

# Nonminimal coupling of perfect fluids to curvature

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In this work, we consider different forms of relativistic perfect fluid Lagrangian densities that yield the same gravitational field equations in general relativity (GR). A particularly intriguing example is the case with couplings of the form  $[1 + f_2(R)]\mathcal{L}_m$ , where  $R$  is the scalar curvature, which induces an extra force that depends on the form of the Lagrangian density. It has been found that, considering the Lagrangian density  $\mathcal{L}_m = p$ , where  $p$  is the pressure, the extra-force vanishes. We argue that this is not the unique choice for the matter Lagrangian density, and that more natural forms for  $\mathcal{L}_m$  do not imply the vanishing of the extra force. Particular attention is paid to the impact on the classical equivalence between different Lagrangian descriptions of a perfect fluid.

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## I. INTRODUCTION

Recently, in the context of  $f(R)$  modified theories of gravity, it was shown that a function of  $R$ -matter coupling induces a nonvanishing covariant derivative of the energy-momentum,  $\nabla_\mu T^{\mu\nu} \neq 0$ . This potentially leads to a deviation from geodesic motion, and consequently the appearance of an extra force [1]. Implications, for instance, for stellar equilibrium in this context have been studied in Ref. [2]. The equivalence with scalar-tensor theories with two scalar fields has been considered in Ref. [3], and a viability stability criterion was also analyzed in Ref. [4]. This novel coupling has attracted some attention and, actually, in a recent paper [5], this possibility has been applied to distinct matter contents, such as a massive scalar field and a dust distribution. Regarding the latter, it was argued that a “natural choice” for the matter Lagrangian density for perfect fluids is  $\mathcal{L}_m = p$ , based on Refs. [6,7], where  $p$  is the pressure. This choice has a particularly interesting application in the analysis of the  $R$ -matter coupling for perfect fluids, which implies in the vanishing of the extra force. However, we point out that despite the fact that  $\mathcal{L}_m = p$  does indeed reproduce the perfect fluid equation of state, it is not unique. Other choices include, for instance,  $\mathcal{L}_m = -\rho$  [7,8], where  $\rho$  is the energy density, or  $\mathcal{L}_m = -na$ , where  $n$  is the particle number density, and  $a$  is the physical free energy defined as  $a = \rho/n - Ts$ ,

with  $T$  being the fluid temperature and  $s$  the entropy per particle. Indeed, all these are on shell representations of a more general Lagrangian density, that is, obtained through back substitution of the equations of motion into the related action (see Ref. [7] for details). Furthermore, this equivalence is established within the framework of general relativity (GR). In this work, we address the issue of the Lagrangian formulation of perfect fluids in the context of the proposed model with a nonminimal coupling of the scalar curvature to matter, as depicted below.

This paper is organized as follows. In Sec. II, we review the equations of motion in a curvature-matter coupling. In Sec. III, we show the nonuniqueness of the relativistic perfect matter Lagrangian densities. In Sec. IV, we analyze the perfect fluid Lagrangian description with a nonminimal scalar curvature coupling. In Sec. V, we present our conclusions. Throughout this work, we use the convention  $\kappa = 8\pi G = 1$  and the metric signature  $(-1, 1, 1, 1)$ .

## II. EQUATION OF MOTION WITH CURVATURE-MATTER COUPLINGS

The action for curvature-matter couplings, in  $f(R)$  modified theories of gravity [1], takes the following form:

$$S = \int \left[ \frac{1}{2} f_1(R) + [1 + \lambda f_2(R)] \mathcal{L}_m \right] \sqrt{-g} d^4x, \quad (1)$$

where  $f_i(R)$  (with  $i = 1, 2$ ) are arbitrary functions of the curvature scalar  $R$ , and  $\mathcal{L}_m$  is the Lagrangian density corresponding to matter.

Varying the action with respect to the metric  $g_{\mu\nu}$  yields the field equations, given by

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$$\begin{aligned}
& F_1 R_{\mu\nu} - \frac{1}{2} f_1 g_{\mu\nu} - \nabla_\mu \nabla_\nu F_1 + g_{\mu\nu} \square F_1 \\
& = (1 + \lambda f_2) T_{\mu\nu} - 2\lambda F_2 \mathcal{L}_m R_{\mu\nu} + 2\lambda (\nabla_\mu \nabla_\nu \\
& \quad - g_{\mu\nu} \square) \mathcal{L}_m F_2,
\end{aligned} \tag{2}$$

where we have denoted  $F_i(R) = f'_i(R)$ , and the prime represents the derivative with respect to the scalar curvature. The matter energy-momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta(g^{\mu\nu})}. \tag{3}$$

Now, taking into account the generalized Bianchi identities, one deduces the following corrected conservation equation

$$\nabla^\mu T_{\mu\nu} = \frac{\lambda F_2}{1 + \lambda f_2} [g_{\mu\nu} \mathcal{L}_m - T_{\mu\nu}] \nabla^\mu R. \tag{4}$$

If one considers the equivalence with a scalar field theory (with two scalar fields,  $\phi = R$  and  $\psi = \mathcal{L}_m$ ) [3], it is clear that the nonminimal coupling between curvature and matter yields an exchange of energy and momentum between the latter and these scalar fields.

In the following, consider the equation of state for a perfect fluid

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}, \tag{5}$$

where  $\rho$  is the energy density and  $p$ , the pressure, respectively. The four-velocity,  $U_\mu$ , satisfies the conditions  $U_\mu U^\mu = -1$  and  $U^\mu U_{\mu;\nu} = 0$ .

Introducing the projection operator  $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$ , gives rise to nongeodesic motion governed by the following equation of motion for a fluid element:

$$\frac{dU^\mu}{ds} + \Gamma_{\alpha\beta}^\mu U^\alpha U^\beta = f^\mu, \tag{6}$$

where the extra force,  $f^\mu$ , is given by

$$f^\mu = \frac{1}{\rho + p} \left[ \frac{\lambda F_2}{1 + \lambda f_2} (\mathcal{L}_m - p) \nabla_\nu R + \nabla_\nu p \right] h^{\mu\nu}. \tag{7}$$

One verifies that the first term vanishes for the specific choice of  $\mathcal{L}_m = p$ , as noted by Ref. [5]. However, as emphasized in the Introduction, this is not the unique choice for the Lagrangian density of a perfect fluid, as we shall outline below.

### III. RELATIVISTIC PERFECT FLUID MATTER LAGRANGIAN DENSITIES

In this section, we follow Ref. [7] closely, and review the Lagrangian formulation of a perfect fluid in the context of GR. The action is presented in terms of Lagrange multipliers along the Lagrange coordinates  $\alpha^A$  in order to enforce specific constraints, and is given by

$$S_m = \int d^4x [-\sqrt{-g} \rho(n, s) + J^\mu (\varphi_{,\mu} + s \theta_{,\mu} + \beta_A \alpha^A_{,\mu})]. \tag{8}$$

Note that the action  $S_m = S(g_{\mu\nu}, J^\mu, \varphi, \theta, s, \alpha^A, \beta_A)$  is a functional of the spacetime metric  $g_{\mu\nu}$ , the entropy per particle  $s$ , the Lagrangian coordinates  $\alpha^A$ , and spacetime scalars denoted by  $\varphi$ ,  $\theta$ , and  $\beta_A$ , where the index  $A$  takes the values 1, 2, 3 (see Ref. [7] for details). The physical interpretation of these parameters is given below.

The vector density  $J^\mu$  is interpreted as the flux vector of the particle number density, and defined as  $J^\mu = \sqrt{-g} n U^\mu$ . The particle number density is given by  $n = |J|/\sqrt{-g}$ , so that the energy density is given as a function  $\rho = \rho(|J|/\sqrt{-g}, s)$ .

Varying the action with respect to the metric, and using the definition given by Eq. (3), provides the stress-energy tensor for a perfect fluid

$$T^{\mu\nu} = \rho U^\mu U^\nu + \left( n \frac{\partial \rho}{\partial n} - \rho \right) (g^{\mu\nu} + U^\mu U^\nu), \tag{9}$$

with the pressure defined as

$$p = n \frac{\partial \rho}{\partial n} - \rho. \tag{10}$$

Note that this definition of pressure is in agreement with the first law of thermodynamics,  $d\rho = \mu dn + n T ds$ . The latter shows that the equation of state can be specified by giving the function  $\rho(n, s)$ , i.e., the energy density as a function of number density and entropy per particle. The quantity  $\mu = \partial \rho / \partial n = (\rho + p)/n$  is defined as the chemical potential, which is the energy per particle required to inject a small amount of fluid into a fluid sample, maintaining a constant sample volume and a constant entropy per particle  $s$ . In addition, when imposing the stress-energy tensor covariant conservation, i.e.,  $T^{\mu\nu}_{;\nu} = 0$ , the perfect fluid also implies the covariant conservation of particle number, given by  $(n U^\mu)_{;\mu} = 0$ .

The variation of the action with respect to  $J^\mu$ ,  $\varphi$ ,  $\theta$ ,  $s$ ,  $\alpha^A$ , and  $\beta_A$ , provides the following equations of motion:

$$\frac{\delta S}{\delta J^\mu} = \mu U_\mu + \varphi_{,\mu} + s \theta_{,\mu} + \beta_A \alpha^A_{,\mu} = 0, \tag{11}$$

$$\frac{\delta S}{\delta \varphi} = -J^\mu_{,\mu} = 0, \tag{12}$$

$$\frac{\delta S}{\delta \theta} = -(s J^\mu)_{,\mu} = 0, \tag{13}$$

$$\frac{\delta S}{\delta s} = -\sqrt{-g} \frac{\partial \rho}{\partial s} + \theta_{,\mu} J^\mu = 0, \tag{14}$$

$$\frac{\delta S}{\delta \alpha^A} = -(\beta_A J^\mu)_{,\mu} = 0, \tag{15}$$

$$\frac{\delta S}{\delta \beta_A} = \alpha^A_{,\mu} J^\mu = 0. \quad (16)$$

The first relationship, Eq. (11), provides the velocity representation of the 4-velocity. The second equation, Eq. (12), reflects the particle number conservation, i.e.,  $(nU^\mu)_{;\mu} = \frac{1}{\sqrt{-g}} J^\mu_{;\mu} = 0$ . Equation (13) translates the entropy exchange constraint. Equation (14) provides the identification of  $T = \theta_{,\mu} U^\mu = \frac{1}{n} \frac{\partial \rho}{\partial s} \Big|_n$  after comparing it with the first law of thermodynamics. Equation (15) reflects the constancy of the parameter  $\beta_A$  along the fluid flow lines, and finally, Eq. (16) restricts the fluid 4-velocity to be directed along the flow lines of constant  $\alpha^A$ .

One may now infer the physical interpretation for the respective parameters. The scalar field  $\varphi$  is interpreted as a potential for the chemical free energy  $f$ , and is a Lagrange multiplier for  $J^\mu_{;\mu}$ , the particle number conservation. The scalar fields  $\beta_A$  are interpreted as the Lagrange multipliers for  $\alpha^A_{,\mu} J^\mu = 0$ , restricting the fluid 4-velocity to be directed along the flow lines of constant  $\alpha^A$ .

Note that taking into account Eq. (11) and the definitions  $J^\mu = \sqrt{-g} n U^\mu$  and  $\mu = (\rho + p)/n$ , the action Eq. (8) reduces to the on shell Lagrangian density  $\mathcal{L}_{m(1)} = p$ , with the action given by

$$S_m = \int d^4x \sqrt{-g} p, \quad (17)$$

which is the form considered in Ref. [6]. One should bear in mind that this on shell Lagrangian density yields the equations of motion (11)–(16) only if one considers that the pressure is functionally dependent on the previously considered fields  $\varphi$ ,  $s$ ,  $\theta$ ,  $\beta_A$ ,  $\alpha^A$ , and on the current density  $J^\mu$ .

Now, it was a Lagrangian density given by  $\mathcal{L}_m = p$  that the authors of Ref. [5] use to obtain a vanishing extra force due to the nontrivial coupling of matter to the scalar curvature  $R$ . For concreteness, replacing  $\mathcal{L}_m = p$  in Eq. (7), one arrives at the general relativistic expression

$$f^\mu = \frac{h^{\mu\nu} \nabla_\nu p}{\rho + p}. \quad (18)$$

However, the on shell degeneracy of the Lagrangian densities arises from adding up surface integrals to the action. For instance, consider the following surface integrals added to the action Eq. (8):

$$\begin{aligned} & - \int d^4x (\varphi J^\mu)_{;\mu}, & - \int d^4x (\theta s J^\mu)_{;\mu}, \\ & - \int d^4x (J^\mu \beta_A \alpha^A)_{;\mu}, \end{aligned}$$

so that the resulting action takes the form

$$\begin{aligned} S = \int d^4x [ & -\sqrt{-g} \rho(n, s) - \varphi J^\mu_{;\mu} - \theta (s J^\mu)_{;\mu} \\ & - \alpha^A (\beta_A J^\mu)_{;\mu} ], \end{aligned} \quad (19)$$

Note that this action reproduces the equations of motion, Eqs. (11)–(16). Taking into account the latter, the action reduces to

$$S_m = - \int d^4x \sqrt{-g} \rho, \quad (20)$$

i.e., the on shell matter Lagrangian density takes the following form  $\mathcal{L}_m = -\rho$ ; as before,  $\rho$  is dependent on the original fields present in the action Eq. (8). This choice is also considered for isentropic fluids, where the entropy per particle is constant  $s = \text{const}$  [7,8]. For the latter, the first law of thermodynamics indicates that isentropic fluids are described by an equation of state of the form  $a(n, T) = \rho(n)/n - sT$  [7] (see Ref. [9] for a bulk-brane discussion of this choice).

For this specific choice of  $\mathcal{L}_{m(2)} = -\rho$  the extra force takes the following form:

$$f^\mu = \left( -\frac{\lambda F_2}{1 + \lambda f_2} \nabla_\nu R + \frac{1}{\rho + p} \nabla_\nu p \right) h^{\mu\nu}. \quad (21)$$

An interesting characteristic is that the term related to the specific curvature-matter coupling is independent on the energy-matter distribution.

Another interesting action functional is given by the equation of state of the physical free energy as a function of the number density and the temperature,  $a(n, T)$ . For this we follow the reasoning of Ref. [7]. For instance, solving Eq. (14) for  $s$  as a function of  $n$  and  $T$  (using the definition  $T = \theta_{,\mu} J^\mu / |J|$ ), and finally eliminating  $s$  from the action Eq. (8), yields

$$S_m = \int d^4x [ -|J| a(n, T) + J^\mu (\varphi_{,\mu} + \beta_A \alpha^A_{,\mu}) ]. \quad (22)$$

Using the definitions  $n = |J| / \sqrt{-g}$  and  $T = \theta_{,\mu} J^\mu / |J|$ , and varying the action with respect to  $J^\mu$ , one ends up with the equation of motion, Eq. (11). The remaining equations of motion are readily obtained by varying  $S$  with respect to  $\varphi$ ,  $\theta^A$ , and  $\beta_A$ . It is simple to show that this action also provides the perfect fluid stress-energy tensor. As before, one may consider the addition of the following surface integrals to Eq. (8):

$$- \int d^4x (\varphi J^\mu)_{;\mu}, \quad - \int d^4x (J^\mu \beta_A \alpha^A)_{;\mu},$$

so that the action takes the following form

$$S_m = \int d^4x (-\sqrt{-g} n a). \quad (23)$$

The matter Lagrangian density is given by  $\mathcal{L}_{m(3)} = -na$ . The extra force in terms of this Lagrangian density yields the following expression:

$$f^\mu = \frac{1}{\rho + p} \left[ -\frac{\lambda F_2}{1 + \lambda f_2} (na + p) \nabla_\nu R + \nabla_\nu p \right] h^{\mu\nu}. \quad (24)$$

Hence, it is clear that no immediate conclusion may be extracted regarding the additional force imposed by the nonminimal coupling of curvature to matter, given the different available choices for the Lagrangian density; moreover, one could doubt the validity of a conclusion that allows for different physical predictions arising from these apparently equivalent Lagrangian densities.

Thus, from the above point of view, there is no particular reason to regard the choice of the on shell Lagrangian density  $\mathcal{L}_{m(1)} = p$  as preferable over the others we have discussed above. However, this degeneracy of the Lagrangian density of a perfect fluid, which does not appear in GR, is rather intriguing and object of further discussion in the next section.

#### IV. PERFECT FLUID LAGRANGIAN DESCRIPTION WITH NONMINIMAL SCALAR CURVATURE COUPLING

There is a caveat in above treatment, which can easily pass ignored: The discussion of the Lagrangian density dependence of the extra force given by Eq. (7), and degeneracy thereof, implicitly admits that the equivalence between different on shell Lagrangian densities holds. However, the latter is established in GR, and may not be valid in the more general model considered here.

Clearly, one may argue that two Lagrangian densities are equivalent if both generate the same energy-momentum tensor, and if variation of the corresponding actions yields the same equations of motion (11)–(16). As has been shown above, the several on shell Lagrangian densities  $\mathcal{L}_{m(1)} = p$ ,  $\mathcal{L}_{m(2)} = -\rho$ ,  $\mathcal{L}_{m(3)} = -na$  are all equivalent to the original, “bare” Lagrangian density,

$$\mathcal{L}_m = -\rho(n, s) + \frac{J^\mu}{\sqrt{-g}} (\varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha_{,\mu}^A). \quad (25)$$

Hence, one must attempt to retrace the derivation of the classical equivalence leading to these on shell quantities. Clearly, if one simply includes the  $[1 + \lambda f_2(R)]$  factor of Eq. (1) into action Eq. (8), that is,

$$S = \int d^4x [1 + \lambda f_2(R)] [-\sqrt{-g} \rho(n, s) + J^\mu (\varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha_{,\mu}^A)], \quad (26)$$

then the equations of motion (11)–(16) are unaffected, as variation with respect to each field yields only a global factor  $[1 + \lambda f_2(R)]$ .

However, the guiding principle behind the proposal first put forward in Ref. [1] is to allow for a nonminimal coupling between curvature and matter. Thus, the modification of the perfect fluid action Eq. (8) should only affect

the terms that show a minimal coupling between curvature and matter, i.e., those multiplied by  $\sqrt{-g}$ . For this reason, the current density term, which is not coupled to curvature, should not be altered. This yields

$$S'_m = \int d^4x [-\sqrt{-g} [1 + \lambda f_2(R)] \rho(n, s) + J^\mu (\varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha_{,\mu}^A)]. \quad (27)$$

The equations of motion (12), (13), (15), and (16) are unchanged, while Eqs. (11) and (14) read

$$\begin{aligned} \frac{\delta S}{\delta J^\mu} &= \mu [1 + f_2(R)] U_\mu + \varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha_{,\mu}^A = 0, \\ \frac{\delta S}{\delta s} &= -\sqrt{-g} [1 + f_2(R)] \frac{\partial \rho}{\partial s} + \theta_{,\mu} J^\mu = 0. \end{aligned} \quad (28)$$

This results from the coupling of the variables  $J^\mu$  and  $s$  with the factor  $[1 + f_2(R)]$  (since  $n = |J|/\sqrt{-g}$ ). Recalling that  $J^\mu = \sqrt{-g} n U^\mu$ , one obtains

$$-[1 + \lambda f_2(R)] \mu U_\mu = \varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha_{,\mu}^A, \quad (29)$$

$$T = \frac{1}{n} \frac{\partial \rho}{\partial s} \Big|_n = \frac{1}{1 + \lambda f_2(R)} \theta_{,\mu} U^\mu, \quad (30)$$

so that both the velocity representation and the temperature reflect the nonminimal coupling of curvature to matter.

One may now proceed and substitute the modified equations of motion into action (27), in order to obtain the new on shell Lagrangian density,

$$S'_m = \int d^4x \sqrt{-g} [1 + \lambda f_2(R)] p. \quad (31)$$

Hence, one concludes that the on shell Lagrangian density  $\mathcal{L}_{m(1)} = p$  is also obtained in the considered scenario. By including extra surface integrals, a similar procedure (not pursued here) also yields the previously discussed forms  $\mathcal{L}_{m(2)} = -\rho$ ,  $\mathcal{L}_{m(3)} = -na$ .

#### A. Gravitational field equations and the nonequivalence between on shell and bare Lagrangian densities

The above discussion confirms that one may adopt any particular on shell Lagrangian density as a suitable functional for describing a perfect fluid, therefore leading to the issue of distinguishing between different predictions for the extra force. However, this is not quite correct: although the above Lagrangian densities  $\mathcal{L}_{m(i)}$  are indeed obtainable from the original action, it turns out that they are not equivalent to the original Lagrangian density  $\mathcal{L}_m$ . Indeed, this equivalence demands that not only the equations of motion of the fields describing the perfect fluid remain invariant, but also that the gravitational field equations do not change.

Recall that the terms in the field Eqs. (2) which depend on  $\mathcal{L}_m$  arise from the presence of the nonminimal coupling  $[1 + \lambda f_2(R)]$ . However, the formulation of a perfect fluid

action functional includes the presence of a current density term, plus eventual surface integral terms  $B_{;\mu}^{\mu}$ . Writing  $\mathcal{L}_c = -\rho(n, s)$ ,  $V_{\mu} \equiv \varphi_{, \mu} + s\theta_{, \mu} + \beta_{\Lambda} \alpha_{, \mu}^{\Lambda}$ , for simplicity, then

$$S'_m = \int d^4x [\sqrt{-g} [1 + \lambda f_2(R)] \mathcal{L}_c + J^{\mu} V_{\mu} + B_{;\mu}^{\mu}], \quad (32)$$

one can see that only the nonminimal coupled term  $\mathcal{L}_c$  appears in the field equations, as variations with respect to  $g^{\mu\nu}$  of the remaining terms vanish:

$$\begin{aligned} F_1 R_{\mu\nu} - \frac{1}{2} f_1 g_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} F_1 + g_{\mu\nu} \square F_1 \\ = (1 + \lambda f_2) T_{\mu\nu} - 2\lambda F_2 \mathcal{L}_c R_{\mu\nu} + 2\lambda (\nabla_{\mu} \nabla_{\nu} \\ - g_{\mu\nu} \square) \mathcal{L}_c F_2. \end{aligned} \quad (33)$$

Clearly, the appropriate energy-momentum tensor is still obtained from  $\mathcal{L}_c$ , definitions (10), and relations  $U^{\mu} U_{\mu} = -1$  and  $J^{\mu} = \sqrt{-g} n U^{\mu}$ .

One arrives not at a paradox, but a tautology: different predictions for nongeodesic motion result from different forms of the gravitational field equations. Therefore, the equivalence between different on shell Lagrangian densities  $\mathcal{L}_{m(i)}$  and the original quantity  $\mathcal{L}_m$  is broken, so that one can no longer freely choose between the available forms.

By the same token, the additional extra force is unique, and obtained by replacing  $\mathcal{L}_c = -\rho$  into Eq. (7), yielding expression (21), here repeated for convenience

$$f^{\mu} = \left( -\frac{\lambda F_2}{1 + \lambda f_2} \nabla_{\nu} R + \frac{1}{\rho + p} \nabla_{\nu} p \right) h^{\mu\nu}. \quad (34)$$

### B. Null dust case

Following Ref. [5], it is interesting to analyze the generalized conservation law, given by Eq. (4), in the case of a null dust matter distribution. The latter is defined as the particular case of a perfect fluid with vanishing pressure,  $p = 0$ . This is usually interpreted as expressing weakly interacting nonrelativistic particles, with  $\rho c^2 \gg p \approx 0$ . However, given the previous discussion of the functional description of a perfect fluid, where the pressure is not an independent quantity, but defined by Eq. (10), a more rigorous (and physically compatible) formulation corresponds to an isentropic ( $s = \text{const}$ ) perfect fluid with an equation of state of the form  $\rho(n) = n\mu$ , with a constant chemical potential  $\mu$ .

The authors of Ref. [5] conclude that the extra force arising due to the nonminimal coupling of dust with the scalar curvature does not lead to nongeodesic motion, as it preserves parallel transport (and only changes the parameterization of the geodesic). However, this result arises from the particular choice  $\mathcal{L}_m = p$ , which is commonly used in the framework of GR. As the previous discussion has

shown, in the context of the considered curvature-matter coupling model, one cannot freely choose between available on shell Lagrangian densities, since these do not lead to the same gravitational field equations [10].

Instead, inserting the component of the original Lagrangian density that couples to the geometry,  $\mathcal{L}_c = -\rho$  into Eq. (4), and considering the energy-momentum tensor  $T_{\mu\nu} = \rho U_{\mu} U_{\nu}$ , one arrives at the following relationship:

$$\begin{aligned} (U^{\mu} \nabla_{\mu} U_{\nu} + U_{\nu} \nabla^{\mu} U_{\mu} + U_{\nu} U_{\mu} \nabla^{\mu}) \rho \\ = -\frac{\lambda F_2 \rho}{1 + \lambda f_2} (g_{\mu\nu} + U_{\mu} U_{\nu}) \nabla^{\mu} R. \end{aligned} \quad (35)$$

Following the notation of Ref. [5], one writes

$$\Theta = \nabla^{\mu} U_{\mu} + \frac{1}{\rho} U_{\mu} \nabla^{\mu} \rho + \frac{\lambda F_2}{1 + \lambda f_2} U_{\mu} \nabla^{\mu} R, \quad (36)$$

obtaining

$$U^{\mu} \nabla_{\mu} U_{\nu} = -\Theta U_{\nu} - \frac{\lambda F_2}{1 + \lambda f_2} \nabla_{\nu} R, \quad (37)$$

which clearly shows that parallel transport is no longer conserved, and one concludes that nongeodesic motion is also followed by pressureless dust.

## V. CONCLUSIONS

In this work we have discussed the degeneracy of Lagrangian densities for a perfect fluid, in the context of a gravity model where matter is coupled nonminimally with the scalar curvature. This degeneracy problem is well known in the context of GR, but in the discussed nonminimally coupled model possesses some new and rather surprising features, such as nongeodesic motion (first discussed in Ref. [1]; see Ref. [11] for a recent review); this dependency on the choice of the Lagrangian density was pointed out in Ref. [5].

We show that this degeneracy does not appear in the considered model, since different on shell Lagrangian densities which are classically equivalent do not yield the same gravitational field equations. Instead, we conclude that only the part of the original Lagrangian density  $\mathcal{L}_m$  that is coupled to the geometry (via the  $\sqrt{-g}$  factor) appears in these field equations. Hence, it follows that the motion of test particles is necessarily nongeodesic, if the nonminimal nature of the coupling between matter and curvature is properly accounted in the onset of fluid treatment.

However, we should point out that this study only solves the issue of an apparent degeneracy due to the classical equivalence between on shell Lagrangian densities. This should not be seen as an exhaustive account of the overall problem, since it only lifts this degeneracy for a particular original Lagrangian density  $\mathcal{L}_m$ . One can take a Lagrangian density different from that of Eq. (8) to begin

with, which also gives a full account of the behavior of a perfect fluid, describing both the correct energy-momentum tensor as well as its thermodynamics. As an example, if the bare Lagrangian density of Ref. [12] is considered (see Ref. [13] for a discussion), one obtains an extra force that is corrected by a factor linearly dependent on the Helmholtz free energy.

If one takes an initial  $\mathcal{L}_m$  that is functionally different from the one adopted in this study, and that still enables a convenient description of a perfect fluid (and is suitably interpreted through the use of the first law of thermodynamics), then one could obtain different results for the predicted extra force. This might lead to two different interpretations: One can conjecture that there must exist an underlying principle or symmetry that yields a unique Lagrangian density for a perfect fluid, so that different extra force predictions stem from an incomplete action description of it; however, one might also posit that different extra forces arising from different Lagrangian densities are physically distinguishable. If so, the model under scrutiny would serve to discriminate between different fluids that share the same energy-momentum tensor (and are thus “perfect”), but have different thermodynamic formulations. In the authors’ opinion, this has not been given due attention in the literature, most likely because arbitrary gravitational field equations depending on the matter Lagrangian have not often been the object of scrutiny.

In fact, to judge which matter Lagrangian density is the “natural” one depends, to some extent, on the independent variables that are considered. In this respect, an interesting avenue for future research would be to consider the non-minimal curvature-matter coupling using velocity potentials [6]. In this case, there are no constraints in the action principle, and one could compare this analysis with the one where the comoving Lagrangian coordinates label the fluid elements and that exhibits constraints [14] (we refer the reader to Ref. [15] for a proof of the need for constraints and related issues). However, this is not a trivial task, as no Lagrangian is unique, even in the presence of the non-minimal coupling, since it is invariant under the addition of a divergence, as mentioned above. Work along these lines is presently underway.

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