n)}}{k^2 - n^2}\right] \ll 1.$$

Equation (44) is independent of both g and G_{xy} and therefore the previous solution for F_{xy}; equation (49), remains valid. To solve (43), we require a γ_x that satisfies the integral equation $\int dx \gamma_x G_{xy} = \lambda \gamma_y$ where λ is a constant. The expression

$$\gamma_x = f_x^{(a)} + \sum_{n\neq a} \epsilon^{(an)} f_x^{(n)}$$

is correct to first order in g. We also need to ensure that E^q remains a constant (i.e., independent of ϕ_x). This requirement is satisfied if we restrict to values of a that are large enough to have $a^2/m^2 \sim 1/g$. In this case, the correction to the quantum energy E^q becomes negligible (of order g^2) and can be ignored, since

$$E^q = \frac{\hbar}{2}\int dx G_{xx} + \sqrt{a^2+m^2}\left(1 + \frac{gI^a}{a^2/m^2+1}\right) \simeq \frac{\hbar}{2}\int dx G_{xx} + \sqrt{a^2+m^2}.$$

Therefore, our solution is correct to first order in g provided the quantized field is in a state of high enough energy.

We now consider the three equations derived from (38): Equations (46) and (48) remain as before, but (47) is replaced by

$$\int dy (\beta_y K_{yz})^{\cdot} + \iint dy dx \beta_y K_{yx} F_{xz} + \frac{\iint dy dx \frac{\delta N^q}{\delta G_{yx}} \frac{\delta G_{yx}}{\delta \phi_z}}{N^q} - \frac{2\iint dy dx \gamma_y \frac{\delta G_{yx}}{\delta \phi_z} \psi_x}{\iint dy dx \gamma_y G_{yx} \psi_x} + \frac{1}{\hbar}\iint dy dx \psi_y \frac{\delta G_{yx}}{\delta \phi_z} \psi_x = 0. \tag{57}$$

The last three terms in (57) are all of order g, and the correction due to these terms becomes negligible (of order g^2) if we restrict to gravitational waves where the constants $\tau_k \sim 1/g$ in Eq. (51). Under this assumption, the expressions for K_{xy}, β_x and N^c given by (51), (54), and (55) all remain valid. Therefore, our solution is correct to first order in g provided we restrict to gravitational waves ϕ_x that are sharply peaked about β_x.

C. Discussion

We have considered a classical gravitational wave interacting with a quantum field in the context of a two-dimensional model and we have derived an explicit solution for this "mixed" classical-quantum system. Our solution is valid to first order in the coupling parameter g and for a particular class of states. While the argument of Eppley and Hannah is based on supposed inconsistencies that would arise when trying to couple a classical gravitational wave to a quantum system, the results of this section suggest that there is no fundamental principle that excludes such systems. This is further evidence that the argument for quantizing gravity cannot be based on the claim that the nonquantization of gravity would lead to logical inconsistencies of this sort. It is of interest to discuss briefly one particular aspect of the solution derived here. In the noninteracting case, the probability $P[\phi, \psi, t)$ of the total system is the product of the probabilities of each of the subsystems, as expressed by (40), and this means that the uncertainty associated with the quantized scalar field cannot affect any averages that we calculate for observables of the classical gravitational wave (in particular, estimates of uncertainties such as the root mean square deviation will be independent of the state of the quantized scalar field). But once the interaction is turned on, and, in particular, as we incorporate corrections that are of higher order in g, the uncertainty of the observables of the classical gravitational wave will depend on the particular state of the quantized scalar field since the probability of the total system will be a rather complicated function of the fields which will no longer factor into a simple product. This confirms one of the results obtained in the previous section, that uncertainty is transferred from the quantum system to the classical system as a result of the interaction. Since the model of interacting classical-quantum systems discussed in this section provides an explicit counterexample to DeWitt's claim that "...the quantization of a given system implies also the quantization of any other system to which it can be coupled [...] therefore, the quantum theory must immediately be extended to all physical systems, including the gravitational field" [3], it will be useful to examine DeWitt's argument to clarify this issue.

V. CRITIQUE OF DEWITT'S ARGUMENT

In order to explain and present DeWitt's argumentation, it is necessary to introduce an appropriate concept used by him: the Peierls brackets. The Peierls brackets are a generalization of the ordinary Poisson brackets. A generalization because they directly follow from the action of the considered system and do not require the canonical coordinates and momenta to be defined in advance. But they are nevertheless identical with the usual Poisson brackets for standard canonical systems (if it is possible to arrive at an unconstrained Hamiltonian formalism corresponding to the initial Lagrangian formalism, which is the case for nonsingular Langrange functions). This fact allows one to apply the correspondence principle, that is, to introduce commutators, to the measurement analysis, which in turn enabled DeWitt to obtain his final result (67) below.

To explain the meaning of the Peierls brackets, it is convenient to recall the formalism of Sec. III. Consider a system $S[\phi]$. Its equation of motion may be solved by ϕ_0. A change of the action functional S will in general lead to an equation of motion whose solution deviates from ϕ_0. Under the requirement that this change is small, it is natural to expand the action functional around the free solution. Omitting higher terms allows one to calculate the deviation $\delta\phi$ of the system variable. The formal solution via the Green-function method reads

$$\delta\phi^j = gG^{ij}\Omega_{,i},$$

where G^{ij} denotes the Green function in the condensed notation. The additional action term not only produces disturbances within the dynamical variable ϕ_0, but also in any observable B built out of these. The expansion of B up to first order in g leads to

$$B[\phi_0 + \delta\phi] \approx B[\phi_0] + B_{,i}\delta\phi^i,$$

where $B_{,i}\delta\phi^i$ is formally given by

$$B_{,i}\delta\phi^i = gB_{,i}G^{ij}\Omega_{,j} \equiv \delta_\Omega B.$$

Before we introduce the Peierls bracket, it is convenient to define the operation

$$D_\Omega B \equiv \lim_{g\to 0}\frac{1}{g}\delta_\Omega B = B_{,i}G^{ij}\Omega_{,j}.$$

To comply with condition (4), one can choose the retarded Green function for the calculation of $\delta\phi$. Doing this leads to the interpretation of $D_\Omega B$ as the retarded change which Ω causes within B.

The Peierls brackets are now defined as the difference between the effects which two quantities, say A and B, have on each other in the sense described above,

$$(A, B) \equiv D_A B - D_B A.$$

A formal calculation for the small disturbance of the apparatus variable θ yields

$$\delta\theta^J = gG^{IJ}(\Omega_{,I} + \Omega_{,Ii}\delta\phi^i)$$

as a solution of (9). To approximate the disturbance of any *apparatus* observable A, one can use a procedure analogous to the one used above for system observables B; that is, approximate $A[\theta] \approx A[\theta_0] + \delta A$ with

$$\begin{aligned}\delta A &= A_{,I}\delta\theta^I = gA_{,I}G^{IJ}(\Omega_{,J} + \Omega_{,Ji}\delta\phi^i)\\ &= gA_{,I}G^{IJ}\Omega_{,J} + gA_{,I}G^{IJ}\Omega_{,Ji}\delta\phi^i\\ &= gD_\Omega A + g^2A_{,I}G^{IJ}\Omega_{,Ji}G^{ij}\Omega_{,j}\\ &= gD_\Omega A + g^2(D_\Omega A)_{,i}G^{ij}\Omega_{,j}\\ &\approx gD_\Omega A + g^2D_\Omega(D_\Omega A),\end{aligned} \tag{58}$$

where the omission of the $(D_\Omega A)_{,I}G^{IJ}\Omega_{,J}$ term in the last line is justified by choosing the apparatus to be "macroscopic" compared to the system (see Sec. III).

Given a particular choice of apparatus observable, one would like to use the expression for δA to express a system observable in terms of the "experimental data." Since this is not possible in general, DeWitt considered the following special case. If the coupling term Ω is chosen so that it satisfies

$$D_\Omega A = s \tag{59}$$

for a particular system observable s, (58) becomes

$$\delta A = gs + g^2D_\Omega s.$$

Then, if $D_\Omega s$ only depends on apparatus variables, s can be expressed in terms of experimental data,

$$s = \frac{\delta A}{g} - gD_\Omega s. \tag{60}$$

Since s is a function of the experimental data, there will be an uncertainty associated with s that reflects limited knowledge of the apparatus quantities. If, say, A and $D_\Omega s$ are only known up to ΔA and $\Delta D_\Omega s$, the uncertainties will propagate as

$$\Delta s^2 = \frac{\Delta A^2}{g^2} + g^2(\Delta D_\Omega s)^2.$$

If this is minimized with respect to the coupling constant g, it becomes

$$\Delta s = \sqrt{2\Delta A\Delta D_\Omega s}. \tag{61}$$

This equation is not very interesting if one remains at the classical level, because both ΔA and $\Delta D_\Omega s$ can be made arbitrarily small and hence s can be measured with arbitrary accuracy. But if one refers to quantum mechanics, limitations for the product of uncertainties of conjugate observables arise. DeWitt noticed how to reformulate (61) to contain exactly such a product; that is, to choose

$$\Omega = sC, \tag{62}$$

where C is an apparatus variable conjugate to A, that is, with $(A, C) = 1$. Using the identity $D_{r(\phi)s(\theta)}t(\phi) = s(\theta) \times (D_{r(\phi)}t(\phi))$ enables (61) to be rewritten as

$$\Delta s = \sqrt{2\Delta A \Delta D_\Omega s} = \sqrt{2\Delta A \Delta D_{sC} s} = \sqrt{2\Delta A \Delta (C D_s s)} \approx \sqrt{2\Delta A \Delta C |D_s s|}, \tag{63}$$

provided one assumes $D_s s$ to be approximately constant. The resulting product of uncertainties ΔA and ΔC leads to an interesting result if one applies a quasiclassical uncertainty principle to it. That is, to require the product of the uncertainties of two conjugated observables to be limited by the quantum-mechanical uncertainty principle $\Delta A \Delta C \geq \hbar/2$, and thus

$$\Delta s \geq \sqrt{\hbar |D_s s|}. \tag{64}$$

A possible interpretation of (64) could be that the achievable accuracy within the measurement of even one single observable is limited, unlike in established quantum theory where every *single* quantity is assumed to be detectable with arbitrary accuracy. This, however, would contradict the commonly accepted principle of the determinability of a single observable. A way of overcoming this apparent contradiction was given by Bohr and Rosenfeld in their work on the measurability of the quantized electromagnetic field [1]. Bohr and Rosenfeld showed that the reason for the apparent contradiction was that the measurement was not performed "carefully enough." They found that the accuracy improved immediately if a particular term was added to the total action: when considering a measurement of the electromagnetic field with the help of a test body, they found that they had to add another force to the system. In the language of DeWitt's approach, this means adding a so-called "compensation term" to the action:

$$S + \Sigma + g\Omega \rightarrow S + \Sigma + g\Omega - \frac{1}{2} g^2 D_\Omega \Omega.$$

In virtue of this compensation mechanism, the second term of (60) drops out and (63) becomes

$$\Delta s = \frac{\Delta A}{g},$$

which contains no fundamental limitation on the measurement of s anymore.

In fact, for the simultaneous measurement of *two* observables nothing essentially or conceptually new has to be added. The only thing that is required is the addition of a further coupling term corresponding to the measurement of the second quantity s_2, say Ω_2, and the corresponding compensation term to the action. The total coupling then reads

$$g(\Omega_1 + \Omega_2) - \underbrace{\frac{1}{2} g^2 D_{\Omega_1+\Omega_2}(\Omega_1 + \Omega_2)}_{\text{compensation term}}.$$

To ensure that the resulting equations are resolvable with respect to experimental data one has, analogously to (59) and (62), to demand

$$\Omega_1 = s_1 B_1 \quad \text{with} \quad (B_1, A_1) = 1,$$
$$\Omega_2 = s_2 B_2 \quad \text{with} \quad (B_2, A_2) = 1,$$
$$D_{\Omega_1} A_1 = s_1,$$
$$D_{\Omega_2} A_2 = s_2.$$

After the introduction of classical uncertainties for the product of Δs_1 and Δs_2 (minimized with respect to g), this leads to

$$\Delta s_1 \Delta s_2 = \frac{1}{2}(\Delta A_1 \Delta(\Omega_1, s_1) + \Delta A_2 \Delta(\Omega_2, s_2)). \tag{65}$$

Together with

$$(A_1, (\Omega_1, s_1)) = -(\Omega_1, (s_1, A_1)) - (s_1, (A_1, \Omega_1)) = (s_1, s_2),$$
$$(A_2, (\Omega_2, s_2)) = -(\Omega_2, (s_2, A_2)) - (s_2, (A_2, \Omega_2)) = (s_2, s_1),$$

it is possible to arrive at a, as DeWitt called it, *universal principle*. This is done by assuming the existence of a *fundamental principle* which limits the accuracy of two, simultaneously considered, apparatus observables by

$$\Delta A \Delta B \geq \frac{\hbar}{2} |(A, B)|. \tag{66}$$

For example, this could be A_1 and A_2 and, if independent of the system trajectories, (Ω_1, s_1) and (Ω_2, s_2). This fundamental principle together with (65) would result in the corresponding principle for the system observables

$$\Delta s_1 \Delta s_2 \geq \frac{\hbar}{2} |(s_1, s_2)|, \tag{67}$$

which could be interpreted as a *universal principle* valid for any two observables of any system coupled to an apparatus obeying (66).

However, one cannot conclude from this inequality that the system that is being measured *has* to be quantized. The only place where quantum-mechanical considerations appear explicitly is in the observation that the quantization rule $[A, B] = i\hbar(A, B)$ applied to apparatus observables automatically results in an uncertainty relation for A and B that satisfies (66), and therefore that an apparatus that is quantum mechanical in nature would satisfy his fundamental principle. DeWitt's derivation, however, makes use of classical Poisson brackets only, which indicates that quantum considerations do not play a fundamental role in his calculation. His argument is, after all, very general: it applies equally well to two classical systems that interact, or to a classical system that interacts with a quantum system, as long as all of his requirements are satisfied. Furthermore, one has to recall that the uncertainty relations have nothing to do with disturbances; they only express the limited applicability of classical concepts [7]. DeWitt's argument therefore does not provide a proof that the quantum theory must be extended to all physical systems.

VI. OUTLOOK

Inspired by the gedanken experiment proposed in [4], we have investigated a model in which a gravitational wave interacts with a mass cube. In order to discuss the mutual interaction, we have applied a DeWitt-type measurement analysis to the coupled system. We have found that if one system possesses some uncertainty, this uncertainty is necessarily transferred to the other system. It does not matter whether the uncertainty is of quantum-mechanical or classical origin. DeWitt's formalism is general enough to encompass both situations. We have also shown that some of the arguments in [4] are incorrect because they do not take into account the quantum entanglement which arises in the situation under study.

We have investigated a second model in which a classical gravitational wave interacts consistently with a quantized scalar field. This hybrid model provides a counterexample to the claim that a system coupled to a quantum system must necessarily also be of quantum nature. We have considered the argument made in [3] that the quantum theory must be extended to all physical systems, and have shown that this conclusion is not justified. The universal validity of quantum theory would in turn necessarily entail the quantization of the gravitational field, cf. also the remarks by Richard Feynman in this context, for example: "...if you believe in quantum mechanics up to any level then you have to believe in gravitational quantization ..." [21]. One cannot, therefore, conclude from these gedanken experiments alone, without assuming the universal validity of quantum theory, that gravity *must* be quantized. This is similar to the old analysis by Bohr and Rosenfeld about the measurability of the quantized electromagnetic field; this analysis does not show that the electromagnetic field must be quantized but that the results of the measurement analysis are in accordance with the quantum electrodynamical commutation relations, cf. [22]. Of course, by empirical arguments (e.g., the observed coupling of photons to matter) one knows that the electromagnetic interaction is of quantum nature. As long as quantum-gravitational experiments are not possible, analogous empirical arguments are unavailable for gravity; there exists, however, an experiment that falsifies (under some assumptions) the standard version of semiclassical gravity where classical gravity is coupled to the expectation value of a quantum energy-momentum tensor for matter [23].

From general arguments (singularity theorems, universality of gravity, unification), the quantization of gravity seems unavoidable [2], although there is no logical proof. The task is then to consider particular approaches to quantum gravity, such as loop quantum gravity or string theory, and to perform a quantum measurement analysis along the lines of [1] or [3]. This is, however, left to future publications.

ACKNOWLEDGMENTS

M. R. would like to thank Michael J. W. Hall for very helpful discussions. M. A. gratefully acknowledges stimulating discussions with Friedemann Queißer and Barbara Sandhöfer. C. K. and M. R. are, moreover, most grateful to Steve Carlip for a helpful and illuminating correspondence.

APPENDIX A: ANALOGY BETWEEN THE CLASSICAL SECTOR OF THE HYBRID MODEL AND THE HARMONIC OSCILLATOR

We consider here the interpretation of the solution $\{S^c[\phi, t), P^c[\phi, t)\}$ for the case $g \to 0$ which we derived in Sec. IV.

It will be helpful to consider first a simpler but closely related system, an ensemble in configuration space for the classical one-dimensional harmonic oscillator. The equations that describe the state of the system are the Hamilton-Jacobi equation and the continuity equation,

$$\dot{S}_{\rm osc} + \frac{1}{2}\left(\frac{\partial S_{\rm osc}}{\partial x}\right)^2 + \frac{1}{2}\omega^2 x^2 = 0,$$
$$\dot{P}_{\rm osc} + \frac{\partial}{\partial x}\left(P\frac{\partial S_{\rm osc}}{\partial x}\right) = 0.$$

The Hamilton-Jacobi equation has a solution of the form

$$S_{\rm osc}(x, t) = -\frac{\omega x^2}{2}\tan(\omega t).$$

Given $S_{\rm osc}$, the most general solution of the continuity equation is given by

$$P = \frac{1}{\cos(\omega t)} f\left(\frac{x}{\cos(\omega t)}\right),$$

where f is an arbitrary function. In particular, there is a Gaussian solution,

$$P_{\rm osc}(x, t) = \sqrt{\frac{\tau}{2\pi \cos^2(\omega t)}} \times \exp\left\{-\frac{1}{2}\frac{\tau}{\cos^2(\omega t)}(x - w\cos(\omega t))^2\right\},$$

where w and τ are constants.

At $t = 0$, the state has average momentum

$$\langle p\rangle(0) \equiv \int dx P(x,0)\frac{\partial S_{\rm osc}}{\partial x}(x,0) = 0$$

and no momentum uncertainty (i.e., $\Delta p(0) = 0$); average position $\langle x\rangle(0) = w$ and position uncertainty $\Delta x(0) = \tau^{-1/2}$. The solution therefore represents a state which is prepared with zero momentum and no momentum uncertainty, but localized in space. Furthermore, when $\omega t \rightarrow \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\}$, $P_{\rm osc}(x,t)$ becomes a delta function and the particle is found at $x = 0$ with probability one. The total energy of the state is given by

$$E = \left\langle -\frac{\partial S_{\rm osc}}{\partial t}\right\rangle = \frac{1}{2}\omega^2\tau^{-1}.$$

A formal solution of the *quantum* analogue of this problem is obtained by taking the limit $\tau \rightarrow 0$. One can check that the wave function $\psi = \sqrt{P_{\rm osc}}e^{iS_{\rm osc}/\hbar}$ is then a solution of the Schrödinger equation, and that it corresponds to a state which is prepared with zero momentum and no momentum uncertainty but completely delocalized.

Consider now the solution $S^c[\phi, t)$ and $P^c[\phi, t)$ of Sec. IV. To see the analogy to the one-dimensional harmonic oscillator, evaluate $S^c[\phi, t)$ and $\frac{\delta S^c[\phi,t)}{\delta\phi_x}$ using the representation $\phi_x = \sum_k a_k f_x^{(k)}$and $F_{yx} = -\sum_k k\tan(kt)f_y^{(k)}f_z^{(k)}$, where the $f_y^{(k)}$ are the basis functions introduced previously. This leads to

$$S^c[\phi, t) = -\sum_k \frac{ka_k^2}{2}\tan(kt)$$
$$\frac{\delta S^c[\phi, t)}{\delta\phi_x} = -\sum_k ka_k\tan(kt)f_x^{(k)}.$$

Furthermore, $P^c[\phi, t)$ is given by

$$P^c[\phi, t) = N^c(t)e^{-(1/2)\iint dadb(\phi_a-\beta_a)K_{ab}(t)(\phi_b-\beta_a)}$$

with

$$K_{yx} = \sum_k \frac{\tau_k}{\cos^2(kt)}f_x^{(k)}f_y^{(k)}, \qquad \beta_x = \sum_k w_k\cos(kt)f_x^{(k)},$$
$$N^c \sim \frac{1}{\Pi_k\cos(kt)}.$$

Comparing these expressions, one can see that the solution $S^c[\phi, t)$ is analogous to $S_{\rm osc}(x,t)$ while $P^c[\phi, t)$ is analogous to $P_{\rm osc}(x,t)$.

Just as in the case of the one-dimensional harmonic oscillator, a formal solution of the corresponding quantized field equations is obtained by taking the limit $\tau \rightarrow 0$.

APPENDIX B: SCATTERING OF CLASSICAL AND QUANTUM NONRELATIVISTIC PARTICLES THAT INTERACT GRAVITATIONALLY

Consider a gedanken experiment that involves the scattering of two nonrelativistic particles, one a classical particle of mass M (the projectile) and the other one a quantum particle of mass m (the target). The interaction is assumed to be caused by the gravitational attraction between the two particles.

This gedanken experiment is most interesting when the initial amplitude for the quantum particle (i.e., as $t \rightarrow -\infty$ when the two particles are very far from each other so that the interaction term can be neglected) has two peaks of equal magnitude, A and B, that are well separated. Then, in a measurement, the quantum particle will be "found" at the location of peak A (with probability 1/2) or at the location of peak B (with probability 1/2). (In the Everett interpretation, the measuring agency is entangled with these separate possibilities.) Consider the case where the classical particle comes very close to peak A and remains at all times at a very large distance from peak B.

What happens when the classical particle scatters from the quantum particle? A "naive approach" suggests three possible mutually exclusive outcomes: (a) the quantum particle is found at the location of peak A and the classical particle comes very close to the quantum particle of mass m: the scattering is very strong; (b) the quantum particle is found at the location of peak B and the classical particle never comes very close to the quantum particle of mass m: the scattering is very weak; (c) as in "semiclassical gravity", cf. [23], the classical particle "sees" a mass $m/2$ at the location of peak A and it "sees" a mass $m/2$ at the location of peak B: the scattering is about one half of what one would calculate under assumption (a).

Each of these possibilities seems unrealistic. The source of difficulties is, clearly, the use of the "naive approach": it is impossible to derive any reasonable conclusion without introducing a concrete model for this mixed classical-quantum system. We show below that none of these outcomes is supported by a more careful analysis.

The formalism of configuration space ensembles allows a general and consistent description of interacting classical-quantum systems [14,24], and we will now apply it to this problem. Using this formalism, it is straightforward to set up the equations that are needed to describe the gedanken experiment. Let q denote the configuration space coordinate of a quantum particle of mass m, and x denote the configuration coordinate of a classical particle of mass M, and consider an interaction term of the form

$$V(q, x) = -G\frac{mM}{|x - q|}.$$

Within this formalism, the equations that describe the system are derived from the Hamiltonian

$$\tilde{H}_{QC}[P, S] = \int dq dx P\left[\frac{|\nabla_q S|^2}{2m} + \frac{|\nabla_x S|^2}{2M} - G\frac{mM}{|x-q|} + \frac{\hbar^2}{4}\frac{|\nabla_q \log P|^2}{2m}\right] \tag{B1}$$

and take the form

$$\frac{\partial P}{\partial t} = -\nabla_q \cdot \left(P\frac{\nabla_q S}{m}\right) - \nabla_x \cdot \left(P\frac{\nabla_x S}{M}\right), \tag{B2}$$

$$-\frac{\partial S}{\partial t} = \frac{|\nabla_q S|^2}{2m} + \frac{|\nabla_x S|^2}{2M} - \frac{\hbar^2}{2m}\frac{\nabla_q^2 P^{1/2}}{P^{1/2}} - G\frac{mM}{|x-q|}. \tag{B3}$$

Equations of this type have been studied in detail in Refs. [14,24], and we refer the reader to these papers; here we summarize a few aspects of the formalism. Moreover, Sec. IV presents a detailed discussion of a model based on this approach.

The state of the mixed classical-quantum system is described by two functions, $P(x, q, t)$ and $S(x, q, t)$, which have the following physical interpretation: P is a probability density defined over configuration space, and $\frac{1}{M}P\nabla_x S$ ($\frac{1}{m}P\nabla_q S$) is a probability current associated with the classical (quantum) particle. Equation (B2) has the form of a continuity equation, and (B3) has the form of a modified Hamilton-Jacobi equation. One may introduce a wave function of the form $\psi = \sqrt{P}e^{iS/\hbar}$ which will satisfy a nonlinear generalization of the Schrödinger equation.

Already the equations show some features that tell us what to expect of the solutions:

(1) When $|\frac{\hbar^2}{2m}\frac{\nabla_q^2 P^{1/2}}{P^{1/2}}| \ll |G\frac{mM}{|x-q|}|$, we can neglect the term proportional to $\frac{\hbar^2}{2m}\frac{\nabla_q^2 P^{1/2}}{P^{1/2}}$ in (B3). This will be the case when the mass m of the quantum system is large enough for this inequality to be valid. But if this is the case, (B3) reduces to the classical Hamilton-Jacobi equation and we end up with the equations of a classical-classical system. This shows that the formalism has the correct classical limit.

(2) If the interaction term $V(q, x) = -G\frac{mM}{|x-q|}$ that appears in (B3) is very small (say at $t \to -\infty$ when the two particles are very far from each other) and there is no initial correlation between the particles, then the nonlinearity in the equations of the quantum particle will amount to only a small perturbation and the superposition principle will be valid for the quantum sector to a very good approximation. This means that the formalism has the correct quantum limit. Notice, however, that the equations are nonlinear when the interaction term is taken into consideration: the quantum superposition principle breaks down when the interaction between the classical and quantum particles is strong.

(3) Suppose that the systems were independent before the interaction (i.e., at $t \to -\infty$ when the two particles are very far from each other). That amounts to postulating initial conditions

$$P^{(-\infty)}(x, q, t) = P_C^{(-\infty)}(x, t)P_Q^{(-\infty)}(q, t)$$
$$S^{(-\infty)}(x, q, t) = S_C^{(-\infty)}(x, t) + S_Q^{(-\infty)}(q, t).$$

Before the interaction, the combined classical-quantum system breaks up naturally into classical and quantum sectors. However, after the interaction, the two systems will not be independent anymore, and we will have

$$P^{(+\infty)}(x, q, t) \neq P_C^{(+\infty)}(x, t)P_Q^{(+\infty)}(q, t)$$
$$S^{(+\infty)}(x, q, t) \neq S_C^{(+\infty)}(x, t) + S_Q^{(+\infty)}(q, t).$$

Now the combined classical-quantum system will no longer have well-defined classical and quantum sectors. This means that a measurement of the position of either the classical or quantum particle will force a change of the fields $\{P(x, q, t), S(x, q, t)\}$ that describe the *total* system. In other words, the classical and quantum sectors have become entangled.

(4) Cases (a) and (b) of the "naive approach" are "either-or" outcomes that implicitly assume that there is no entanglement in the case of mixed classical-quantum systems. But we have just shown that a consistent theory of interacting classical-quantum systems leads to final states that *are* entangled. We have to conclude therefore that these two outcomes are not supported by a more careful analysis and that they must be rejected. Furthermore, the theory that we have used is fundamentally different from standard "semiclassical gravity." Therefore, case (c) of the "naive approach" is also excluded.

The predictions of the mixed classical-quantum system described by (B2) and (B3) differ then substantially from the outcomes predicted using the "naive approach." The qualitative features of the solution can be determined without carrying out a detailed calculation. To do this, we introduce center-of-mass and relative coordinates

$$\bar{x} := \frac{mq + Mx}{m + M}, \qquad r := q - x,$$

and the total mass M_T and relative mass μ defined by

$$M_T := m + M, \qquad \mu := \frac{mM}{m + M}.$$

Rewriting the Hamiltonian (B1) in terms of these new coordinates leads to [14]

$$\tilde{H}_{QC} = \int d\bar{x} dr P\left[\frac{|\nabla_{\bar{x}} S|^2}{2M_T} + \frac{\hbar^2 m}{4(m+M)}\frac{|\nabla_{\bar{x}} \log P|^2}{2M_T}\right] + \int d\bar{x} dr P\left[\frac{|\nabla_r S|^2}{2\mu} + \frac{\hbar^2 M}{4(m+M)}\frac{|\nabla_r \log P|^2}{2\mu} - G\frac{\mu M_T}{|r|}\right] - \frac{\hbar^2}{4(m+M)}\int d\bar{x} dr \frac{\nabla_{\bar{x}} P . \nabla_r P}{P}. \tag{B4}$$

To interpret this expression, compare to the Hamiltonian of a purely quantum system, which is of the general form [14]

$$\tilde{H}_Q[P, S] = \int dqP\left[\frac{|\nabla S|^2}{2m} + \frac{\hbar^2}{4}\frac{|\nabla \log P|^2}{2m} + V(q)\right].$$

We see then that the Hamiltonian $\tilde{H}_{QC}$ in (B4) is the sum of three terms: (i) a quantumlike term corresponding to free center-of-mass motion but with a rescaled Planck constant

$$\hbar_{\bar{X}} := [m/(m+M)]^{1/2}\hbar;$$

(ii) a quantumlike term corresponding to relative motion in a potential $V(r) = -G\frac{\mu M_T}{|r|}$ but with a rescaled Planck constant

$$\hbar_R := [M/(m+M)]^{1/2}\hbar;$$

and (iii) an intrinsic interaction term. One would expect then a solution with *qualitative* features that resemble those of a purely quantum system, however with important modifications induced by the (iii) term and the rescaling of the Planck constant in (i) and (ii). Regarding conservation laws, we point out that the equations of motion (B2) and (B3) are invariant under Galilean transformations [14]; therefore the usual conservation laws that follow from Galilean invariance apply. A more detailed analysis of this particular mixed classical-quantum system is in preparation.

(5) As mentioned before, the wave function ψ associated with this mixed classical-quantum system obeys a nonlinear generalization of the Schrödinger equation and the theory can be seen as a nonlinear modification of quantum mechanics. Therefore, the concepts that are used to describe measurements in quantum theory will have their counterparts in this formalism, although perhaps with limited validity. For example, if we want to modify standard quantum theory by introducing a wave function collapse (i.e., a discontinuous change in the wave function due to an observation), then we will need to assume that there is something equivalent here that applies to the *whole* system (i.e., a discontinuous change in $\{P(x, q), S(x, q)\}$ due to an observation).

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