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Homogeneous noncommutative quantum cosmology

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Using the Groenewold-Moyal product, the noncommutative Bianchi IX model is constructed by imposing commutation relations on the minisuperspace variables $(\Omega, \beta_+, \beta_-)$. A noncommutative "wormhole" solution to the corresponding Wheeler-DeWitt equation is constructed and its behavior at fixed Ω is analyzed.

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I. INTRODUCTION

In this work we consider quantum cosmological models for Bianchi type IX models in the generalized vacuum scalar-tensor theories of gravity defined by the action

$$A = \int d^4x\sqrt{-g}R. \tag{1}$$

The line element for the class of spatially homogeneous space-times is given by

$$ds^2 = -dt^2 + h_{ab}\omega^a\omega^b, \qquad a, b = 1, 2, 3, \tag{2}$$

where h_{ab} is a function of cosmic time t and represents the metric on the surfaces of homogeneity and ω^a are one-forms. The three-metric may be parametrized by $h_{ab} = R_0^2 e^{-2\Omega}(e^{2\beta(t)})_{ab}$, where $e^{-3\Omega}$ represents the effective volume of the universe and

$$\beta_{ab} := \mathrm{diag}(\beta_+ + 2\sqrt{3}\beta_-, \beta_+ - 2\sqrt{3}\beta_-, -2\beta_+) \tag{3}$$

is a traceless matrix that determines the anisotropy of the universe.

According to Misner [1,2], the corresponding Hamiltonian is

$$H^2 = p_+^2 + p_-^2 - 24\pi^{2(3)}Rg \tag{4}$$

where

$$^{(3)}Rg = -\tfrac{3}{32}R_0^4 e^{-4\Omega}(V-1) \tag{5}$$

and

$$\begin{aligned} V(\Omega, \beta_+, \beta_-) &:= 1 + \tfrac{1}{3}\mathrm{Tr}(e^{4\beta} - 2e^{-2\beta}) \\ &= 1 + \tfrac{2}{3}e^{4\beta_+}[\cosh(4\sqrt{3}\beta_-) - 1] + \tfrac{1}{3}e^{-8\beta_+} \\ &\quad - \tfrac{4}{3}e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-). \end{aligned} \tag{6}$$

From Eq. (4) we obtain then

$$[-p_\Omega^2 + p_+^2 + p_-^2 + e^{-4\Omega}(V-1)]\Psi(\Omega, \beta_+, \beta_-) = 0 \tag{7}$$

where we have set $H^2 = p_\Omega^2$ and the scale factor R_0 has also been chosen as $R_0 = \sqrt{\frac{2}{3\pi}}$.

Equation (7) admits the rewriting [3]

$$[-p_\Omega^2 + p_+^2 + p_-^2 - \Phi_{,\Omega}^2 + \Phi_{,\beta_+}^2 + \Phi_{,\beta_-}^2] \times \Psi(\Omega, \beta_+, \beta_-) = 0 \tag{8}$$

where

$$\begin{aligned} \Phi(\Omega, \beta_+, \beta_-) &:= \tfrac{1}{6}e^{-2\Omega}\,\mathrm{Tr}(e^{2\beta}) \\ &= \tfrac{1}{6}e^{-2\Omega}[2e^{2\beta_+}\cosh(2\sqrt{3}\beta_-) + e^{-4\beta_+}]. \end{aligned} \tag{9}$$

Several solutions are known to Eq. (8) in the commutative case [4–8]. The simplest solution is obtained by rewriting Eq. (8) in the form

$$[(\partial_\Omega - \Phi_{,\Omega})(\partial_\Omega + \Phi_{,\Omega}) - (\partial_+ - \Phi_{,\beta_+})(\partial_+ + \Phi_{,\beta_+}) - (\partial_- - \Phi_{,\beta_-})(\partial_- + \Phi_{,\beta_-})]\Psi(\Omega, \beta_+, \beta_-) = 0 \tag{10}$$

after factorization followed of the quantization ($\hbar = 1$)

$$p_\Omega := -i\partial_\Omega, \qquad p_+ := -i\partial_{\beta_+} =: -i\partial_+, \qquad p_- := -i\partial_{\beta_+} =: -i\partial_-. \tag{11}$$

One obtains thus the set of equations

$$(\partial_\Omega + \Phi_{,\Omega})\Psi = 0, \qquad (\partial_+ + \Phi_{,\beta_+})\Psi = 0, \qquad (\partial_- + \Phi_{,\beta_-})\Psi = 0 \tag{12}$$

leading to the "wormhole" solution [6]

$$\Psi_{\mathrm{W.H.}} = \exp[-\Phi]. \tag{13}$$

A straightforward calculation shows that from Eq. (10) we can write the Wheeler-DeWitt equation for Bianchi IX cosmology as

$$[\partial_\Omega^2 - \partial_+^2 - \partial_-^2 - 12\Phi + e^{-4\Omega}(V-1)]\Psi(\Omega, \beta_+, \beta_-) = 0. \tag{14}$$

The term proportional to Φ arises due to factor ordering [3].

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II. NONCOMMUTING CASE

Several models [9–11] involving noncommutative deformations have been studied in the last few years based on the use of the Groenewold-Moyal (GM) product [12,13], which is defined as

$$(f \star g)(x) := \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu\partial'_\nu\right)f(x)g(x')|_{x=x'}.$$

The GM star product acting on the variables Ω, β_+, β_- of minisuperspace will be defined with the help of the following commutators

$$[\Omega, \beta_+]_{\star_1} = i\theta^1, \qquad [\Omega, \beta_-]_{\star_2} = i\theta^2, \qquad [\beta_+, \beta_-]_{\star_3} = i\theta^3. \tag{15}$$

In the above we have chosen the symbols $\star_i$ to denote the different star products.

We shall now introduce noncommutativity in the model presented in Sec. I. Our starting point will be the equivalent of Eq. (10) before quantization, namely

$$[(ip_\Omega - \Phi_{,\Omega})\star(ip_\Omega + \Phi_{,\Omega}) - (ip_+ - \Phi_{,\beta_+})\star(ip_+ + \Phi_{,\beta_+}) - (ip_- - \Phi_{,\beta_-})\star(ip_- + \Phi_{,\beta_-})]\star\Psi(\Omega, \beta_+, \beta_-) = 0. \tag{16}$$

Here $\star := \star_1 \circ \star_2 \circ \star_3$.

The GM star product affects effectively in Eq. (16) the terms involving the function Φ. However, it is known that the commutation relations given by Eq. (15) can be implemented in terms of commutative variables and ordinary product of functions through the replacements

$$\begin{aligned}\Omega &\to \Omega - \tfrac{1}{2}\theta^1 p_+ - \tfrac{1}{2}\theta^2 p_-,\\ \beta_+ &\to \beta_+ + \tfrac{1}{2}\theta^1 p_\Omega - \tfrac{1}{2}\theta^3 p_-,\\ \beta_- &\to \beta_- + \tfrac{1}{2}\theta^2 p_\Omega + \tfrac{1}{2}\theta^3 p_+\end{aligned} \tag{17}$$

where

$$[\Omega, p_\Omega] = i, \qquad [\beta_+, p_+] = i, \qquad [\beta_-, p_-] = i. \tag{18}$$

This way of proceeding is known generally as a Bopp shift map [14–16]: it will modify the dependence of Φ (and V) on the variables Ω, β_+, β_-, which are now commuting variables.

A. Noncommutative "wormhole" solution

The simplest solution to look for is the GM deformed one generalizing Eq. (13). Actually, in the spirit of the commutative "wormhole" solution we obtain the constraints

$$(ip_\Omega + \Phi_{,\Omega})\star\Psi = 0, \qquad (ip_+ + \Phi_{,\beta_+})\star\Psi = 0, \qquad (ip_- + \Phi_{,\beta_-})\star\Psi = 0. \tag{19}$$

After the quantization

$$p_\Omega := -i\partial_\Omega, \qquad p_+ := -i\partial_+, \qquad p_- := -i\partial_-, \tag{20}$$

we finally arrive to the set of equations

$$(\partial_\Omega + \Phi_{,\Omega})\star\Psi = 0, \qquad (\partial_+ + \Phi_{,\beta_+})\star\Psi = 0, \qquad (\partial_- + \Phi_{,\beta_-})\star\Psi = 0. \tag{21}$$

This system of equations clearly generalizes the commutative one in Eq. (12). Using now the Bopp shifts defined previously in Eq. (17) and taking into account Eq. (9) we have explicitly

$$\begin{aligned}\partial_\Omega\Psi &= +\frac{1}{3}e^{-2\Omega - i\theta^1\partial_+ - i\theta^2\partial_-}[2e^{2\beta_+ - i\theta^1\partial_\Omega + i\theta^3\partial_-}\cosh(2\sqrt{3}\beta_- - i\sqrt{3}\theta^2\partial_\Omega - i\sqrt{3}\theta^3\partial_+) + e^{-4\beta_+ + 2i\theta^1\partial_\Omega - 2i\theta^3\partial_-}]\Psi,\\ \partial_+\Psi &= -\frac{1}{6}e^{-2\Omega - i\theta^1\partial_+ - i\theta^2\partial_-}[4e^{2\beta_+ - i\theta^1\partial_\Omega + i\theta^3\partial_-}\cosh(2\sqrt{3}\beta_- - i\sqrt{3}\theta^2\partial_\Omega - i\sqrt{3}\theta^3\partial_+) - 4e^{-4\beta_+ + 2i\theta^1\partial_\Omega - 2i\theta^3\partial_-}]\Psi,\\ \partial_-\Psi &= -\frac{2}{\sqrt{3}}e^{-2\Omega - i\theta^1\partial_+ - i\theta^2\partial_-}e^{2\beta_+ - i\theta^1\partial_\Omega + i\theta^3\partial_-}\sinh(2\sqrt{3}\beta_- - i\sqrt{3}\theta^2\partial_\Omega - i\sqrt{3}\theta^3\partial_+)\Psi.\end{aligned} \tag{22}$$

The system just written has a rather complicated look. It is therefore desirable to find a suitable way to find a solution to it. Consider then the system of equations in Eq. (21), which we write as

$$\vec{\nabla}\Psi = -\vec{\nabla}\Phi\star\Psi, \qquad \vec{\nabla} := \begin{pmatrix}\partial_\Omega\\ \partial_+\\ \partial_-\end{pmatrix}. \tag{23}$$

We recall now the definition of the *path-ordered exponential*

$$Pe(t) := P_\star \exp\left[\int_0^t dt A(t)\right] \tag{24}$$

in the 1-dimensional case via its initial value problem

$$\frac{\partial Pe(t)}{\partial t} = A(t)\star Pe(t), \qquad Pe(0) = 1. \tag{25}$$

The path ordering P in Eq. (24) is done with respect of the $\star$-product. In terms of a Taylor series development one has the expression

$$Pe(t) = 1 + \int_0^t dt_1 A(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 A(t_1) \star A(t_2) + \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 A(t_1) \star A(t_2) \star A(t_3) + \ldots \tag{26}$$

where ... means higher order terms constructed in an obvious way. It is then clear that for our problem under study we can write the following expression as a solution to Eq. (23)

$$\Psi(\Omega, \beta_+, \beta_-) = P_\star \exp\left[-\int_\Gamma \vec{ds}\cdot\vec{\nabla}\Phi\right] = P_\star \exp\left[-\int_0^1 d\sigma \frac{\vec{ds}}{d\sigma}\cdot\vec{\nabla}\Phi\right] \tag{27}$$

where

$$\vec{ds} := \begin{pmatrix} d\Omega \\ d\beta_+ \\ d\beta_- \end{pmatrix}$$

and Γ denotes a path in parameter space starting at the origin and ending at an arbitrary point $(\Omega, \beta_+, \beta_-)$ so that $\Psi(0, 0, 0) = 1$. In the second equality we have chosen to introduce a variable σ parametrizing the path Γ.

We should mention that equations of the same type as those in the system Eq. (21) have appeared in the context of noncommutative gauge theories [17–20]. In this case the relevant object is given by the expression

$$W(p, \vec{A}) = \int d^4x e^{-ip\cdot x} P_\star \exp\left[ie\int_0^1 d\sigma \frac{\vec{d\xi}}{d\sigma}\cdot \vec{A}(x + \xi(\sigma))\right] \tag{28}$$

with $\xi^i(0) = 0$, $\xi^i(1) = l^i$. This defines an open Wilson line and the choice of parametrization $\xi^i(\sigma) = \theta^{ij}p_j\sigma$ corresponds to a *straight* Wilson line [18].

III. LOWEST ORDER CONTRIBUTIONS

From the formal point of view our solution Eq. (27) has the same functional form as the commutative one. Two new features are present however, the path ordering prescription and the star product in the integrals associated to it. In contrast to previous works, a closed expression for Ψ seems to be rather difficult to write. We shall thus calculate the difference

$$\delta\Psi(\Phi, \vec{\theta}) = P_\star \exp\left[-\int_\Gamma \vec{ds}\cdot\vec{\nabla}\Phi\right] - \exp[-\Phi] \tag{29}$$

to first order on the deformation parameters $\vec{\theta} = (\theta^1, \theta^2, \theta^3)$. Using Eq. (26) we have then

$$\begin{aligned}\delta\Psi(\Phi, \vec{\theta}) &= \int_\Gamma ds_1^i \int_\Gamma ds_2^j (\Phi_{,i}\star\Phi_{,j} - \Phi_{,i}\Phi_{,j}) = \int_\Gamma ds_1^i \int_\Gamma ds_2^j (\Phi_{,i}^{B.\text{shift}} - \Phi_{,i})\Phi_{,j} \\ &= \int_\Gamma d\Omega_1 (\Phi_{,\Omega_1}^{B.\text{shift}} - \Phi_{,\Omega_1})\Phi + \int_\Gamma d\beta_{+1}(\Phi_{,+1}^{B.\text{shift}} - \Phi_{,+1})\Phi + \int_\Gamma d\beta_{-1}(\Phi_{,-1}^{B.\text{shift}} - \Phi_{,-1})\Phi \\ &= \int_\Gamma (d\Omega_1 \hat{S}_{\Omega_1}\Phi + d\beta_{+1}\hat{S}_{\beta_{+1}}\Phi + d\beta_{-1}\hat{S}_{\beta_{-1}}\Phi),\end{aligned} \tag{30}$$

where we have used the Bopp shifts in the second equality and introduced the following operators

$$\hat{S}_\Omega := -\frac{1}{3}e^{-2\Omega}\vec{A}_1\cdot\vec{X}, \qquad \hat{S}_{\beta_+} := +\frac{2}{3}e^{-2\Omega}\vec{A}_2\cdot\vec{X}, \qquad \hat{S}_{\beta_-} := +\frac{2}{\sqrt{3}}e^{-2\Omega}\vec{A}_3\cdot\vec{X} \tag{31}$$

with

$$\vec{A}_1 := \begin{pmatrix} 2e^{2\beta_+}\cosh(2\sqrt{3}\beta_-) + e^{-4\beta_+} \\ 2e^{2\beta_+}\cosh(2\sqrt{3}\beta_-) - 2e^{-4\beta_+} \\ 2\sqrt{3}e^{2\beta_+}\sinh(2\sqrt{3}\beta_-) \end{pmatrix}, \tag{32}$$

$$\vec{A}_2 := \begin{pmatrix} e^{2\beta_+}\cosh(2\sqrt{3}\beta_-) - e^{-4\beta_+} \\ e^{2\beta_+}\cosh(2\sqrt{3}\beta_-) + 2e^{-4\beta_+} \\ \sqrt{3}e^{2\beta_+}\sinh(2\sqrt{3}\beta_-) \end{pmatrix}, \tag{33}$$

$$\vec{A}_3 := \begin{pmatrix} e^{2\beta_+}\sinh(2\sqrt{3}\beta_-) \\ e^{2\beta_+}\sinh(2\sqrt{3}\beta_-) \\ \sqrt{3}e^{2\beta_+}\cosh(2\sqrt{3}\beta_-) \end{pmatrix}, \tag{34}$$

and

$$\vec{X} := \begin{pmatrix} \theta^1 p_+ + \theta^2 p_- \\ \theta^1 p_\Omega - \theta^3 p_- \\ \theta^2 p_\Omega + \theta^3 p_+ \end{pmatrix} = i\begin{pmatrix} -\theta^1\partial_+ - \theta^2\partial_- \\ -\theta^1\partial_\Omega + \theta^3\partial_- \\ -\theta^2\partial_\Omega - \theta^3\partial_+ \end{pmatrix}. \tag{35}$$

The three different vectors $\vec{A}_i$ encode in their expressions part of the structure of the auxiliary potential Φ while the vector $\vec{X}$ contains indeed the pure first order noncommutative corrections.

To proceed further, we shall now use the expression for Φ in Eq. (9) in order to carry out the integrals involved in the calculation of $\delta\Psi(\Phi, \vec{\theta})$ in Eq. (30). After a straightforward calculation we obtain the following results

$$\hat{S}_{\Omega}\Phi = 0, \qquad \hat{S}_{\beta_+}\Phi = 2e^{-4\Omega-2\beta_+}\left[\theta^1\cosh(2\sqrt{3}\beta_-) + \frac{1}{\sqrt{3}}\theta^2\sinh(2\sqrt{3}\beta_-) + \frac{2}{\sqrt{3}}\theta^3\sinh(2\sqrt{3}\beta_-)\right],$$
$$\hat{S}_{\beta_-}\Phi = \frac{2}{\sqrt{3}}e^{-4\Omega-2\beta_+}\left[\theta^1\sinh(2\sqrt{3}\beta_-) + \frac{1}{\sqrt{3}}\theta^2(\cosh(2\sqrt{3}\beta_-) + 2e^{6\beta_+}) + \frac{2}{\sqrt{3}}\theta^3(\cosh(2\sqrt{3}\beta_-) - e^{6\beta_+})\right]. \tag{36}$$

It is interesting to note that the contribution from the operator $\hat{S}_{\Omega}$ vanishes identically. With the previous expressions we finally have

$$\delta\Psi(\Phi,\vec{\theta}) = \frac{1}{3}ie^{-4\Omega-2\beta_+}\left[\frac{e^{6\beta_+}\beta_-(\theta^2-\theta^3)}{-\Omega+\beta_+} - \cosh(2\sqrt{3}\beta_-)\frac{3(2\Omega\beta_+ + \beta_+^2 + \beta_-^2)\theta^1 + 2(\Omega+2\beta_+)\beta_-(\theta^2+2\theta^3)}{(2\Omega+\beta_+)^2 - 3\beta_-^2}\right.$$
$$\left. - \sqrt{3}\sinh(2\sqrt{3}\beta_-)\frac{2(\Omega+2\beta_+)\beta_-\theta^1 + (2\Omega\beta_+ + \beta_+^2 + \beta_-^2)(\theta^2+2\theta^3)}{(2\Omega+\beta_+)^2 - 3\beta_-^2}\right] \tag{37}$$

where we have used the parametrization $s^i(t) = ts^i$.

However, due to the fact that $\delta\Psi(\Phi,\vec{\theta})$ is pure imaginary, the modulus

$$\left|P_\star \exp\left[-\int_\Gamma \vec{d}s\cdot\vec{\nabla}\Phi\right]\right|^2 = \exp[-2\Phi] = |\Psi_{\mathrm{W.H.}}|^2 \tag{38}$$

to first order on $\vec{\theta}$ (see [21] for a detailed discussion on the probabilistic interpretation of the wave functional Ψ). From this result it would seem that in order to see the effects due to noncommutativity we should consider either higher orders on the expansion Eq. (26) or a numerical analysis. In the next section we shall show that we can indeed say something.

IV. NONCOMMUTATIVITY ON THE (β_+, β_-)-PLANE AT FIXED Ω

In this section we shall consider the simplified case $\vec{\theta} = (0, 0, \theta)$. This choice is motivated by the form of the potential Φ since its dependence on the variable Ω is just through a multiplicative factor and furthermore, an interesting structure is present in the coordinate plane (β_+, β_-) [1,22].

Let us then fix Ω to be a constant and make a Taylor series development of Eq. (9) around the origin in the (β_+, β_-)-plane. The function Φ (and thus the potential V) has a very simple symmetric structure

$$\Phi = \tfrac{1}{2}e^{-2\Omega}[1 + 4(\beta_+^2 + \beta_-^2)]. \tag{39}$$

In the commutative case then we obtain the following set of equations

$$\partial_+\Psi = -4e^{-2\Omega}\beta_+\Psi, \qquad \partial_-\Psi = -4e^{-2\Omega}\beta_-\Psi. \tag{40}$$

This system leads in the noncommutative case to the following equations

$$\partial_+\Psi = -4e^{-2\Omega}\beta_+\star\Psi, \qquad \partial_-\Psi = -4e^{-2\Omega}\beta_-\star\Psi. \tag{41}$$

Using now the Bopp shifts of Eq. (17), these equations can be rewritten as

$$\partial_+\Psi + 2i\theta e^{-2\Omega}\partial_-\Psi = -4e^{-2\Omega}\beta_+\Psi,$$
$$\partial_-\Psi - 2i\theta e^{-2\Omega}\partial_+\Psi = -4e^{-2\Omega}\beta_-\Psi. \tag{42}$$

If we now introduce new variables u and v through the relations

$$\beta_+ =: u - 2i\theta e^{-2\Omega}v, \qquad \beta_- =: v + 2i\theta e^{-2\Omega}u \tag{43}$$

we have immediately

$$\partial_u \ln\Psi = -4e^{-2\Omega}u + 2(2e^{-2\Omega})^2 i\theta v,$$
$$\partial_v \ln\Psi = -4e^{-2\Omega}v - 2(2e^{-2\Omega})^2 i\theta u. \tag{44}$$

It follows from them that

$$\partial_v\partial_u \ln\Psi = 2(2e^{-2\Omega})^2 i\theta,$$
$$\partial_u\partial_v \ln\Psi = -2(2e^{-2\Omega})^2 i\theta. \tag{45}$$

This shows that due to noncommutativity $\ln\Psi$ may not be well defined and something interesting occurs.

In spite of the above, however, it is really the density probability $|\Psi|^2$ that we are looking after. From Eq. (42) it follows that

$$\partial_+\bar{\Psi} - 2i\theta e^{-2\Omega}\partial_-\bar{\Psi} = -4e^{-2\Omega}\beta_+\bar{\Psi},$$
$$\partial_-\bar{\Psi} + 2i\theta e^{-2\Omega}\partial_+\bar{\Psi} = -4e^{-2\Omega}\beta_-\bar{\Psi}. \tag{46}$$

The system formed from Eq. (42) and (46) is equivalent to the equations

$$\partial_+ \ln\Psi\bar{\Psi} - 2i\theta e^{-2\Omega}\partial_- \ln\frac{\bar{\Psi}}{\Psi} = -8e^{-2\Omega}\beta_+,$$
$$\partial_- \ln\Psi\bar{\Psi} + 2i\theta e^{-2\Omega}\partial_+ \ln\frac{\bar{\Psi}}{\Psi} = -8e^{-2\Omega}\beta_-,$$
$$\partial_+ \ln\frac{\bar{\Psi}}{\Psi} - 2i\theta e^{-2\Omega}\partial_- \ln\Psi\bar{\Psi} = 0,$$
$$\partial_- \ln\frac{\bar{\Psi}}{\Psi} + 2i\theta e^{-2\Omega}\partial_+ \ln\Psi\bar{\Psi} = 0 \tag{47}$$

In consequence

$$\begin{aligned}[1 - 4\theta^2 e^{-4\Omega}]\partial_+ \ln\Psi\bar{\Psi} &= -8e^{-2\Omega}\beta_+,\\ [1 - 4\theta^2 e^{-4\Omega}]\partial_- \ln\Psi\bar{\Psi} &= -8e^{-2\Omega}\beta_-\end{aligned} \tag{48}$$

with solution

$$|\Psi|^2_{\rm NC} = \exp\biggl[-\frac{4e^{-2\Omega}}{1 - 4\theta^2 e^{-4\Omega}}(\beta_+^2 + \beta_-^2) - c_0\biggr] \tag{49}$$

where c_0 is a constant. We should stress that we have not made any assumption on θ but to be a constant.

To find the value of c_0 it suffices to recall that in the commutative case

$$|\Psi|^2_C = \exp[-2\Phi] = \exp[-4e^{-2\Omega}(\beta_+^2 + \beta_-^2) - e^{-2\Omega}] \tag{50}$$

for small β_+ and β_-. Thus we see immediately that $c_0 = e^{-2\Omega}$.

An interesting feature arises from the solution given in Eq. (49): there *do exist* critical values

$$\theta_c^\pm = \pm\tfrac{1}{2}e^{2\Omega} \tag{51}$$

for which $|\Psi|^2_{\rm NC} = 0$. For values of $\theta_c^- < \theta < \theta_c^+$, the probability $|\Psi|^2_{\rm NC}$ behaves as a decreasing exponential, while for values in the range $\theta_c^+ < \theta$ or $\theta < \theta_c^-$ its behavior is dictated by an increasing exponential. The presence of θ^2 in the denominator inside the brackets in the solution Eq. (49) also exemplifies why no first order correction on θ appears on $|\Psi|^2_{\rm NC}$ as shown previously in Sec. III.

Figure 1 illustrates the behavior of $|\Psi|^2_{\rm NC}$ in the neighborhood of the origin on the (β_+, β_-)-plane for several values of θ at $\Omega = 1$.

V. CONCLUSIONS

We have shown that when introducing commutation relations through the GM product among the variables $(\Omega, \beta_+, \beta_-)$ used to describe the Bianchi IX model according to Misner's parametrization, a deformed solution similar in structure to the "wormhole" solution of the commutative case can be obtained. The solution has been expressed in the form of an open Wilson line by using the path-ordered exponential, path-ordered with respect to the star product.

Even though a closed expression has not been possible to attain for the full set of equations, due in part to the fact that the potential has a non trivial structure as encoded in the supersymmetric potential Φ, the lowest order deviations of the commutative case can be calculated. Furthermore, a nontrivial structure appears even in the simplified case $\vec{\theta} = (0, 0, \theta)$. By considering a neighborhood of the origin, where the potential V in Eq. (6) is known to posses a highly symmetric structure in the (β_+, β_-)-plane, we have been able to show the existence of a critical value θ_c given by Eq. (51) for which $|\Psi|^2_{\rm NC}$ exhibits a sort of transition in its behavior as seen from the explicit expression obtained in Eq. (49).

It is clear that the most interesting deviation of the commutative case happens when $\theta = \theta_c$ since $|\Psi|^2_{\rm NC}$ totally vanishes. Even though this result is valid in a region around the origin, it means that contrary to previous studies on this subject something drastic happens: the probability has a peak (local maximum strictly positive) at the origin but as $\theta \to \theta_c^+$ from the left, for example, it becomes narrower until it completely disappears and immediately

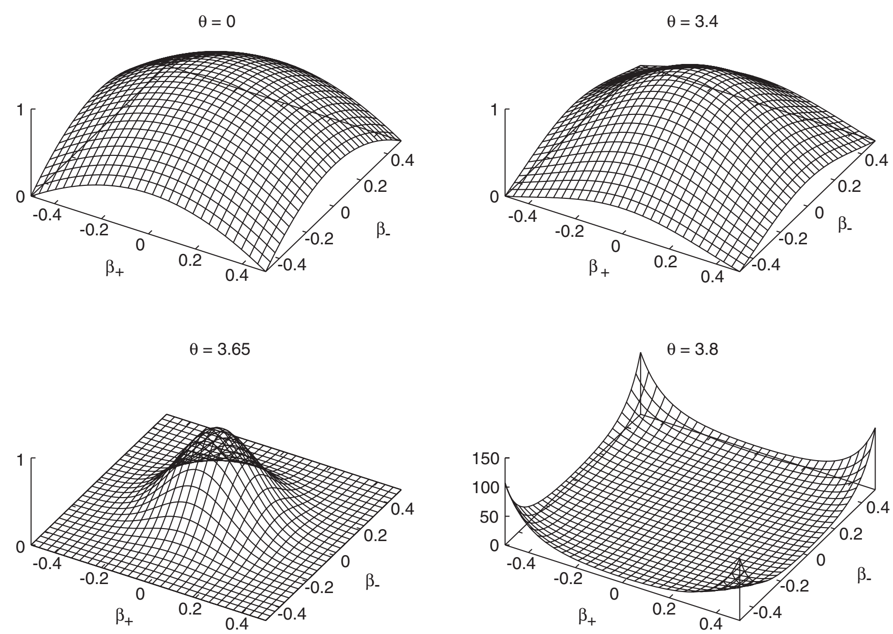

FIG. 1. Probability $|\Psi(1, \beta_+, \beta_-)|^2_{\rm NC}$ of the state function of the universe with $|\theta_c^\pm| = \frac{e^2}{2} \approx 3.69$.

after the value θ_c^+, a valley (local minimum strictly positive) pops up.

One should take into account that in the commutative case $|\Psi|_C^2$ vanishes when $\beta_+, \beta_- \longrightarrow \pm\infty$ and something similar is to be expected in the noncommutative case. This means then for $\theta_c^+ < \theta$ or $\theta < \theta_c^-$, the state defined by the wave functional at the origin is certainly not longer the preferred one. Our result is different from previous known works where the peak of the commutative solution is shifted from its original place, but continues to be the global maximum.

It is worthwhile to mention that the presence of a transition phase has also been discussed in the context of quantum mechanics on the noncommutative plane in the presence of a magnetic field [23,24]. In this case two very distinct behaviors for the model are observed and this feature in the context of quantum gravity has also been discussed [25]. We should also mention that when studying the canonical formulation of general relativity using minisuperspace models, the appearance of different regimes for the behavior of the wave function of states has been observed [26]. The setting there however is completely commutative.

It has been argued recently in an interesting work [27] that it would be more natural to consider the expression

$$\Psi_{\rm NC}(\Omega, \beta_+, \beta_-) \star \bar{\Psi}_{\rm NC}(\Omega, \beta_+, \beta_-) \qquad (52)$$

as the relevant quasiprobability distribution. This form is very appealing due to the fact that we are considering noncommutative objects, but we feel that it also greatly depends on finding a well-behaved function $\Psi_{\rm NC}$. As mentioned before, in most cases of physical systems there will be only one (partial) differential equation to be GM deformed and no problem would arise. However, since the Bopp shifts make use of commutative variables, it would seem more natural to consider $\Psi_{\rm NC}\bar{\Psi}_{\rm NC}$ as the appropriate quantity. This is certainly an issue to be further discussed.

On the mathematical side of the problem treated in this work, we would like to say also a few words. It is known that the expressions

$$\Psi_{NB} = e^{-\Phi} \exp[\tau_1 + \tau_2 + \tau_3] \qquad (53)$$

and

$$\Psi = e^{-\Phi} \exp[\tau_i - \tau_j - \tau_k], \qquad \epsilon_{ijk} = 1 \qquad (54)$$

with $\tau_i = c_0 e^{2\Omega - 2\beta_i}$ are also solutions of the Wheeler-DeWitt equation [3] and it would be desirable to construct similar solutions in the noncommutative case. The shift by the factor $e^{-\Phi}$ can be performed in a straightforward way leading to an expression differing slightly from the commutative one. However, one of the serious difficulties that arises when trying to continue with this program is an ambiguity inherent to the definition of the variables τ_i: in the noncommutative case we may define τ_1 as $\tau_1 = c_0 e^{-2\Omega} \star e^{-2\beta_+} \star e^{-2\sqrt{3}\beta_-}$ or perhaps as $\tau_1 = P_\star e^{-2\Omega - 2\beta_+ - 2\sqrt{3}\beta_-}$, for example. This leads in turn to an ambiguity in the expressions one may obtain using a chain rule appropriate to the star product. How to proceed is under current study.

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