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Nonlinear parent action and dual gravity

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We give a reformulation of nonlinear Einstein gravity, which contains the dual graviton together with the ordinary metric and a shift-gauge field. The metric does not enter through a ''kinetic'' Einstein-Hilbert term, but via topological couplings, and so the theory does not lead to a doubling of degrees of freedom. The field equations take the form of first-order duality relations. We analyze the gauge symmetries and comment on their meaning with regard to the E_{11} proposal.

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I. INTRODUCTION

It is a classic result that Kaluza-Klein reduction of 11-dimensional supergravity gives rise to exceptional hidden symmetries. Based on this observation it has been conjectured that the infinite-dimensional Kac-Moody algebra E_{11} is a symmetry of supergravity or possibly even M theory [1]. Part of the evidence for this conjecture consists of the fact that the level decompositions of E_{11} with respect to the $SL(D)$ subgroups precisely reproduce the field content of maximal supergravity in D dimensions. On the supergravity side this identification requires that one adds to each field its dual. For instance, E_{11} predicts not only the 3-form of 11-dimensional supergravity, but also a 6-form. Moreover, at higher level fields appear that transform in mixed Young tableaux representations, and the lowest of these can be interpreted as the dual of the metric (''dual graviton'').

At the free linearized level, Einstein gravity with metric $h_{\mu\nu}$ can be equivalently formulated in terms of the dual mixed Young tableaux field $C_{\mu_1\cdots\mu_{D-3}|\nu}$. To see this, one may choose light-cone gauge and dualize one index on the metric tensor by means of the epsilon tensor of the little group $SO(D-2)$, resulting in the dual metric with mixed symmetries [2]. Afterwards, the dual metric can be elevated to a space-time covariant object with an associated gauge symmetry [1,3–10], whose covariant action has been given by Curtright [11]. However, this dualization is problematic once the nonlinear theory is considered. The no-go theorems of [12] prove that there is no local, manifestly Poincaré-invariant, non-Abelian deformation of the Curtright action, and so there is no consistent non-Abelian self-interaction of the dual graviton. One way to circumvent this no-go theorem would be to give up space-time covariance and/or locality. In fact, if one is willing to do so, dualization is trivially possible. One simply has to replace inside the Einstein-Hilbert action in light-cone gauge [13]—which is neither local nor covariant—the graviton by (the Hodge-dual of) the dual graviton. A nontrivial way would be to give up covariance, but keeping locality, as it happens naturally in the E_{10} σ-model of [14]. In contrast, an essential feature of the E_{11} proposal is precisely its space-time covariance in that it reproduces the supergravity spectra in their covariant form. So at first sight there seems to be no way to preserve E_{11} beyond the ''dual graviton barrier.''

One may still hope to avoid the no-go theorem of [12], which considers only pure gravity, by taking other fields into account, as for instance 3- and 6-form of $D = 11$ supergravity, or the original metric itself. The former possibility seems to be unlikely since the Kac-Moody approach actually applies not only to maximal supergravity, but, in particular, also to pure gravity (then based on the Kac-Moody algebra A_{D-3}^{+++}), where these fields are not available. The idea of adding to the action of the dual graviton the original Einstein-Hilbert term, in order to possibly obtain consistent cross interactions, is equally unpromising since, even supposing the existence of such cross interactions, it would double the degrees of freedom, in contrast to the expectation that we should ultimately recover ordinary (super)gravity.

So the question we should really ask is a different one, namely, whether there exists a theory, which is

(i) classically equivalent to nonlinear Einstein gravity,
(ii) contains besides the metric the dual metric, and
(iii) is covariant and local.

The idea of a formulation in which the metric and its dual appear simultaneously in itself is not new. However, while so far these attempts abandoned space-time covariance and/or locality [15,16], we will see below, that it is surprisingly straightforward to satisfy all of the requirements (i)–(iii). For this we will mimic an approach, which has recently been proven to be very fruitful in the context of gauged supergravity [17–20] (see also [21] and references therein). Specifically, we will start from a certain covariantization of the Curtright action and add a topological (Chern-Simons like) term containing the original metric.

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The resulting theory is then proven to be equivalent to Einstein gravity.

This paper is organized as follows. In Sec. II we review the dualization of the graviton in the linearization, and discuss the symmetries of the covariant action for the dual graviton. We turn to nonlinear gravity in Sec. III, where we first explain our strategy with a toy model of vector-scalar duality in $D = 3$. This is then used to derive a nonlinear action, called "parent action," which simultaneously contains the graviton and its dual. We comment on its symmetry structure in view of the E_{11} proposal and conclude in Sec. IV.

II. LINEARIZED DUAL GRAVITY

We start by reviewing the dualization of the graviton at the linearized level, as given, for instance, in [1,8]. For this one uses that the Einstein-Hilbert action based on the vielbein $e_\mu{}^a$ can be written, up to boundary terms, as [22][1]

$$S_{\rm EH} = -\int d^D x e(\Omega^{abc}\Omega_{abc} + 2\Omega^{abc}\Omega_{acb} - 4\Omega_{ab}{}^b\Omega^{ac}{}_c), \tag{2.1}$$

where

$$\Omega_{ab}{}^c = e_a{}^\mu e_b{}^\nu(\partial_\mu e_\nu{}^c - \partial_\nu e_\mu{}^c) \tag{2.2}$$

are the coefficients of anholonomy. This form of the Einstein-Hilbert action can be recast into first-order form by introducing an auxiliary field $Y_{ab|c} = -Y_{ba|c}$,

$$S[Y, e] = -2\int d^D x e\Big(Y^{ab|c}\Omega_{abc} - \frac{1}{2}Y_{ab|c}Y^{ac|b} + \frac{1}{2(D-2)}Y_{ab|}{}^b Y^{ac|}{}_c\Big). \tag{2.3}$$

The field equation of Y can be used to solve for it in terms of Ω,

$$Y_{ab|c} = \Omega_{abc} - 2\Omega_{c[ab]} + 4\eta_{c[a}\Omega_{b]d}{}^d. \tag{2.4}$$

After reinserting (2.4) into (2.3), one precisely recovers the Einstein-Hilbert action in the form (2.1). In fact, the action (2.3) coincides with the standard first-order action with the spin connection as an independent field, up to a mere field redefinition, which replaces the spin connection by $Y_{ab|c}$. For later use we note that (2.3) has the same symmetries as the original Einstein-Hilbert action. First, it is manifestly diffeomorphism invariant. Moreover, the invariance of the second-order action (2.1) under the local Lorentz group can be elevated to a symmetry of the first-order action by requiring that the auxiliary $Y_{ab|c}$ transforms as

$$\delta_\Lambda Y_{ab|c} = -2e_c{}^\mu\partial_\mu\Lambda_{ab} - 4\eta_{c[a}e^{\mu d}\partial_\mu\Lambda_{b]d} - 2\Lambda^d{}_{[a}Y_{b]d|c} + \Lambda^d{}_c Y_{ab|d}. \tag{2.5}$$

In order to obtain the dual graviton from (2.3) we have to consider the linearized theory and vary with respect to the metric. Before we linearize, it turns out to be convenient to first rewrite the action in terms of the Hodge dual of $Y^{ab|c}$,

$$Y^{ab|c} = \frac{1}{(D-2)!}\epsilon^{abc_1\cdots c_{D-2}}Y_{c_1\cdots c_{D-2}|}{}^c. \tag{2.6}$$

This yields

$$S = -\frac{2}{(D-2)!}\int d^D x e\Big(\epsilon^{abc_1\ldots c_{D-2}}Y_{c_1\ldots c_{D-2}|}{}^c\Omega_{abc} + \frac{D-3}{2(D-2)}Y^{c_1\ldots c_{D-2}|b}Y_{c_1\ldots c_{D-2}|b} - \frac{D-2}{2}Y^{c_1\ldots c_{D-3}a|}{}_a Y_{c_1\ldots c_{D-3}b|}{}^b + \frac{1}{2}Y^{c_1\ldots c_{D-3}a|b}Y_{c_1\ldots c_{D-3}b|a}\Big). \tag{2.7}$$

In the linearization around flat space, $e_\mu{}^a = \delta_\mu{}^a + \kappa h_\mu{}^a$, we can ignore the distinction between flat and curved indices. In particular, we have $\Omega_{\mu\nu\rho} = 2\partial_{[\mu}h_{\nu]\rho}$, where the field $h_{\mu\nu}$ has no symmetry. The field equation for $h_{\mu\nu}$ is

$$\partial_{[\mu_1}Y_{\mu_2\ldots\mu_{D-1}]|\nu} = 0. \tag{2.8}$$

The Poincaré lemma then implies that Y is the curl of a potential $C_{\mu_1\ldots\mu_{D-3}|\nu}$ (the "dual graviton"), which is completely antisymmetric in its first $D-3$ indices,

$$Y_{\mu_1\ldots\mu_{D-2}|\nu} = \partial_{[\mu_1}C_{\mu_2\ldots\mu_{D-2}]|\nu}. \tag{2.9}$$

Inserting this back into (2.7) yields a consistent action $S[C]$ for the dual graviton.

Up to now, $C_{\mu_1\ldots\mu_{D-3}|\nu}$ as defined by (2.9) does not transform in an irreducible $GL(D)$ representation since also Y does not possess a specific Young-diagram symmetry. However, one may check [8] that, after inserting (2.9) into the linearization of (2.7), the resulting action $S[C]$ is invariant under the following Stückelberg symmetry:

$$\delta_\Lambda C_{\mu_1\ldots\mu_{D-3}|\nu} = -\Lambda_{\mu_1\ldots\mu_{D-3}\nu}, \tag{2.10}$$

with completely antisymmetric shift parameter. Therefore, the totally antisymmetric part of $C_{\mu_1\ldots\mu_{D-3}|\nu}$ can be gauge-fixed to zero inside $S[C]$, giving rise to the dual graviton with a $(D-3, 1)$ Young-diagram symmetry.[2] In other words, in the action $S[C]$ the dual graviton appears in the so-called framelike formulation. The latter is the analogue of the vielbein formalism, in which the linearized Lorentz transformations act as Stückelberg transformations, and

[1] We choose the space-time signature to be $(- + \cdots +)$. The epsilon symbol is defined by $\varepsilon^{012\cdots} = +1$, i.e. as a density, such that $e^{-1}\varepsilon^{\mu_1\ldots\mu_D}$ transforms as a tensor.

[2] In this paper we denote by (p, q) two-column Young diagrams in the antisymmetric basis with p boxes in the first column and q boxes in the second column.

which can be generalized to arbitrary-spin fields [23–25] (more recently, see also [26–29]).

Let us stress that even though (2.3) and thus (2.7) are first-order formulations of *nonlinear* Einstein gravity, the identification of the dual graviton in (2.9) is only possible in the linearization, since in the full theory the integrability condition (2.8) is violated [8]. This is in agreement with the fact that there is no non-Abelian self-interacting theory for the dual graviton [12].

Before we proceed, let us examine the free theory of the dual graviton in more detail. In order to indicate that the field now carries a specific Young-diagram symmetry, we denote it by $D_{\mu_1\ldots\mu_{D-3}|\nu}$. The characteristics of those mixed Young tableaux fields have been studied independently in [11,30]. First of all, it transforms under two types of gauge transformations,

$$\delta D_{\mu_1\cdots\mu_{D-3}|\nu} = \partial_{[\mu_1}\alpha_{\mu_2\cdots\mu_{D-3}]|\nu} + \partial_{[\mu_1}\beta_{\mu_2\cdots\mu_{D-3}]\nu} - (-1)^{D-3}\partial_\nu\beta_{\mu_1\cdots\mu_{D-3}}. \quad (2.11)$$

Here, α possesses the $(D-4, 1)$ Young-diagram symmetry, and β is completely antisymmetric. Consequently, (2.11) is consistent with the Young tableau symmetry of $D_{\mu_1\ldots\mu_{D-3}|\nu}$. The β-transformations are the "dual" diffeomorphisms. For instance, in $D=4$, where the metric is self-dual, (2.11) reads $\delta_\beta D_{\mu\nu} = \partial_\mu\beta_\nu + \partial_\nu\beta_\mu$. In analogy to the ordinary graviton, there is no invariant field strength which is first order in derivatives, but only a second-order Riemann-tensorlike object. However, for the α-transformations an invariant field strength is simply given by

$$F_{\mu_1\cdots\mu_{D-2}|\nu} = \partial_{[\mu_1}D_{\mu_2\cdots\mu_{D-2}]|\nu}. \quad (2.12)$$

An invariant action (the Curtright action) can then be written as $S[C] = \int d^Dx \mathcal{L}_C(F)$, where

$$\mathcal{L}_C(F) = \frac{D-3}{2(D-2)}F^{\mu_1\cdots\mu_{D-2}|\nu}F_{\mu_1\cdots\mu_{D-2}|\nu} - \frac{1}{2}(D-2)F^{\mu_1\cdots\mu_{D-3}\rho|}{}_\rho F_{\mu_1\cdots\mu_{D-3}\lambda|}{}^\lambda + F^{\mu_1\cdots\mu_{D-3}\nu|\rho}F_{\mu_1\cdots\mu_{D-3}\rho|\nu}. \quad (2.13)$$

Here the coefficients are fixed by requiring gauge invariance under β-transformations. Up to a global prefactor, this is precisely the action one obtains by inserting (2.9) into (2.7). And, in fact, the distinction between C and D becomes redundant, since due to the symmetry (2.10), in the action the antisymmetric part of C drops out. To be more precise, the Lagrangian $\mathcal{L}_C(F)$ given above is invariant under (2.10) up to a total derivative.

III. COVARIANT THEORY OF NONLINEAR DUAL GRAVITY

In this section we are going to propose a nonlinear theory featuring the dual graviton, which still contains the original metric via a topological term. The resulting theory will be equivalent to ordinary general relativity. In order to motivate our approach, we first recall a nontrivial duality for non-Abelian gauge vectors encountered in gauged supergravity.

A. A toy model: Dualizing non-Abelian vectors

As is well known, in $D=3$ a free theory of Abelian Maxwell vectors is dual to a free theory of massless scalars. However, once the gauge vectors are promoted to non-Abelian Yang-Mills gauge fields, or if they are coupled to charged matter, this duality breaks down. As has been shown in [19,20], it is nevertheless possible to assign all propagating degrees of freedom to scalar fields, while the gauge vectors appear only through topological Chern-Simons terms. In other words, besides the dual scalars the action still contains the (non-Abelian) gauge vectors.

To illustrate this, let us start directly from the nonlinear action, whose corresponding Lagrangian is given by

$$\mathcal{L}_g(\varphi, A, B) = \tfrac{1}{2}(\kappa^{ab}\mathcal{D}_\mu\varphi_a\mathcal{D}^\mu\varphi_b + \varepsilon^{\mu\nu\rho}B_{\mu a}\mathcal{F}^a_{\nu\rho}), \quad (3.1)$$

which depends on scalars φ_a and gauge vectors A^a_μ, $B_{\mu a}$. Here the covariant derivatives and non-Abelian field strengths are defined by

$$\mathcal{D}_\mu\varphi_a = \partial_\mu\varphi_a + gf_{ab}{}^cA^b_\mu\varphi_c + B_{\mu a}, \quad (3.2)$$

$$\mathcal{F}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{bc}{}^aA^b_\mu A^c_\nu, \quad (3.3)$$

where $f_{ab}{}^c$ are the structure constants of a compact semisimple real Lie algebra with invariant Cartan-Killing form $\kappa^{ab} \propto \delta^{ab}$. Therefore, (3.1) is manifestly invariant under the gauge symmetries

$$\delta\varphi_a = -\Sigma_a - gf_{ab}{}^c\epsilon^b\varphi_c, \quad (3.4)$$

$$\delta A^a_\mu = \partial_\mu\epsilon^a + gf_{bc}{}^aA^b_\mu\epsilon^c, \quad (3.5)$$

$$\delta B_{\mu a} = \partial_\mu\Sigma_a + gf_{ab}{}^cA^b_\mu\Sigma_c - gf_{ab}{}^c\epsilon^bB_{\mu c}. \quad (3.6)$$

Even though this theory describes charged scalars and non-Abelian gaugings, it is still possible to dualize the scalars to vectors. To see this, we observe that due to the presence of a Chern-Simons term, the field equations of the gauge vectors are duality relations between vectors and scalars. Specifically, varying with respect to $B_{\mu a}$ gives

$$\mathcal{D}^\mu\varphi_a = -\tfrac{1}{2}\kappa_{ab}\varepsilon^{\mu\nu\rho}\mathcal{F}^b_{\nu\rho}. \quad (3.7)$$

This can be used to solve for $B_{\mu a}$ in terms of A^a_μ and φ_a. After reinsertion into (3.1), one recovers precisely the non-Abelian Yang-Mills Lagrangian,

$$\mathcal{L}_g(A) = -\tfrac{1}{4}\kappa_{ab}\mathcal{F}^{\mu\nu a}\mathcal{F}^b_{\mu\nu}. \quad (3.8)$$

(More conveniently, one may first use the shift symmetry spanned by Σ_a in order to gauge-fix φ_a to zero. Then, on shell, $B_{\mu a}$ is entirely expressed in terms of A^a_μ.)

In the ungauged limit $g \to 0$, the covariant derivatives reduce to mere Stückelberg derivatives, $D_\mu \varphi_a = \partial_\mu \varphi_a + B_{\mu a}$, while the Chern-Simons term becomes Abelian. In this limit the symmetries reduce to the Abelian

$$\delta\varphi_a = -\Sigma_a, \qquad \delta B_{\mu a} = \partial_\mu \Sigma_a, \qquad \delta A^a_\mu = \partial_\mu \epsilon^a, \tag{3.9}$$

and integrating out $B_{\mu a}$ results into the (positive) sum of Maxwell actions, of which (3.8) provides a consistent nonlinear deformation.[3]

Let us finally analyze the deformation of the gauge symmetries in more detail. At first sight, the gauging deforms the Abelian gauge transformations (3.9) for the ϵ^a as well as for the Σ_a in that the latter transform nontrivially under the former [see Eq. (3.4)]. In fact, the gauge transformations close according to

$$[\delta_\epsilon, \delta_\Sigma] = \delta_{\tilde{\Sigma}}, \qquad \tilde{\Sigma}_a = g f_{ab}{}^c \epsilon^b \Sigma_c, \tag{3.10}$$

indicating a semidirect product between the Yang-Mills gauge group and the translations. However, it is possible to show that the only true deformation of the gauge algebra concerns the Yang-Mills transformations spanned by ϵ^a. More precisely, one can redefine the parameters and the fields in such a way that the seemingly semidirect product (3.10) trivializes, leaving separate Yang-Mills transformations and Abelian translations. To show this we redefine the shift parameter according to

$$\bar{\Sigma}_a = \Sigma_a + g f_{ab}{}^c \epsilon^b \varphi_c, \tag{3.11}$$

and the gauge field $B_{\mu a}$ by

$$\bar{B}_{\mu a} = B_{\mu a} + g f_{ab}{}^c A^b_\mu \varphi_c. \tag{3.12}$$

After this redefinition, in total the fields transform as

$$\delta\varphi_a = -\bar{\Sigma}_a, \qquad \delta\bar{B}_{\mu a} = \partial_\mu \bar{\Sigma}_a - g f_{ab}{}^c \epsilon^b \mathcal{D}_\mu \varphi_c. \tag{3.13}$$

In other words, the gauge transformations on φ_a and $B_{\mu a}$ are as in the free case (3.9), up to a correction by the gauge-covariant derivative $\mathcal{D}_\mu \varphi_a$. However, as the latter is shift invariant, one finds that the commutator (3.10) indeed trivializes, $[\delta_\epsilon, \delta_{\bar{\Sigma}}] = 0$.

Before we proceed with the dual graviton, let us briefly comment on the properties of this theory in view of the E_{11} proposal. The reader might be disturbed by the fact that the enhancement of symmetries has been achieved through the introduction of a simple shift invariance, expressing a trivial product structure. However, this is in precise correspondence to what happens in the relation between E_{11} and ordinary p-form gauge symmetries [31]. For instance, a 2-form is taken to transform as $\delta B_{\mu\nu} = \partial_{[\mu} \Lambda A_{\nu]}$, for which the algebra closes according to the (p-form truncation of the) E_{11} algebra. This transformation can in turn be redefined such that $\delta B_{\mu\nu} = -\Lambda F_{\mu\nu}$, with the gauge-invariant field strength $F_{\mu\nu}$. Therefore, the commutator vanishes, hence trivializing the algebra. Given these similarities, we apply the presented scheme of "non-Abelian dualization" to the dual graviton and comment on the supergravity/Kac-Moody correspondence later on.

[3]In gauged supergravity it is usually convenient to have a different dependence on the gauge coupling g, which is such that the Chern-Simons term vanishes for $g \to 0$ [17]. The chosen assignment of the deformation parameter here is necessary in order to have the same "duality-covariant" form in the ungauged theory as well.

B. Linear dual gravity and its symmetries

In the last section we have seen that in $D = 3$ even the non-Abelian, that is, self-interacting Yang-Mills theory, can be dualized to a scalar theory, which then contains both the field and its dual. Consequently, this amounts to an enhancement of the gauge symmetry, since the action (3.1) exhibits besides the standard Yang-Mills symmetry additional local symmetries spanned by Σ_a (even though, as we have seen, their product structure is trivial). As we have argued in the introduction, we expect something similar for gravity. By strict analogy, we are looking for a nonlinear and covariant theory with kinetic terms for the dual graviton, but which still contains topological terms for the original graviton.

Let us start with the free theory in framelike formulation, with kinetic terms for the dual graviton $C_{\mu_1 \dots \mu_{D-3}}{}^a$. In addition, we introduce a Stückelberg gauge field $Y_{\mu_1 \cdots \mu_{D-2}}{}^a$ and a shift-invariant form $\hat{F}_{\mu_1 \cdots \mu_{D-2}}{}^a$ of the field strength $F_{\mu_1 \cdots \mu_{D-2}}{}^a = \partial_{[\mu_1} C_{\mu_2 \cdots \mu_{D-2}]}{}^a$

$$\hat{F}_{\mu_1 \cdots \mu_{D-2}}{}^a = F_{\mu_1 \cdots \mu_{D-2}}{}^a + Y_{\mu_1 \cdots \mu_{D-2}}{}^a. \tag{3.14}$$

The field strength $\hat{F}$ is invariant under

$$\begin{aligned} \delta Y_{\mu_1 \cdots \mu_{D-2}}{}^a &= \partial_{[\mu_1} \Sigma_{\mu_2 \cdots \mu_{D-2}]}{}^a, \\ \delta C_{\mu_1 \cdots \mu_{D-3}}{}^a &= -\Sigma_{\mu_1 \cdots \mu_{D-3}}{}^a. \end{aligned} \tag{3.15}$$

In order to make the transition to the nonlinear theory in the next section more transparent, we have kept the formal distinction between flat and curved indices, which are related by the trivial background vielbein $\bar{e}_\mu{}^a = \delta_\mu{}^a$. We recall that the vierbein is expanded, around flat space-time, as $e_\mu{}^a = \bar{e}_\mu{}^a + \kappa h_\mu{}^a$. Here we have taken all fields to be in reducible representations, i.e., the fields C and Y as well as the transformation parameter Σ possess an antisymmetric part, after converting all the indices into curved indices. In total, we consider the action

$$S = \int d^D x \mathcal{L}(h, C, Y), \tag{3.16}$$
$$\mathcal{L}(h, C, Y) = \mathcal{L}_C(\hat{F}) + 2\varepsilon^{\mu_1 \cdots \mu_{D-2}\nu\rho} Y_{\mu_1 \cdots \mu_{D-2}}{}^a \partial_\nu h_{\rho a},$$

where we added in complete analogy to (3.1) a topological term containing the ordinary graviton $h_\mu{}^a$. Let us stress that here also $h_{\mu\nu}$ is not in an irreducible Young tableau, but carries an antisymmetric part.

The physical content of (3.16) can be analyzed as follows. Varying with respect to $h_\mu{}^a$ yields $\partial_{[\mu_1} Y_{\mu_2 \ldots \mu_{D-1}]}{}^a = 0$, i.e. the shift-gauge field is pure gauge and can therefore be gauged to zero by virtue of (3.15). The action for the remaining field $C_{\mu_1 \ldots \mu_{D-3}}{}^a$ is then precisely the Curtright action for the dual graviton. On the other hand, varying with respect to Y one obtains a "duality relation" between h and $\hat{F}$. Integrating out Y yields the linearized action for gravity, where the antisymmetric part of $h_{\mu\nu}$ appears in the corresponding Lagrangian only through total derivatives. This is essentially the same calculation as the one which led from the first-order, quadratic action (2.7) back to the quadratic part of the Einstein-Hilbert action (2.1), the only difference being the presence of C in the field strength (3.14). However, the latter cancels out, as it should be due to the shift invariance (3.15). To summarize, the parent action based on (3.16) contains both the graviton and its dual and consistently describes the free dynamics of either of them.

Let us briefly analyze the symmetries of the free theory (3.16), apart from the manifest shift symmetry (3.15). The diffeomorphisms and local Lorentz transformations on $h_\mu{}^a$ read

$$\delta h_\mu{}^a = \partial_\mu \xi^a - \Lambda^a{}_\mu, \tag{3.17}$$

while all other fields are invariant under ξ^a. The dual diffeomorphisms and α-transformations "unify" to one symmetry, given by

$$\delta_\gamma C_{\mu_1 \ldots \mu_{D-3}}{}^a = \partial_{[\mu_1} \gamma_{\mu_2 \ldots \mu_{D-3}]}{}^a. \tag{3.18}$$

More precisely, γ carries the Young-diagram symmetries

$$(D-4) \otimes (1) = (D-4, 1) \oplus (D-3), \tag{3.19}$$

whose irreducible parts are identified with α and β, respectively. That both symmetries are manifest is due to the framelike formulation. In fact, instead of the dual diffeomorphisms it is now the local Lorentz symmetry which acts nontrivially and fixes the relative coefficients in $\mathcal{L}(h, C, Y)$. It reads

$$\delta_\Lambda^{(0)} Y_{\mu_1 \cdots \mu_{D-2}|a} = \partial_{[\mu_1} (\bar{e}_{\mu_2}{}^{b_2} \ldots \bar{e}_{\mu_{D-2}]}{}^{b_{D-2}} \tilde{\Lambda}_{b_2 \cdots b_{D-2} a}), \tag{3.20}$$

$$\delta_\Lambda^{(0)} C_{\mu_1 \cdots \mu_{D-3}|a} = \bar{e}_{\mu_1}{}^{b_1} \ldots \bar{e}_{\mu_{D-3}}{}^{b_{D-3}} \tilde{\Lambda}_{b_1 \cdots b_{D-3} a}, \tag{3.21}$$

where $\tilde{\Lambda}$ is proportional to the Hodge dual of Λ

$$\tilde{\Lambda}_{a_1 \ldots a_{D-2}} = \tfrac{1}{2}(-1)^{D-3}(D-2)\epsilon_{a_1 \ldots a_{D-2} bc} \Lambda^{bc}. \tag{3.22}$$

Thus, the Lorentz parameter can be used to gauge away either the antisymmetric part of the metric or of its dual (but not simultaneously). Such gauge-fixing requires compensating gauge transformations for the symmetries (3.17) and (3.18), which in turn reintroduces the nonmanifest invariance of the action either under the diffeomorphisms $\delta_\xi h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ or under their dual (2.11).

C. Nonlinear dual gravity

We turn now to the nonlinear theory. We proceed again in analogy to the vector-scalar example (3.1), where the step from the linear to the nonlinear theory was simply given by covariantizing the field strengths and derivatives with respect to the Yang-Mills gauge group. Thus, here we are going to make the action invariant under the full diffeomorphism group by introducing the dynamical metric in the kinetic terms for the dual graviton.

The action reads

$$S[e, C, Y] = \int d^D x [\mathcal{L}_C(e, \hat{F}) + 2\kappa^{-1} \varepsilon^{\mu_1 \ldots \mu_{D-2}\nu\rho} Y_{\mu_1 \ldots \mu_{D-2}|a} \partial_\nu e_\rho{}^a], \tag{3.23}$$

where we introduced the "covariantized" Curtright Lagrangian

$$\mathcal{L}_C(e, \hat{F}) = \frac{D-3}{2(D-2)} e \hat{F}^{\mu_1 \ldots \mu_{D-2}|a} \hat{F}_{\mu_1 \ldots \mu_{D-2}|a} - \frac{D-2}{2} e e_\nu{}^a e_b{}^\rho \hat{F}^{\mu_1 \ldots \mu_{D-3}\nu|}{}_a \hat{F}_{\mu_1 \ldots \mu_{D-3}\rho|}{}^b + \frac{1}{2} e e_\nu{}^b e_a{}^\rho \hat{F}^{\mu_1 \ldots \mu_{D-3}\nu|a} \hat{F}_{\mu_1 \ldots \mu_{D-3}\rho|b}.$$

Here, all curved indices are raised and lowered with the metric $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ derived from $e_\mu{}^a = \bar{e}_\mu{}^a + \kappa h_\mu{}^a$, and we introduced κ in the topological term, such that we recover the free Lagrangian (3.16) in the limit $\kappa \to 0$. The shift-invariant field strength is not modified and still given by (3.14). Because of the appearance of inverse vielbeins and the determinant e this action is indeed a nonlinear deformation (in κ) of (3.16). The action (3.23) is equivalent to the nonlinear Einstein-Hilbert action, which can be reobtained by integrating out Y. In fact, this can be made completely manifest by gauge-fixing the shift symmetry such that $C = 0$ and then converting all indices into flat ones. The resulting action then coincides with the first-order form (2.7).

Let us now turn to the nonlinear symmetries of (3.23). First, it is manifestly diffeomorphism invariant due the presence of a dynamical metric (and for the topological term anyway). In particular, due to the framelike formulation, we do not need to introduce Christoffel connections, since the (curved) space-time indices are totally antisymmetric. The dual diffeomorphisms together with the α-transformations (both parametrized by γ^a) act as in the linearized theory according to (3.18), leaving the field strength (3.14) manifestly invariant. The shift symmetries are still given by (3.15).

The only nontrivial symmetry is the Lorentz symmetry, which we assume to act in the standard way on the vielbein,

$$\delta_\Lambda e_\mu{}^a = -\Lambda^a{}_b e_\mu{}^b. \tag{3.24}$$

This is only a symmetry if suitable transformations are assigned to C and Y. On the C field we take the direct nonlinear covariantization of (3.21):

$$\delta_\Lambda C_{\mu_1\cdots\mu_{D-3}|a} = \tilde{\Lambda}_{\mu_1\cdots\mu_{D-3}a} - \Lambda_{ab} C_{\mu_1\ldots\mu_{D-3}}{}^b, \tag{3.25}$$

with the dual Lorentz parameter

$$\tilde{\Lambda}_{\mu_1\cdots\mu_{D-3}a} = e_{\mu_1}{}^{b_1}\ldots e_{\mu_{D-3}}{}^{b_{D-3}}\tilde{\Lambda}_{b_1\cdots b_{D-3}a} \tag{3.26}$$

introduced in (3.22). In the gauge-fixed formulation where $C=0$, the corresponding variation for Y can simply be determined by applying (2.5) to (2.6). Then, in the full theory, a correction term containing C has to be added. In total, we find the nonlinear transformations

$$\begin{aligned}\delta_\Lambda Y_{\mu_1\ldots\mu_{D-2}}{}^a = {}& \partial_{[\mu_1}\tilde{\Lambda}_{\mu_2\ldots\mu_{D-2}]}{}^a \\ & - (D-3)\Omega_{[\mu_1\mu_2}{}^\rho\tilde{\Lambda}_{|\rho|\mu_3\ldots\mu_{D-2}]}{}^a \\ & - \Lambda^a{}_b Y_{\mu_1\ldots\mu_{D-2}}{}^b \\ & + (-1)^{D-3} C_{[\mu_1\ldots\mu_{D-3}}{}^b\partial_{\mu_{D-2}]}\Lambda^a{}_b.\end{aligned} \tag{3.27}$$

Let us note that invariance of the action (3.23) under these Lorentz transformations can be most easily checked in flat indices, for which the correction term in (3.27) proportional to $\Omega_{\mu\nu}{}^\rho$ is not required. Actually, the role of the second and fourth terms in (3.27) is to make the total gauge transformation of the shift-invariant field strength $\hat{F}$ simple:

$$\begin{aligned}(\delta_\Lambda + \delta_\gamma)\hat{F}_{\mu_1\ldots\mu_{D-2}}{}^a = {}& -\Lambda^a{}_b\hat{F}_{\mu_1\ldots\mu_{D-2}}{}^b + 2(-1)^{D-3} \\ & \times e_{[\mu_1}{}^{b_1}\ldots e_{\mu_{D-3}}{}^{b_{D-3}}\partial_{\mu_{D-2}]} \\ & \times \tilde{\Lambda}_{b_1\cdots b_{D-3}}{}^a.\end{aligned} \tag{3.28}$$

The gauge transformations take a somewhat unconventional form, as for instance the presence of the dual Lorentz parameter (3.22). Moreover, the partial derivative on γ^a in (3.18) is not Lorentz covariant, and so at first sight the dual diffeomorphisms will not close with the local Lorentz group. However, it turns out that closure is ensured by virtue of the additional local shift symmetry in that

$$\begin{aligned}[\delta_\gamma, \delta_\Lambda]C_{\mu_1\ldots\mu_{D-3}}{}^a &= \delta_\Sigma C_{\mu_1\ldots\mu_{D-3}}{}^a, \\ \Sigma_{\mu_1\ldots\mu_{D-3}}{}^a &= \Lambda^a{}_b\partial_{[\mu_1}\gamma_{\mu_2\ldots\mu_{D-3}]}{}^b,\end{aligned} \tag{3.29}$$

and similarly on Y. Moreover, one finds off-shell closure for the local Lorentz group itself,

$$\begin{aligned}[\delta_{\Lambda_1}, \delta_{\Lambda_2}] &= \delta_{[\Lambda_1,\Lambda_2]}, \\ [\Lambda_1, \Lambda_2]^{ab} &= \Lambda_1{}^a{}_c\Lambda_2{}^{cb} - \Lambda_2{}^a{}_c\Lambda_1{}^{cb}.\end{aligned} \tag{3.30}$$

In order to verify this, it is again more convenient to work in flat indices or, otherwise, to keep in mind that the definition of the parameter $\tilde{\Lambda}$ in (3.27) involves the vielbein.

Let us now turn to the equations of motion, specifically to the duality relation between the metric and its dual. As in the toy model discussed in Sec. III A, by virtue of the topological term in (3.23), the duality relation follows from the action by varying with respect to the gauge field Y. One finds

$$\begin{aligned}e^{-1}\varepsilon^{\mu_1\ldots\mu_{D-2}\nu\rho}\Omega_{\nu\rho}{}^a = {}& -\frac{D-3}{D-2}\hat{F}^{\mu_1\ldots\mu_{D-2}|a} \\ & + (-1)^{D-3}(D-2)e_{\rho b} \\ & \times e^{a[\mu_1}\hat{F}^{\mu_2\ldots\mu_{D-2}]\rho|b} \\ & - (-1)^{D-3}e_\rho{}^a e_b^{[\mu_1}\hat{F}^{\mu_2\ldots\mu_{D-3}]\rho|b}.\end{aligned} \tag{3.31}$$

As a consistency check one may now verify that this nonlinear duality relation is completely gauge covariant. In particular, due to the presence of Y, it transforms covariantly under the local Lorentz group. The field equations for C can be obtained from (3.31) by acting with a derivative. In order to obtain the Einstein equation, we have to use the field equation for $e_\mu{}^a$, which also takes a first-order form,

$$e^{-1}\varepsilon^{\mu\mu_1\ldots\mu_{D-1}}\partial_{\mu_1}Y_{\mu_2\ldots\mu_{D-1}|a} = \frac{1}{2}e^{-1}\frac{\delta\mathcal{L}_C(e,\hat{F})}{\delta e_\mu{}^a}. \tag{3.32}$$

In this sense, the full set of field equations—and so the nonlinear Einstein equations—can be written as first-order duality relations. Moreover, it follows that even in the presence of the dual graviton arbitrary matter couplings can be introduced, simply by adding to (3.23) the matter action. This, in fact, leaves the first duality relation unchanged, but adds to the second duality relation (3.32) the standard energy-momentum tensor $T^\mu{}_a \sim \delta\mathcal{L}_M/\delta e_\mu{}^a$, which in turn appears in the Einstein equation in the usual way. Equivalently, since the shift-gauge field Y will not contribute to possible matter terms added to (3.23), it can be integrated out as before, leading to the Einstein-Hilbert action augmented by these matter couplings. This circumvents the negative findings of [32], where it has been shown that in presence of matter the elimination of the graviton in favor of its dual is problematic even if gravity is treated linearly.

D. Symmetries and their deformation

In this section we would like to discuss to what extent the gauge symmetries of the nonlinear theory (3.23) represent deformations of the symmetries of the free theory

(3.16). Analogously to what happens in the vector-scalar example presented before, we expect that the nonlinear action can be obtained from the free one by a deformation that does not affect the gauge algebra, apart from diffeomorphisms and Lorentz transformations.

To see this, let us recover (3.27) from a different perspective. First we deform the free Lagrangian and the corresponding Abelian gauge transformations in such a way that

$$\delta_\Lambda^{(0)} Y_{\mu_1\ldots\mu_{D-2}|a} \longrightarrow \delta_\Lambda^{(0)} Y_{\mu_1\ldots\mu_{D-2}|a} + \hat{F}_{\mu_1\ldots\mu_{D-2}|b}\Lambda^b{}_a. \tag{3.33}$$

This deformation does not change the gauge algebra involving Σ and γ due to the shift invariance of $\hat{F}$. Then, we redefine the gauge parameter $\Sigma_{\mu_1\ldots\mu_{D-3}|a}$ by

$$\Sigma_{\mu_1\ldots\mu_{D-3}|a} \longrightarrow \Sigma_{\mu_1\ldots\mu_{D-3}|a} - C_{\mu_1\ldots\mu_{D-3}|b}\Lambda^b{}_a. \tag{3.34}$$

This procedure generates the algebra (3.29) and gives the gauge transformation (3.27), apart from the second term therein, that reflects the Lorentz and diffeomorphism deformations.

Finally, let us briefly comment on the connection between the discussed symmetries of the dual graviton theory and the hidden symmetries found in dimensional reductions. Since the appearance of the latter symmetries relies on the dualization of certain fields, one might expect that, after introducing the dual graviton, they are at least partially present already in the higher-dimensional theory. For instance, in the reduction of pure gravity from $D = 4$ to $D = 3$ a hidden $SL(2, \mathbb{R})$ appears, which acts nonlinearly on scalars ϕ and φ, which are the dilaton arising from the metric and the dual of the Kaluza-Klein vector. (For a review see, e.g., [33].) Specifically, among the $SL(2, \mathbb{R})$ generators h, e, and f in the standard Chevalley basis, h originates from the higher-dimensional diffeomorphism invariance and acts linearly, while e and f correspond to nonlinear symmetries [33],

$$\begin{gathered}\delta_\lambda(e)\varphi = \lambda, \qquad \delta_\alpha(f)\phi = 2\alpha\phi\varphi,\\ \delta_\alpha(f)\varphi = \alpha(\varphi^2 - \phi^2).\end{gathered} \tag{3.35}$$

In the reformulation given in Sec. III C, there are additional Kaluza-Klein components originating from the dual graviton $C_\mu{}^a$, whose "dilaton" component $C_3{}^3$ one might identify with φ.[4] Therefore, the dual diffeomorphisms (3.17) give rise to an additional symmetry, $\delta_\gamma C_3{}^3 = \partial_3\gamma^3$, which for $\gamma^3 = x^3\lambda$ implies the global shift symmetry $\delta_\lambda\varphi = \lambda$ in the dimensionally reduced theory. Thus, the e transformations have been uplifted to $D = 4$. Unfortunately, the more interesting symmetries given by f still seem not to correspond to any invariance of the action (3.23), in agreement with the essentially trivial deformation of the gauge algebra analyzed above.

[4]Besides, the theory contains separately the Kaluza-Klein vector, but in the reformulation (3.23) it appears only topologically.

IV. COMMENTS AND OUTLOOK

In this paper we have constructed a nonlinear theory involving the dual graviton. Instead of aiming at a non-Abelian theory for the dual graviton only—which cannot exist in a local and covariant fashion [12],—we derived a parent action, which still contains the original metric. The latter guarantees invariance under the full diffeomorphism group. However, this does not lead to a doubling of degrees of freedom since there is no "kinetic" Einstein-Hilbert term, while the metric enters through a topological Chern-Simons-like term. Moreover, due to this topological term, the theory can be shown to be classically equivalent to nonlinear Einstein-Hilbert gravity. It exhibits an enhanced gauge symmetry, which contains not only the usual space-time symmetries, but also "dual" diffeomorphisms and a local shift invariance. By virtue of the shift-gauge field, the nonlinear duality relations between the metric and its dual are fully gauge covariant.

Thus, in total, we established the existence of a nontrivial theory for the dual graviton, satisfying the requirements (i)–(iii) raised in the introduction. One might wonder whether the necessity of introducing a gauge field, which is a $(D-2)$-form with a Lorentz index, has a natural interpretation within E_{11}. An inspection of the relevant tables reveals that E_{11} in the $SL(11)$ decomposition indeed has a $(D-2, 1)$ Young tableau at level 7 [34], but that, at least at low levels, similar objects seem not to appear for other decompositions or different Kac-Moody algebras (as A_{D-3}^{+++} in case of pure gravity) [35]. Thus, it is most likely that the shift-gauge fields have to be viewed as external quantities. This is not an entirely unsuspected feature in that something similar happens for the correspondence between gauged supergravity and E_{11}. In fact, gauged supergravity requires the so-called embedding tensor, which in turn is not predicted by E_{11}, but appears only through its dual $(D-1)$-forms [31,36,37]. While the latter, together with the D- or top-form potentials, encode all constraints imposed by gauged supergravity, the embedding tensor is nevertheless indispensable in order to construct an action [31,38].

Unfortunately, the presented theory does not seem to fully uplift the "hidden symmetries" of Kaluza-Klein reductions to the original theory. This can be traced back to the fact that only the usual diffeomorphisms are truly nonlinear—giving rise to the $SL(d)$ symmetry for reductions on d-tori,—while the dual diffeomorphisms are still Abelian. Therefore, the symmetry enhancement $SL(d) \longrightarrow SL(d+1)(\longrightarrow E_{d(d)})$ taking place for reductions of (maximal super)gravity can be elevated to the higher-dimensional theory only for the positive-level "shift" transformations. However, this is not different from the correspondence between ordinary p-forms and Kac-

Moody algebras. (See the discussion in Sec. III A) The results of this paper therefore show that in this respect gravity is not special.

It would be interesting to extend this research into the following directions. First of all, one might speculate that a true uplifting of all hidden symmetries requires abandoning space-time covariance as in [39]. Moreover, even though we have seen that generic matter couplings are compatible with the presented parent action for dual gravity, it would be interesting to see whether for special cases, like 3- and 6-form in $D = 11$, an enhancement of symmetries is possible such that the dual graviton starts transforming under lower-level gauge transformations.

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