

Numerical performance of the parabolized ADM formulation of general relativityVasileios Paschalidis,¹ Jakob Hansen,¹ and Alexei Khokhlov^{1,2}¹*Department of Astronomy and Astrophysics, The University of Chicago, 5640 S. Ellis Avenue, Chicago, Illinois 60637, USA*²*Enrico Fermi Institute, The University of Chicago, 5640 S. Ellis Avenue, Chicago, Illinois 60637, USA*

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In a recent paper [Vasileios Paschalidis, *Phys. Rev. D* **78**, 024002 (2008).], the first coauthor presented a new parabolic extension (PADM) of the standard $3 + 1$ Arnowitt, Deser, Misner (ADM) formulation of the equations of general relativity. By parabolizing first-order ADM in a certain way, the PADM formulation turns it into a well-posed system which resembles the structure of mixed hyperbolic—second-order parabolic partial differential equations. The surface of constraints of PADM becomes a local attractor for all solutions and all possible well-posed gauge conditions. This paper describes a numerical implementation of PADM and studies its accuracy and stability in a series of standard numerical tests. Numerical properties of PADM are compared with those of standard ADM and its hyperbolic Kidder, Scheel, Teukolsky (KST) extension. The PADM scheme is numerically stable, convergent, and second-order accurate. The new formulation has better control of the constraint-violating modes than ADM and KST.

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I. INTRODUCTION

For years people have tried to obtain analytic solutions of the complex field equations of Einstein's general theory of relativity (GR). Apart from a few cases where symmetry is invoked, it is almost impossible to analyze the complicated dynamics in the strong gravitational field regime as described by GR. Approximation methods have been developed over the course of time, but the most promising tool for tackling problems such as gravitational waves arising from binary black hole (BBH) or binary neutron star mergers, gravitational collapse, etc., is numerical relativity.

In the past few years remarkable progress has been made towards achieving long term and stable evolution of the Einstein equations. Recently, particular cases of the BBH problem were solved [1–7]. Despite this remarkable achievement, the general problem of long term and stable evolution of the GR equations remains open and there is still much work left to be done. There is no theory or prescription to choose what formulation(s) and what gauge conditions are suitable for the numerical solution of a given problem. For example, it is well known that the Baumgarte, Shapiro, Shibata, Nakamura (BSSN) formulation [8], although successful with BBHs, has not been successful with certain spacetimes [9]. In addition to that, there does not seem to exist a definitive explanation of why the approaches of [1,2,5] perform so well when contrasted to previous efforts. Furthermore, there are astrophysical and theoretical problems of great interest for which the formulations above have not been applied yet and it is not known whether they will prove successful in such cases. Such problems are the study of the internal structure of black holes and astrophysical phenomena, where except for black holes matter is also involved.

The numerical integration of the Einstein equations is not an easy task because the computations can become unstable and an exponential blowup of the numerical error may occur, even when the formulation employed admits a well-posed initial value problem. If the numerical techniques employed and gauge and boundary conditions chosen do not suffer from pathologies, perhaps the most important source that can potentially lead to instabilities during a free evolution is the growth of the constraint-violating modes. Over the years, several methods have been proposed to deal with this last type of instability [10–23].

Of all these approaches, the one which has attracted most attention in recent years takes advantage of the fact that, in the ideal case where the constraint equations are satisfied, one has the freedom to add combinations of the constraint equations to the right-hand side (RHS) of the evolution equations of a given formulation. By virtue of this freedom, it is also possible to introduce in the system of evolution equations terms which act as “constraint drivers” and turn the constraint surface into an attractor. This technique is nowadays usually termed as “constraint damping.”

The goal of many formulations in numerical relativity has been to incorporate such drivers in symmetric or strongly hyperbolic systems without changing the principal part of the evolution equations [17,18]. However, not all formulations with such drivers have been as successful as the generalized harmonic decomposition [1]. A possible explanation is that the constraint satisfying modes evolve differently with different but equivalent formulations. Another explanation may be that some formulations may require more efficient damping of the constraint-violating perturbations that are present in numerical simulations and lead to all sort of instabilities.

One way to achieve efficient damping is to construct formulations of GR that under free evolution force all constraint-violating modes to evolve according to parabolic equations. Parabolic equations are known for their damping and smoothing properties [24] and this is of extreme importance in numerical relativity. In [25] this goal was achieved by the construction of an evolution system based on the first-order form of the standard Arnowitt, Deser, Misner (ADM) formulation [26], through the addition of appropriate combinations of the derivatives of the constraints and the constraints themselves at the RHS of the ADM evolution equations. We call this evolution system the parabolized ADM (PADM) formulation throughout this work. It was shown in [25] that the evolution of the constraint equations with the PADM formulation are second-order parabolic, independently of the gauge conditions employed. This in turn implies that the constraint surface becomes a local attractor. It was finally proved, that for a certain family of algebraic gauges the PADM system admits a well-posed Cauchy problem.

The purpose of this work is to describe a numerical scheme for solving the PADM equations, test the accuracy and stability of the PADM system, and compare the PADM formulation with the first-order ADM and the Kidder, Scheel, Teukolsky (KST) [27] formulations.

The first-order ADM system in conjunction with harmonic or “1 + log” slicing is strongly hyperbolic for the set of one-dimensional solutions considered in this work. Therefore, we choose the first-order ADM formulation as a basis for comparison. We choose the KST formulation because it is strongly hyperbolic and because the KST and the PADM systems are both extensions of the ADM evolution system, and we want to study how different extensions of the same base system perform numerically. Furthermore, the only method, to our knowledge, which has been employed to integrate the KST evolution equations, in the series of tests considered here, is pseudospectral methods. This work implements and demonstrates the numerical performance of the KST formulation with finite difference methods. A case study of the KST system in conjunction with finite difference methods has also been carried out in [28].

The comparison between the three formulations is carried out in a series of standard one- and two-dimensional tests, usually referred to as the “Apples with Apples” tests or the “Mexico City” tests [9]. There are four basic tests: (a) The evolution of small initial noise on flat spacetime data; (b) the evolution of one-dimensional and two-dimensional gauge waves; (c) the evolution of a small amplitude 1D and 2D gravitational waves; and (d) the evolution of polarized Gowdy waves.

This paper is organized as follows. In Sec. II, we briefly describe the first-order ADM, KST, and PADM formulations. In Sec. III we describe the numerical scheme we developed. In Sec. IV we describe the standard tests,

present the results of the numerical simulations we carried out, and compare the numerical performance of the aforementioned formulations. In Sec. V we study the convergence of the numerical schemes. We conclude this paper in Sec. VI.

II. FORMULATIONS

The four-dimensional theory of general relativity can be cast into a 3 + 1 decomposition of spacetime [26] by assuming that the spacetime can be foliated by a one-parameter family of spacelike hypersurfaces. The spacetime metric is then written in the following form:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (1)$$

where γ_{ij} is the positive definite 3-metric on the $t = \text{const}$ hypersurfaces, α is the lapse function, β^i the shift vector, and x^i are the spatial coordinates, $i = 1, 2, 3$.

A. The first-order ADM formulation

The ADM formulation consists of two subsets of equations. The first subset is that of the evolution equations, which describe how the dynamical variables evolve in time. The second subset is that of the constraint equations, which have to be satisfied for all times. The standard second-order ADM formulation [26] has as dynamical variables the 3-metric γ_{ij} and the extrinsic curvature K_{ij} of the 3D spacelike hypersurfaces.

The first-order ADM formulation is derived from the second-order one [25] by introducing additional dynamical variables,

$$D_{kij} \equiv \partial_k \gamma_{ij}, \quad (2)$$

and then deriving the evolution equations for D_{kij} . The dynamical variables of the first-order ADM system evolve according to the following equations:

$$\hat{\partial}_o \gamma_{ij} = -2\alpha K_{ij}, \quad (3)$$

$$\hat{\partial}_o K_{ij} = -\nabla_i \nabla_j \alpha + \alpha(R_{ij} + K K_{ij} - 2\gamma^{mn} K_{im} K_{jn}), \quad (4)$$

$$\hat{\partial}_o D_{kij} = -2\alpha \partial_k K_{ij} - 2K_{ij} \partial_k \alpha, \quad (5)$$

where γ_{ij} is the 3-metric, K_{ij} and $K = \gamma^{mn} K_{mn}$ are the extrinsic curvature and its trace respectively, ∇_i is the covariant derivative operator associated with the 3-metric, R_{ij} is the Ricci tensor associated with the 3-metric, and $\hat{\partial}_o = \partial_t - \mathcal{L}_\beta$, with $\mathcal{L}_\beta$ the Lie derivative along the shift vector β^i . In Eq. (4), all partial derivatives of the 3-metric have been replaced by D_{kij} .

The Lie derivatives of the dynamical variables are

$$\mathcal{L}_\beta \gamma_{ij} = \nabla_i \beta_j + \nabla_j \beta_i, \quad (6)$$

$$\mathcal{L}_\beta K_{ij} = (\nabla_i \beta^m) K_{mj} + (\nabla_j \beta^m) K_{mi} + \beta^m \nabla_m K_{ij}, \quad (7)$$

and

$$\begin{aligned} \mathcal{L}_\beta D_{kij} = & \beta^m \partial_m D_{kij} + D_{mij} \partial_k \beta^m + 2D_{km(i} \partial_{j)} \beta^m \\ & + 2\gamma_{m(i} \partial_{j)} \partial_k \beta^m. \end{aligned} \quad (8)$$

The set of constraint equations is

$$\mathcal{H} \equiv R + K^2 - K_{mn} K^{mn} = 0, \quad (9)$$

$$\mathcal{M}_i \equiv \nabla_m K^m_i - \nabla_i K = 0, \quad i = 1, 2, 3, \quad (10)$$

$$\mathcal{C}_{kij} \equiv \partial_k \gamma_{ij} - D_{kij} = 0, \quad (11)$$

where R is the trace of the three-Ricci tensor. Just like in the evolution equations, all partial derivatives of the 3-metric in (9) and (10) are replaced by D_{kij} . $\mathcal{H}$ is the Hamiltonian constraint, $\mathcal{M}_i$ are the momentum constraints, and $\mathcal{C}_{kij}$ are new constraints due to the introduction of D_{kij} . The equations as presented above do not include matter terms, because in this work we focus on vacuum solutions of the Einstein equations.

At this point we must make an important remark concerning the geometrical interpretation of the $\mathcal{C}_{kij}$ constraints. The covariant derivative of any purely spatial tensor A_{ij} is

$$\nabla_k A_{ij} = \partial_k A_{ij} - \Gamma^s_{ik} A_{sj} - \Gamma^s_{kj} A_{is}, \quad (12)$$

where, in the context of the first-order ADM formulation, Γ^k_{ij} is given by

$$\Gamma^k_{ij} = \frac{1}{2} \gamma^{kn} (D_{jin} + D_{ijn} - D_{nij}). \quad (13)$$

The covariant derivative of the 3-metric must be zero, but by virtue of (12) and (13) it is straightforward to show that

$$\nabla_k \gamma_{ij} = \mathcal{C}_{kij}. \quad (14)$$

Therefore, the geometrical interpretation of the $\mathcal{C}_{kij}$ constraints is that the covariant derivative is metric compatible, if and only if $\mathcal{C}_{kij} = 0$.

To close the system, Eqs. (3)–(5) must be supplemented with the gauge equations for the lapse function and the shift vector. Following [29], we write them in a general form as

$$\begin{aligned} F_a(x^b, \alpha, \beta^i, \partial_b \alpha, \partial_b \partial_c \alpha, \dots, \partial_b \beta^i, \dots, \gamma_{ij}, \partial_b \gamma_{ij}, \dots) = 0, \\ a, b, c = 0, \dots, 3, \quad i, j = 1, 2, 3. \end{aligned} \quad (15)$$

In this work we limit our consideration to algebraic gauges where the lapse function is of the form

$$\alpha = \alpha(\gamma), \quad (16)$$

where γ is the determinant of the 3-metric γ_{ij} , and the shift vector β^i may be either constant or a fixed function of the spacetime coordinates (t, x^i) . In [29,30] it was shown that

(16) makes the evolution well posed on the surface of constraints, if $A = \partial \ln \alpha / \partial \ln \gamma > 0$. Working with (16), we have the flexibility to use either “ $1 + \log$ ” slicing

$$\alpha = 1 + \ln \gamma \quad (17)$$

or a densitized lapse

$$\alpha = Q \gamma^\sigma, \quad (18)$$

where Q is constant or a fixed function of t, x^i , and σ the densitization parameter. If (17) is used, $A = 1/\alpha$. If (18) is employed, $A = \sigma$.

An important property of a formulation of GR is the structure of evolution equations of the constraints. The ADM constraint propagation system is weakly hyperbolic, which in turn implies that it admits growing polynomial modes [20].

B. The KST formulation

The strongly hyperbolic, four-parameter KST modification to the first-order ADM formulation [27] is

$$\partial_t K_{ij} = (\text{ADM}) + \rho \alpha \gamma_{ij} \mathcal{H} + \psi \alpha \gamma^{ab} \mathcal{C}_{a(ij)b}, \quad (19)$$

$$\partial_t D_{kij} = (\text{ADM}) + \eta \alpha \gamma_{k(i} \mathcal{M}_{j)} + \chi \alpha \gamma_{ij} \mathcal{M}_k, \quad (20)$$

where (ADM) stands for the RHS of the first-order ADM evolution equations and ρ, ψ, η , and χ are four parameters of the formulation. In addition to those modifications, Kidder, Scheel, and Teukolsky used a densitized lapse gauge (18).

In this work we use the KST system in conjunction with the more general gauge condition (16). We do so because in [25] it was shown that the KST formulation with (16) is strongly hyperbolic, if the KST parameters are

$$\eta = \frac{6}{5}, \quad \psi = -\frac{5}{6}, \quad \chi = -\frac{2}{5}, \quad \rho \neq 0. \quad (21)$$

It was also shown that for the particular choice of $\rho = -1/3$ the number of flat space modes which violate the Hamiltonian constraint is smaller than for any other value of ρ . Thus, the values of the KST parameters that we will use throughout this work are those in (21) with $\rho = -1/3$. Our numerical experiments seem to show that our flat space simulations with the KST system are consistently better with the former choice than with any other choice we attempted.

The constraint propagation equations of the KST formulation are strongly hyperbolic whenever the evolution equations of the dynamical variables are strongly hyperbolic [27]. An analysis of the KST constraint propagation system for perturbations about flat space shows that the system is strongly hyperbolic without any low order terms [25]. This in turn implies that in the weak field limit the amplitude of the violations of the constraints should remain constant with time. However, this should not be the case when one considers the full nonlinear equations or strong

field cases, because the KST formulation has no constraint damping.

C. The PADM formulation

The PADM system is obtained from the first-order ADM formulation by addition of constraints and their derivatives to the RHS of the ADM evolution equations. It has six parameters and is given by

$$\partial_t \gamma_{ij} = (\text{ADM}) + \lambda \gamma^{ab} \partial_b \mathcal{C}_{a ij}, \quad (22)$$

$$\partial_t K_{ij} = (\text{ADM}) + \phi \gamma_{ij} \gamma^{ab} \partial_a \mathcal{M}_b + \theta \partial_{(i} \mathcal{M}_{j)}, \quad (23)$$

$$\partial_t D_{kij} = (\text{ADM}) + \epsilon \gamma^{ab} \partial_a \mathcal{C}_{b kij} + \xi \gamma_{ij} \partial_k \mathcal{H} + \zeta \mathcal{C}_{kij}, \quad (24)$$

where λ , ϕ , θ , ϵ , ξ , ζ are the six parameters of the formulation. We refer to the added terms of the PADM formulation as the constraint driver. It was shown in [25] that for gauges which do not introduce second-order derivatives of the dynamical variables in the evolution equations the PADM system satisfies the Petrovskii condition for well posedness [31], provided that

$$\lambda > 0, \quad \epsilon > 0, \quad \xi = \phi < 0, \quad \theta > 0. \quad (25)$$

Therefore, in this work we use the PADM equations in conjunction with the algebraic gauge condition (16).

When the aforementioned conditions are satisfied, the PADM system has the following properties:

- (a) The PADM evolution equations with (16) resemble the structure of mixed hyperbolic–second-order parabolic quasilinear (MHSP) partial differential equations (for the definition of MHSP equations see Appendix A). By this we mean that the evolution equations have a set of parabolic modes and a set of hyperbolic modes just as MHSP equations do. As demonstrated in [25], the hyperbolic modes correspond to constraint satisfying degrees of freedom and the parabolic ones to constraint-violating ones. The parabolic character of the equations can be most easily seen by the evolution equation of the 3-metric, whose principal part is given by

$$\partial_t \gamma_{ij} \simeq \lambda \gamma^{ab} \partial_a \partial_b \gamma_{ij}, \quad (26)$$

where $\simeq$ implies equal to the principal part.

- (b) The evolution equations of the constraint variables become second-order parabolic partial differential equations (PDEs) independently of the spacetime geometry and the gauge conditions employed, hence the name parabolized-ADM formulation. For the definition of parabolic equations see Appendix A. Parabolic equations admit a well-posed initial value problem and have damping properties [24]. Therefore, the constraint propagation equations of

the PADM system admit a well-posed Cauchy problem and damp the constraints.

- (c) The constraint surface becomes a local attractor because of the parabolic structure of the evolution equations of the constraints. In particular, all small amplitude, high-frequency constraint-violating perturbations are exponentially damped in time as $\exp(-\lambda \kappa^2 t)$, where κ is the magnitude of the wave vector of the perturbations. This in turn implies that the hazardous high-frequency constraint-violating perturbations must be damped very efficiently. This is a property of the system independently of either the spacetime studied or the gauge conditions chosen [25].

We expect from a theoretical standpoint that, if the PADM parameters are chosen large enough, the PADM system will have better control of the constraint violations than both the ADM and the KST formulations, no matter what the KST parameters are, because the PADM formulation inherently has constraint damping. This prediction is numerically verified in the following sections.

In this work we use the PADM formulation with the following set of parameters:

$$\zeta = 1, \quad \lambda = \epsilon = \theta = -2\xi = -2\phi = 0.02 \quad (27)$$

in all our simulations. We do this for two reasons: (a) It was shown in [25] that the PADM evolution equations admit a well-posed initial value problem when

$$\lambda = \epsilon = -2\xi = -2\phi \neq \theta/2 \quad (28)$$

and (b) when using explicit numerical schemes to carry out the integration of the PADM evolution equations, the values of the PADM parameters given in (27) are small enough to allow for large enough time steps.

This choice for the parameters of the PADM formulation is not special. We have run simulations with efficient damping even for parameters $(\phi, \xi, \lambda, \epsilon, \theta)$ of the order of 10^{-3} . The parameters used in the current work are a compromise between very efficient damping and large enough time stepping. We choose them large enough because we want to demonstrate the power of the PADM system. However, we have not fully explored the parameter space of the PADM system and it may as well be that the values of the parameters need not be as large as in (27). This will be the subject of a future paper.

Finally, note that care must be taken in the choice of the parameters of the PADM system for backwards in time evolutions, e.g., the collapsing Gowdy spacetimes. The backwards in time Cauchy problem for a parabolic equation is ill posed, because of the existence of exponentially growing modes. The same is true for the PADM formulation if the parameters of the formulation satisfy (25). To overcome this problem we simply have to reverse the signs of the PADM parameters in (25).

III. NUMERICAL SCHEME

We use a finite difference method to carry out computations. For all formulations we use a staggered spatial mesh in which γ_{ij} and K_{ij} are located at the cell centers and D_{kij} are staggered half a grid point in all spatial directions. Moreover, all variables are defined at the same time layer. For ADM and KST, we use a third-order Runge-Kutta method to do the integration in time, as this scheme has shown to possess desirable dispersion and dissipation properties compared to other commonly used numerical schemes [32]. We use second-order accurate centered derivative operators to calculate spatial derivatives and a 3rd order accurate parabolic interpolation operator whenever staggered values of the dynamic variables are needed.

The PADM formulation has both hyperbolic and parabolic terms in the RHS of its evolution equations. Since we have implemented an explicit algorithm, if we were to deal with the integration of the PADM evolution equations in the same way as for the KST and ADM systems, we could soon face strict limitations on the speed of the computations because of limitations on the maximum allowed time step. We deal with this limitation by combining the methods of operator-split and fractional steps.

The basic idea of operator splitting was originally introduced in [33]. Here we omit all the details and present the technique in its simplest form. Let

$$\frac{\partial u}{\partial t} = \mathcal{L}u \quad (29)$$

be a set of PDEs, where u is the column vector of the unknown variables and $\mathcal{L}$ is a differential operator. Assume further that $\mathcal{L}$ can be written as a sum of two operators, i.e.,

$$\mathcal{L}u = \mathcal{L}_1u + \mathcal{L}_2u. \quad (30)$$

Let us now consider the individual equations

$$\partial_t u = \mathcal{L}_1u, \quad \partial_t u = \mathcal{L}_2u \quad (31)$$

and let $C(\Delta t)$ be a finite difference operator for Eq. (29), i.e., the solution at time-step $n+1$ is given by $u^{n+1} = C(\Delta t)u^n$. Also, let C_1 and C_2 be corresponding finite difference operators for the individual equations of (31). If the operators C_1 and C_2 are second-order accurate and stable for time-step Δt , then it can be shown that the approximation

$$C(2\Delta t) = C_1(\Delta t)C_2(\Delta t)C_2(\Delta t)C_1(\Delta t) \quad (32)$$

provides a second-order accurate and stable scheme for (29) for a time-step $2\Delta t$.

In the discussion above, we made no explicit mention to the exact form of the finite difference operators C_1 and C_2 . Therefore, one is free to use any finite difference operators, as long as those are stable and second-order accurate for a time-step Δt . This means that for any of the operators C_1 and C_2 we can use the scheme known as fractional steps

[34]. In this scheme, instead of advancing the solution forward a whole time-step Δt , one advances successively q times, each with a time step of $\Delta t/q$. Thus, if we choose to do fractional stepping for the C_2 operator, we can approximate $C_2(\Delta t)$ as

$$C_2(\Delta t) = \underbrace{C_2(\Delta t/q)C_2(\Delta t/q)\dots C_2(\Delta t/q)}_{q\text{-times}}. \quad (33)$$

One may naturally think of the PADM formulation as consisting of two operators which act additively on the dynamical variables, such as those described by Eqs. (29) and (30). The first operator is that of the RHS of the evolution equations of the ADM formulation and the second one is the constraint driver. There is no unique way to split the entire operator of the RHS of the evolution equations of the PADM formulation, but because of its mathematical structure and for computational efficiency, the most straightforward way to proceed with splitting is to set the $\mathcal{L}_1$ operator equal to the ADM operator including the low order terms of the constraint driver, and set $\mathcal{L}_2$ equal to the higher order terms of the constraint driver. We call this $\mathcal{L}_1$ operator the hyperbolic operator and $\mathcal{L}_2$ the parabolic operator. If the splitting is done in this way, then the $\mathcal{L}_2$ operator has a smaller number of operations on the dynamical variables compared to any other choice of splitting. This feature is important for computational efficiency as it will be evident shortly.

For the PADM formulation, we use the techniques described above in the following way. We operator split Eqs. (22)–(24) as is described in the previous paragraph. For the time integration of the hyperbolic terms, we use the same technique as that for the ADM and KST formulations. For the time integration of the parabolic terms, we use a second-order accurate iterative Crank-Nicolson scheme with two iterations. Furthermore, whenever it is required by the numerical stability criterion, we use the fractional steps method to integrate the parabolic terms. We do this by finding the smallest positive integer number p , defined by

$$\Delta t_p = \frac{\Delta t}{p}, \quad (34)$$

for which the parabolic time-step Δt_p satisfies the stability condition (see discussion on numerical stability below) and then proceed with the integration of the parabolic terms as is dictated by (33). In Eq. (34), Δt is the time step of the hyperbolic terms. The advantage of this algorithm is that, even though we may have to use a large p for high resolutions, and thus have to calculate the parabolic terms p times for each hyperbolic time step, it reduces the computational cost significantly, because the hyperbolic terms have larger computational overhead than the overhead of the parabolic terms.

An important issue which has to be addressed at this point is the numerical stability criterion of the algorithm

for the PADM formulation. Since there is a hyperbolic and a parabolic part in the equations, there arise two stability conditions: (a) A condition from the hyperbolic part, and (b) a condition from the parabolic part. Therefore, we must always make sure that we satisfy both the hyperbolic and the parabolic stability criteria [34].

The stability criterion for general hyperbolic equations with anisotropic speed of propagation of information is

$$\max\left(\frac{c_x \Delta t}{\Delta x}, \frac{c_y \Delta t}{\Delta y}, \frac{c_z \Delta t}{\Delta z}\right) = cfl \leq c_1, \quad (35)$$

where c_x , c_y , and c_z are the speeds of propagation of information in the x , y , and z directions respectively, cfl is the Courant-Friedrichs-Levy number, and c_1 is a constant which depends on the numerical scheme. For example, for a scalar wave equation with speed unity in conjunction with a 3rd-order Runge-Kutta time integrator and second-order accurate centered spatial derivative operators, the constant is $c_1 = \sqrt{3/4}$ [32].

For most applications in numerical relativity the speed of propagation of information is isotropic and the choice

$$\frac{\Delta t}{\Delta x} = 0.25 \quad (36)$$

satisfies condition (35). For this reason in this work we set the time step for the hyperbolic part of PADM via (36). We do the same for all computations with the ADM and KST formulations.

Now we turn our attention to the stability criterion of the parabolic part. A Von Neumann analysis of Eq. (26) shows that in order to preserve numerical stability the explicit numerical scheme we use must satisfy

$$\lambda \Delta t_p \left(\frac{\gamma^{11}}{(\Delta x)^2} + \frac{\gamma^{22}}{(\Delta y)^2} + \frac{\gamma^{33}}{(\Delta z)^2} \right) \leq \frac{1}{2}. \quad (37)$$

Although condition (37) results from the stability analysis of the principal part of the evolution equation of the 3-metric, all our numerical experiments confirm that if we choose the PADM parameters according to

$$\lambda = \epsilon = \theta = -2\xi = -2\phi, \quad (38)$$

then if (37) is satisfied the computations remain stable, and if it is violated the computations soon become unstable.

Taking all these facts into consideration, we conclude that, if condition (38) is satisfied, the overall numerical stability criterion of the numerical scheme for the PADM system is

$$p > \frac{\lambda(\gamma^{11} + \gamma^{22} + \gamma^{33})}{2\Delta x}, \quad (39)$$

with $p = 1$ the minimum value p can obtain. In the derivation of the last equation we combined (34), (36), and (37).

At this point we would like address the question of whether our numerical PADM scheme is essential to obtain the results of the following sections.

First, it is important to note that the PADM system has been formulated in order to deal with the growth of the constraint-violating modes at the analytical/continuum limit. Theoretical arguments why this is so have been given in [25]. As such, any consistent and stable (in the sense of Lax [34]) numerical scheme should be convergent and, hence, should reproduce the analytical damping and smoothing properties of the PADM system.

For this reason, the numerical methods described in this section should not be crucial to obtain the present results. This is further strengthened by the fact that we have experimented with a 3rd and 4th-order Runge-Kutta time integration methods as well as an iterative Crank-Nicolson scheme and obtained similar results as far as constraint damping, stability, and convergence are concerned. However, we did not test these integration methods as exhaustively as the ones which the present results have been obtained with.

What we have found to be crucial for the stability of the integration in gauge wave simulations, see Sec. IV B, is staggering of the mesh. The gauge wave problem admits a continuous family of exponentially growing solutions when a harmonic gauge choice is made. Unstaggered meshes seem to excite those solutions much faster than staggered meshes. This is true for all three formulations. At present, we do not know whether this is caused by the introduction of greater truncation errors or the lack of numerical stability of unstaggered schemes. To determine this, a Von Neumann stability analysis [34] of the full numerical scheme is required, which is a task of great complexity and beyond the scope of the current work.

Finally, the operator-split method is not essential to control stability. The reason why we introduced it is for computational efficiency only.

IV. NUMERICAL TESTS

In this section we briefly describe the standard tests we used and the results of the numerical comparison of the three formulations presented in Sec. II. As is usual in the literature, we set up tests which probe both the linear and the nonlinear regime of the GR equations. We closely follow [9] in the setup of the tests, but also modify some of them in order to make the distinction among the three formulations more apparent.

For the evaluation of errors we follow [35] and check the accuracy of the integration with a single number given by the following L_2 norm:

$$\|C\|_2 = \sqrt{\frac{1}{\text{vol}} \int C^2 \sqrt{\gamma} d^3x}, \quad (40)$$

where $\text{vol} = \int \sqrt{\gamma} d^3x$, and

$$\mathcal{C} = \sqrt{(\mathcal{H})^2 + (\mathcal{M}_i)^2 + (\mathcal{C}_{kij})^2}. \quad (41)$$

The squared quantities are equal to the norms of the quantities with respect to the numerical 3-metric tensor, e.g., $(\mathcal{C}_{kij})^2 = \mathcal{C}^{kij}\mathcal{C}_{kij}$, where all indices are raised with the numerical metric. It is clear from Eq. (40) that $\|\mathcal{C}\|_2 = 0$, if and only if all the constraints are satisfied. $\|\mathcal{C}\|_2$ is called the “constraint energy.”

Similarly, we monitor the accuracy of the numerical solution as compared with the analytic one, by using the “error energy” which is given by

$$\|\delta\mathcal{U}\|_2 = \sqrt{\frac{1}{\text{vol}} \int \delta\mathcal{U}^2 \sqrt{\gamma} d^3x}, \quad (42)$$

where

$$\delta\mathcal{U} = \sqrt{(\delta\gamma_{ij})^2 + (\delta K_{ij})^2 + (\delta D_{kij})^2}, \quad (43)$$

and where all “delta” dynamical variables constitute the difference between the numerical and the analytic solution, e.g. $\delta\gamma_{ij} = \gamma_{ij}^{\text{analytic}} - \gamma_{ij}^{\text{numerical}}$. As in the constraint energy, the indices in the error energy are raised with the numerical metric. The error energy vanishes, if and only if the numerical solution matches the analytical one.

Just like in [35], for the Gowdy wave tests we plot the normalized version of those quantities, i.e., $\|\delta\mathcal{U}\|_2/\|\mathcal{U}\|_2$ and $\|\mathcal{C}\|_2/\|\partial\mathcal{U}\|_2$, where

$$\mathcal{U} = \sqrt{(\gamma_{ij})^2 + (K_{ij})^2 + (D_{kij})^2}, \quad (44)$$

and

$$\partial\mathcal{U} = \sqrt{(\partial_k\gamma_{ij})^2 + (\partial_k K_{ij})^2 + (\partial_s D_{kij})^2}, \quad (45)$$

and where the squared quantities here are with respect to the analytic 3-metric tensor. The reason for plotting the normalized errors is that the error depends on the magnitude of the solution.

We set up all tests on a 3D mesh with periodic boundary conditions. We perform all one-dimensional tests on a cell-centered domain, such that $x \in [0, 1]$, with $x_n = (n - \frac{1}{2})\Delta x$, where $n = 1, 2, \dots, 2^r$ and $\Delta x = \Delta y = \Delta z = 2^{-r}$, where $r \in \mathbb{N}$. In addition, we follow [36] and set up the domain with four points in the y and z directions. In the case of two-dimensional tests the domain is similar. In particular, $x \in [0, 1]$, $y \in [0, 1]$, $x_n = (n - \frac{1}{2})\Delta x$, $y_n = (n - \frac{1}{2})\Delta y$, where again $n = 1, 2, \dots, 2^r$, $\Delta x = \Delta y = \Delta z = 2^{-r}$, and have four points in the z direction.

A. Robust stability test

The robust stability test is based on small random perturbations about Minkowski spacetime. The amplitude of the perturbations is sufficiently small so that the evolution remains in the linearized regime unless any instabilities arise. The initial data are given by

$$\gamma_{ij} = \delta_{ij} + \epsilon_{ij}^1, \quad K_{ij} = \epsilon_{ij}^2, \quad D_{kij} = \epsilon_{kij}^3, \quad (46)$$

where the random numbers $\{\epsilon_{ij}^1, \epsilon_{ij}^2, \epsilon_{kij}^3\}$ are uniformly distributed in $[-10^{-10}/\rho, 10^{-10}/\rho]$, where $\rho = 2^{r-4}$. Note that, instead of rescaling by ρ^2 , as is suggested in [9], here we rescale by ρ . We do this because the constraints for the formulations we are considering are first order, and in order to achieve the same initial violation on the constraints we have to rescale by ρ . We call the above test the “standard” robust test.

Another version of this test requires perturbation by noise not only at the initial time step, but at every time step throughout the entire simulation. We call this test the “strong” robust stability test. We run both versions of the robust test with $r = 4, 5, 6$ for 1000 light crossings or until the code crashes.

At this point we ought to define what we mean by the expression “the code crashes” in order to avoid confrontations with results that have already been published in the literature. Independently of the formulation we use, when we say that the simulations crash, we henceforth mean that the simulations with finite differences crash.

1. Standard robust stability test

The results of this test ran with harmonic slicing are summarized in Fig. 1, where we have plotted the constraint energy for the ADM, KST, and PADM formulations as a function of time. The ill-posed nature of the ADM formulation is clearly seen in this figure. The simulations crash fairly quickly and most importantly at earlier times for higher resolutions, as expected for ill-posed systems.

Figure 1 also shows that the three error energies of the KST runs practically overlap. This is a result of rescaling

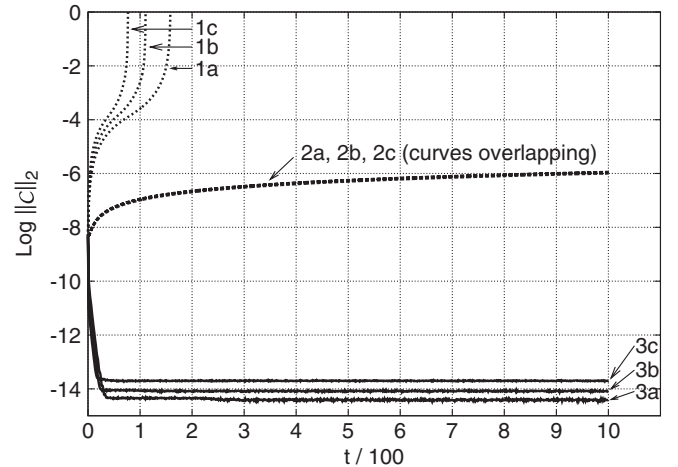


FIG. 1. Constraint energies as functions of coordinate time t for the standard robust test with $\alpha = \gamma^{1/2}$. The letters a, b, c correspond to resolutions $r = 4, 5, 6$, respectively. Curves 1a, 1b, and 1c correspond to the ADM formulation, curves 2a, 2b, and 2c correspond to the KST formulation, and curves 3a, 3b, and 3c correspond to the PADM formulation.

the amplitude of the random noise with resolution, so that the initial violation of the constraints is of the same magnitude for all resolutions. The KST simulations exhibit noticeable growth in the constraint energy. Close investigation shows that $\mathcal{C}_{kij}$ grow linearly with time, whereas $\mathcal{H}$ and $\mathcal{M}_i$ stay roughly constant throughout the entire simulation. Therefore, the growth of the constraint energy comes from $\mathcal{C}_{kij}$.

The linear growth with time of $\mathcal{C}_{kij}$ has a very simple explanation. The evolution equations of the $\mathcal{C}_{kij}$ constraints with the KST formulation are

$$\hat{\partial}_0 \mathcal{C}_{kij} = -\eta \alpha \gamma_{k(i} \mathcal{M}_{j)} - \chi \alpha \gamma_{ij} \mathcal{M}_k. \quad (47)$$

This last equation implies that any small violation of the momentum constraints will be spilled into the $\mathcal{C}_{kij}$ constraints and cause them to evolve with time. Our results of the robust test with the KST formulation show that the RHS of Eq. (47) is roughly equal to a nonzero constant. This implies that the $\mathcal{C}_{kij}$ constraints must grow linearly with time.

We will now show that the linear growth with time of $\mathcal{C}_{111}$ with the KST formulation is due to violations of $\mathcal{M}_x$. We will do this by using our results of the $r = 6$ runs of the robust test. Via a least squares fit of the infinity norm of $\mathcal{C}_{111}$ as a function of time, we find that its observed growth rate is $\partial_t |\mathcal{C}_{111}|_\infty^{\text{observed}} \approx 6.5773 \times 10^{-10}$.

To show that the linear growth of $|\mathcal{C}_{111}|_\infty$ is due to violations of $\mathcal{M}_x$, we need to demonstrate that the observed growth rate agrees with the predicted growth rate from Eq. (47). Keeping in mind that the lapse function and γ_{11} are practically equal to unity throughout the entire run, and using (21) and (47) we find that the absolute value of the time derivative of $\mathcal{C}_{111}$ is

$$\partial_t |\mathcal{C}_{111}| \approx \frac{3}{5} |\mathcal{M}_x|. \quad (48)$$

Using the data from the same simulation, we find that the infinity norms of $|\mathcal{M}_x|_\infty$ stays roughly constant during the entire run and that its time average is such that $\partial_t |\mathcal{C}_{111}|_\infty^{\text{predicted}} \approx 6.162 \times 10^{-10}$. The growth rate predicted by Eq. (47) agrees with the observed one within 6.3%.

A consequence of this pathology of the KST system is that, given sufficient time or stronger initial perturbations, the linear growth will eventually terminate any KST simulation.

Figure 1 shows that the PADM formulation has drastically different behavior than the ADM and KST systems. The constraint energy decreases very rapidly by many orders of magnitude and then reaches a plateau. This formulation damps any small violations of the constraint equations and pushes the evolution back onto the constraint surface. The plateau can be explained as follows. Because of roundoff errors, random noise of the order of 10^{-16} is introduced into the system at every time step. This round-

off error noise causes violations of the constraints of the order of $10^{-16}/dx$ because all our constraints contain first-order derivatives of the dynamical variables. However, dx is roughly of the order of 10^{-2} and hence the average magnitude of the constraints when they reach the plateau is expected to lie at roughly 10^{-14} . This is in accordance with what we observe in Fig. 1.

The plateau occurs at the point where the damping of the constraints essentially balances the constraint violations caused by the roundoff errors. There is a slight difference in the plateau at which the constraint energy flattens out for different resolutions. It is seen that, for the highest resolution $r = 6$ (curve 3c), the plateau lies higher than those for resolutions $r = 4$ and $r = 5$ (curves 3b and 3a, respectively). This is because the roundoff errors are on average of the same magnitude and hence the roundoff error perturbations are not scaled as $1/\rho$, as is the case with the initially added random noise. Consequently, the plateau of simulations of the robust test with the PADM system will lie higher with increasing resolutions.

In Fig. 2 we explore the damping properties of the PADM formulation. The plot in the figure was created as follows. Since the simulation is quasi-one-dimensional, we focus on a one-dimensional line located along $x = [0, 1]$, $y = \Delta y/2$, $z = \Delta z/2$. Let the constraint variables be denoted by $\mathcal{C}_i$, where $i = 1, 2, \dots, 18$. Along this line and at a given time t we perform a discrete Fourier transform of the $N = 2^r$ values of all the constraint variables $\mathcal{C}_{i,n}$, $n = 1, 2, \dots, N$, so that

$$A_{i,k} = \sum_{n=1}^N \mathcal{C}_{i,n} e^{-i(2\pi/N)kn}, \quad k = 0, 1, \dots, N-1, \quad (49)$$

where $A_{i,k}$ denotes the Fourier harmonic k of the constraint $\mathcal{C}_i$. For a given harmonic, we then sum the amplitudes of all

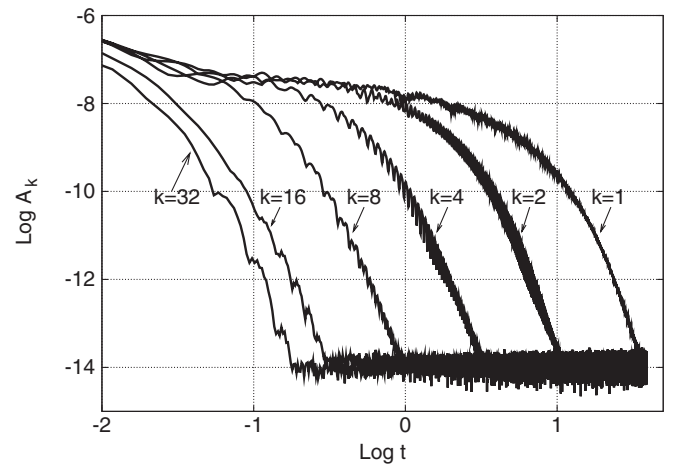


FIG. 2. Cumulative amplitude A_k [see Eq. (50)] of the Fourier transform of the constraint variables as a function of coordinate time t , for harmonics $k = 1, 2, 4, 8, 16, 32$. The plot corresponds to the run of the standard robust test using $\alpha = \gamma^{1/2}$, with the PADM formulation, for resolution $r = 6$.

the Fourier pairs of the constraints, i.e.,

$$A_k = \sum_{i=1}^{18} |A_{i,k}|. \quad (50)$$

A_k is zero, only if all constraints have zero contribution to the specific harmonic k .

The quantity A_k as a function of time is plotted in Fig. 2 for the standard robust test with the PADM formulation. A_k provides us with a measure which describes the cumulative amplitude of a specific Fourier mode (k) over all the constraints. The set of A_k also carries the information of whether any instability arises and on which end of the frequency spectrum it occurs. Figure 2 shows that the PADM formulation damps the highest frequencies (higher k) the most. This is precisely what was predicted in [25] for the PADM system. It is worth noting that the Nyquist frequency (32nd harmonic) constraint violations are damped away in less than one light crossing time and all the constraint-violating modes are damped away after approximately 25 light crossings.

Finally, we also carried out the standard robust stability test with $1 + \log$ slicing and with a $\sigma = 1.0$ densitized lapse and we find similar behavior for all formulations. For brevity we only show the results for $1 + \log$ slicing (Fig. 3). It is again worth noting that the PADM formulation damps the constraints, indicating that the damping properties of the formulation are independent of the gauge choice, as long as the gauge chosen is well posed according to [30]. This again is in excellent accordance with the predictions of [25].

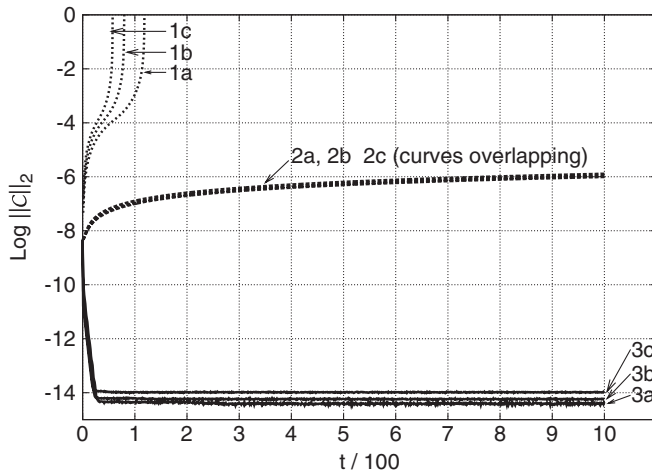


FIG. 3. Constraint energies as functions of coordinate time t for the standard robust test in $1 + \log$ slicing. The letters a, b, c correspond to resolutions $r = 4, 5, 6$, respectively. Curves 1a, 1b, and 1c correspond to the ADM formulation, curves 2a, 2b, and 2c correspond to the KST formulation, and curves 3a, 3b, and 3c correspond to the PADM formulation.

2. Strong robust stability test

This test is much more challenging than the standard robust test and is much closer to reality because in numerical simulations, truncation error will unavoidably be introduced into the solution of the PDE system at every time step.

Simulations of this test with the ADM formulation crash almost immediately. For this reason we only show the results for the KST and PADM formulations. In Fig. 4 we show the constraint energies resulting from the strong robust tests for the KST and PADM systems for various noise amplitudes. All plots in this figure correspond to resolution $r = 6$. For the KST formulation we see that at amplitude $A = 10^{-10}$ (curve 1a), the constraint energy grows more rapidly than in the standard robust test (Fig. 1). Nevertheless, the simulation can be run for 1000 light crossing times. However, it is clear that, given enough time, the constraint energy will eventually grow so large that the simulation will be terminated. This can be seen more clearly in curve 1b, which shows the constraint energy for noise amplitude $A = 10^{-8}$. It is evident that in this case the KST runs manage to complete only approximately 600 light crossings before the simulation is so polluted that it crashes.

The results of the PADM runs are dramatically different. Figure 4 shows no sign of increase of the PADM constraint energy for the first 1000 light crossings. Naturally we do not see the damping behavior to the level shown in Fig. 1, because strong noise is being pumped into the system at every time step. However, it is impressive that even with random noise of amplitude $A = 10^{-6}$ the constraint energy does not grow throughout 1000 light crossings and the simulations show no signs of pathologies.

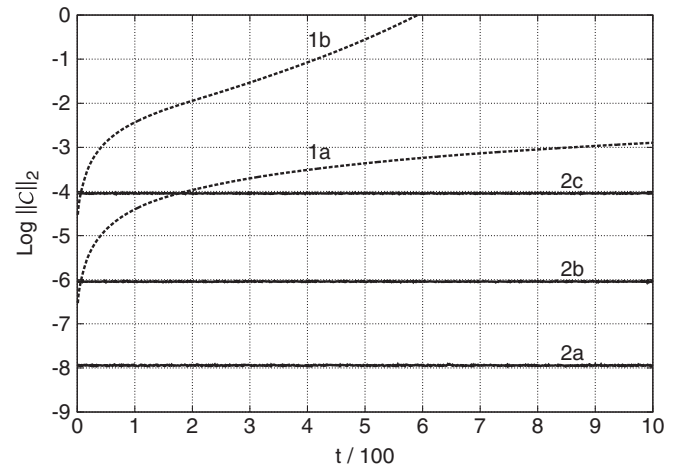


FIG. 4. Constraint energies as functions of coordinate time t for the strong robust test, with $\alpha = \gamma^{1/2}$ and resolution $r = 6$. The letters a, b, c correspond to noise amplitudes $A = 10^{-10}$, $A = 10^{-8}$ and $A = 10^{-6}$ respectively. Curves 1a and 1b correspond to the KST formulation. Curves 2a, 2b, and 2c correspond to the PADM formulation.

B. Gauge-wave test

The metric of the one-dimensional gauge wave test [9] is

$$ds^2 = -h dt^2 + h dx^2 + dy^2 + dz^2, \quad (51)$$

where

$$h = 1 + A \sin\left(\frac{2\pi(x-t)}{d}\right), \quad (52)$$

where A is the amplitude of the gauge wave and d the size of the evolution domain. For the first-order ADM formulation (51) and (52) is equivalent to

$$\begin{aligned} \gamma_{11} &= 1 + A \sin\left(\frac{2\pi(x-t)}{d}\right), \\ K_{11} &= -\frac{\pi A}{d\gamma_{11}} \cos\left(\frac{2\pi(x-t)}{d}\right) \\ D_{111} &= \frac{2\pi A}{d} \cos\left(\frac{2\pi(x-t)}{d}\right), \end{aligned} \quad (53)$$

$\gamma_{22} = \gamma_{33} = 1$, and all other dynamical variables are zero. A further coordinate transformation,

$$x = \frac{1}{\sqrt{2}}(x' - y'), \quad y = \frac{1}{\sqrt{2}}(x' + y'), \quad (54)$$

yields the two-dimensional diagonally traveling gauge wave.

The gauge wave stability test has proven to be one of the most challenging tests for most evolution schemes mainly due to the existence of a one-parameter family of exponential, harmonic gauge solutions [37] which correspond to the metric

$$ds^2 = e^{\mu t} h(-dt^2 + dx^2) + dy^2 + dz^2, \quad (55)$$

for arbitrary μ . The gauge wave test (51) corresponds to $\mu = 0$, but in any numerical implementation of harmonic gauge conditions the exponential modes, $\mu \neq 0$, are likely to be excited by truncation error. This eventually causes the numerical solution to veer off the analytic one.

The growth of these exponential gauge modes can be avoided by the incorporation of a set of semidiscrete conservation laws into the principal part of the evolution equations of a harmonic formulation [37]. However, these conservation laws are specific to the gauge wave solution and it is not known whether similar conservation laws exist for arbitrary spacetimes.

The lapse function for the gauge wave test is harmonic and we use $\alpha = \gamma^{1/2}$ in our simulations. Since these coordinates are harmonic and we do not incorporate any conservation laws in our scheme, we do expect to excite the exponential gauge modes. We run both the 1D and the 2D gauge wave tests with amplitudes $A = 0.01, 0.1$ and resolutions $r = 5, 6, 7$. The 1D gauge wave is run for 1000 light crossings or until the code crashes and the 2D gauge wave is run for 100 light crossings or until the code crashes.

Below we describe the results of this test for each formulation separately.

1. The low amplitude 1D gauge wave

The $A = 0.01$ 1D gauge wave presented no apparent difficulty for any of the formulations used in this work. For brevity, we only describe the results of this test because the high amplitude gauge wave, discussed directly below, is much more interesting. The error energies as functions of time, with all three formulations, completely overlap. However, while the KST and ADM constraint energies overlap, the PADM constraint energy lies at least 2 orders of magnitude below those two. The PADM formulation damps the constraints very fast and reaches the plateau value $\|C\|_2 = 3.17 \times 10^{-8}$ within three light crossing times.

2. The high amplitude 1D gauge wave

Our results for the high amplitude 1D gauge wave are summarized in Fig. 5, where we show the error and constraint energies as functions of the coordinate time t for resolution $r = 7$.

(a) *The behavior of the ADM formulation.*—Figure 5(a) shows that the constraint energy in the high amplitude gauge wave runs with the ADM formulation does not grow. However, Fig. 5(b) shows that the ADM error energy grows quickly with time and overlaps with the PADM error energy. For both systems, the source of this growth is the excitation of exponential harmonic gauge modes. We do not study this subject here because we address it thoroughly below in the context of the PADM results of the 1D gauge wave test.

(b) *The behavior of the KST formulation.*—Fig. 5 shows that the KST system cannot pass the high amplitude gauge wave test because the computations do not complete 1000 light crossing times. The KST constraint energy suddenly begins to grow exponentially at $t \approx 160$, and later on at $t \approx 200$ the KST error energy blows up.

We will now show that the unexpected behavior of the KST system in the simulations of the $A = 0.1$ gauge wave is due to excitation of unwanted degrees of freedom by violations of the momentum constraints. If there is no noise introduced in an evolution system, the only evolving dynamical variables for the one-dimensional gauge wave must be γ_{11} , K_{11} , and D_{111} . This implies that the only constraint that evolves is C_{111} . Furthermore, $\mathcal{H}$ and $\mathcal{M}_x$ must evolve due to roundoff error, but must be of the order of 10^{-16} . However, this is not the case with the KST system. It is straightforward to show by use of (20) that the KST equations excite D_{122} , D_{133} , D_{212} , and D_{313} because $\mathcal{M}_x$ is not exactly satisfied. These dynamical variables in turn excite γ_{22} , γ_{33} , γ_{12} , γ_{13} , K_{22} , K_{33} , K_{12} , K_{13} and eventually all the constraints these variables are involved in, i.e., C_{122} , C_{133} , C_{112} , C_{113} , $\mathcal{M}_i$, and $\mathcal{H}$, grow rapidly with time. The constraint violations soon become very

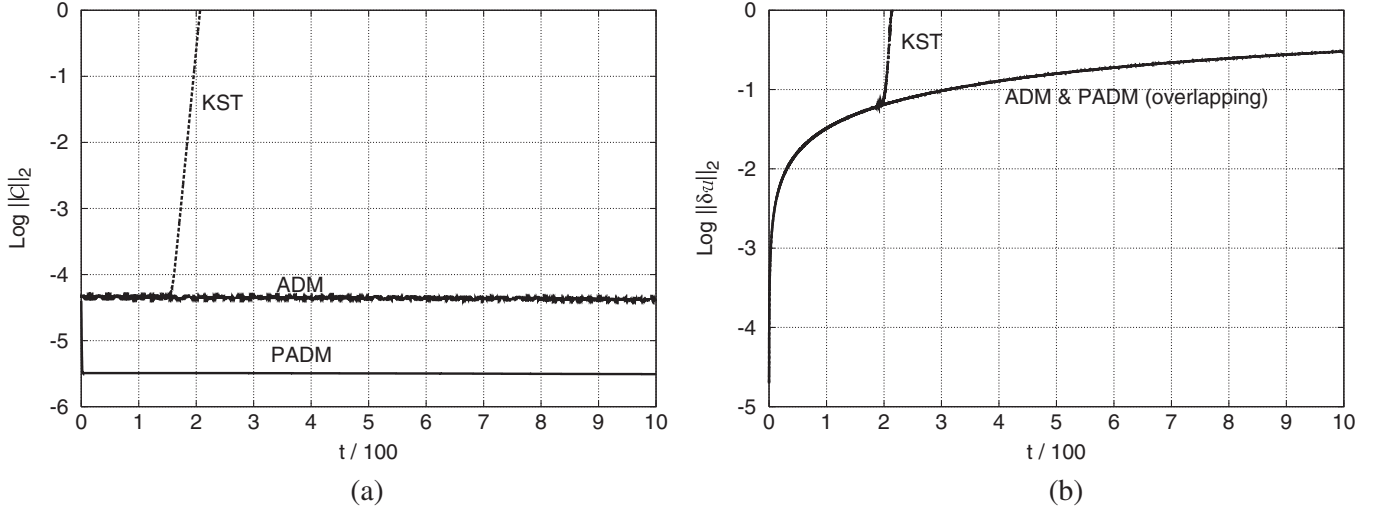


FIG. 5. Errors as functions of coordinate time t for the 1D gauge wave, with $A = 0.1$ and resolution $r = 7$. On the left panel (a) we show the constraint energies and on the right panel (b) we show the error energies for the ADM, KST, and PADM formulations. The PADM formulation damps the constraints within three light crossing times and reaches the plateau value $||C||_2 = 3.11 \times 10^{-6}$, while the ADM constraint energy at $t = 1000$ is equal to $||C||_2 = 4.15 \times 10^{-5}$.

large that a nonlinear instability develops and the evolution is terminated.

The excitation of unwanted degrees of freedom can be seen in Fig. 6, where we show the infinity norms of C_{111} , C_{122} , $\mathcal{M}_x$, and $\mathcal{H}$ as functions of the coordinate time t . An instability causes C_{122} , $\mathcal{M}_x$, and $\mathcal{H}$ to grow exponentially in time from very early on in the evolution. When the violations of C_{122} , $\mathcal{M}_x$, and $\mathcal{H}$ grow large enough, the solution gets spoiled, and at $t \approx 200$ the numerical solution is so polluted that the simulation crashes.

To prove that the instability of the evolutions of the gauge wave test with the KST formulation is caused by the excitation of D_{122} , D_{133} , D_{212} , and D_{313} by roundoff

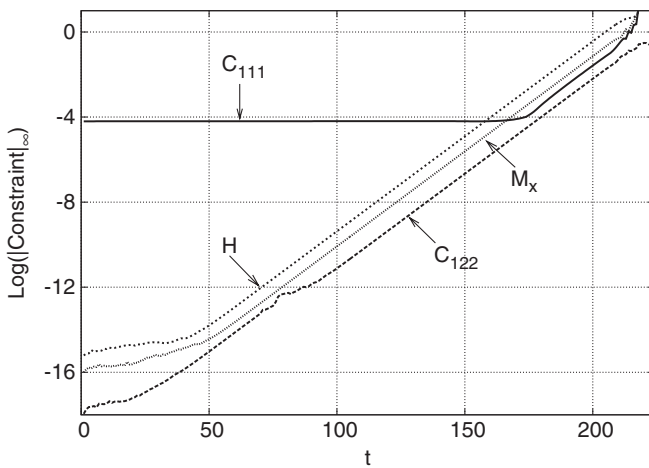


FIG. 6. Infinity norms of the C_{111} , $\mathcal{H}$, $\mathcal{M}_x$, and C_{122} constraints as functions of the coordinate time t . The plot corresponds to the KST run for the gauge wave test, with $A = 0.1$ and resolution $r = 7$.

error in $\mathcal{M}_x$, we also ran a “partially constrained” evolution of the gauge wave with the KST system, where we set the D_{122} , D_{133} , D_{212} , and D_{313} equal to the exact values these should have at every time step, i.e., 0. The result of these runs was that the KST formulation can evolve the gauge wave for 1000 light crossing times without any problems. Furthermore, we find that the KST error and constraint energies completely overlap with those of the ADM formulation. We do not show plots of these simulations, but any plots can be made available by the authors upon request.

These results indicate that simulations of the gauge wave with formulations which add multiples of the constraints to the RHS of evolution equations are likely to be terminated by the same type of instability that terminates corresponding simulations with the KST formulation. An example of such formulation is the BSSN system and it is known that until now no BSSN based numerical scheme has been able to give satisfactory gauge wave simulations with $A = 0.1$ [36,38,39]. Moreover, the hyperbolic system studied in [40] demonstrated similar behavior. We do not analyze this subject any further, as it is beyond the scope of this paper and will be addressed in a future work.

(c) *The behavior of the PADM formulation.*—Fig. 5(a) shows that the PADM constraint energy does not grow with time and that it is over an order of magnitude less than the corresponding ADM constraint energy. However, Fig. 5(b) shows that the PADM error energy grows quickly in time. We will now show that the source of this growth in the gauge wave simulations is the excitation of exponential harmonic gauge modes (55).

To obtain direct evidence that exponential harmonic gauge modes are excited during a simulation we must show that at any given time t ,

$$\gamma_{xx} = f(t)\gamma_{11}, \quad (56)$$

where γ_{xx} is the numerical solution corresponding to γ_{11} of Eq. (53), and where $f(t)$ is a normalization factor to be determined. We do not assume that $f(t) = e^{\mu t}$ with fixed μ because truncation errors are continuously introduced into the numerical solution and hence different values of μ are likely to be excited as the evolution proceeds.

We will now show how one can calculate $f(t)$ numerically. The spatial average of γ_{11} is unity, because $\int_0^1 \sin(2\pi(x-t)/d)dx = 0$. Therefore, for an exponential harmonic gauge mode, $\gamma_{11}(\mu, x, t) = e^{\mu t}\gamma_{11}(x, t)$, the analytical counterpart $f_a(t)$ of $f(t)$ is

$$f_a(t) = \int_0^1 \gamma_{11}(\mu, x, t)dx. \quad (57)$$

The result of the integration of Eq. (57) is $f_a(t) = e^{\mu t}$ as expected. To find $f(t)$ all we must do is to replace $\gamma_{11}(\mu, x, t)$ in (57) by γ_{xx} . $f(t)$ must be different for different resolutions, because γ_{xx} changes with resolution. For this reason we denote the normalization factor of (56) at the n th time step by f_r^n and we calculate it as

$$f_r^n = \frac{1}{2^r} \sum_{i=1}^{2^r} \gamma_{xx,i}^n \quad (58)$$

where $\gamma_{xx,i}^n$ denotes the value of the numerical solution at the i th grid point and time t_n .

To demonstrate that Eq. (56) holds true in our simulations of the gauge wave, we use the data of the gauge wave runs with $A = 0.5$. In Fig. 7 we show a snapshot of γ_{xx} , γ_{xx}/f_r^n , and γ_{11} at $t = 300$. The continuous curve corresponds to the numerical solution with the PADM formulation and the dashed curve to the analytical one. The

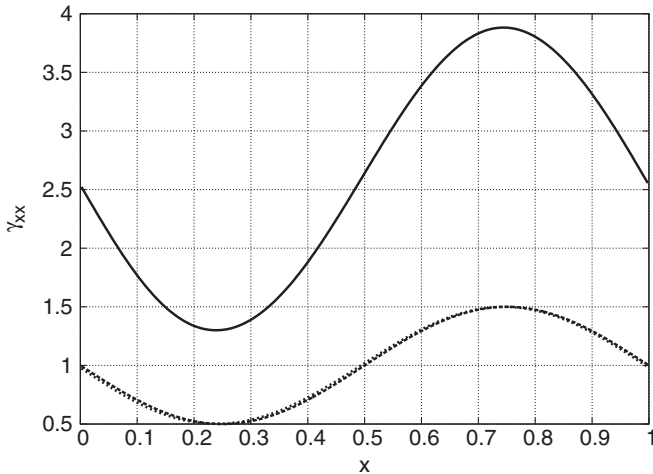


FIG. 7. Snapshot of the solution of the γ_{xx} component for the 1D gauge wave, with $A = 0.5$, resolution $r = 8$, and $t = 300$. The continuous curve corresponds to the PADM numerical solution, while the dashed curve is the exact solution. The dotted curve corresponds to the normalized PADM numerical solution, which is essentially overlapping with the exact solution.

rescaled numerical solution corresponds to the dotted curve, but is essentially overlapping with the exact solution. We find that the rescaled numerical solution overlaps with the analytic one not only at $t = 300$, but also at all times. This proves that (56) holds true in our simulations and in turn provides us with direct evidence that the main source of the growth of the error is the excitation of exponential harmonic gauge modes by truncation error. The same analysis as presented above shows that the same source of error causes the growth of the PADM error energy seen in Fig. 5(b).

We now turn our attention to the properties of function $f(t)$ and we will show that in our simulations $f(t) = e^{at^2}$, $a > 0$. Figure 8 shows the natural logarithm of f_r^n as a function of time for the gauge wave runs with $A = 0.5$ and resolutions $r = 7, 8$. If truncation errors excited a given μ of the solution (55), the curve $t - \ln f_r^n$ should be a straight line with the slope equal to μ . However, the growing slope of the curves in Fig. 8 indicates that the normalization factor grows faster than exponentially. If we interpret the local in time slope of these curves as the value of μ , the former implies that μ grows as time goes by.

In order to find how μ evolves with time, we assume that $f(t) = e^{\mu t}$ with time varying μ . If this is the case, $\mu = \ln f(t)/t$. Thus, to find $\mu(t)$ we need to study how $\ln f_r^n/t$ evolves with time. We do not show $t - \ln f_r^n/t$ plots, but our results indicate that $\ln f_r^n/t$ grows linearly with time. This implies that $\mu(t) = a \cdot t$ and thus $f(t) = e^{at^2}$. For example, by using the $r = 8$ data we find via a least squares fit that $a = 1.05915 \times 10^{-5}$. As the resolution increases, a decreases. This is also clear in Fig. 8, because at any given time $f_8^n < f_7^n$. This is consistent with the fact that with increasing resolution the truncation errors become smaller and hence the solution approaches $\mu = 0$.

The existence of exponential harmonic gauge modes is known to be a property of harmonic coordinates only.

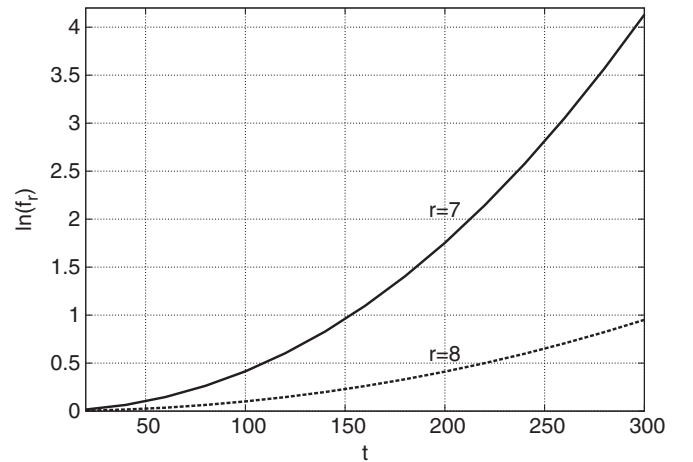


FIG. 8. Normalization factor f_r^n as a function of time. The plot corresponds to the PADM simulations of the 1D gauge wave, with amplitude $A = 0.5$ and resolutions $r = 7, 8$.

Thus, we anticipate that simulations of the gauge wave would not demonstrate the same error growth in other gauges. This analysis will be the subject of future work and for now we conclude that the performance of the PADM formulation in the 1D gauge wave test is better than those of the ADM and KST formulations, since the overall PADM error is less than the overall ADM error and much less than overall KST error.

3. The low amplitude 2D gauge wave

The simulations of the 2D gauge wave with $A = 0.01$ are not as interesting as those of the 2D strong gauge wave. For this reason we only describe the results of the weak 2D gauge wave here. The PADM and KST formulations have no problem completing 100 light crossing times without any pathologies in the constraint error. The constraint energies of the simulations with PADM are damped and lie lower than the simulations with KST. On the other hand, both the PADM and KST systems experience growth in the error energy due to the excitation of exponential harmonic gauge modes. In contrast to the KST and PADM results, the ADM formulation cannot evolve the initial data for more than 66 light crossing times.

4. The high amplitude 2D gauge wave

Our results of the strong two-dimensional gauge wave are summarized in Fig. 9, where we show the error and constraint energies for $r = 7$ for the three evolution systems considered here.

The results with the ADM formulation are drastically different than the corresponding results in the 1D case. In the gauge condition we employed and in 1D, the ADM evolution equations can be shown to be strongly hyperbolic, but in 2D they are only weakly hyperbolic.

Therefore, due to the ill-posed nature of the ADM formulation the simulations of the 2D gauge wave crash quickly, and most importantly they crash faster with increasing resolution. We find that for $r = 5, 6, 7$ the ADM runs crash after 16, 9, 5 light crossing times, respectively.

The KST evolution equations are strongly hyperbolic, and hence the KST runs of the 2D gauge wave are longer than those of the ADM system. However, the simulations with the KST system crash very early on and cannot complete more than 30 crossing times. We have not been able to understand the mechanism via which the KST formulation crashes in this test. Similarly to the 1D case, there are C_{3ij} constraints which are excited because of violation of the momentum constraints, and then grow exponentially with time. However, when the computations stop the values of C_{3ij} are of the order of 10^{-10} , and this level of violation is not strong enough to explain the unexpected behavior of the KST system. The violations of the momentum constraints introduce significant error into the C_{1ij} and C_{2ij} , but all constraints seem to grow exponentially at the same time, and thus we have not been able to attribute the blowup of the simulations to the same source as in the 1D gauge wave. Finally, we note that the instability which causes the evolutions to crash occurs at a time scale which is resolution independent.

The results of the gauge wave simulations with the PADM formulation are drastically different. Figure 9 shows that PADM successfully completes 100 light crossing times. In addition to that and as expected, the PADM formulation has better control of the constraints than the ADM and KST formulations, see Fig. 9(a). Figure 9(b) shows that the PADM error energy experiences rapid growth. The same is true for the ADM and PADM error energies until the ADM and KST runs crash. This growth

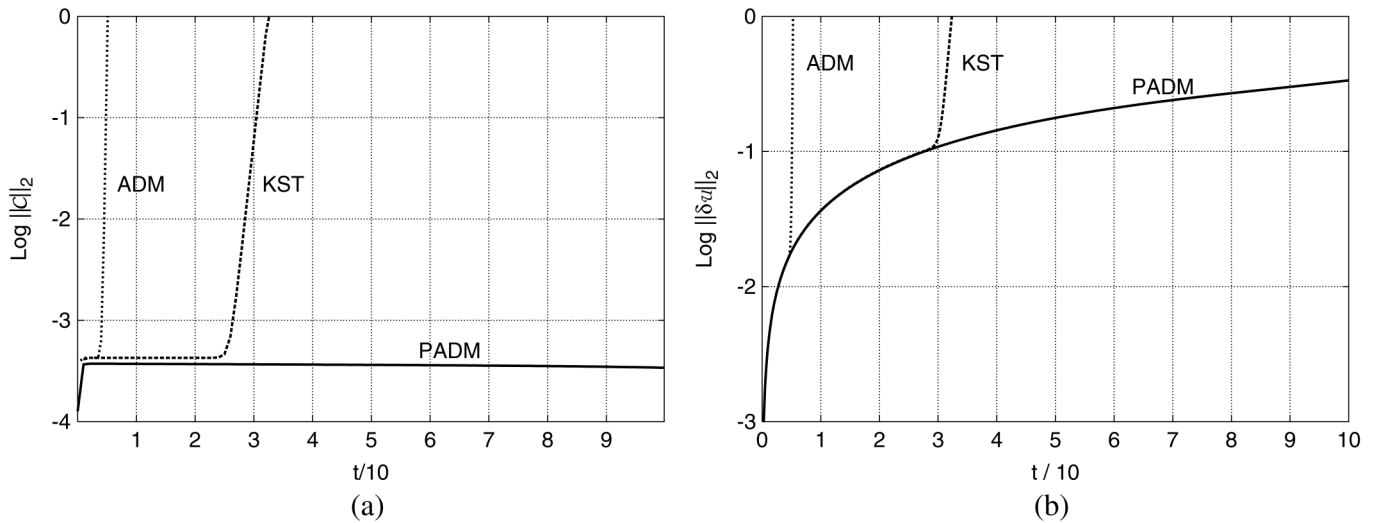


FIG. 9. Errors as functions of coordinate time t in units of light crossing times for the strong two-dimensional gauge wave and resolution $r = 7$. On the left panel (a) we show the constraint energies and on the right panel (b) we show the error energies for the ADM, KST, and PADM formulations.

results because of excitation of exponential harmonic gauge modes as in the 1D gauge wave.

Finally, we note that for resolutions $r = 5$ and $r = 6$ the PADM system manages to evolve the 2D strong gauge wave initial data for 56 and 98 light crossing times, respectively. We have not been able to understand the mechanism via which the simulations of the 2D gauge wave with the PADM system crash for low resolutions, but it seems to be correlated with the exponential growth of the solution. Nevertheless, the indisputable conclusion of the gauge wave simulations is that the numerical performance of the PADM system is much better than those of the KST and ADM formulations.

C. The linear wave test

The linear wave test is a traveling gravitational wave of amplitude, small enough that the evolution remains in the linear regime. The metric is given by

$$ds^2 = -dt^2 + dx^2 + (1+b)dy^2 + (1-b)dz^2, \quad (59)$$

where

$$b = A \sin\left(\frac{2\pi(x-t)}{d}\right), \quad (60)$$

where A is the amplitude of the wave and d the size of the evolution domain. The suggested amplitude for this test is $A = 10^{-8}$ [9]. For the first-order ADM formulation (59) and (60) is equivalent to

$$\gamma_{11} = 1, \quad \gamma_{22} = 1 + b, \quad \gamma_{33} = 1 - b, \quad (61)$$

$$K_{22} = -K_{33} = -\frac{\pi A}{d} \cos\left(\frac{2\pi(x-t)}{d}\right), \quad (62)$$

$$D_{122} = -D_{133} = \frac{2\pi A}{d} \cos\left(\frac{2\pi(x-t)}{d}\right), \quad (63)$$

and all other dynamical variables are zero.

Equation (59) suggests geodesic slicing for the lapse function, but the test can be run both with $\alpha = \gamma^{1/2}$ and $\alpha = 1 + \ln \gamma$. This is so because for (59) $\gamma = 1 - b^2$. However, b is so small that $\alpha = \gamma^{1/2} \approx 1 - b^2/2 \approx 1 - 5 \times 10^{-17} \approx 1$. Thus, because of roundoff error a computer cannot “see” the difference between geodesic slicing and harmonic slicing. Similarly, it can be shown that “1 + log” slicing is compatible with (59). As in the gauge wave case, the 2D linear wave can be obtained from (59) by the coordinate transformation (54).

We carry out simulations of both a one-dimensional and a two-dimensional wave with harmonic slicing $\alpha = \gamma^{1/2}$ and resolutions $r = 5, 6, 7$. The simulation time is 1000 light crossings in the 1D case and 100 light crossings in the 2D case. The results of our runs are summarized in Figs. 10 and 11 where we show the constraint energies for the three formulations in the 1D and 2D version of the test, respec-

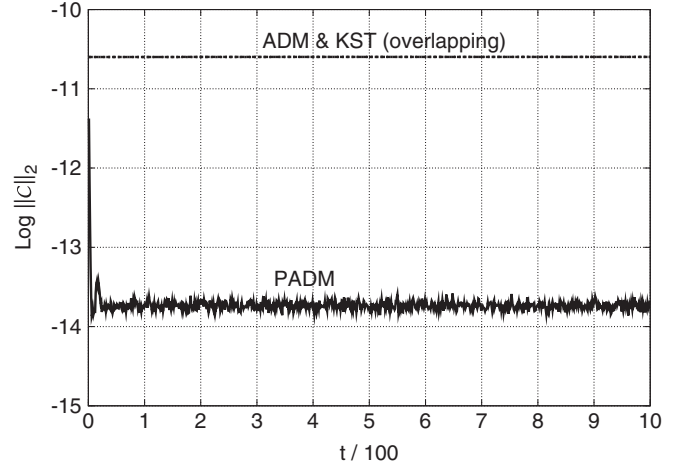


FIG. 10. Constraint energies as functions of coordinate time t for the 1D linear wave for resolution $r = 6$.

tively. For brevity we do not show the error energies, because all formulations can easily handle this test and the essential difference between them lies in the behavior of the constraint energies.

In the 1D simulations (Fig. 10) the constraint energies of the KST and ADM formulations overlap and stay constant throughout the 1000 light crossings of the simulation. On the other hand, the PADM formulation pushes the constraint energy down close to the roundoff error level. There is a difference of at least 3 orders of magnitude between the constraint energy of the PADM system and those of the KST and ADM systems.

In Fig. 11 we show the constraint energies of the three formulations considered in this work for the 2D linear wave test. In the case of the PADM formulation we output the data at every light crossing time. However, outputting the constraint energies for the KST and ADM formulations every light crossing time, results in a false picture of the

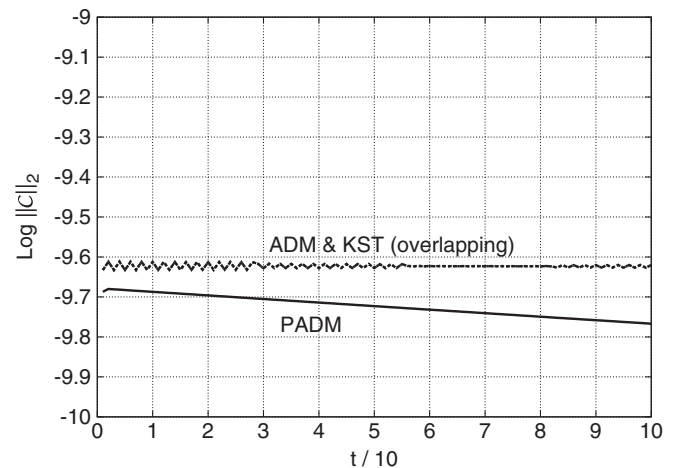


FIG. 11. Constraint energies as functions of coordinate time t for the 2D linear wave test for resolution $r = 6$.

behavior of the KST and ADM formulations for this test, as the constraint energies can be seen to grow in time. What really happens is that the constraint energies wildly oscillate roughly around the value 2.4×10^{-10} and they show no sign of growth in time. For this reason, instead of outputting the constraint energy at every light crossing time, we output at every time step and average the constraint energy through each light crossing. Then this average value is assigned to be the value of the constraint energy at every light crossing time, which is what we have plotted in Fig. 11 for the KST and ADM systems.

From this figure it is clear that the ADM and KST constraint energies stay roughly constant throughout the entire run. For the 2D high amplitude gauge wave, the damping properties of the PADM formulation are still evident, but not at the same level as in the 1D case. Nevertheless, the PADM system controls the constraint violations better than the KST and ADM systems.

D. Gowdy spacetimes

The polarized Gowdy T3 spacetimes are solutions of the Einstein equations which describe an expanding (or contracting) universe containing plane polarized gravitational waves, see for example [9] and references therein.

1. Expanding solution

The expanding Gowdy metric is usually written as

$$ds^2 = t^{-1/2} e^{\lambda/2} (-dt^2 + dz^2) + tdw^2, \quad (64)$$

where

$$dw^2 = e^P dx^2 + e^{-P} dy^2. \quad (65)$$

The solution which is considered standard is the one with

$$\begin{aligned} \lambda(z, t) = & 2\pi^2 t^2 (J_0(2\pi t)^2 + J_1(2\pi t)^2) \\ & - 2\pi t J_0(2\pi t) J_1(2\pi t) \cos(2\pi z) \\ & + \pi J_0(2\pi) J_1(2\pi) - 2\pi^2 (J_0(2\pi)^2 + J_1(2\pi)^2) \end{aligned} \quad (66)$$

and

$$P(z, t) = J_0(2\pi t) \cos(2\pi z), \quad (67)$$

where J_n denotes the Bessel function of the first kind of order n .

The nonzero dynamical variables in the context of the first-order ADM formulation can be directly derived from (64) and they are presented in Appendix B. By use of Eqs. (64) and (65), it is straightforward to show that the lapse is harmonic and that

$$\alpha = t^{-1} \gamma^{1/2}. \quad (68)$$

The expanding Gowdy wave test has proved to be one of the most challenging problems in numerical relativity. This

is because the metric components grow exponentially with time and hence the truncation errors introduced in the numerical solution grow with time, if the resolution is fixed. Consequently, the numerical solution soon veers off the analytical one. Another result of the exponential growth is that very soon the dynamical variables grow so large that the computations cannot be handled by using standard 64-bit floating-point arithmetic. Therefore, all simulations of the expanding Gowdy solution are expected to be terminated [9].

We start our expanding Gowdy wave simulations at $t = 1$ and evolve the initial data forward in time using the densitized lapse (68). We run the test for resolutions $r = 6, 7, 8$ and until the code crashes.

The results of the simulations are summarized in Fig. 12, where we plot the constraint energy [Fig. 12(a)] and error energy [Fig. 12(b)] for the ADM, KST, and PADM systems for resolution $r = 8$. These plots show that the runs with the KST formulation explode very quickly, whereas the ADM and PADM formulations can run the test for significantly longer times. The error energy and constraint energy of the PADM system lies lower than that of the ADM system and for this reason the PADM formulation extends the lifetime of the simulation by 12 light crossing times.

We checked that the simulations with the KST system do not crash because of violation of the numerical stability criterion. Instead, like in the gauge wave and robust tests, the error growth of the Gowdy simulations with KST system can be explained by the mixing and excitation of unwanted degrees of freedom the KST equations cause, because of violations of the momentum constraints. However, the situation with the expanding polarized Gowdy wave is much more severe than the gauge wave. This is simple to understand by a glimpse at the evolution equation of $\mathcal{C}_{333}$ which is

$$\partial_t \mathcal{C}_{333} = -\frac{3}{5} \alpha \gamma_{33} \mathcal{M}_3. \quad (69)$$

Equation (69) shows that, because of the exponential growth of both the lapse function and the 3-metric with time, the $\mathcal{C}_{333}$ constraint is bound to grow exponentially with time. The RHS of (69) varies roughly as $t^{-3} e^{\lambda/4} \mathcal{M}_3$. By using the analytic solution we can estimate the RHS of (69). For example, at $t = 50$ even if $\mathcal{M}_3$ were $\mathcal{M}_3 = 10^{-16}$ the RHS of Eq. (69) is of the order of 10^{46} .

Although we are able to evolve expanding Gowdy wave initial data with the ADM and PADM systems to very late times, it is seen in Fig. 12(b) that at the highest resolution the numerical solution matches the analytical solution up to approximately 110 light crossings with the ADM formulation and up to 122 light crossings with the PADM formulation. At these times the normalized error energy becomes of order 0.1. The accuracy with the PADM system is completely lost after roughly $t = 275$ and with the ADM system after roughly $t = 258$. We note here that in [35] the 1D (not 3D) simulations of the expanding Gowdy wave

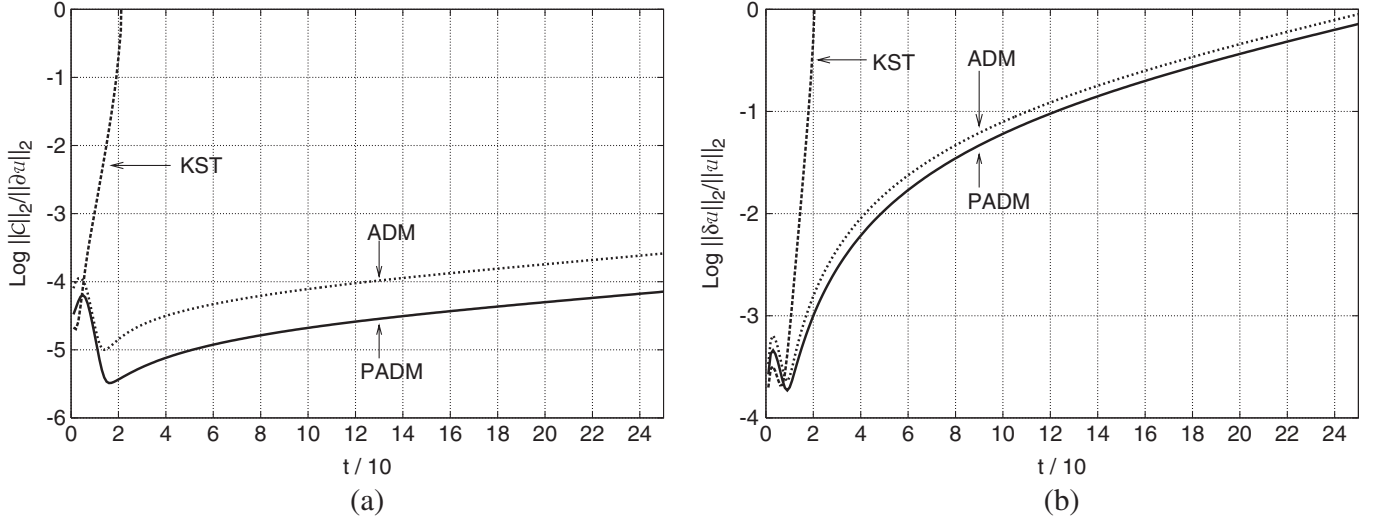


FIG. 12. Errors as functions of coordinate time t for the expanding Gowdy wave for resolution $r = 8$. On the left panel (a) we show the normalized constraint energies and on the right panel (b) we show the normalized error energies for the ADM, KST, and PADM formulations.

with the KST system completely lose accuracy after 150 light crossing times, even though the authors use a much more sophisticated and accurate method of integration (i.e., pseudospectral methods) in order to perform their simulations. For resolution $r = 9$ we can accurately evolve the solution with the PADM system up to approximately 200 light crossings.

Taking all these facts into account, we conclude that the comparison of the three formulations in the expanding Gowdy wave test bed shows that the PADM formulation performs better than the ADM formulation and significantly better than the KST formulation.

2. Collapsing Gowdy cosmology

Unlike the expanding solution, which moves away from the $t = 0$ singularity, the collapsing solution will reach the singularity very soon if integrated in the coordinate time t as is dictated by the metric (64). For this reason a coordinate transformation is implemented by asking for a new coordinate time τ , such that $t = F(\tau) = ke^{c\tau}$ with k, c constants given by $c \approx 0.002119511921460752$ and $k \approx 9.670769812764059$. In this new coordinate time the singularity is reached at $\tau = -\infty$ and the numerical integration is much more tractable. The metric in the new coordinates is given by

$$ds^2 = -\alpha(\tau)^2 d\tau^2 + F(\tau)^{-1/2} e^{\lambda(z, F(\tau))/2} dz^2 + \tau dw^2, \quad (70)$$

where

$$\alpha(\tau) = cF(\tau)^{3/4} e^{\lambda(z, F(\tau))/4}, \quad (71)$$

and

$$dw^2 = e^{P(z, F(\tau))} dx^2 + e^{-P(z, F(\tau))} dy^2. \quad (72)$$

The nonzero dynamical variables in the context of the first-order ADM formulation can be easily derived by the use of (70) and they are presented in Appendix C. The lapse function is harmonic and it is straightforward to show that

$$\alpha = c\gamma^{1/2}. \quad (73)$$

We start the simulations at the 20th zero of the $J_0(2\pi t)$ Bessel function, i.e., at $\tau_o \approx 9.87532058290983$ and evolve the initial data backwards in time using (73). We run the test for $r = 6, 7, 8$ and for 1000 light crossing times with all formulations considered in this work. Finally, we note that for this test the values of the PADM parameters are opposite to those of Eq. (27).

The results of our simulations are summarized in Fig. 13 where we show the constraint energies and the error energies for all formulations considered here. We can see in 13(a) that the ADM constraint energy grows in time and becomes larger than the PADM and KST constraint energies very early on. At 1000 light crossing times, the ADM constraint energy is about 2 orders of magnitude larger than the PADM and KST constraint energies. However, we see in Fig. 13(b) that the ADM error energy lies below the PADM and KST error energies for up to about 300 light crossing times. After 500 crossing times, the ADM error grows faster than the PADM and KST error and at 1000 crossing times there is roughly 1 order of magnitude difference between the ADM error energy and the corresponding error energies of the PADM and KST formulations.

Figure 13 shows that the PADM and KST systems perform equally well. However, for the most part of the 1000 crossing times the PADM constraint and error energies are less than those of the runs with the KST system. Towards the end of the simulation, the KST error energy is

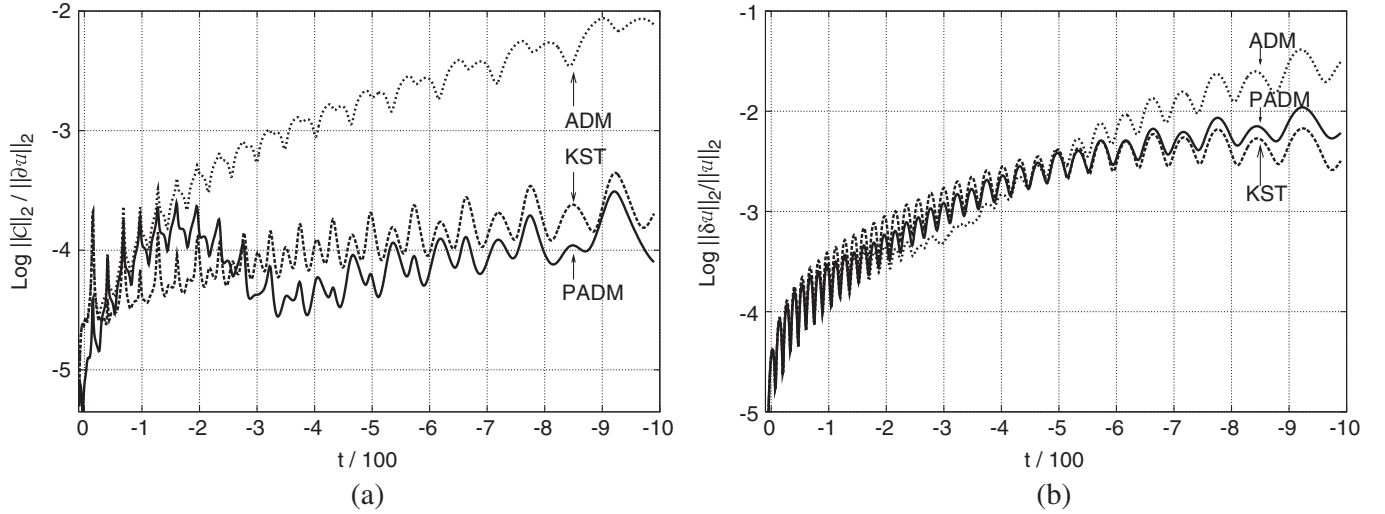


FIG. 13. Errors as functions of coordinate time t for the collapsing Gowdy case for resolution $r = 7$. On the left panel (a) we show the normalized constraint energies and on the right panel (b) we show the normalized error energies for the ADM, KST, and PADM formulations.

slightly less than the PADM error energy, while the PADM constraint energy is slightly less than the KST constraint energy.

V. CONVERGENCE

In this section we test the convergence and accuracy of our numerical schemes. In order to test the order of convergence, we need to use exact solutions to the sets of PDEs which we solve. Of the four tests we performed for the comparison of the ADM, KST, and PADM formulations, only the gauge wave and polarized Gowdy spacetimes are exact solutions of the Einstein equations and only these can form a real basis for convergence testing.

Unfortunately, the gauge wave results suffer by the excitation of exponential harmonic gauge modes, and for this reason they cannot serve as a basis for convergence testing for long time. This is so because of the properties of the solutions and not because of our numerical scheme. We demonstrated in Sec. IV B that for different resolutions different exponential modes are excited and at different time scales. Thus, convergence testing with the gauge wave can be done only for short periods of time before exponential modes become noticeable. We have been able to show that our code is indeed convergent and second-order accurate for short periods of time in the gauge wave test.

The expanding Gowdy test cannot serve as a basis for convergence testing for long times either because the test cannot be run for very long times. For example, the KST formulation cannot even complete 22 light crossing times in the expanding Gowdy cosmology. Again this is because of the properties of the solution and not because of our numerical scheme. Nevertheless, we have been able to show that our code is convergent and second-order accurate in simulations of the expanding Gowdy wave with the

ADM, KST, and PADM systems up until the simulations crash. In order to keep this discussion short, we omit the convergence plots for the expanding Gowdy and gauge wave tests, but they can be available by the authors upon request.

In order to test long term convergence of our code, we will use our results of the collapsing Gowdy wave test bed, where all three formulations successfully complete 1000 light crossing times. The convergence plots can be seen in Figs. 14–16 for the ADM, KST, and PADM formulations respectively, where it is evident that with increasing resolution the numerical solution converges to the analytical one for all formulations.

It is also important to demonstrate that the order of convergence of our code is 2. To do this we use our data

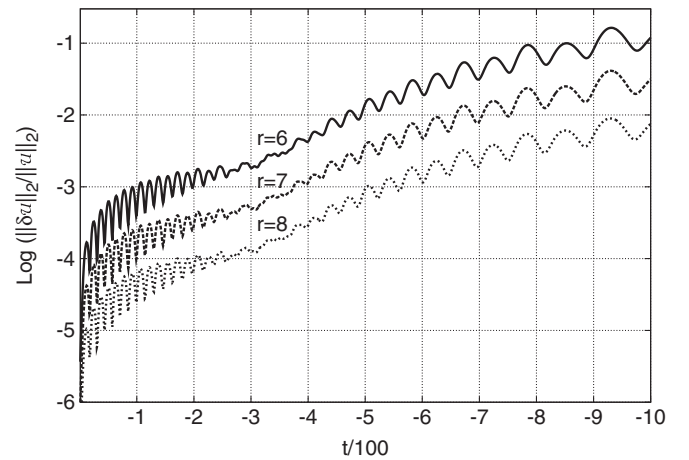


FIG. 14. Normalized error energies as functions of coordinate time t , for resolutions $r = 6, 7, 8$. The plot corresponds to the simulations of the collapsing Gowdy spacetime with the ADM formulation.

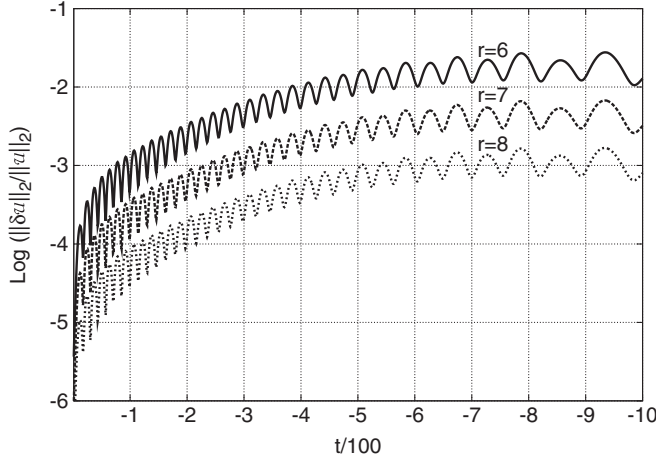


FIG. 15. Normalized error energies as functions of coordinate time t , for resolutions $r = 6, 7, 8$. The plot corresponds to the simulations of the collapsing Gowdy spacetime with the KST formulation.

for the error energies and calculate the order of convergence ($\mathcal{F}$), which is given by

$$\mathcal{F}(t_n, r_1, r_2) = \frac{1}{r_2 - r_1} \log_2 \left(\frac{\|\delta \mathcal{U}_N(t_n, r_1)\|}{\|\delta \mathcal{U}_N(t_n, r_2)\|} \right), \quad (74)$$

where r_1 and r_2 two arbitrary resolutions, t_n a given time step, and $\|\delta \mathcal{U}_N(t_n, r)\|$ is the normalized error energy. Exact second-order accuracy means that $\mathcal{F} = 2$. In Table I we show the time average of $\mathcal{F}$ for the collapsing Gowdy spacetime with all three formulations and resolutions $\{r_1, r_2\} = \{5, 6\}$, $\{r_1, r_2\} = \{6, 7\}$, and $\{r_1, r_2\} = \{7, 8\}$. The time-average order of convergence, $\bar{\mathcal{F}}$, is

$$\bar{\mathcal{F}}(r_1, r_2) = \frac{1}{N} \sum_{n=0}^N \mathcal{F}(t_n, r_1, r_2), \quad (75)$$

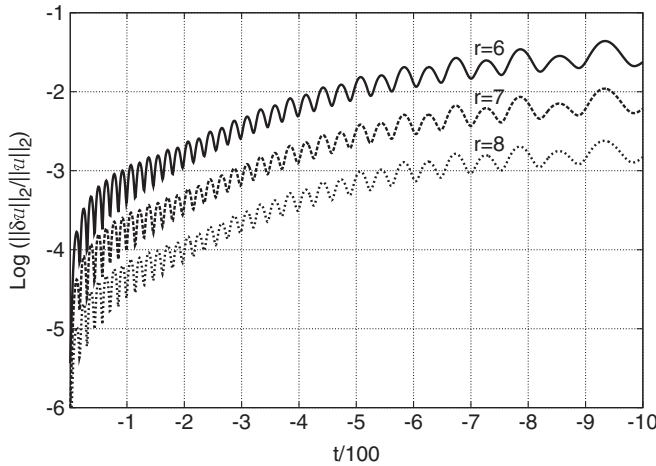


FIG. 16. Normalized error energies as functions of coordinate time t , for resolutions $r = 6, 7, 8$. The plot corresponds to the simulations of the collapsing Gowdy spacetime with the PADM formulation.

TABLE I. Convergence order table. The first column shows the resolutions which are involved in the calculation of the order of convergence and the second to fourth columns show the order of convergence of our code with the ADM, KST, and PADM formulations, respectively.

r_1, r_2	ADM	KST	PADM
5, 6	1.9057	2.0461	1.9617
6, 7	1.9793	2.0146	1.9945
7, 8	2.0296	2.0032	2.0144

where in our case $t_n = n$ and $N = 1001$. Table I and Figs. 14–16 clearly demonstrate that our code is convergent and second-order accurate.

VI. CONCLUSIONS

We have described a stable, convergent, and second-order accurate numerical scheme for solving the equations of the recently proposed PADM formulation of GR; we have tested the accuracy and stability of the PADM system and compared it with the first-order ADM and KST evolution systems.

The ADM system is generally ill posed, but in conjunction with a densitized lapse or “1 + log” slicing the 1D ADM evolution equations are strongly hyperbolic and can be compared to other well-posed systems. The KST evolution system in conjunction with either a densitized lapse or “1 + log” slicing is strongly hyperbolic. The well-posed PADM equations have the structure of a mixed hyperbolic–second-order parabolic set of PDEs. PADM possesses the desirable property of constraint damping. Unlike all other systems used in numerical relativity with constraint damping, the PADM system damps the short wavelength constraint-violating modes more efficiently than the long wavelength ones.

The series of tests which we used to compare the ADM, KST, and PADM systems is known as the “Apples with Apples” or the “Mexico City” tests. These tests are designed to probe both the linear and the nonlinear regime of the Einstein equations and they are called the robust stability test, the gauge wave stability test, the linear wave stability test, and the Gowdy wave stability test. The first three can be considered as perturbations about flat space-time, so they probe the weak field limit of the Einstein equations, whereas the last one probes the strong field regime.

We ran the robust stability test with a densitized lapse with $\sigma = 0.5, 1$ and “1 + log” slicing and we demonstrated that the PADM formulation performs much better than both the ADM and KST systems. The constraint violations with PADM are about 7 orders of magnitude smaller than the constraint violations with the KST system. PADM damps the constraints very quickly and the shorter the wavelength of the constraint violations the faster they are damped. This is in complete agreement with the theo-

retical properties of PADM. Finally, the ADM formulation crashes very quickly and faster for higher resolutions. Finally, we demonstrated that the behavior of all three systems in this test is similar in the other gauge conditions we employed. This confirms the theoretical prediction that the PADM formulation damps the constraints independently of the gauge chosen.

The simulations of the high amplitude 1D gauge wave test clearly demonstrate that the PADM formulation performs better than the ADM formulation, while the KST system cannot evolve the initial data for more than 200 crossing times. The reason for this behavior of the KST system is that the dynamics of the formulation is such that it excites unwanted degrees of freedom, due to error in the momentum constraints. These degrees of freedom are not part of the dynamics of the gauge wave test bed and after they are excited, they grow exponentially with time and lead the computations to an end.

The simulations of the high amplitude 2D gauge wave test also demonstrate that the numerical performance of the PADM formulation is much better than those of the ADM and KST systems. In this case it is the ADM simulations which crash very quickly. The runs with the KST equations cannot evolve the initial data for more than 30 crossing times no matter how large the resolution is. Finally, the runs with PADM can successfully complete 100 light crossing times and show that the PADM system damps the constraints with time.

The linear wave test bed presented no essential difficulty for any of the three formulations, but both the 1D and the 2D linear wave simulations show that the PADM formulation performs better than the ADM and KST formulations. In the 1D case the constraint error with the PADM system is quickly damped and is 3 orders of magnitude less than the constraint error with the ADM and KST systems. We observed similar behavior in the 2D linear wave, but the difference in the constraint error among the three formulations is not as large.

The polarized Gowdy wave test bed has two versions: the expanding one and the collapsing one. In the collapsing Gowdy wave, the KST and PADM systems perform equally well, but much better than the ADM one. At the end of the simulations the ADM constraint error is at least 2 orders of magnitude larger than the constraint error of the PADM and KST systems and the error in the solution with the ADM system is 1 order of magnitude larger than the corresponding errors with the KST and PADM systems.

The expanding Gowdy wave was by far the strongest field solution we simulated. The crash of simulations of the expanding Gowdy solution is naturally expected for any evolution system because the metric grows exponentially with time. The results of this test show that the PADM formulation performs better than the ADM formulation and dramatically better than the KST formulation. The expanding Gowdy wave initial data cannot be evolved

with the KST system for more than 25 crossing times at the highest resolution we attempted. The cause for the sudden breakdown of the KST system is the same as the cause for the sudden breakdown of the KST system in the 1D gauge wave. In contrast, the ADM and PADM systems can evolve the expanding Gowdy wave solution for significantly longer times, until the error in the solution is so polluted that the simulations terminate. Nevertheless, the PADM errors are less than those of the ADM and for this reason the PADM system can prolong the simulations by about 12 light crossing times.

An important comment on the tests considered in this work regards the generality of our results with respect to the parameters chosen. One may object to the results of our KST simulations because we focused on a certain set of parameters for this formulation. However, we believe that our results are quite general for the following reasons: (a) Our experiments with the KST formulation showed that the flat space simulations with the parameters presented here were better than any other choice we attempted. This is also supported by the theoretical argument that the KST parameters chosen in this paper should result in better simulations because this choice minimizes the number of flat space modes which violate the Hamiltonian constraint. (b) In addition, the PADM system should have better control of the constraint violations than both the KST and ADM systems because it has constraint damping.

Finally, all tests considered in this work describe vacuum spacetimes. One cannot help but wonder whether the properties of the PADM system will remain the same when tested in nonvacuum spacetimes. In nonvacuum spacetimes errors will unavoidably be introduced in the gravity equations by matter terms. Fortunately, matter terms come as low order terms in the evolution equations and hence they cannot destroy the well posedness of the system. The parabolic nature of the constraint propagation equations holds true whether vacuum or nonvacuum cases are considered, and hence constraint dissipation should be present in nonvacuum spacetimes as well. However, the study of such spacetimes is complicated and unless detailed numerical simulations are carried out it is almost impossible to theoretically analyze the exact behavior of the PADM system. A study of nonvacuum cases is beyond the scope of the current paper and will be the subject of a future work.

Taking all these facts into accounts, we conclude that PADM successfully passes the standard tests of numerical relativity and works equally well with a variety of algebraic gauges. Via the comparison of the numerical performance of the PADM formulation and those of the ADM and the KST formulations, we conclude that PADM has better control of the constraint violations than both ADM and KST. The PADM system performs better than the ADM one and in most tests better than the KST system.

APPENDIX A: PARABOLIC AND MIXED HYPERBOLIC-SECOND-ORDER PARABOLIC EQUATIONS

In this Appendix we present some ideas and theorems concerning quasilinear parabolic partial differential equations and quasilinear mixed hyperbolic-second-order parabolic partial differential equations. For a more complete description of such equations see [24], where the concepts of this Appendix have been taken from.

Consider a quasilinear even-order operator of the form

$$P(\partial_x) = P_{2m}(\partial_x) + Q(\partial_x), \quad (\text{A1})$$

where

$$P_{2m}(\partial_x) = \sum_{|\nu|=2m} A_\nu \partial_1^{\nu_1} \partial_2^{\nu_2} \cdots \partial_d^{\nu_d} \quad (\text{A2})$$

and

$$Q(\partial_x) = \sum_{|\nu| \leq 2m-1} A_\nu \partial_1^{\nu_1} \partial_2^{\nu_2} \cdots \partial_d^{\nu_d}, \quad (\text{A3})$$

where $\sum_{i=1,d} \nu_i = \nu$, $\partial_s^{\nu_s}$ denotes the partial derivative with respect to x^s of order ν_s , $s = 1, \dots, d$ with d the number of spatial dimensions, u is the column vector of the n dynamical variables, and A_ν are $n \times n$ matrix functions of the dynamical variables but not their derivatives.

Definition 1.—The equation $\partial_t u = P(\partial_x)u$ is called parabolic, if for all $\kappa \in \mathbf{R}^d$ the eigenvalues $\omega_j(\kappa)$, $j = 1, \dots, n$ of $P_{2m}(i\kappa) = -\sum_{\nu=2m} A_{2m} \kappa_1^{\nu_1} \cdots \kappa_d^{\nu_d}$ satisfy

$$\text{Re}(\omega_j(\kappa)) \leq -\delta |\kappa|^{2m}, \quad j = 1, \dots, n, \quad (\text{A4})$$

with some $\delta > 0$ independent of κ , where in the symbol $P_{2m}(i\kappa)$, $i^2 = -1$.

Theorem 1.—The Cauchy problem for a parabolic system $\partial_t u = P(\partial_x)u$ is well-posed.

Assume now that a system of partial differential equations can be written in the following form:

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} P_2 & 0 \\ 0 & P_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & R_{12} \\ R_{21} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (\text{A5})$$

where u and v are dynamical variables, $P_2 = P_2(\partial_x)$ is a second-order differential operator, $P_1 = P_1(\partial_x)$ is a first-order operator, and the coupling terms $R_{12} = R_{12}(\partial_x)$ and $R_{21} = R_{21}(\partial_x)$ are general first-order differential operators. Equation (A5) may or may not have zeroth-order couplings. Further assume that the uncoupled systems

$$\partial_t u = P_2(\partial_x)u \quad \partial_t v = P_1(\partial_x)v \quad (\text{A6})$$

are second-order parabolic and first-order strongly hyperbolic (for definitions of hyperbolic equation see [31]), respectively.

Definition 2.—The coupled system (A5) is called mixed strongly hyperbolic-second-order parabolic.

The coupling terms R_{12} and R_{21} or any low order terms do not destroy the well posedness of the system and the following theorem can be proved.

Theorem 2.—The Cauchy problem for mixed second-order parabolic-strongly hyperbolic systems (A5) is well posed.

APPENDIX B: EXPANDING GOWDY SOLUTION

In this Appendix we present the nonzero dynamical variables, in the context of the first-order ADM formulation, for the expanding Gowdy solution. Those can be directly derived by use of Eqs. (2), (3), and (64)–(67) and they are given by

$$\gamma_{xx} = te^P, \quad \gamma_{yy} = te^{-P}, \quad \gamma_{zz} = t^{-1/2} e^{\lambda/2}, \quad (\text{B1})$$

$$K_{xx} = -\frac{1}{2\alpha} g_{xx}(t^{-1} + \partial_t P),$$

$$K_{yy} = -\frac{1}{2\alpha} g_{yy}(t^{-1} - \partial_t P), \quad (\text{B2})$$

$$K_{zz} = \frac{1}{4\alpha} g_{zz}(t^{-1} - \partial_t \lambda),$$

where

$$\alpha = \sqrt{\gamma_{zz}}, \quad (\text{B3})$$

$$\partial_t \lambda(z, t) = 2\pi^2 t [(\cos(4\pi z) + 1)J_1(2\pi t)^2 + 2J_0(2\pi t)^2 \sin^2(2\pi z)] \quad (\text{B4})$$

and

$$\partial_t P(z, t) = -2\pi J_1(2\pi t) \cos(2\pi z). \quad (\text{B5})$$

We also have

$$D_{zxx} = \gamma_{xx} \partial_z P, \quad D_{zyy} = -\gamma_{yy} \partial_z P, \quad (\text{B6})$$

$$D_{zzz} = \frac{1}{2} \gamma_{zz} \partial_z \lambda,$$

where

$$\partial_z P(z, t) = -2\pi J_0(2\pi t) \sin(2\pi z), \quad (\text{B7})$$

$$\partial_z \lambda(z, t) = 4\pi^2 t J_0(2\pi t) J_1(2\pi t) \sin(4\pi z).$$

APPENDIX C: COLLAPSING GOWDY SOLUTION

In this Appendix we present the nonzero dynamical variables, in the context of the first-order ADM formulation, for the collapsing Gowdy solution. Those can be directly derived by use of Eqs. (2), (3), and (70)–(72) and they are given by

$$\gamma_{xx} = F(\tau) e^{P(z, F(\tau))}, \quad \gamma_{yy} = F(\tau) e^{-P(z, F(\tau))}, \quad (\text{C1})$$

$$\gamma_{zz} = F(\tau)^{-1/2} e^{\lambda(z, F(\tau))/2},$$

and

$$\begin{aligned}
K_{xx} &= -\frac{c}{2\alpha} \gamma_{xx} \left(1 + F \frac{\partial P}{\partial F}\right), \\
K_{yy} &= -\frac{c}{2\alpha} \gamma_{yy} \left(1 - F \frac{\partial P}{\partial F}\right), \\
K_{zz} &= \frac{c}{4\alpha} \gamma_{zz} \left(1 - F \frac{\partial \lambda}{\partial F}\right),
\end{aligned} \tag{C2}$$

where the lapse function α is given by (71). In addition, we have

$$\begin{aligned}
\frac{\partial \lambda(z, F)}{\partial F} &= 2\pi^2 F [(\cos(4\pi z) + 1)J_1(2\pi F)^2 \\
&\quad + 2J_0(2\pi F)^2 \sin^2(2\pi z)],
\end{aligned} \tag{C3}$$

and

$$\frac{\partial P(z, F)}{\partial F} = -2\pi J_1(2\pi F) \cos(2\pi z). \tag{C4}$$

Finally the nonzero components of the D_{kij} variables are given by

$$\begin{aligned}
D_{zxx} &= \gamma_{xx} \partial_z P, & D_{zyy} &= -\gamma_{yy} \partial_z P, \\
D_{zzz} &= \frac{1}{2} \gamma_{zz} \partial_z \lambda,
\end{aligned} \tag{C5}$$

where

$$\begin{aligned}
\partial_z P(z, F) &= -2\pi J_0(2\pi F) \sin(2\pi z), \\
\partial_z \lambda(z, F) &= 4\pi^2 F J_0(2\pi F) J_1(2\pi F) \sin(4\pi z).
\end{aligned} \tag{C6}$$

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