

Generating technique for $U(1)^3$ 5D supergravityDmitri V. Gal'tsov* and Nikolai G. Scherbluk⁺*Department of Theoretical Physics, Moscow State University, 119899, Moscow, Russia*

(Received 27 May 2008; published 12 September 2008)

We develop a generating technique for solutions of $U(1)^3$ 5D supergravity via dimensional reduction to three dimensions. This theory, which recently attracted attention in connection with black rings, can be viewed as a consistent truncation of the T^6 compactification of the 11-dimensional supergravity. Its further reduction to three dimensions accompanied by dualization of the vector fields leads to a 3D gravity coupled sigma model on the homogeneous space $SO(4, 4)/SO(4) \times SO(4)$ or $SO(4, 4)/SO(2, 2) \times SO(2, 2)$ depending on the signature of the three-space. We construct a 8×8 matrix representation of these cosets in terms of lower-dimensional blocks. Using it we express a solution generating transformations in terms of potentials and identify those preserving asymptotic conditions relevant to black holes and black rings. As an application we derive the doubly rotating black hole solution with three independent charges. A suitable contraction of the above cosets is used to construct a new representation of the coset $G_{2(2)}/(SL(2, R) \times SL(2, R))$ relevant for minimal five-dimensional supergravity.

DOI: 10.1103/PhysRevD.78.064033

PACS numbers: 04.20.Jb, 04.65.+e

I. INTRODUCTION

Since the discovery of black rings [1] in five dimensions, a variety of solution generation methods was developed to derive these solutions in a regular way and to construct their generalizations [2–18]. These methods allow one to find solutions possessing a certain number of isometries and they can be combined into three groups: (i) applications of T -duality symmetries acting on scalar and vector fields in any dimensions, (ii) the derivation of three-dimensional sigma models on homogeneous spaces, (iii) further reduction to two dimensions to apply soliton techniques. Usually the third approach involves the second one as an intermediate step. Dimensional reduction of higher-dimensional supergravity theories to three dimensions accompanied by dualization of the vector fields leads to the enhanced U -duality symmetries (hidden symmetries) which contain transformations useful for generating purposes. So far, only a restricted class of hidden symmetries of five-dimensional supergravity [the vacuum $SL(3, R)$ subgroup [19,20]] was applied to the black ring problems. Nevertheless, an application of the level (iii) technique based on this subgroup has led to impressive new results for *vacuum* black ring solutions [5–15].

For *charged* black rings, only the T duality at level (i) was used until recently to generate such solutions from the uncharged ones. To proceed further to level (ii), one has to specify the five-dimensional Lagrangian containing vector fields. Pure Einstein-Maxwell theory in five dimensions fails to produce a three-dimensional sigma model on a symmetric target space. Adding the Chern-Simons term, as prescribed by the minimal five-

dimensional supergravity [21,22], one obtains a more symmetric three-dimensional sigma model with an exceptional group $G_{2(2)}$ acting as the target-space isometry [23–35]. The corresponding generating technique was recently developed in [36,37]. It amounts to using the 7×7 matrix representation of the coset $G_{2(2)}/(SL(2, R) \times SL(2, R))$ and opens a way to construct the most general five-parametric black ring solution of the minimal five-dimensional supergravity as well as its possible generalizations such as charged Saturns.

The purpose of this paper is to generalize the same approach to $U(1)^3$ five-dimensional supergravity with three vector fields. This theory can be regarded as a truncated toroidal compactification of the 11D supergravity:

$$I_{11} = \frac{1}{16\pi G_{11}} \int \left(R_{11} \star_{11} \mathbf{1} - \frac{1}{2} F_{[4]} \wedge \star_{11} F_{[4]} - \frac{1}{6} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \right), \quad (1)$$

where $F_{[4]} = dA_{[3]}$, according to an ansatz

$$ds_{11}^2 = ds_5^2 + X^1(dz_1^2 + dz_2^2) + X^2(dz_3^2 + dz_4^2) + X^3(dz_5^2 + dz_6^2), \quad (2)$$

$$A_{[3]} = A^1 \wedge dz_1 \wedge dz_2 + A^2 \wedge dz_3 \wedge dz_4 + A^3 \wedge dz_5 \wedge dz_6.$$

Here z^i , $i = 1, \dots, 6$ are the coordinates parametrizing the torus T^6 . The three scalar moduli X^I ($I = 1, 2, 3$) and the three one-forms A^I depend only on the five coordinates entering ds_5^2 . The moduli X^I satisfy the constraint $X^1 X^2 X^3 = 1$, implying that the five-dimensional metric ds_5^2 is the Einstein-frame metric. The reduced five-dimensional action reads

*galtsov@phys.msu.ru
⁺shcherbluck@mail.ru

$$I_5 = \frac{1}{16\pi G_5} \int \left(R_5 \star_5 \mathbf{1} - \frac{1}{2} G_{IJ} dX^I \wedge \star_5 dX^J \right. \\ \left. - \frac{1}{2} G_{IJ} F^I \wedge \star_5 F^J - \frac{1}{6} \delta_{IJK} F^I \wedge F^J \wedge A^K \right),$$

$$G_{IJ} = \text{diag}((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}),$$

$$F_I = dA_I, \quad I, J, K = 1, 2, 3,$$

where the totally symmetric Chern-Simons coefficients $\delta_{IJK} = 1$ for the indices I, J, K being a permutation of 1, 2, 3, and zero otherwise. Supersymmetric solutions to this theory were studied in a number of papers [38–48]. The most general ring solution to this theory constructed so far [49] is a family of nonsupersymmetric rings parametrized by three conserved charges Q_I , three dipole charges q_I , and a radius of S^1 , with the mass M and the two angular momenta J_ψ, J_ϕ being some functions of these seven free parameters. An existence of a larger family of nonsupersymmetric black rings with nine independent parameters ($M, J_\psi, J_\phi, Q_I, q_I$) is expected, which reduces to the solutions of [45–48] in the supersymmetric limit. The generating technique developed in the present paper provides a sufficient number of parameters to construct the nine-parametric solution.

It is worth noting that the ansatz (2) is far from being the general toroidal compactification of the 11D supergravity. The generic toroidal reduction leads to the five-dimensional theory with 27 vector fields and 42 scalar moduli, parametrizing a coset $E_{6(6)}/USp(8)$. Correspondingly, the general black ring must contain 27 conserved charges and 27 dipole charges. More accurate analysis [50] shows that 24 conserved charges can be generated from the above three by duality transformations, while the number of independent dipole charges is 15 (the number of independent four-cycles of T^6).

Contraction of the above theory to minimal 5D supergravity is effected via an identification of the vector fields:

$$A^1 = A^2 = A^3 = \frac{1}{\sqrt{3}} A,$$

and freezing out the moduli: $X^1 = X^2 = X^3 = 1$. This leads to the Lagrangian

$$\mathcal{L}_5 = R_5 \star_5 \mathbf{1} - \frac{1}{2} F \wedge \star_5 F - \frac{1}{3\sqrt{3}} F \wedge F \wedge A.$$

In this case our results go back to those of Refs. [36,37]. However, our matrix representation of the coset $SO(4,4)/SO(4) \times SO(4)$ leads upon this contraction to a new representation of the coset $G_{2(2)}/(SL(2, R) \times SL(2, R))$, different and somewhat simpler than given in [36,37].

II. 3D SIGMA MODEL

A. Dimensional reduction

To perform dimensional reduction of the 11D theory to three dimensions, we follow an approach of Ref. [51] (keeping all basic notations of that paper) which has an advantage to provide the roots of the hidden symmetry group directly in terms of the so called *dilaton vectors* (coefficients in the dilaton exponentials entering the reduced action). For this purpose we go back to 11 dimensions and consider compactification of 11-dimensional supergravity on the torus $T^8 = T^6 \times T^2$ parametrized by coordinates z^1, \dots, z^8 . It will be convenient to distinguish the six coordinates on the torus T^6 , corresponding to the reduction to five dimensions, $z^i, i = 1, \dots, 6$, from those on T^2 , corresponding to the reduction from five to three dimensions, which will be denoted by elder indices $p, q = 7, 8$. The decomposition of the 11-dimensional metric in terms of the five-dimensional and three-dimensional metrics incorporating a diagonal ansatz (2) on T^6 (sector z^i) and the Kaluza-Klein mechanism (KK) ansatz on T^2 (sector z^p) then reads in the notation of [51]:

$$ds_{11}^2 = e^{\vec{s} \cdot \vec{\sigma}} ds_5^2 + \sum_{k=1}^6 e^{2\vec{\gamma}_k \cdot \vec{\sigma}} (dz^k)^2 \\ = e^{\vec{s} \cdot \vec{\theta}} ds_3^2 + e^{\vec{s} \cdot \vec{\sigma}} [e^{2\vec{\gamma}_7 \cdot \vec{\sigma}} (dz^7 + \mathcal{A}^7 + \chi dz_8)^2 \\ + \kappa e^{2\vec{\gamma}_8 \cdot \vec{\sigma}} (dz^8 + \mathcal{A}^8)^2] + \sum_{k=1}^6 e^{2\vec{\gamma}_k \cdot \vec{\sigma}} (dz^k)^2, \quad (3)$$

where $\mathcal{A}^7, \mathcal{A}^8$ are three-dimensional Kaluza-Klein one-forms from the reduction of the five-dimensional metric on T^2 , χ is an axion arising in the reduction of the four-dimensional one-form $\hat{\mathcal{A}}^7$ on the second cycle of T^2 : $\hat{\mathcal{A}}^7 = \mathcal{A}^7 + \chi dz^8$ (in the notation of the Ref. [51] $\chi = \mathcal{A}_8^7$), the factor $\kappa = \pm 1$ is responsible for the signature: $\kappa = 1$ for a spacelike z^8 , and $\kappa = -1$ for a timelike z^8 . The eight-dimensional dilaton $\vec{\theta}$ is split into the sum of the T^6 and T^2 components:

$$\vec{\theta} = \vec{\sigma} + \vec{\varphi}, \quad \vec{\sigma} = (\sigma_1, \dots, \sigma_6, 0, 0), \\ \vec{\varphi} = (0, \dots, 0, \varphi_1, \varphi_2),$$

and the dilaton vectors can be presented as

$$\vec{s} = (s_1, \dots, s_8), \quad s_k = \sqrt{2/((10-k)(9-k))}, \\ \vec{\gamma}_k = \frac{1}{2}(\vec{s} - \vec{f}_k), \\ \vec{f}_k = \underbrace{(0, \dots, 0)}_{k-1}, (10-k)s_k, s_{k+1}, s_{k+2}, \dots, s_8), \quad (4) \\ k = 1, \dots, 8.$$

The relation between the dilatons and the moduli X^I with account for the constraint $X^1 X^2 X^3 = 1$ is given by

$$\vec{s} \cdot \vec{\varphi} = \frac{1}{\sqrt{3}} \varphi_1 + \varphi_2, \quad \vec{s} \cdot \vec{\sigma} = 0 \Leftrightarrow X^1 X^2 X^3 = 1, \quad (5)$$

$$X^1 = e^{2\vec{\gamma}_1 \cdot \vec{\sigma}} = e^{2\vec{\gamma}_2 \cdot \vec{\sigma}}, \quad X^2 = e^{2\vec{\gamma}_3 \cdot \vec{\sigma}} = e^{2\vec{\gamma}_4 \cdot \vec{\sigma}}, \quad (6)$$

$$X^3 = e^{2\vec{\gamma}_5 \cdot \vec{\sigma}} = e^{2\vec{\gamma}_6 \cdot \vec{\sigma}},$$

$$2\vec{\gamma}_7 \cdot \vec{\varphi} = -\frac{2}{\sqrt{3}} \varphi_1, \quad 2\vec{\gamma}_8 \cdot \vec{\varphi} = \frac{1}{\sqrt{3}} \varphi_1 - \varphi_2, \quad (7)$$

$$(\partial \vec{\sigma})^2 = \sum_{l=1}^3 \left(\frac{\partial X^l}{X^l} \right)^2.$$

To rewrite the ansatz for the 11-dimensional three-form potential $A_{[3]}$ in the notation of Ref. [51], we will use the pairwise indices ii', jj', \dots taking three values $ii' = (12, 34, 56)$, together with the indices on T^2 : $p = 7, 8$, so that

$$A_{[3]} = \frac{1}{2} A_{ii'} \wedge dz^i \wedge dz^{i'} + \frac{1}{6} A_{ii'p} dz^i \wedge dz^{i'} \wedge dz^p.$$

Here the one-forms $A_{12} = A^1, A_{34} = A^2, A_{56} = A^3$ are the pullback of the five-dimensional one-forms introduced in (2) onto the three-space (assuming $A_{ii'} = -A_{i'i}$), and the scalars $A_{ii'p} = (A_{127}, A_{347}, A_{567}, A_{128}, A_{348}, A_{568})$ are axions arising in the reduction of the five-dimensional one-forms on T^2 .

Using the result of Ref. [51], we obtain the following three-dimensional Lagrangian (for $\kappa = 1$):

$$e_3^{-1} \mathcal{L}_3 = R_3 - \frac{1}{2} (\partial \vec{\theta})^2 - \frac{1}{4} \sum_{i < i'} e^{\vec{a}_{ii'} \cdot \vec{\theta}} (F_{ii'})^2$$

$$- \frac{1}{4} \sum_p e^{\vec{b}_p \cdot \vec{\varphi}} (\mathcal{F}^p)^2 - \frac{1}{2} \sum_{i < i', p} e^{\vec{a}_{ii'p} \cdot \vec{\theta}} (F_{ii'p})^2$$

$$- \frac{1}{2} e^{\vec{b}_{78} \cdot \vec{\varphi}} (\partial \chi)^2 + e_3^{-1} \mathcal{L}_{\text{CS}}, \quad (8)$$

where the field strength two-forms are defined as

$$\mathcal{F}^7 = d\mathcal{A}^7 - d\chi \wedge \mathcal{A}^8, \quad \mathcal{F}^8 = d\mathcal{A}^8,$$

$$F_{ii'} = dA_{ii'} - dA_{ii'p} \gamma^p \wedge \mathcal{A}^q, \quad \gamma^7 \gamma_7 = \gamma^8 \gamma_8 = 1,$$

$$\gamma^7 \gamma_8 = -\chi, \quad F_{ii'p} = \gamma^q_p dA_{ii'q}.$$

The newly introduced dilaton vectors are related to the quantities defined in (4) via

$$\vec{a}_{ii'} = \vec{f}_i + \vec{f}_{i'} - 3\vec{s}, \quad \vec{a}_{ii'p} = \vec{f}_i + \vec{f}_{i'} + \vec{f}_p - 3\vec{s},$$

$$\vec{b}_p = -\vec{f}_p, \quad \vec{b}_{78} = \vec{f}_8 - \vec{f}_7. \quad (9)$$

Finally, the Chern-Simons term reads

$$\mathcal{L}_{\text{CS}} = -\frac{1}{144} \epsilon^{ii'pkk'qjj'} dA_{ii'p} \wedge dA_{kk'q} \wedge A_{jj'},$$

where an eight-dimensional (T^8) Levi-Civita symbol is used.

In our truncated toroidal reduction, the six-dimensional part of the dilaton $\vec{\sigma}$ is effectively two-dimensional in view of the four constraints (5) and (6). So, together with the two-dimensional T^2 -dilaton $\vec{\varphi}$, one gets instead of the 8-dimensional vector $\vec{\theta}$ the four-dimensional vector $\vec{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$ with the following components:

$$\phi_1 = \frac{1}{\sqrt{2}} \left(-\ln(X^3) + \frac{1}{\sqrt{3}} \varphi_1 + \varphi_2 \right), \quad (10)$$

$$\phi_2 = \frac{1}{\sqrt{2}} \left(\ln(X^3) - \frac{1}{\sqrt{3}} \varphi_1 + \varphi_2 \right),$$

$$\phi_3 = \frac{1}{\sqrt{2}} \left(\ln(X^3) + \frac{2}{\sqrt{3}} \varphi_1 \right), \quad \phi_4 = \frac{1}{\sqrt{2}} \ln \frac{X^1}{X^2}. \quad (11)$$

It has the same norm:

$$\sum_{k=1}^8 \partial \theta_k^2 = \sum_{k=1}^4 \partial \phi_k^2,$$

and the corresponding exponents convert to the four-dimensional ones as follows:

$$e^{\vec{a}_{12} \cdot \vec{\theta}} \rightarrow e^{\vec{a}_{12} \cdot \vec{\phi}}, \quad e^{\vec{b}_7 \cdot \vec{\varphi}} \rightarrow e^{\vec{b}_7 \cdot \vec{\phi}}, \dots,$$

$$\vec{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4),$$

while the four-dimensional dilaton coefficient vectors in the new basis read

$$\vec{a}_{12} = \sqrt{2}(-1, 0, 0, -1), \quad \vec{a}_{34} = \sqrt{2}(-1, 0, 0, 1), \quad \vec{a}_{56} = \sqrt{2}(0, -1, -1, 0),$$

$$\vec{a}_{127} = \sqrt{2}(0, 0, 1, -1), \quad \vec{a}_{347} = \sqrt{2}(0, 0, 1, 1), \quad \vec{a}_{567} = \sqrt{2}(1, -1, 0, 0),$$

$$\vec{a}_{128} = \sqrt{2}(0, 1, 0, -1), \quad \vec{a}_{348} = \sqrt{2}(0, 1, 0, 1), \quad \vec{a}_{568} = \sqrt{2}(1, 0, -1, 0),$$

$$\vec{b}_7 = \sqrt{2}(-1, 0, -1, 0), \quad \vec{b}_8 = \sqrt{2}(-1, -1, 0, 0), \quad \vec{b}_{78} = \sqrt{2}(0, 1, -1, 0).$$

B. T^2 -covariant form

It will be useful to perform reduction on a torus T^2 in the 2D-covariant form decomposing the five-metric as

$$ds_5^2 = \lambda_{pq}(dz^p + a^p)(dz^q + a^q) - \kappa\tau^{-1}ds_3^2,$$

where the 2D metric λ_{pq} ($p = 7, 8$) is introduced:

$$\lambda = e^{-(2/\sqrt{3})\varphi_1} \begin{pmatrix} 1 & \chi \\ \chi & \chi^2 + \kappa e^{\sqrt{3}\varphi_1 - \varphi_2} \end{pmatrix}, \quad (12)$$

$$\det\lambda \equiv -\tau = \kappa e^{-(1/\sqrt{3})\varphi_1 - \varphi_2},$$

and a^p are the Kaluza-Klein one-forms: $a^7 = \mathcal{A}^7 - \chi\mathcal{A}^8$, $a^8 = \mathcal{A}^8$. For the moduli X^I , we have the following expressions in terms of τ and the dilatons ϕ_1, ϕ_4 :

$$(X^1)^2 = e^{\sqrt{2}\phi_4}(X^3)^{-1}, \quad (X^2)^2 = e^{-\sqrt{2}\phi_4}(X^3)^{-1}, \quad (13)$$

$$X^3 = \tau^{-1}e^{-\sqrt{2}\phi_1}.$$

Using the relations (10) and (11) we can rewrite the metric G_{IJ} of the moduli space as

$$G_{IJ} = -\frac{\kappa}{\tau} \text{diag}(e^{\tilde{a}_{12}\cdot\tilde{\phi}}, e^{\tilde{a}_{34}\cdot\tilde{\phi}}, e^{\tilde{a}_{56}\cdot\tilde{\phi}}). \quad (14)$$

C. Dualization

To obtain a purely scalar 3D Lagrangian we have to perform *dualization* of the 2-forms $F_{ii'}$ and \mathcal{F}^p . First of all, we change notation from that of Ref. [51] replacing the pairwise indices $ii' = 12, 34, 56$ by a capital Roman index $I = 1, 2, 3$, and relabeling the axions similarly:

$$F^I = (F_{12}, F_{34}, F_{56}), \quad u^I = (A_{127}, A_{347}, A_{567}),$$

$$v^I = (A_{128}, A_{348}, A_{568}).$$

It is important to realize that the indices I are the vector indices in the moduli space endowed with the metric G_{IJ} . We also combine u^I and v^I into the T^2 -covariant doublet $\psi_p^I = (u^I, v^I)$ with the index p relative to the metric λ_{pq} , or, in the matrix form,

$$\psi^I = \begin{pmatrix} u^I \\ v^I \end{pmatrix}.$$

In what follows the summation over all the repeated indices is understood. In this notation the field strength tensors will read

$$F^I = dA^I - d\psi_p^I \wedge a^p, \quad \mathcal{F}^7 = da^7 + \chi da^8, \quad (15)$$

$$\mathcal{F}^8 = da^8.$$

To perform dualization along the lines of [51], we introduce into the Lagrangian (8) three Lagrange multipliers μ_I ensuring the Bianchi identities for the two-forms $F^I - \psi_p^I da^p = dA^I - d(\psi_p^I a^p)$ and two Lagrange multipliers ω_p ensuring the Bianchi identities for the two-forms $da^p = \gamma^p{}_q \mathcal{F}^q$. We also rewrite the Chern-Simons term as

follows [see Eq. (3.29) in [51]]:

$$\mathcal{L}_{\text{CS}} = \frac{1}{2}\delta_{IJK}\epsilon^{pq}(d\psi_p^I \psi_q^J \wedge F^K + \frac{1}{3}d\psi_p^I \psi_q^J \psi_r^K \gamma^r{}_s \wedge \mathcal{F}^s),$$

with $\epsilon^{pq} = -\epsilon^{qp}$, $\epsilon^{78} = 1$. Integrating by parts we can present the Lagrangian (8) as

$$\mathcal{L}_3 = R_3 \star \mathbf{1} - \frac{1}{2} \star d\tilde{\phi} \wedge d\tilde{\phi} - \frac{1}{2} \sum_{i < i', p} e^{\tilde{a}_{ii'}\cdot\tilde{\phi}} \star F_{ii'p}$$

$$\wedge F_{ii'p} - \frac{\tau\kappa}{2} G_{IJ} \star F^I \wedge F^J + G_I \wedge F^I - \frac{1}{2} e^{\tilde{b}_7\cdot\tilde{\phi}}$$

$$\star \mathcal{F}^7 \wedge \mathcal{F}^7 - \kappa \frac{1}{2} e^{\tilde{b}_8\cdot\tilde{\phi}} \star \mathcal{F}^8 \wedge \mathcal{F}^8 + G_p \wedge \mathcal{F}^p,$$

where the one-forms G_I, G_p are related to the scalars μ_I, ω_p as follows:

$$G_I = d\mu_I + \frac{1}{2}\delta_{IJK}d\psi_p^I \psi_q^K \epsilon^{pq}, \quad G_7 = V_7,$$

$$G_8 = V_8 - \chi V_7,$$

$$V_p = d\omega_p - \psi_p^I (d\mu_I + \frac{1}{6}\delta_{IJK}d\psi_q^I \psi_r^K \epsilon^{qr}),$$

Then, eliminating the initial two-forms F^I, \mathcal{F}^p via the equations of motion

$$F^I = \tau^{-1}G^{IJ} \star G_J, \quad \mathcal{F}^7 = -\kappa e^{-\tilde{b}_7\cdot\tilde{\phi}} \star G_7, \quad (16)$$

$$\mathcal{F}^8 = -e^{-\tilde{b}_8\cdot\tilde{\phi}} \star G_8,$$

we obtain the Lagrangian in the dual terms:

$$\mathcal{L}_3 = R_3 \star \mathbf{1} - \frac{1}{2} \star d\tilde{\phi} \wedge d\tilde{\phi} - \frac{1}{2} \sum_{i < i', p} e^{\tilde{a}_{ii'}\cdot\tilde{\phi}} \star F_{ii'p}$$

$$\wedge F_{ii'p} + \frac{1}{2} \tau^{-1} G^{IJ} \star G_I \wedge G_J - \kappa \frac{1}{2} e^{-\tilde{b}_7\cdot\tilde{\phi}} \star G_7$$

$$\wedge G_7 - \frac{1}{2} e^{-\tilde{b}_8\cdot\tilde{\phi}} \star G_8 \wedge G_8,$$

where G^{IJ} is the inverse moduli metric G_{IJ} . Note that the signs in the dilaton exponents were inverted under dualization. Equation (15) together with the relations $e^{\tilde{b}_7\cdot\tilde{\phi}} = -\kappa\tau\lambda_{77}$, $e^{\tilde{b}_8\cdot\tilde{\phi}} = \tau(\chi\lambda_{78} - \lambda_{88})$, which follow from the definitions (10)–(12), enable us to rewrite Eq. (16) as the dualization equations covariant with respect to all indices:

$$\tau\lambda_{pq}da^q = \star V_p, \quad dA^I = d\psi_p^I \wedge a^p + \tau^{-1}G^{IJ} \star G_J,$$

or, explicitly:

$$\lambda_{pq}\partial^{[i}a^{j]q} = \frac{1}{2\tau\sqrt{h}}\epsilon^{ijk}\left[\partial_k\omega_p\right. \\ \left.- \psi_p^I\left(\partial_k\mu_I + \frac{1}{6}\delta_{IJK}\partial_k\psi_r^I\psi_s^K\epsilon^{rs}\right)\right], \quad (17)$$

$$\begin{aligned} \partial^{[i} A^{j]l} &= a^{p[j} \partial^{i]} \psi_p^l + \frac{1}{2\tau\sqrt{h}} \epsilon^{ijk} G^{lj} \\ &\times \left(\partial_k \mu_j + \frac{1}{2} \delta_{JKL} \partial_k \psi_p^K \psi_q^L \epsilon^{pq} \right), \end{aligned} \quad (18)$$

where the antisymmetrization is assumed with $1/2$.

Combining all the above formulas, we can present the dualized action as that of a 3D gravity coupled sigma model:

$$I_3 = \frac{1}{16\pi G_3} \int \sqrt{|h|} \left(R_3 - \mathcal{G}_{AB} \frac{\partial \Phi^A}{\partial x^i} \frac{\partial \Phi^B}{\partial x^j} h^{ij} \right) d^3x,$$

where h^{ij} is the inverse metric of the three-space, R_3 is the corresponding Ricci scalar, and $\mathcal{G}_{AB}(\Phi^A)$ is the metric of the target space parametrized by 16 scalar variables $\Phi^A = (\vec{\phi}, \psi^I, \mu_I, \chi, \omega_p)$, which can be read off from the following line element:

$$\begin{aligned} dl^2 &= \mathcal{G}_{AB} d\Phi^A d\Phi^B \\ &= \frac{1}{2} ((d\vec{\phi})^2 + \kappa e^{\sqrt{2}(\phi_1+\phi_4)} (G_1)^2 + \kappa e^{\sqrt{2}(\phi_1-\phi_4)} (G_2)^2 \\ &\quad + \kappa e^{\sqrt{2}(\phi_3+\phi_2)} (G_3)^2 + e^{\sqrt{2}(\phi_3-\phi_4)} (du^1)^2 \\ &\quad + e^{\sqrt{2}(\phi_4+\phi_3)} (du^2)^2 + e^{\sqrt{2}(\phi_1-\phi_2)} (du^3)^2 \\ &\quad + \kappa e^{\sqrt{2}(\phi_2-\phi_4)} (dv^1 - \chi du^1)^2 + \kappa e^{\sqrt{2}(\phi_4+\phi_2)} (dv^2 \\ &\quad - \chi du^2)^2 + \kappa e^{\sqrt{2}(\phi_1-\phi_3)} (dv^3 - \chi du^3)^2 \\ &\quad + \kappa e^{\sqrt{2}(\phi_1+\phi_3)} (G_7)^2 + e^{\sqrt{2}(\phi_1+\phi_2)} (G_8)^2 \\ &\quad + \kappa e^{\sqrt{2}(\phi_2-\phi_3)} d\chi^2). \end{aligned} \quad (19)$$

This line element can be more concisely rewritten in the T^2 -covariant form:

$$\begin{aligned} dl^2 &= \frac{1}{2} G_{IJ} (dX^I dX^J + d\psi^{I'} \lambda^{-1} d\psi^{J'}) - \frac{1}{2} \tau^{-1} G^{IJ} G_I G_J \\ &\quad + \frac{1}{4} \text{Tr}(\lambda^{-1} d\lambda \lambda^{-1} d\lambda) + \frac{1}{4} \tau^{-2} d\tau^2 - \frac{1}{2} \tau^{-1} V^T \lambda^{-1} V. \end{aligned}$$

D. Hidden symmetry

The set of the dilaton vectors $\vec{a}_{ii'}$, $\vec{a}_{ii'p}$, \vec{b}_p , \vec{b}_{78} is directly related to the root system of the isometry algebra of the target space [51]. Enumerating them as

$$\begin{aligned} -\vec{a}_{ii'} &= (\vec{e}_1, \vec{e}_2, \vec{e}_3); & \vec{a}_{ii'7} &= (\vec{e}_4, \vec{e}_5, \vec{e}_6); \\ \vec{a}_{ii'8} &= (\vec{e}_7, \vec{e}_8, \vec{e}_9); & -\vec{b}_p &= (\vec{e}_{10}, \vec{e}_{11}); & \vec{b}_{78} &= \vec{e}_{12}, \end{aligned}$$

one can easily see that these 12 four-dimensional vectors form the system of positive roots of the algebra $so(8)$. Indeed, from the relations (9) and the property

$$\sum_{k=1}^8 \vec{f}_k = 9\vec{s},$$

one can express $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_7, \vec{e}_8, \vec{e}_9, \vec{e}_{10}, \vec{e}_{11}$ in terms of $\vec{e}_4, \vec{e}_5, \vec{e}_6, \vec{e}_{12}$ as follows:

$$\begin{aligned} \vec{e}_I &= \sum_{K \neq I} \vec{e}_{K+3} + \vec{e}_{12}, & \vec{e}_{I+6} &= \vec{e}_{I+3} + \vec{e}_{12}, \\ \vec{e}_{10} &= \sum_K \vec{e}_{K+3} + \vec{e}_{12}, & \vec{e}_{11} &= \sum_K \vec{e}_{K+3} + 2\vec{e}_{12}. \end{aligned}$$

It is clear then that the vectors $\vec{e}_4, \vec{e}_5, \vec{e}_6, \vec{e}_{12}$ are the simple roots forming the Dynkin diagram of $so(8)$ [52].

The signature of the target space is $+16$ for $\kappa = 1$ (dimensional reduction in all spacelike directions) and $(+8, -8)$ $\kappa = -1$ (one of the reduced dimensions is time-like). Then it is easy to recognize that the isometry group is the noncompact form $SO(4, 4)$ of the $SO(8)$, whose Killing metric has the signature $(-12, +16)$, while the target space is the coset $SO(4, 4)/SO(4) \times SO(4)$ for $\kappa = 1$ and $SO(4, 4)/SO(2, 2) \times SO(2, 2)$ for $\kappa = -1$. For both these symmetric spaces the scalar curvature is negative:

$$\mathcal{R} = -96.$$

Denoting the four-dimensional Cartan subalgebra of $so(4, 4)$ as \vec{H} , and the generators corresponding to the nonzero roots $\pm \vec{e}_k$, $k = 1, \dots, 12$ as $P^{\pm I}$, $W_{\pm I}$, $Z_{\pm I}$, $\Omega^{\pm p}$, X^{\pm} , we will have the relations:

$$\begin{aligned} P^{\pm I} &\leftrightarrow \pm \vec{e}_I, & W_{\pm I} &\leftrightarrow \pm \vec{e}_{I+3}, & Z_{\pm I} &\leftrightarrow \pm \vec{e}_{I+6}, \\ \Omega^{\pm p} &\leftrightarrow \pm \vec{e}_{p+3}, & X^{\pm} &\leftrightarrow \pm \vec{e}_{12}. \end{aligned}$$

The commutators of these generators with the Cartan subalgebra \vec{H} read

$$\begin{aligned} [\vec{H}, X^{\pm}] &= \pm \vec{e}_{12} X^{\pm}, & [\vec{H}, \Sigma_{\pm I}^p] &= \pm \vec{e}_{I+3(p-6)} \Sigma_{\pm I}^p, \\ [\vec{H}, \Omega^{\pm p}] &= \pm \vec{e}_{p+3} \Omega^{\pm p}, & [\vec{H}, P^{\pm I}] &= \pm \vec{e}_I P^{\pm I}, \end{aligned} \quad (20)$$

where we have arranged W_I, Z_I into a column vector

$$\Sigma_I = \begin{pmatrix} W_I \\ Z_I \end{pmatrix}.$$

The remaining nonzero commutators are obtained from the relations between the root vectors

$$\begin{aligned} \vec{e}_{I+3} + \vec{e}_{J+6} &= \vec{e}_K, & \vec{e}_{I+3} + \vec{e}_{12} &= \vec{e}_{I+6}, \\ \vec{e}_{I+3(a+1)} + \vec{e}_I &= \vec{e}_{a+10} & (a = 0, 1), \\ \vec{e}_{12} + \vec{e}_{10} &= \vec{e}_{11}, \end{aligned}$$

where in the first equations I, J, K are all different. One finds

$$\begin{aligned}
[\Sigma_{\pm I}^p, \Sigma_{\pm J}^q] &= \mp \epsilon^{pq} \delta_{IK} P^{\pm K}, \\
[\Sigma_{\pm I}^p, \Sigma_{\pm J}^q] &= \pm \epsilon^{pq} \delta_{IJ} X^{\pm}, \\
[X^{\pm}, W_{\pm I}] &= \mp Z_{\pm I}, \\
[X^{\mp}, Z_{\pm I}] &= \mp W_{\pm I}, \\
[\Sigma_{\pm I}^p, P^{\pm J}] &= \mp \delta_I^J \Omega^{\pm p}, \\
[\Sigma_{\pm I}^p, P^{\mp J}] &= \pm \epsilon_{pq}^p \delta_{IK} \delta^{KJL} \Sigma_{\mp L}^q, \\
[X^{\pm}, \Omega^{\pm 7}] &= \mp \Omega^{\pm 8}.
\end{aligned} \tag{21}$$

We will give the generators of $SO(4,4)$ as differential operators acting on the target-space manifold in what follows.

III. COSET REPRESENTATIVE

A. The strategy

As a convenient representative of the coset, one can choose the upper-triangular matrix \mathcal{V} which transforms under the global action of the symmetry group G by the right multiplication and under the local action of the isotropy group H by the left multiplication:

$$\mathcal{V} \rightarrow \mathcal{V}' = h(\Phi) \mathcal{V} g, \quad g \in G, \quad h \in H.$$

Given this representative, one can construct the H -invariant matrix

$$\mathcal{M} = \mathcal{V}^T K \mathcal{V}, \tag{22}$$

where K is an involution matrix invariant under H :

$$h(\Phi)^T K h(\Phi) = K \tag{23}$$

(dependent on the coset signature parameter κ). Then the transformation of \mathcal{M} under G will be

$$\mathcal{M} \rightarrow \mathcal{M}' = g^T \mathcal{M} g. \tag{24}$$

The target-space metric (19) in terms of the matrix \mathcal{M} will read

$$dl^2 = -\frac{1}{8} \text{tr}(d\mathcal{M}d\mathcal{M}^{-1}). \tag{25}$$

The desired upper-triangular matrix \mathcal{V} can be constructed by an exponentiation of the Borel subalgebra of the Lie algebra of G consisting of the Cartan H and the positive-root E_+ generators (in what follows we omit the sign $+$ in the indices):

$$\mathcal{V} = \mathcal{V}_H \mathcal{V}_{E_+} = \mathcal{V}_H \mathcal{V}_X \mathcal{V}_\Psi \mathcal{V}_\Omega \mathcal{V}_P,$$

where the matrices \mathcal{V}_H , \mathcal{V}_X , \mathcal{V}_Ψ , \mathcal{V}_Ω , \mathcal{V}_P are the exponentials:

$$\begin{aligned}
\mathcal{V}_H &= e^{(1/2)\vec{\phi} \cdot \vec{H}}, & \mathcal{V}_X &= e^{\chi X}, & \mathcal{V}_{\Psi} &= e^{\psi^I \Sigma_I}, \\
\mathcal{V}_\Omega &= e^{\omega_p \Omega^p}, & \mathcal{V}_P &= e^{\mu_I P^I}.
\end{aligned} \tag{26}$$

Using (25), one can rewrite the target-space metric in terms of the matrix current $\mathcal{J} = d\mathcal{V}\mathcal{V}^{-1}$ as follows:

$$dl^2 = \frac{1}{4} \text{tr}(\mathcal{J}^2) + \frac{1}{4} \text{tr}(\mathcal{J}^T K \mathcal{J} K^{-1}).$$

Using Eq. (26) and the commutators (20) and (21) for the positive-root generators, one can show that the matrix current one-form \mathcal{J} is spanned by the Borel subalgebra generators as follows:

$$\begin{aligned}
\mathcal{J} &= d\mathcal{V}\mathcal{V}^{-1} \\
&= \frac{1}{2} d\vec{\phi} \cdot \vec{H} + e^{1/2\vec{e}_{12} \cdot \vec{\phi}} d\chi X + \sum_I e^{1/2\vec{e}_{I+3} \cdot \vec{\phi}} du^I W_I \\
&\quad + \sum_I e^{1/2\vec{e}_{I+6} \cdot \vec{\phi}} (dv^I - \chi du^I) Z_I \\
&\quad + \sum_p e^{1/2\vec{e}_{p+3} \cdot \vec{\phi}} G_p \Omega^p + \sum_I e^{1/2\vec{e}_I \cdot \vec{\phi}} G_I P^I.
\end{aligned}$$

B. Matrix representation

We use the 8×8 matrix representation of the $so(4,4)$ algebra given in Appendix A. The exponentiation of the Borel subalgebra gives the coset representatives in the following block form:

$$\mathcal{V} = \begin{pmatrix} S & R \\ 0 & \tilde{S} \end{pmatrix},$$

where S and R are 4×4 matrices which in turn have the block structure:

$$S = \begin{pmatrix} s_{ij} & a_i \\ b_j & s \end{pmatrix}, \quad R = \begin{pmatrix} a'_i & r_{ij} \\ s' & b'_j \end{pmatrix}, \quad i, j = 1, \dots, 3.$$

In what follows we use the symbol \hat{T} to denote transposition with respect to the minor diagonal and the symbol \tilde{A} to denote $\tilde{A} = -A^{\hat{T}}$ for a degenerate matrix A ($\det A = 0$) and $\tilde{A} = (A^{-1})^{\hat{T}}$ for a nondegenerate A ($\det A \neq 0$). In particular,

$$\begin{aligned}
(\psi^I)^T &= (u^I, v^I), & \tilde{\psi}^I &= -(v^I, u^I), \\
(\tilde{\psi}^I)^T &= -\begin{pmatrix} v^I \\ u^I \end{pmatrix}.
\end{aligned}$$

When applied to a matrix written in the block form, this means

$$\tilde{S} = \begin{pmatrix} s^{-1} & \tilde{a}_i \\ \tilde{b}_j & \tilde{s}_{ij} \end{pmatrix}.$$

In the above blocks we use the following 3×3 matrix potentials:

$$\hat{\phi} = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix}, \quad \hat{\chi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \chi \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Psi_3 = \begin{pmatrix} 0 & u^3 & -v^3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\omega} = \begin{pmatrix} \omega_7 & \omega_8 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\mu}_3 = \begin{pmatrix} 0 & 0 & 0 \\ -\mu_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the 3-columns:

$$\Psi_a = \begin{pmatrix} u^a \\ v^a \\ 0 \end{pmatrix}, \quad \hat{\mu}_a = \begin{pmatrix} \mu_a \\ 0 \\ 0 \end{pmatrix}.$$

An explicit exponentiation in (26) gives the following 4×4 blocks for the partial coset representatives:

$$\mathcal{V}_H: S_H = \begin{pmatrix} e^{(1/\sqrt{2})\hat{\phi}} & 0 \\ 0 & e^{(1/\sqrt{2})\phi_4} \end{pmatrix}, \quad R_H \equiv 0,$$

$$\mathcal{V}_X: S_X = \begin{pmatrix} e^{-\hat{\chi}} & 0 \\ 0 & 1 \end{pmatrix}, \quad R_X \equiv 0,$$

$$\mathcal{V}_\Psi: S_\Psi = \begin{pmatrix} e^{\Psi_3} & e^{(1/2)\Psi_3}\tilde{\Psi}_1^T \\ 0 & 1 \end{pmatrix},$$

$$R_\Psi = \begin{pmatrix} e^{(1/2)\Psi_3}\tilde{\Psi}_2^T & \Psi \\ 0 & \Psi_2^T e^{(1/2)\tilde{\Psi}_3} \end{pmatrix},$$

where

$$S = \begin{pmatrix} e^{(1/\sqrt{2})\hat{\phi}} e^{-\hat{\chi}} e^{\Psi_3} & e^{(1/\sqrt{2})\hat{\phi}} e^{-\hat{\chi}} \hat{\rho}_{21} \\ 0 & e^{(1/\sqrt{2})\phi_4} \end{pmatrix}, \quad \hat{\rho}_{12} = \hat{\mu}_1 + e^{(1/2)\Psi_3}\tilde{\Psi}_2^T, \quad \hat{\rho}_{21} = \hat{\mu}_2 + e^{(1/2)\Psi_3}\tilde{\Psi}_1^T,$$

$$R = \begin{pmatrix} e^{(1/\sqrt{2})\hat{\phi}} e^{-\hat{\chi}} \hat{\rho}_{12} & e^{(1/\sqrt{2})\hat{\phi}} e^{-\hat{\chi}} (e^{\Psi_3}(\Omega - \Pi - \hat{\mu}_2 \otimes \tilde{\mu}_1) + e^{(1/2)\Psi_3} \sum_{a=1}^2 \tilde{\Psi}_a^T \otimes \tilde{\mu}_a + \Psi) \\ 0 & e^{(1/\sqrt{2})\phi_4} \tilde{\rho}_{12} \end{pmatrix}.$$

Finally, we construct the gauge-invariant representative of the coset (22):

$$\mathcal{M} = \begin{pmatrix} \mathcal{P} & \mathcal{P}\mathcal{Q} \\ \mathcal{Q}^T \mathcal{P} & \tilde{\mathcal{P}} + \mathcal{Q}^T \mathcal{P}\mathcal{Q} \end{pmatrix}, \quad \mathcal{Q} = S^{-1}R, \quad \mathcal{P} = S^T \mathcal{E} S, \quad \tilde{\mathcal{P}} = \tilde{S}^T \tilde{\mathcal{E}} \tilde{S},$$

with the block components $\mathcal{Q} = -\mathcal{Q}^T$:

$$\mathcal{Q} = \begin{pmatrix} \hat{\mu}_1 + e^{-(1/2)\Psi_3}\tilde{\Psi}_2^T & -\Pi + \frac{1}{2}(\Psi_{21} - \Psi_{12}) + \Omega - \hat{\mu}_2 \otimes \Psi_2^T + \frac{1}{3}\Psi_3(\frac{1}{2}\Psi_{12} - \Psi_{21}) - \tilde{\Psi}_2^T \otimes \tilde{\mu}_2 - \frac{1}{3}(\frac{1}{2}\Psi_{12} - \Psi_{21})\tilde{\Psi}_3 \\ 0 & \tilde{\mu}_1 + \Psi_2^T e^{-(1/2)\tilde{\Psi}_3} \end{pmatrix}. \quad (27)$$

and \mathcal{P} :

$$\mathcal{P} = \mathcal{P}^T = \begin{pmatrix} e^{\Psi_3^T} \Lambda e^{\Psi_3} & e^{\Psi_3^T} \Lambda \hat{\rho}_{21} \\ \hat{\rho}_{21}^T \Lambda e^{\Psi_3} & \hat{\rho}_{21}^T \Lambda \hat{\rho}_{21} + e^{\sqrt{2}\phi_4} \end{pmatrix}. \quad (28)$$

The matrix Λ entering this expression reads

$$\Psi = \frac{1}{2} \left(\Psi_{12} + \frac{1}{3}(\Psi_3 \Psi_{12} + \Psi_{12} \tilde{\Psi}_3) + \frac{1}{12} \Psi_3 \Psi_{12} \tilde{\Psi}_3 \right) + \frac{1}{2}(1 \leftrightarrow 2).$$

The remaining exponentials are

$$\mathcal{V}_\Omega: S_\Omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_\Omega = \begin{pmatrix} 0 & \Omega \\ 0 & 0 \end{pmatrix},$$

$$\Omega = \hat{\omega} + \tilde{\omega}.$$

$$\mathcal{V}_P: S_P = \begin{pmatrix} 1 & \hat{\mu}_2 \\ 0 & 1 \end{pmatrix},$$

$$R_P = \begin{pmatrix} \hat{\mu}_1 & -\Pi - \hat{\mu}_2 \otimes \tilde{\mu}_1 \\ 0 & \tilde{\mu}_1 \end{pmatrix}, \quad \Pi = \hat{\mu}_3 + \tilde{\mu}_3.$$

For the K -involution matrix we have the following block representation:

$$S_K = \mathcal{E}, \quad \mathcal{E} = \begin{pmatrix} \hat{\kappa} & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\kappa} = \text{diag}(\kappa, \kappa, 1),$$

$$\tilde{\mathcal{E}} = \kappa \mathcal{E}, \quad R_K \equiv 0.$$

Multiplying all the matrices \mathcal{V}_H , \mathcal{V}_X , \mathcal{V}_Ψ , \mathcal{V}_Ω , and \mathcal{V}_P , we obtain for the coset representative \mathcal{V} :

$$\mathcal{V} = \begin{pmatrix} S & R \\ 0 & \tilde{S} \end{pmatrix}, \quad \tilde{S} = (S^{-1})^{\hat{T}},$$

where

$$\Lambda = e^{-\hat{\chi}^T} e^{(1/\sqrt{2})\hat{\phi}} \hat{\kappa} e^{(1/\sqrt{2})\hat{\phi}} e^{-\hat{\chi}} = \begin{pmatrix} \kappa e^{\sqrt{2}\phi_1} & 0 \\ 0 & \tilde{\lambda}^0 \end{pmatrix},$$

where the 2×2 matrix λ^0 , related to λ as $\lambda^0 = (X^3)^{-1} \lambda$, is given by

$$\begin{aligned}\lambda^0 &= e^{-\sqrt{2}\phi_3} \begin{pmatrix} 1 & \chi \\ \chi & \chi^2 + \kappa e^{\sqrt{2}(\phi_3 - \phi_2)} \end{pmatrix}, \\ \tilde{\lambda}^0 &= \kappa e^{\sqrt{2}\phi_2} \begin{pmatrix} 1 & -\chi \\ -\chi & \chi^2 + \kappa e^{\sqrt{2}(\phi_3 - \phi_2)} \end{pmatrix}.\end{aligned}\quad (29)$$

The following relations are useful:

$$\begin{aligned}\tilde{\lambda}_{77}^0 &= -\frac{\lambda_{77}^0}{\tau_0}, & \tilde{\lambda}_{78}^0 &= \frac{\lambda_{78}^0}{\tau_0}, & \tilde{\lambda}_{88}^0 &= -\frac{\lambda_{88}^0}{\tau_0}, \\ \tau_0 &= -\det(\lambda^0) = -\kappa e^{-\sqrt{2}(\phi_2 + \phi_3)}, & \tilde{\tau}_0 &= \tau_0^{-1}.\end{aligned}$$

IV. ISOMETRIES OF THE TARGET SPACE

A. Transformation of the coset

To classify 28 isometry transformations $SO(4, 4)$ of the target space, we consider the action of one-parameter subgroups generated by

$$\tilde{H}, \quad P^{\pm I}, \quad W_{\pm I}, \quad Z_{\pm I}, \quad \Omega_{\pm p}, \quad X^{\pm}.\quad (30)$$

In terms of the gauge-independent coset matrix \mathcal{M} , the isometries are represented by (24). In conformity with the matrix representation (A1) we can distinguish three types of the $SO(4, 4)$ matrices g :

- (i) The ‘‘right’’ upper-triangular matrices generated by the B -type elements of $so(4, 4)$:

$$\begin{aligned}g_R &= e^{\alpha C} = \begin{pmatrix} 1 & e^{\alpha B} \\ 0 & 1 \end{pmatrix}, \\ C &= P^1, P^3, W_2, Z_2, \Omega^7, \Omega^8,\end{aligned}$$

whose action on the coset components \mathcal{P} and \mathcal{Q} consists in the shift

$$\mathcal{P} \rightarrow \mathcal{P}' = \mathcal{P}, \quad \mathcal{Q} \rightarrow \mathcal{Q}' = \mathcal{Q} + \alpha B.\quad (31)$$

These correspond to gauge transformations.

- (ii) The ‘‘central’’ block-diagonal matrices with the upper-triangular blocks

$$\begin{aligned}g_{Su} &= e^{\alpha C} = \begin{pmatrix} e^{\alpha A} & 0 \\ 0 & e^{\alpha \tilde{A}} \end{pmatrix}, \\ C &= P^2, W_1, W_3, Z_1, Z_3, X,\end{aligned}$$

with the lower-triangular blocks

$$\begin{aligned}g_{Sd} &= e^{\alpha C} = \begin{pmatrix} e^{\alpha A^T} & 0 \\ 0 & e^{\alpha \tilde{A}^T} \end{pmatrix}, \\ C &= P^{-2}, W_{-1}, W_{-3}, Z_{-1}, Z_{-3}, X^-, \end{aligned}$$

and with the diagonal blocks

$$\begin{aligned}g_S &= e^{\alpha C} = \begin{pmatrix} e^{\alpha A^T_{H_i}} & 0 \\ 0 & e^{\alpha \tilde{A}^T_{H_i}} \end{pmatrix}, \\ C &= H_i, \quad i = 1 \dots 4.\end{aligned}$$

These act on the \mathcal{P} and \mathcal{Q} blocks as follows:

$$\begin{aligned}\mathcal{P} &\rightarrow \mathcal{P}' = e^{\alpha M^T} \mathcal{P} e^{\alpha M}, \\ \mathcal{Q} &\rightarrow \mathcal{Q}' = e^{-\alpha M} \mathcal{Q} e^{\alpha \tilde{M}},\end{aligned}\quad (32)$$

where $M = A, A^T$ or A_{H_i} for $g = g_{Su}, g_{Sd}$ or g_S respectively.

- (iii) The ‘‘left’’ lower-triangular type matrices

$$g_L = \begin{pmatrix} 1 & 0 \\ e^{\alpha B^T} & 1 \end{pmatrix},$$

whose action on the \mathcal{P} and \mathcal{Q} is highly nontrivial

$$\begin{aligned}\mathcal{P}' &= \mathcal{P} + \alpha \mathcal{P} \mathcal{Q} B^T + \alpha B \mathcal{Q}^T \mathcal{P} \\ &\quad + \alpha^2 B (\tilde{\mathcal{P}} + \mathcal{Q}^T \mathcal{P} \mathcal{Q}) B^T, \\ \mathcal{P}' \mathcal{Q}' &= \mathcal{P} \mathcal{Q} + \alpha B (\tilde{\mathcal{P}} + \mathcal{Q}^T \mathcal{P} \mathcal{Q}), \\ \tilde{\mathcal{P}} + \mathcal{Q}^T \mathcal{P} \mathcal{Q} &= \text{inv}.\end{aligned}$$

It is this part of isometries which contains the charging transformations.

Meanwhile, there exists a reflection symmetry of the root diagram interchanging positive and negative roots. This enables us to reparametrize the target-space metric introducing the dual coordinates Φ_d^A in which the positive-root generators look the same as the negative roots generators in terms of the initial coordinates Φ^A . Thus, the dual coset matrix $\mathcal{M}_d(\Phi_d^A)$ constructed from the dual potentials will be transformed under the action of the lower-triangular generators in the same way as the coset matrix $\mathcal{M}(\Phi^A)$ under the gauge transformation g_R . We find that the dual coset matrix is an inverse of the initial coset matrix:

$$\begin{aligned}\mathcal{M}_d &= \begin{pmatrix} \mathcal{P}_d & \mathcal{P}_d \mathcal{Q}_d \\ \mathcal{Q}_d^T \mathcal{P}_d & \tilde{\mathcal{P}}_d + \mathcal{Q}_d^T \mathcal{P}_d \mathcal{Q}_d \end{pmatrix} = \mathcal{M}^{-1} \\ &= \begin{pmatrix} \mathcal{P}^{-1} + \mathcal{Q} \tilde{\mathcal{P}}^{-1} \mathcal{Q}^T & -\mathcal{Q} \tilde{\mathcal{P}}^{-1} \\ -\tilde{\mathcal{P}}^{-1} \mathcal{Q}^T & \tilde{\mathcal{P}}^{-1} \end{pmatrix}.\end{aligned}$$

The transformation of the \mathcal{M} components under g_L is then described as the shift of the dual matrix \mathcal{Q}_d

$$\tilde{\mathcal{P}}_d^{-1} \rightarrow (\tilde{\mathcal{P}}_d^{-1})' = \tilde{\mathcal{P}}_d^{-1}, \quad \mathcal{Q}_d \rightarrow \mathcal{Q}'_d = \mathcal{Q}_d - \alpha B.$$

B. Finite transformations explicitly

From now on we will assume $\kappa = -1$. Using Eqs. (31) and (32) it is easy to find the finite actions of the Cartan and the positive-root transformations. The diagonal ones give

$$\begin{aligned}
 H_1: \phi'_1 &= \phi_1 + 2\alpha_1, & \omega'_p &= \omega_p e^{-\alpha_1}, \\
 \psi'_3 &= \psi_3 e^{-\alpha_1}, & \mu'_a &= \mu_a e^{-\alpha_1}, \\
 H_2: \phi'_2 &= \phi_2 + 2\alpha_2, & \omega'_8 &= \omega_8 e^{-\alpha_2}, \\
 v'_1 &= v_1 e^{-\alpha_2}, & v'_2 &= v_2 e^{-\alpha_2}, & u'_3 &= u_3 e^{\alpha_2}, \\
 \mu'_3 &= \mu_3 e^{-\alpha_2}, & \chi' &= \chi e^{-\alpha_2}, \\
 H_3: \phi'_3 &= \phi_3 + 2\alpha_3, & \omega'_7 &= \omega_7 e^{-\alpha_3}, \\
 u'_1 &= u_1 e^{-\alpha_3}, & u'_2 &= u_2 e^{-\alpha_3}, & v'_3 &= v_3 e^{\alpha_3}, \\
 \mu'_3 &= \mu_3 e^{-\alpha_3}, & \chi' &= \chi e^{\alpha_3}, \\
 H_4: \phi'_4 &= \phi_4 + 2\alpha_4, & \psi'_1 &= \psi_1 e^{\alpha_4}, & \psi'_2 &= \psi_2 e^{-\alpha_4}, \\
 \mu'_1 &= \mu_1 e^{-\alpha_4}, & \mu'_2 &= \mu_2 e^{\alpha_4}.
 \end{aligned}$$

The upper-triangular matrices (positive roots) produce gauge transformations of the potentials Φ^A . Namely, the generators P^I and Ω_p give simple gauge shifts of the potentials μ_I and the KK potentials ω_p respectively:

$$\mu'_I = \mu_I + \alpha^I_\mu, \quad \omega'_p = \omega_p + \alpha^p_\omega, \quad \text{other potentials invariant.}$$

The generators W_I and Z_I of the electromagnetic sector produce the gauge shift of the axions u^I and v^I respectively, changing also some other potentials:

$$\begin{aligned}
 W_I: (u^I)' &= u^I + \alpha^I_u, & \mu'_J &= \mu_J + \frac{1}{2} \delta_{IJK} \alpha^I_u v^K, & \text{no sum over } I, K \\
 \omega'_7 &= \omega_7 + \alpha^I_u \mu_I + \frac{1}{6} \sum_{J,K} \delta_{IJK} \alpha^I_u u^J v^K, & \omega'_8 &= \omega_8 + \frac{1}{6} \sum_{J,K} \delta_{IJK} \alpha^I_u v^J v^K, & \text{no sum over } I.
 \end{aligned}$$

$$\begin{aligned}
 Z_I: (v^I)' &= v^I + \alpha^I_v, & \mu'_J &= \mu_J - \frac{1}{2} \delta_{IJK} \alpha^I_v u^K, & \text{no sum over } I, K \\
 \omega'_8 &= \omega_8 + \alpha^I_v \mu_I - \frac{1}{6} \sum_{J,K} \delta_{IJK} \alpha^I_v u^J v^K, & \omega'_7 &= \omega_7 - \frac{1}{6} \sum_{J,K} \delta_{IJK} \alpha^I_v u^J u^K, & \text{no sum over } I.
 \end{aligned}$$

Finally, the generators X and X^- lead to finite transformations:

$$\begin{aligned}
 X: (v^I)' &= v^I + \alpha_\chi u^I, & \omega'_8 &= \omega_8 + \alpha_\chi \omega_7, \\
 \lambda'^0_{77} &= \lambda^0_{77}, & \lambda'^0_{78} &= \lambda^0_{78} + \alpha_\chi \lambda^0_{77}, & \lambda'^0_{88} &= \lambda^0_{88} + 2\alpha_\chi \lambda^0_{78} + \alpha_\chi^2 \lambda^0_{77} \\
 X^-: (u^I)' &= u^I + \alpha_{-\chi} v^I, & \omega'_7 &= \omega_7 + \alpha_{-\chi} \omega_8, \\
 \lambda'^0_{88} &= \lambda^0_{88}, & \lambda'^0_{78} &= \lambda^0_{78} + \alpha_{-\chi} \lambda^0_{88}, & \lambda'^0_{77} &= \lambda^0_{77} + 2\alpha_{-\chi} \lambda^0_{78} + \alpha_{-\chi}^2 \lambda^0_{88},
 \end{aligned}$$

with the remaining potentials invariant.

C. Killing vectors

To find the differential operators generating isometries (the Killing vectors X_k of the target space) from the finite transformations, one can use the defining equations:

$$X_k = \left. \frac{\partial \Phi^{A'}}{\partial \alpha_k} \right|_{\alpha_k=0} \frac{\partial}{\partial \Phi^A},$$

where $\Phi^{A'} = \Phi^A(\Phi^B, \alpha_k)$ are the potentials transformed under the action of the one-parametric subgroups. Here we give X_k corresponding to the Cartan, positive-root, and X^- generators. The other Killing vectors are much more complicated and we will present them in the next section only for the vacuum seed potentials. Enumerating the potentials as $\Phi^A = (X^1, X^2, \lambda^0_{pq}, \psi^I, \mu_I, \omega_p)$ and the parameters as $\alpha_k = (\alpha_1, \dots, \alpha_4, \alpha^I_u, \alpha^I_v, \alpha^I_\mu, \alpha^p_\omega, \alpha_\chi, \alpha_{-\chi})$, we find the following set of the Killing vectors:

$$\begin{aligned}
M_p{}^q &= 2\lambda_{pr}^0 \frac{\partial}{\partial \lambda_{rq}^0} + \omega_p \frac{\partial}{\partial \omega_q} + \delta_p^q \omega_r \frac{\partial}{\partial \omega_r} + \delta_p^q \mu_I \frac{\partial}{\partial \mu_I} + \psi_p^I \frac{\partial}{\partial \psi_q^I}, \\
\frac{H_1 + H_4}{\sqrt{2}} &= 2(X^1)^2 X^2 \frac{\partial}{\partial X^1} + \psi_p^1 \frac{\partial}{\partial \psi_p^1} - \psi_p^2 \frac{\partial}{\partial \psi_p^2} - \psi_p^3 \frac{\partial}{\partial \psi_p^3} - 2\mu_1 \frac{\partial}{\partial \mu_1} - \omega_p \frac{\partial}{\partial \omega_p}, \\
\frac{H_1 - H_4}{\sqrt{2}} &= 2(X^2)^2 X^1 \frac{\partial}{\partial X^2} + \psi_p^2 \frac{\partial}{\partial \psi_p^2} - \psi_p^1 \frac{\partial}{\partial \psi_p^1} - \psi_p^3 \frac{\partial}{\partial \psi_p^3} - 2\mu_2 \frac{\partial}{\partial \mu_2} - \omega_p \frac{\partial}{\partial \omega_p}, \\
\Sigma_I^p &= \frac{\partial}{\partial \psi_p^I} + \frac{1}{2} \delta_{IJK} \epsilon^{pq} \psi_q^K \frac{\partial}{\partial \mu_J} + \mu_I \frac{\partial}{\partial \omega_p} + \frac{1}{6} \delta_{IJK} \epsilon^{pq} \psi_q^J \psi_r^K \frac{\partial}{\partial \omega_r}, \\
\Omega^p &= \frac{\partial}{\partial \omega_p}, \quad P^I = \frac{\partial}{\partial \mu_I},
\end{aligned}$$

where $M_p{}^q$ are given by

$$\begin{aligned}
M_7^7 &= -\frac{1}{\sqrt{2}}(H_1 + H_3), & M_8^8 &= -\frac{1}{\sqrt{2}}(H_1 + H_2), \\
M_7^8 &= X, & M_8^7 &= X^-.
\end{aligned}$$

V. SOLUTION GENERATING TECHNIQUE

The use of the target-space isometries for generating purposes consists in three steps. First, one has to choose the seed solution and to find the corresponding target-space potentials. This involves solving the (differential) dualization equations. Then the isometry transformations are applied to get the target-space potentials of the new solution. Finally, one has to solve the dualization equations (18) to obtain a new solution in terms of the metric and the matter fields. The three-dimensional metric h_{ij} remains essentially the same.

Since the dimensional reduction from eleven to five dimensions does not involve dualization, the identification of solution in five-dimensional or in 11-dimensional terms is a matter of choice. Five target-space variables $\phi_1, \phi_2, \phi_3, \phi_4, \chi$ enter the 11-dimensional metric algebraically, via the moduli X^I, λ_{pq} :

$$\begin{aligned}
ds_{11}^2 &= \sum_{I,i,i'} X^I ((dz^i)^2 + (dz^{i'})^2) \\
&\quad + \lambda_{pq} (dz^p + a^p)(dz^q + a^q) + \tau^{-1} h_{ij} dx^i dx^j,
\end{aligned}$$

while the KK vectors a^p in the T^2 sector are related to the target-space potentials ω_p via dualization. In the form-field sector,

$$\begin{aligned}
A_{[3]} &= (A^1 + \psi_p^1 dz^p) \wedge dz^1 \wedge dz^2 + (A^2 + \psi_p^2 dz^p) \wedge dz^3 \\
&\quad \wedge dz^4 + (A^3 + \psi_p^3 dz^p) \wedge dz^5 \wedge dz^6,
\end{aligned}$$

the six quantities ψ_p^I are the target-space potentials, while the remaining one-forms A^I are related to the potentials μ_I via dualization.

A. Asymptotic conditions

To find a proper direction in the target space which would lead to the solution with desired properties is a nontrivial task, and usually it invokes an identification of the subgroups of the isometry group preserving certain asymptotic conditions for the metric (and/or the form field). For the black hole/black ring applications several such conditions are of interest.

(i) Minkowskian metric

Consider the 11-dimensional Minkowski metric in the Cartesian coordinates (assuming $\kappa = -1$)

$$ds_{11}^2 = \sum_{k=1}^7 (dz^k)^2 - (dz_8)^2 + \sum_{i=1}^3 (dx^i)^2, \quad A_{[3]} = 0. \quad (33)$$

This corresponds to $\lambda_{88} = -1, \lambda_{77} = 1$ and all other potentials zero. Consequently, the coset matrix $\mathcal{M}_{\text{as}} = K$. By virtue of Eq. (23), such an asymptotic is preserved under isometries belonging to the isotropy subgroup H of the $SO(4, 4)$:

$$\begin{aligned}
P^I + P^{-I}, & \quad Z_I + Z_{-I}, & W_I - W_{-I}, \\
X + X^-, & \quad \Omega^7 + \Omega^{-7}, & \Omega^8 - \Omega^{-8}.
\end{aligned}$$

(ii) Flat metric in the $S^2 \times S^1$ fibration

Another useful form of the flat metric appropriate to the five-dimensional ring problem is

$$\begin{aligned}
ds_5^2 &= -(dt)^2 + r^2 \cos^2 \theta (d\psi)^2 + dr^2 \\
&\quad + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\end{aligned}$$

and the reduction is performed along t, ψ where ψ is the angular variable along S^1 in the ring $S^2 \times S^1$ fibration. We identify $z^7 = \psi, z^8 = t$, then the target-space variables $\lambda_{88} = -1, \lambda_{77} = \tau = r^2 \cos^2 \theta$. Preservation of the above line element is more restrictive: from an analysis of an infinitesimal action of generators (30) we find the only combination of the Killing vectors

$$Z_I + Z_{-I}.$$

Mathematically, this is related to the coordinate dependence of the asymptotical matrix \mathcal{M}_{as} .

(iii) *Guisto-Saxena coordinates*

However, one can perform dimensional reduction with respect to the combinations $\phi_{\pm} = \frac{1}{2}(\phi \pm \psi)$ instead of ψ , as suggested in Ref. [53]. In this case the coset matrix \mathcal{M}_{as} will be coordinate independent. The target-space potentials then read

$$\lambda_{77} = \tau, \quad \lambda_{88} = -1, \quad \omega_7 = \tau, \quad \tau = r^2 \quad (34)$$

(other potentials zero), so the coset matrix \mathcal{M}_{as} will be given through the following constant 4×4 blocks:

$$\mathcal{P}_{\text{as}} = \begin{pmatrix} -\tau^{-1} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \tau^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{Q}_{\text{as}} = \begin{pmatrix} 0 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tau \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To find the appropriate combinations of the Killing vectors consider their vacuum form setting all electromagnetic potentials to zero and taking $X^I = 1$ (in this case $\lambda^0 = \lambda$):

$$M_p^q = 2\lambda_{pr} \frac{\partial}{\partial \lambda_{rq}} + \omega_p \frac{\partial}{\partial \omega_q} + \delta_p^q \omega_r \frac{\partial}{\partial \omega_r},$$

$$\frac{H_1 + H_4}{\sqrt{2}} = 2 \frac{\partial}{\partial X^1} - \omega_p \frac{\partial}{\partial \omega_p},$$

$$\frac{H_1 - H_4}{\sqrt{2}} = 2 \frac{\partial}{\partial X^2} - \omega_p \frac{\partial}{\partial \omega_p},$$

$$\Sigma_I^p = \frac{\partial}{\partial \psi_p^I},$$

$$\Sigma_{-I}^p = \omega_p \frac{\partial}{\partial \mu_I} + \lambda_{pq} \frac{\partial}{\partial \psi_q^I},$$

$$P^I = \frac{\partial}{\partial \mu_I},$$

$$P^{-I} = -\omega_p \frac{\partial}{\partial \psi_p^I},$$

$$\Omega^p = \frac{\partial}{\partial \omega_p},$$

$$\Omega^{-p} = -2\lambda_{pr} \omega_s \frac{\partial}{\partial \lambda_{rs}} - (\omega_p \omega_r + \tau \lambda_{pr}) \frac{\partial}{\partial \omega_r}.$$

Conditions (34) then give the following linear combinations preserving \mathcal{M}_{as} :

$$Z_I + Z_{-I}, \quad W_{-I} + P^{-I}, \quad X - \Omega^{-8}.$$

More general physically interesting asymptotic conditions may be encountered for five-dimensional type D metrics as discussed in [54].

VI. THREE-CHARGE BLACK HOLE WITH TWO ANGULAR MOMENTA

Five-dimensional stationary charged black holes were investigated for different couplings of vector fields to gravity both for nonsupersymmetric [55–67] and supersymmetric [68,69] configurations (for a recent review see [70]). Somewhat surprisingly, the simplest Einstein-Maxwell theory in five dimensions does not possess analytic solutions like the Kerr-Newman black hole in four dimensions. This is related to the lack of hidden symmetries which are enhanced in the supergravity action due to Chern-Simons term. The enhancement endows us with charging symmetries which open the way to construct the three-charge doubly rotating black hole solution from the 5D vacuum Myers-Perry metric. We assume the following choice of coordinates: $z_7 = \psi$, $z_8 = t$, r , θ , ϕ , and denote the rotation parameters as a , b . The seed solution then reads

$$ds^2 = -dt^2 + \frac{\rho^2 r^2}{\Delta} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2$$

$$+ (r^2 + b^2) \cos^2 \theta d\psi^2 + p(dt + a \sin^2 \theta d\phi$$

$$+ b \cos^2 \theta d\psi)^2. \quad (35)$$

Using the relations between the metric components and the target-space potentials:

$$g_{tt} = \lambda_{88}, \quad g_{t\psi} = \lambda_{78}, \quad g_{\psi\psi} = \lambda_{77},$$

$$g_{t\phi} = \lambda_{88} a_\phi^8 + \lambda_{78} a_\phi^7, \quad g_{\phi\psi} = \lambda_{78} a_\phi^8 + \lambda_{77} a_\phi^7,$$

$$g_{\phi\phi} = \lambda_{88} (a_\phi^8)^2 + 2\lambda_{78} a_\phi^7 a_\phi^8 + \lambda_{77} (a_\phi^7)^2 + \tau^{-1} h_{\phi\phi},$$

we find the σ -model variables:

$$\lambda_{88} = -1 + p, \quad \lambda_{78} = p b \cos^2 \theta,$$

$$\lambda_{77} = (r^2 + b^2) \cos^2 \theta + p b^2 \cos^4 \theta,$$

$$\tau = (r^2 + b^2 - r_0^2 + p a^2 \cos^2 \theta) \cos^2 \theta,$$

$$a_\phi^7 = \tau^{-1} p a b \sin^2 \theta \cos^2 \theta,$$

$$a_\phi^8 = -\tau^{-1} p (r^2 + b^2) a \sin^2 \theta \cos^2 \theta,$$

$$\omega_7 = -p a b \cos^4 \theta, \quad \omega_8 = -p a \cos^2 \theta.$$

The invariant three-metric reads

$$h_{ij}dx^i dx^j = \tau \left(\frac{\rho^2 r^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Delta}{\tau} \sin^2 \theta \cos^2 \theta d\phi^2 \right), \quad \sqrt{h} = \frac{1}{2} \tau \rho^2 \sin \theta \cos \theta,$$

where

$$p = \frac{r_0^2}{\rho^2}, \quad \rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \Delta = (r^2 + a^2)(r^2 + b^2) - (r_0 r)^2.$$

To equip this vacuum solution with three electric charges, we perform the following transformation:

$$\mathcal{M}' = \Pi^T \mathcal{M} \Pi, \quad \Pi = \prod_I e^{\alpha_I (Z_I + Z_{-I})},$$

where the product of one-parametric exponentials reads explicitly:

$$\Pi = \begin{pmatrix} c_3 & 0 & -s_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 c_2 & 0 & -s_1 c_2 & -c_1 s_2 & 0 & -s_1 s_2 & 0 \\ -s_3 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -s_1 c_2 & 0 & c_1 c_2 & s_1 s_2 & 0 & c_1 s_2 & 0 \\ 0 & -c_1 s_2 & 0 & s_1 s_2 & c_1 c_2 & 0 & s_1 c_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_3 & 0 & s_3 \\ 0 & -s_1 s_2 & 0 & c_1 s_2 & s_1 c_2 & 0 & c_1 c_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_3 & 0 & c_3 \end{pmatrix}, \quad c_I \equiv \cosh(\alpha_I), \quad s_I \equiv \sinh(\alpha_I).$$

Using formulas of Appendix C, one can extract the transformed potentials in the following form:

$$\begin{aligned} g'_{\psi\psi} &= \lambda'_{77} = D^{-2/3} \cos^4 \theta \left(pb^2 - p^2 \left(s^2 (a^2 + b^2) + 2abcs + a^2 \sum_{I < J} s_I^2 s_J^2 \right) - (a^2 - b^2) \left(1 + p \sum_I s_I^2 \right) \right) + D^{1/3} \rho^2 \cos^2 \theta, \\ g'_{t\psi} &= \lambda'_{78} = D^{-2/3} \cos^2 \theta p (bc + sa - psa), \quad g'_{tt} = \lambda'_{88} = D^{-2/3} (p - 1), \\ \tau' &= D^{-1/3} \tau, \quad (X^I)' = \frac{D^{1/3}}{D_I}. \\ (u^I)' &= \frac{p \cos^2 \theta}{D_I} (ac_I s_J s_K + bs_I c_J c_K), \quad (v^I)' = \frac{pc_I s_I}{D_I}, \\ (\mu_I)' &= -\frac{D_J + D_K}{2D_J D_K} p \cos^2 \theta (as_I c_J c_K + bc_I s_J s_K), \quad I \neq J \neq K. \end{aligned}$$

$$\begin{aligned} \omega'_7 &= \omega_7 - \frac{p^2 \cos^4 \theta}{D} \left\{ cs(a^2 + b^2) + 2abs^2 + ab \sum_{I < J} s_I^2 s_J^2 \right. \\ &\quad \left. + \frac{1}{3} p \left(cs(a^2 + b^2) \sum_I s_I^2 + ab \left(\sum_{I < J} s_I^2 s_J^2 (s_I^2 + s_J^2) - \sum_{I < J} s_I^2 s_J^2 + 3s^2 + 2s^2 \sum_I s_I^2 \right) \right) + p^2 abs^2 \right\}, \\ \omega'_8 &= c\omega_8 + \frac{p \cos^2 \theta}{D} \left\{ bs + psb \left(3 + \sum_I s_I^2 \right) + \frac{1}{3} p^2 \left((ca + sb) \sum_{I < J} s_I^2 s_J^2 + 3sb \sum_I s_I^2 \right) + p^3 acs^2 \right\}, \end{aligned}$$

where

$$c \equiv \prod_I c_I, \quad s \equiv \prod_I s_I, \quad D \equiv \prod_I D_I, \quad D_I = 1 + ps_I^2.$$

Note that metric (35) is invariant under an interchange of

the angular coordinates ψ and ϕ together with the corresponding rotation parameters $\phi \leftrightarrow \psi$, $\theta \leftrightarrow \theta + \pi/2$, $a \leftrightarrow b$, namely, $g_{tt} \leftrightarrow g_{tt}$, $g_{t\psi} \leftrightarrow g_{t\phi}$, $g_{\psi\psi} \leftrightarrow g_{\phi\phi}$, $g_{\psi\phi} \leftrightarrow g_{\psi\phi}$, $A^I_\psi \leftrightarrow A^I_\phi$. We assume that this symmetry remains valid for the charged solution (35) as well. This simplifies an ex-

traction of the final form of the solution avoiding the inverse dualization. As a result we obtain the five-dimensional one-forms A^I and the metric components as follows:

$$\begin{aligned}
 A^I &= \frac{p}{D_I} (s_I c_I dt + (bc_I s_J s_K + as_I c_J c_K) \sin^2 \theta d\phi \\
 &\quad + (ac_I s_J s_K + bs_I c_J c_K) \cos^2 \theta d\psi), \quad I \neq J \neq K \\
 g'_{\phi\phi} &= D^{-2/3} \sin^4 \theta \left(pa^2 - p^2 \left(s^2 (a^2 + b^2) + 2abc s \right. \right. \\
 &\quad \left. \left. + b^2 \sum_{I < J} s_I^2 s_J^2 \right) + (a^2 - b^2) \left(1 + p \sum s_I^2 \right) \right) \\
 &\quad + D^{1/3} \rho^2 \sin^2 \theta, \\
 g'_{t\phi} &= D^{-2/3} \sin^2 \theta p (ac + sb - psb), \\
 g'_{\psi\phi} &= D^{-2/3} \cos^2 \theta \sin^2 \theta p \left(ab - p \left(ab \sum_{I < J} s_I^2 s_J^2 \right. \right. \\
 &\quad \left. \left. + (a^2 + b^2) cs + 2abs^2 \right) \right), \\
 h'_{ij} &= h_{ij}, \quad g'_{rr} = D^{1/3} \frac{\rho^2 r^2}{\Delta}, \quad g'_{\theta\theta} = D^{1/3} \rho^2.
 \end{aligned}$$

This solution can be compared with the limiting form of the solution found by Cvetic, Lü, and Pope [57] within the gauged 5D supergravity for vanishing gauge-coupling constant $g = 0$. Our solution reduces to the latter under an identification of the rotation parameters $a = b = l$ and relabeling $s_I \rightarrow -s_I$, $\psi, \phi \rightarrow -\psi, -\phi$, $A^I \rightarrow -A^I$. Note that a close three-charge doubly rotating black hole solution within the compactified heterotic theory was given earlier in [55].

VII. $G_{2(2)}$ EMBEDDED IN $SO(4, 4)$

The present model reduces to minimal five-dimensional supergravity under the following identifications:

$$\begin{aligned}
 \psi^1 = \psi^2 = \psi^3 = \psi, \quad \mu_1 = \mu_2 = \mu_3 = \mu, \\
 \lambda^0 = \lambda, \quad X^1 = X^2 = X^3 = 1,
 \end{aligned} \quad (36)$$

leading to the relations:

$$\begin{aligned}
 \phi_1 &= \frac{1}{\sqrt{2}} \left(\varphi_2 + \frac{1}{\sqrt{3}} \varphi_1 \right), \quad \phi_2 = \frac{1}{\sqrt{2}} \left(\varphi_2 - \frac{1}{\sqrt{3}} \varphi_1 \right), \\
 \phi_3 &= \sqrt{\frac{2}{3}} \varphi_1, \quad \phi_4 = 0.
 \end{aligned}$$

In this case the target-space metric of the three-dimensional sigma model will read:

$$\begin{aligned}
 dl^2 &= \frac{1}{2} (d\varphi_1^2 + d\varphi_2^2 + 3\kappa e^{\varphi_2 + (1/\sqrt{3})\varphi_1} G^2 + 3e^{(2/\sqrt{3})\varphi_1} du^2 \\
 &\quad + 3\kappa e^{\varphi_2 - (1/\sqrt{3})\varphi_1} (dv - \chi du)^2 + \kappa e^{\varphi_2 + \sqrt{3}\varphi_1} G_7^2 \\
 &\quad + e^{2\varphi_2} G_8^2 + \kappa e^{\varphi_2 - \sqrt{3}\varphi_1} d\chi^2),
 \end{aligned} \quad (37)$$

where the one-forms are

$$\begin{aligned}
 G &= d\mu + vdu - udv, \\
 G_7 &= V_7, \\
 G_8 &= V_8 - \chi V_7, \\
 V_p &= d\omega_p - \psi_p (3d\mu + \epsilon^{qt} d\psi_q \psi_t).
 \end{aligned}$$

This manifold is invariant under the $G_{2(2)}$ subgroup of the $SO(4, 4)$. Dimensional reduction of the 5D minimal supergravity to three dimensions was recently studied in [36,37]. In the notation of [36,37] the indices p, q, t take the values 0, 1, the coordinate z^7 is timelike, and the matrix λ is related to the present one by transposition with respect to the minor diagonal. In matrix terms the target-space metric (37) reads

$$\begin{aligned}
 dl^2 &= \frac{1}{4} \text{Tr}(\lambda^{-1} d\lambda \lambda^{-1} d\lambda) + \frac{1}{4} \tau^{-2} d\tau^2 + \frac{3}{2} d\psi^T \lambda^{-1} d\psi \\
 &\quad - \frac{1}{2} \tau^{-1} V^T \lambda^{-1} V - \frac{3}{2} \tau^{-1} G^2.
 \end{aligned}$$

This coincides with the result of [36,37] for the Euclidean signature of the three-space ($\kappa = -1$).

A. $\mathfrak{g}_{2(2)}$ subalgebra of $\mathfrak{so}(4, 4)$

The above contraction to $G_{2(2)}$ can be described in terms of the root space as follows. Consider the root vectors of the $\mathfrak{so}(4, 4)$ algebra in the following basis:

$$\begin{aligned}
 \vec{e}_1 &= \left(s - \frac{1}{2}, s + \frac{1}{2}, \frac{1}{2}, s \right), \\
 \vec{e}_2 &= \left(1, -1, \frac{1}{2}, s \right), \\
 \vec{e}_3 &= \left(-s - \frac{1}{2}, \frac{1}{2} - s, \frac{1}{2}, s \right), \\
 \vec{e}_4 &= \left(\frac{1}{2} - s, -s - \frac{1}{2}, 1, 0 \right), \\
 \vec{e}_5 &= (-1, 1, 1, 0), \\
 \vec{e}_6 &= \left(s + \frac{1}{2}, s - \frac{1}{2}, 1, 0 \right), \\
 \vec{e}_7 &= \left(\frac{1}{2} - s, -s - \frac{1}{2}, -\frac{1}{2}, s \right), \\
 \vec{e}_8 &= \left(-1, 1, -\frac{1}{2}, s \right), \\
 \vec{e}_9 &= \left(s + \frac{1}{2}, s - \frac{1}{2}, -\frac{1}{2}, s \right), \\
 \vec{e}_{10} &= \left(0, 0, \frac{3}{2}, s \right), \quad \vec{e}_{11} = (0, 0, 0, 2s), \\
 \vec{e}_{12} &= \left(0, 0, -\frac{3}{2}, s \right), \quad s = \frac{\sqrt{3}}{2}.
 \end{aligned}$$

Examination of this pattern shows that the following combinations of the triplets of the $\mathfrak{so}(4, 4)$ root vectors:

$$\begin{aligned}\tilde{\alpha}_{\pm 4} &= \frac{1}{3} \sum_I \tilde{e}_{\pm I}, & \tilde{\alpha}_{\pm 1} &= \frac{1}{3} \sum_I \tilde{e}_{\pm(I+3)}, \\ \tilde{\alpha}_{\pm 3} &= \frac{1}{3} \sum_I \tilde{e}_{\pm(I+6)},\end{aligned}$$

together with

$$\tilde{\alpha}_{\pm 5} = \tilde{e}_{\pm 10}, \quad \tilde{\alpha}_{\pm 6} = \tilde{e}_{\pm 11}, \quad \tilde{\alpha}_{\pm 2} = \tilde{e}_{\pm 12},$$

form the standard set of the g_2 roots satisfying the relations:

$$\begin{aligned}\tilde{\alpha}_{\pm 3} &= \pm(\tilde{\alpha}_1 + \tilde{\alpha}_2), & \tilde{\alpha}_{\pm 4} &= \pm(2\tilde{\alpha}_1 + \tilde{\alpha}_2), \\ \tilde{\alpha}_{\pm 5} &= \pm(3\tilde{\alpha}_1 + \tilde{\alpha}_2), & \tilde{\alpha}_{\pm 6} &= \pm(3\tilde{\alpha}_1 + 2\tilde{\alpha}_2).\end{aligned}$$

The corresponding generators read

$$\begin{aligned}M_1 &= \frac{\sqrt{2}}{3}(H_1 - H_2 + 2H_3), & M_2 &= \sqrt{\frac{2}{3}}(H_1 + H_2), \\ P^\pm &= \frac{1}{\sqrt{3}} \sum P^{\pm I}, & Z_\pm &= \frac{1}{\sqrt{3}} \sum Z_{\pm I}, \\ W_\pm &= \frac{1}{\sqrt{3}} \sum W_{\pm I}, & \Omega^{\pm p}, & X^\pm.\end{aligned}$$

They obey the following commutation relations in the Cartan-Weyl form:

$$\begin{aligned}[P^+, P^-] &= \frac{1}{2}M_1 + \frac{\sqrt{3}}{2}M_2, \\ [W_+, W_-] &= M_1, \\ [Z_+, Z_-] &= -\frac{1}{2}M_1 + \frac{\sqrt{3}}{2}M_2, \\ [\Omega^7, \Omega^{-7}] &= \frac{3}{2}M_1 + \frac{\sqrt{3}}{2}M_2, \\ [\Omega^8, \Omega^{-8}] &= \frac{\sqrt{3}}{2}M_2, \\ [X^+, X^-] &= -\frac{3}{2}M_1 + \frac{\sqrt{3}}{2}M_2, \\ [W_\pm, P^\pm] &= \mp \Omega^{\pm 7}, \\ [Z_\pm, P^\pm] &= \mp \Omega^{\pm 8}, \\ [W_\pm, Z_\pm] &= \mp P^\pm, \\ [X^\pm, W_\pm] &= \mp Z^\pm, \\ [X^\pm, \Omega^{\pm 7}] &= \mp \Omega^{\pm 8},\end{aligned}$$

and so on.

B. 8×8 matrix representation for the coset $G_{2(2)}/(SL(2, R) \times SL(2, R))$

Contracting the set of the potentials Φ^A according to the conditions (36), we obtain the following representation for the coset blocks \mathcal{P} and \mathcal{Q} :

$$\mathcal{Q} = \begin{pmatrix} \mu, & \omega - \mu\psi^T, & 0 \\ \tilde{\psi}^T, & \mu\sigma_3, & \tilde{\omega} - \mu\tilde{\psi}^T \\ 0, & \psi^T, & -\mu \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\mathcal{P} = -\tau^{-1} \begin{pmatrix} 1, & \eta^T, & \mu \\ \eta, & \eta\eta^T - \tau\tilde{\lambda}, & \eta\mu - \tau\tilde{\lambda}\tilde{\psi}^T \\ \mu, & \eta^T\mu - \tau\tilde{\psi}\tilde{\lambda}, & \mu^2 - \tau - \tau\tilde{\psi}\tilde{\lambda}\tilde{\psi}^T \end{pmatrix},$$

$$\eta = \sigma_3\psi.$$

This gives a 8×8 representation of the coset $G_{2(2)}/(SL(2, R) \times SL(2, R))$ of the minimal five-dimensional supergravity reduced to three dimensions, alternative to the 7×7 one given in [36,37].

VIII. CONCLUSIONS

In this paper we have constructed a generating technique for the $U(1)^3$ 5D supergravity with two commuting Killing symmetries. This theory is reduced to the three-dimensional gravity coupled sigma model on symmetric spaces $SO(4, 4)/SO(4) \times SO(4)$ or $SO(4, 4)/SO(2, 2) \times SO(2, 2)$ depending on the signature of the three-space. The classical U -duality group of the three-dimensional theory is the 28-parametric noncompact group $SO(4, 4)$ which acts transitively on the target space. This enables one to generate new five-dimensional solutions with the same three-metric from the seed ones. We were able to obtain finite transformations in terms of the target-space potentials, and, in addition, we constructed the 8×8 matrix representation of the coset, which is convenient for performing the transformations explicitly. Particular combinations of transformations were identified which preserve asymptotic conditions relevant for the black hole and the black ring problems. We presented the action of charging transformations on a neutral seed, assuming the dimensional reduction in terms of Guisto-Saxena coordinates.

As an application, we have constructed a new rotating five-dimensional black hole with three independent charges and two rotation parameters. Our technique allows in principle to generate black rings with the maximal number of parameters (a mass, two rotation parameters, three electric charges, and three magnetic dipole moments), but so far our attempts to find such a solution in a concise form were unsuccessful.

An identification of the three vector fields and freezing out the two scalar moduli reduce the present theory to minimal five-dimensional supergravity with the three-dimensional U -duality group $G_{2(2)}$, which was extensively studied recently along similar lines [36,37]. For this limiting case we have presented a new matrix representation for

the coset $G_{2(2)}/(SL(2, R) \times SL(2, R))$ in terms of the 8×8 matrices.

ACKNOWLEDGMENTS

The authors thank Gerard Clement and Paul Sorba (LAPTH, Annecy) and Chiang-Mei Chen (NCU, Taiwan) for helpful suggestions. The work was supported by the RFBR Grant No. 08-02-01398-a.

APPENDIX A: 8×8 MATRIX REPRESENTATION

We choose the following 8×8 matrix representation of the $so(4, 4)$ algebra:

$$E = \begin{pmatrix} A & B \\ C & -A^{\hat{T}} \end{pmatrix}, \quad (A1)$$

where A, B, C are the 4×4 matrices, A, B being antisymmetric, $B = -B^T$, $C = -C^T$, and the symbol \hat{T} in $A^{\hat{T}}$ means transposition with respect to the minor diagonal. The diagonal matrices \vec{H} are given by the following A -type matrices (with $B = 0 = C$):

$$\begin{aligned} A_{H_1} &= \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & A_{H_2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ A_{H_3} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & A_{H_4} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}. \end{aligned}$$

Twelve generators corresponding to the positive roots are given by the upper-triangular matrices E_k , $k = 1, \dots, 12$. From these the generators labeled by $k = 2, 2, 4, 6, 7, 9, 12$ are of pure A -type (with $B = 0 = C$):

$$\begin{aligned} A_{E_2} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & A_{E_4} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ A_{E_6} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & A_{E_7} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ A_{E_9} &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & A_{E_{12}} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

while the other six are of pure B type (with $A = 0 = C$):

$$\begin{aligned} B_{E_1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & B_{E_3} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ B_{E_5} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & B_{E_8} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ B_{E_{10}} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & B_{E_{11}} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The correspondence with the previously introduced generators is as follows ($I = 1, 2, 3, p = 7, 8$):

$$\begin{aligned} P^I &\leftrightarrow E_I, & W_I &\leftrightarrow E_{I+3}, & Z_I &\leftrightarrow E_{I+6}, \\ \Omega^p &\leftrightarrow E_{p+3}, & X &\leftrightarrow E_{12}. \end{aligned}$$

In this representation, the matrices corresponding to the negative roots,

$$\begin{aligned} P^{-I} &\leftrightarrow E_{-I}, & W_{-I} &\leftrightarrow E_{-(I+3)}, & Z_{-I} &\leftrightarrow E_{-(I+6)}, \\ \Omega^{-p} &\leftrightarrow E_{-(p+3)}, & X^- &\leftrightarrow E_{-12}, \end{aligned}$$

are transposed with respect to the positive roots matrices:

$$E_{-k} = (E_k)^T.$$

The following normalization conditions are assumed:

$$\begin{aligned} \text{tr}(H_i, H_j) &= 4\delta_{ij}, & i, j &= 1 \dots 4, \\ \text{tr}(E_k, E_{-k}) &= 2, \end{aligned}$$

and the involution matrix K is chosen as

$$K = \text{diag}(\kappa, \kappa, 1, 1, 1, 1, \kappa, \kappa).$$

The generators of the isotropy subgroup are selected by Eq. (23). They are given by the following linear combinations of the generators:

$$\begin{aligned} P^I - \kappa P^{-I}, & & Z_I - \kappa Z_{-I}, & & W_I - W_{-I}, \\ X - \kappa X^-, & & \Omega^7 - \kappa \Omega^{-7}, & & \Omega^8 - \Omega^{-8}. \end{aligned}$$

APPENDIX B: DETAILS OF THE COSET MATRICES

The block matrices entering S_Ψ and R_Ψ , being expressed through the target-space potentials, read

$$\begin{aligned}
e^{1/2\Psi_3}\tilde{\Psi}_a^T &= \begin{pmatrix} \frac{1}{2}(u^a v^3 - u^3 v^a) \\ -v^a \\ -u^a \end{pmatrix}, \quad a = 1, 2; \quad \Psi_{12} = -\begin{pmatrix} 0 & 0 & 0 \\ v^1 u^2 & v^1 v^2 & 0 \\ u^1 u^2 & u^1 v^2 & 0 \end{pmatrix}, \\
\Psi_3 \Psi_{12} + \Psi_{12} \tilde{\Psi}_3 &= \begin{pmatrix} v^3 u^1 u^2 - u^3 v^1 u^2, & -u^3 v^1 v^2 + v^3 v^2 u^1 & 0 \\ 0 & 0 & u^3 v^1 v^2 - v^3 v^2 u^1 \\ 0 & 0 & -v^3 u^1 u^2 + u^3 v^1 u^2 \end{pmatrix}, \\
\Psi_3 \Psi_{12} \tilde{\Psi}_3 &= \begin{pmatrix} 0 & 0 & (u^3)^2 v^1 v^2 - u^3 v^3 v^1 u^2 - u^3 v^3 v^2 u^1 + (v^3)^2 u^1 u^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{B1}$$

The explicit form for \mathcal{Q} is

$$\mathcal{Q} = \begin{pmatrix} \mu_1 + \frac{u_3 v_2 - v_3 u_2}{2} & \omega_7 - \frac{u_3 v_1 u_2 - 2u_3 v_2 u_1 + v_3 u_1 u_2}{6} - u_2 \mu_2, & \omega_8 + \frac{v_3 u_1 v_2 - 2v_3 u_2 v_1 + u_3 v_1 v_2}{6} - v_2 \mu_2 & 0 \\ -v_2 & -\mu_3 + \frac{v_1 u_2 - u_1 v_2}{2} & 0 & \\ -u_2 & 0 & & \\ 0 & & & \end{pmatrix},$$

and the blocks entering \mathcal{P} are

$$e^{\Psi_3^T} \Lambda = \begin{pmatrix} \kappa e^{\sqrt{2}\phi_1} & 0 \\ \kappa e^{\sqrt{2}\phi_1} \eta & \tilde{\lambda}^0 \end{pmatrix}, \quad \eta = \begin{pmatrix} u^3 \\ -v^3 \end{pmatrix}, \quad a = 1, 2e^{\Psi_3^T} \Lambda e^{\Psi_3} = \kappa e^{\sqrt{2}\phi_1} \begin{pmatrix} 1 \\ \eta \end{pmatrix} \otimes (1, \eta^T) + \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\lambda}^0 \end{pmatrix}.$$

APPENDIX C: TRANSFORMATIONS PRESERVING ASYMPTOTIC FLATNESS

In this Appendix we exhibit the action of the transformations generated by linear combinations of generators

$$Z_I + Z_{-I}, \quad W_{-I} + P^{-I}, \quad X - \Omega^{-8}$$

on a neutral seed:

$$\begin{aligned}
X^1 = X^2 = X^3 = 1, \quad \phi_1 &= \frac{1}{\sqrt{2}} \left(\varphi_2 + \frac{1}{\sqrt{3}} \varphi_1 \right), \quad \phi_2 = \frac{1}{\sqrt{2}} \left(\varphi_2 - \frac{1}{\sqrt{3}} \varphi_1 \right), \quad \phi_3 = \frac{\sqrt{6}}{3} \varphi_1, \quad \phi_4 = 0, \\
\Lambda &= \begin{pmatrix} -\tau^{-1} & 0 \\ 0 & \tilde{\lambda} \end{pmatrix}, \quad \lambda = \lambda^0 \neq 0, \quad \tau = \tau_0 = e^{-\sqrt{2}\phi_1}, \quad \omega_p \neq 0, \quad \psi^I = 0, \quad \mu_I = 0.
\end{aligned}$$

The relation $e^{-\sqrt{2}\phi_1} = \tau$ is a consequence of the condition $X^3 = 1$ [see Eq. (13)]. Such a seed may describe a neutral black ring or a black hole with one or two independent rotation parameters. The three above transformations preserve asymptotic flatness with the Guisto-Saxena choice of coordinates and generate some combinations of charges. For this seed the coset blocks simplify to

$$\mathcal{P}_0 = \begin{pmatrix} \Lambda & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{Q}_0 = \begin{pmatrix} 0 & \Omega \\ 0 & 0 \end{pmatrix},$$

and we obtain the following transformations of the potentials:

1. $Z_I + Z_{-I}$

$$\lambda'_{77} = D^{-2/3}(\lambda_{77} c^2 - \tau s^2), \quad \lambda'_{78} = D^{-2/3} c \lambda_{78}, \quad \lambda'_{88} = D^{-2/3} \lambda_{88}, \quad \tau' = \tau D^{-1/3},$$

$$e^{\sqrt{2}\phi'_I} = \begin{cases} \tau^{-1}, & I = 1, 2 \\ D\tau^{-1}, & I = 3, \end{cases}, \quad e^{\sqrt{2}\phi'_I} = \begin{cases} D^{-1}, & I = 1 \\ D & I = 2 \\ 1, & I = 3 \end{cases}$$

$$(v^J)' = \delta^{IJ} \frac{sc(1 + \lambda_{88})}{D}, \quad (u^J)' = \delta^{IJ} \frac{s\lambda_{78}}{D}, \quad (X^J)' = D^{-2/3} \delta^{IJ} + D^{1/3}(1 - \delta^{IJ}).$$

$$\mu'_I = 0, \quad \omega'_7 = \omega_7, \quad \omega'_8 = c\omega_8,$$

with

$$D = c^2 + s^2\lambda_{88}, \quad c \equiv \cosh(\alpha), \quad s \equiv \sinh(\alpha).$$

2. $W_{-I} + P^{-I}$

$$\tilde{\lambda}'_{77} = \begin{cases} \tilde{\lambda}_{77}, & I = 1, 2 \\ \frac{D_2}{D_1} \tilde{\lambda}_{77}, & I = 3 \end{cases}, \quad \tilde{\lambda}'_{78} = \begin{cases} \tilde{\lambda}_{78}(1 - \xi_2) - \tilde{\lambda}_{77}\xi_3, & I = 1, 2 \\ \frac{D_2}{D_1}(\tilde{\lambda}_{78}(1 - \xi_2) - \tilde{\lambda}_{77}\xi_3), & I = 3 \end{cases},$$

$$\tilde{\lambda}'_{88} = \begin{cases} \tilde{\lambda}_{77}\xi_3^2 - 2\tilde{\lambda}_{78}\xi_3(1 - \xi_2) + \tilde{\lambda}_{88}(1 - \xi_2)^2 + \alpha^2(1 - \frac{1}{2}\xi_1), & I = 1, 2 \\ \frac{D_2}{D_1}(\tilde{\lambda}_{77}\xi_3^2 - 2\tilde{\lambda}_{78}\xi_3(1 - \xi_2) + \tilde{\lambda}_{88}(1 - \xi_2)^2 + \alpha^2(1 - \frac{1}{2}\xi_1)), & I = 3 \end{cases},$$

$$\tau' = \begin{cases} \tau D_1^{-1}, & I = 1, 2 \\ D_1 D_2^{-2} \tau, & I = 3 \end{cases}, \quad e^{\sqrt{2}\phi'_I} = \begin{cases} D_2 \tau^{-1}, & I = 1, 2 \\ D_1 \tau^{-1}, & I = 3 \end{cases}, \quad e^{\sqrt{2}\phi'_4} = \begin{cases} D_2/D_1, & I = 1 \\ D_1/D_2, & I = 2 \\ 1, & I = 3 \end{cases}$$

$$\omega'_7 = D_2^{-1}(\omega_7(1 + \xi_2) - \lambda_{77}\xi_1), \quad \omega'_8 = D_2^{-1}(\omega_8(1 + \xi_2) - \lambda_{78}\xi_1),$$

$$(v^J)' = \delta^{JJ} D_1^{-1} \alpha (\lambda_{77}\xi_3 + \lambda_{78}(1 + \xi_2 - \xi_1) - \omega_8(1 - \xi_2)), \quad (u^J)' = \delta^{JJ} D_1^{-1} \alpha (\lambda_{77}(1 - \xi_1) - \omega_7(1 - \xi_2)),$$

$$\mu'_J = \delta_{JJ} D_2^{-1} \alpha (\omega_7(1 + \xi_2) - \lambda_{77}\xi_1 - \tau), \quad (X^J)' = \left(\frac{D_2}{D_1}\right)^{-2/3} \delta^{JJ} + \left(\frac{D_2}{D_1}\right)^{1/3} (1 - \delta^{JJ}).$$

where

$$\xi_1 = \frac{\alpha^2 \tau}{2}, \quad \xi_2 = \frac{\alpha^2 \omega_7}{2}, \quad \xi_3 = \frac{\alpha^2 \omega_8}{2}, \quad \xi_4 = \frac{\alpha^2 \lambda_{77}}{2}.$$

$$D_1 = (1 - \xi_2)^2 + 2\xi_4 \left(1 - \frac{1}{2}\xi_1\right), \quad D_2 = (1 + \xi_2)^2 - 2\xi_1 \left(1 + \frac{1}{2}\xi_4\right).$$

3. $X - \Omega^{-8}$

$$\tilde{\lambda}'_{77} = -\alpha^2 \tau + \tilde{\lambda}_{77}\xi_1^2 - 2\tilde{\lambda}_{78}\alpha\omega_7\xi_1 + \tilde{\lambda}_{88}\alpha^2\omega_7^2,$$

$$\tilde{\lambda}'_{78} = \frac{1}{2}\alpha^3\tau - \frac{1}{2}\tilde{\lambda}_{77}\alpha\xi_1(1 + \xi_1) + \tilde{\lambda}_{78}(\xi_1 + \xi_2 + 2\xi_1\xi_2) - \tilde{\lambda}_{88}\alpha\omega_7(1 + \xi_2),$$

$$\tilde{\lambda}'_{88} = -\frac{1}{4}\alpha^4\tau + \frac{1}{4}\tilde{\lambda}_{77}\alpha^2(1 + \xi_1)^2 - \tilde{\lambda}_{78}\alpha(1 + \xi_1)(1 + \xi_2) + \tilde{\lambda}_{88}(1 + \xi_2)^2,$$

$$\phi'_4 = 0, \quad e^{\sqrt{2}\phi'_I} = D_\omega e^{\sqrt{2}\phi_1}, \quad \tau' = D_\omega^{-1}\tau,$$

$$\omega'_7 = D_\omega^{-1} \left(\omega_7(\xi_1 - \xi_2) + \alpha \left(\frac{1}{2} \alpha \lambda_{77} + \lambda_{78} \right) \tau \right),$$

$$\omega'_8 = D_\omega^{-1} \left((\omega_8 + \alpha\omega_7)(\xi_1 - \xi_2) + \alpha \left(\frac{1}{2} \alpha^2 \lambda_{77} + \frac{3}{2} \alpha \lambda_{78} + \lambda_{88} \right) \tau \right),$$

$$(\psi^I)' = 0, \quad \mu'_I = 0, \quad (X^I)' = 1,$$

$$D_\omega = \left(1 - \alpha\omega_8 - \frac{1}{2}\alpha^2\omega_7 \right)^2 - \alpha^2 \left(\frac{1}{4}\alpha^2\lambda_{77} + \alpha\lambda_{78} + \lambda_{88} \right) \tau, \quad \xi_1 = 1 - \alpha\omega_8, \quad \xi_2 = \frac{1}{2}\alpha^2\omega_7.$$

APPENDIX D: POTENTIALS IN TERMS OF \mathcal{P} AND \mathcal{Q}

Here we give the explicit expressions for the target-space potentials in terms of the components of the matrix \mathcal{P} and \mathcal{Q} :

$$\psi^1 = \frac{1}{\mathcal{P}_{11}D_{11,22,33}} \begin{pmatrix} D_{11,23}D_{11,24} - D_{11,22}D_{11,34} \\ D_{11,23}D_{11,34} - D_{11,24}D_{11,33} \end{pmatrix}, \quad \psi^2 = \begin{pmatrix} \mathcal{Q}_{42} \\ \mathcal{Q}_{43} \end{pmatrix}, \quad \psi^3 = \begin{pmatrix} \mathcal{P}_{12}/\mathcal{P}_{11} \\ -\mathcal{P}_{13}/\mathcal{P}_{11} \end{pmatrix},$$

where $D_{ij,kl}$ denotes the determinant of the 2×2 matrix constructed from \mathcal{P}_{ij} :

$$D_{ij,kl} = \mathcal{P}_{ij}\mathcal{P}_{kl} - \mathcal{P}_{ik}\mathcal{P}_{jl},$$

and $D_{11,22,33}$ is the determinant of the 3×3 minor of the 4×4 matrix \mathcal{P} with the diagonal $\mathcal{P}_{11}, \mathcal{P}_{22}, \mathcal{P}_{33}$. The remaining quantities read

$$\kappa e^{\sqrt{2}\phi_1} = \mathcal{P}_{11}, \quad \tilde{\lambda}^0 = \frac{1}{\mathcal{P}_{11}} \begin{pmatrix} D_{11,22} & D_{11,23} \\ D_{11,23} & D_{11,33} \end{pmatrix}, \quad \mu_1 = \mathcal{Q}_{11} - \frac{1}{2\mathcal{P}_{11}}(\mathcal{P}_{12}\mathcal{Q}_{43} + \mathcal{P}_{13}\mathcal{Q}_{42}),$$

$$\mu_2 = \frac{\mathcal{P}_{14}}{\mathcal{P}_{11}} + \frac{1}{2}(u^3v^1 - v^3u^1), \quad \mu_3 = \mathcal{Q}_{33} + \frac{1}{2}(v^1u^2 - v^2u^1).$$

$$e^{\sqrt{2}\phi_4} = \mathcal{P}_{44} - (\mu_2 + \frac{1}{2}(v_3u_1 - v_1u_3))^2\mathcal{P}_{11} - \tilde{\lambda}_{77}^0v_1^2 - 2\tilde{\lambda}_{78}^0v_1u_1 - \tilde{\lambda}_{88}^0u_1^2,$$

$$\omega_7 = \mathcal{Q}_{12} + u_2\mu_2 + \frac{1}{6}u_1u_2v_3 - \frac{1}{3}u_3u_1v_2 + \frac{1}{6}u_3u_2v_1,$$

$$\omega_8 = \mathcal{Q}_{13} + v_2\mu_2 - \frac{1}{6}u_3v_1v_2 + \frac{1}{3}v_3v_1u_2 - \frac{1}{6}v_3v_2u_1.$$

-
- [1] R. Emparan and H. S. Reall, Phys. Rev. Lett. **88**, 101101 (2002); Classical Quantum Gravity **23**, R169 (2006).
- [2] H. Elvang, Phys. Rev. D **68**, 124016 (2003).
- [3] R. Emparan, J. High Energy Phys. 03 (2004) 064.
- [4] S. S. Yazadjiev, Phys. Rev. D **73**, 104007 (2006); J. High Energy Phys. 07 (2006) 036; Gen. Relativ. Gravit. **39**, 601 (2007); arXiv:hep-th/0805.1600 [Phys. Rev. D (to be published)].
- [5] T. Harmark, Phys. Rev. D **70**, 124002 (2004).
- [6] H. Iguchi and T. Mishima, Phys. Rev. D **73**, 121501 (2006); S. Tomizawa, H. Iguchi, and T. Mishima, Phys. Rev. D **74**, 104004 (2006); H. Iguchi and T. Mishima, Phys. Rev. D **75**, 064018 (2007).
- [7] S. Tomizawa and M. Nozawa, Phys. Rev. D **73**, 124034 (2006).
- [8] T. Koikawa, Prog. Theor. Phys. **114**, 793 (2005).
- [9] A. A. Pomeransky, Phys. Rev. D **73**, 044004 (2006).
- [10] T. Azuma and T. Koikawa, Prog. Theor. Phys. **116**, 319 (2006).
- [11] T. Mishima and H. Iguchi, Phys. Rev. D **73**, 044030 (2006).
- [12] S. Tomizawa, Y. Morisawa, and Y. Yasui, Phys. Rev. D **73**, 064009 (2006); Phys. Rev. D **77**, 064019 (2008).
- [13] H. Elvang and P. Figueras, J. High Energy Phys. 05 (2007) 050.
- [14] A. A. Pomeransky and R. A. Sen'kov, arXiv:hep-th/0612005.
- [15] J. Evslin and C. Krishnan, arXiv:0706.1231.
- [16] S. S. Yazadjiev, Phys. Rev. D **76**, 064011 (2007).
- [17] H. Elvang and M. J. Rodriguez, J. High Energy Phys. 04 (2008) 045.
- [18] H. Elvang, R. Emparan, and P. Figueras, J. High Energy Phys. 05 (2007) 056.
- [19] D. Maison, Gen. Relativ. Gravit. **10**, 717 (1979).
- [20] V. Belinsky and R. Ruffini, Phys. Lett. **89B**, 195 (1980).
- [21] E. Cremmer, in *Superspace and Supergravity*, edited by S. W. Hawking and M. Roček (Cambridge University Press, Cambridge, 1981), p. 267.
- [22] A. H. Chamseddine and H. Nicolai, Phys. Lett. **96B**, 89 (1980).
- [23] S. Cecotti, S. Ferrara, and L. Girardello, Int. J. Mod. Phys. A **4**, 2475 (1989).
- [24] M. Bodner and A. C. Cadavid, Classical Quantum Gravity **7**, 829 (1990).
- [25] B. de Wit, F. Vanderseypen, and A. Van Proeyen, Nucl. Phys. **B400**, 463 (1993).
- [26] B. de Wit and A. Van Proeyen, Int. J. Mod. Phys. D **3**, 31 (1994).
- [27] S. Mizoguchi and N. Ohta, Phys. Lett. B **441**, 123 (1998).
- [28] E. Cremmer, B. Julia, H. Lü, and C. N. Pope, arXiv:hep-th/9909099.
- [29] S. Mizoguchi and G. Schröder, Classical Quantum Gravity **17**, 835 (2000).
- [30] M. Possel, Ph.D. thesis, Uneversität Hamburg, 2003.
- [31] M. Possel and S. Silva, Phys. Lett. B **580**, 273 (2004).
- [32] M. Gunaydin, A. Neitzke, O. Pavlyk, and B. Pioline, arXiv:0707.1669.
- [33] N. G. Scherbluk, M.S. thesis, Moscow State University, 2006.
- [34] K. Dasgupta, V. Hussin, and A. Wissanji, Nucl. Phys. **B793**, 34 (2008).

- [35] M. Berkooz and B. Pioline, *J. High Energy Phys.* **05** (2008) 045.
- [36] A. Bouchareb, G. Clément, C.-M. Chen, D. V. Gal'tsov, N. G. Scherbluk, and Th. Wolf, *Phys. Rev. D* **76**, 104032 (2007).
- [37] G. Clément, *J. Math. Phys. (N.Y.)* **49**, 042503 (2008).
- [38] D. Mateos and P. K. Townsend, *Phys. Rev. Lett.* **87**, 011602 (2001).
- [39] S. Giusto, S. D. Mathur, and A. Saxena, *Nucl. Phys.* **B701**, 357 (2004).
- [40] S. Giusto, S. D. Mathur, and Y. K. Srivastava, *Nucl. Phys.* **B763**, 60 (2007).
- [41] R. Emparan, D. Mateos, and P. K. Townsend, *J. High Energy Phys.* **07** (2001) 011.
- [42] H. Elvang and R. Emparan, *J. High Energy Phys.* **11** (2003) 035.
- [43] I. Bena and P. Kraus, *Phys. Rev. D* **70**, 046003 (2004).
- [44] I. Bena, *Phys. Rev. D* **70**, 105018 (2004).
- [45] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, *J. High Energy Phys.* **02** (2005) 031.
- [46] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, *Phys. Rev. D* **71**, 024033 (2005).
- [47] I. Bena and N. P. Warner, *Adv. Theor. Math. Phys.* **9**, 667 (2005).
- [48] J. P. Gauntlett and J. B. Gutowski, *Phys. Rev. D* **71**, 045002 (2005).
- [49] H. Elvang, R. Emparan, and P. Figueras, *J. High Energy Phys.* **02** (2005) 031.
- [50] F. Larsen, *J. High Energy Phys.* **10** (2005) 100.
- [51] E. Cremmer, B. Julia, H. Lü, and C. N. Pope, *Nucl. Phys.* **B523**, 73 (1998); **B535**, 242 (1998).
- [52] R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications* (John Wiley & Sons, New York, 1974); L. Frappat, A. Sciarrino, and P. Sorba, *Dictionary on Lie Algebras and Superalgebras* (Academic Press, London, 2000).
- [53] S. Giusto and A. Saxena, *Classical Quantum Gravity* **24**, 4269 (2007).
- [54] V. Pravda, A. Pravdova, and M. Ortogio, *Classical Quantum Gravity* **24**, 4407 (2007).
- [55] M. Cvetič and D. Youm, *Nucl. Phys.* **B476**, 118 (1996).
- [56] D. Ida and Y. Uchida, *Phys. Rev. D* **68**, 104014 (2003).
- [57] M. Cvetič, H. Lü, and C. N. Pope, *Phys. Rev. D* **70**, 081502 (2004).
- [58] Y. Morisawa and D. Ida, *Phys. Rev. D* **69**, 124005 (2004).
- [59] J. Kunz and F. Navarro-Lerida, *Phys. Rev. Lett.* **96**, 081101 (2006).
- [60] A. N. Aliev, *Phys. Rev. D* **74**, 024011 (2006).
- [61] J. Kunz, F. Navarro-Lerida, and A. K. Petersen, *Phys. Lett. B* **614**, 104 (2005).
- [62] J. Kunz and F. Navarro-Lerida, *Mod. Phys. Lett. A* **21**, 2621 (2006).
- [63] J. Kunz and F. Navarro-Lerida, *Phys. Lett. B* **643**, 55 (2006).
- [64] J. Kunz, D. Maison, F. Navarro-Lerida, and J. Viebahn, *Phys. Lett. B* **639**, 95 (2006).
- [65] J. Kunz, F. Navarro-Lerida, and J. Viebahn, *Phys. Lett. B* **639**, 362 (2006).
- [66] J. Mei and C. N. Pope, *Phys. Lett. B* **658**, 64 (2007).
- [67] B. Kleihaus, J. Kunz, and F. Navarro-Lerida, *AIP Conf. Proc.* **977**, 94 (2008).
- [68] S. L. Cacciatori and A. Celi, arXiv:hep-th/0405284.
- [69] J. Bellorin, P. Meessen, and T. Ortin, *J. High Energy Phys.* **01** (2007) 020.
- [70] R. Emparan and H. S. Reall, arXiv:hep-th/0801.3471.