

Relativistic stars in $f(R)$ gravity, and absence thereof

Tsutomu Kobayashi^{1,*} and Kei-ichi Maeda^{1,2,+}

¹*Department of Physics, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan*

²*Advanced Research Institute for Science and Engineering, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan*

(Received 22 July 2008; published 8 September 2008)

Several $f(R)$ modified gravity models have been proposed which realize the correct cosmological evolution and satisfy solar system and laboratory tests. Although nonrelativistic stellar configurations can be constructed, we argue that relativistic stars cannot be present in such $f(R)$ theories. This problem appears due to the dynamics of the effective scalar degree of freedom in the strong gravity regime. Our claim thus raises doubts on the viability of $f(R)$ models.

DOI: [10.1103/PhysRevD.78.064019](https://doi.org/10.1103/PhysRevD.78.064019)

PACS numbers: 04.50.Kd, 04.40.Dg, 95.36.+x

I. INTRODUCTION

The current accelerated expansion of the Universe is one of the deepest mysteries in cosmology [1]. This acceleration may be due to some unknown energy-momentum component having the equation of state $p/\rho \approx -1$, or may be due to a modification of general relativity. In this paper, we are interested in the latter possibility. The simplest phenomenological way of modifying gravity is to consider a gravitational action described by a function of the Ricci scalar, $f(R)$, instead of the Einstein-Hilbert action. An early attempt is found, e.g., in [2], where a modification like $f(R) = R + R^2/\mu^2$ was used to explain the accelerated expansion in the *early* Universe. More recently, $f(R)$ modified gravity theories are often considered as a possible origin of the *current* acceleration of the Universe [3].

Any modified theories of gravity must account for the late-time cosmology which is well established by observations, and at the same time must be consistent with solar system and laboratory tests of gravity. However, since $f(R)$ gravity has the equivalent description in terms of the Brans-Dicke theory with the Brans-Dicke parameter $\omega = 0$ [4], naively constructed models would result in violation of the above requirements [5–7]. For example, the original proposal of [8] employs $f(R) = R - \mu^4/R$, which admits an acceleratedly expanding solution even in the absence of a dark energy component. In order to accommodate a late-time acceleration, however, one must introduce a very low energy scale, $\mu \sim H_0$ (the present Hubble scale), leading to a very light scalar field, which predicts the parametrized post-Newtonian (PPN) parameter $\gamma = (1 + \omega)/(2 + \omega) = 1/2$. Obviously, this result contradicts the observational constraint $|\gamma - 1| \lesssim 10^{-4}$ [9]

To circumvent this difficulty, it is important to notice that the presence of matter may affect the dynamics of the extra scalar degree of freedom. The key idea is essentially

the same as that of the “chameleon” model [10–12], in which the effective mass of the scalar field depends on the local matter density. In particular, the scalar field is very light for the cosmological density and is heavy for the solar system density, though the actual mechanism is slightly more complicated. The most successful class of $f(R)$ models [13–21] incorporates this chameleon mechanism to evade local gravity tests. The experimental and observational consequences of this kind of $f(R)$ models are found in Refs. [22–24]. (See also [25].)

In this paper, we consider the strong gravity regime of the carefully constructed models of [19–21]. The strong gravity aspects of $f(R)$ theories have not been explored so much before. Recently, Frolov suggested that such $f(R)$ models generically suffer from the problem of curvature singularities which can be easily accessed by the field dynamics *in the presence of matter* [26]. In other words, a curvature singularity may be caused not by diverging gravitational potential depth, $|\Phi| = \infty$, but rather by a slightly large gravitational field, $|\Phi| \lesssim 1/2$. This motivates us to study relativistic stars in the context of $f(R)$ gravity. Spherically symmetric stars in $f(R)$ gravity have been investigated so far in [27–30]. (We confine ourselves to a *metric* theory of $f(R)$ gravity. Using the *Palatini* formalism, polytropic stars have been studied in [31].) We shall show, both analytically and numerically, that stellar solutions with relatively strong gravitational fields cannot be constructed. Using the specific example of relativistic stars, we clarify how the singularity problem arises in the strong gravity regime of $f(R)$ theories. The singularity problem was also identified in [32] in a cosmological setting.

This paper is organized as follows. In the next section we derive equations of motion for $f(R)$ modified gravity, and define the specific model we study. In Sec. III, we reinterpret the problem of finding the desired stellar configuration as the problem of the particle motion in classical mechanics. We give some analytic arguments in Sec. IV, and then we present our numerical results in Sec. V. Finally, we draw our conclusions in Sec. VI.

*tsutomu@gravity.phys.waseda.ac.jp

+maeda@waseda.jp

II. PRELIMINARIES

A. $f(R)$ gravity

The action we consider is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{f(R)}{16\pi G} + \mathcal{L}_m \right], \quad (1)$$

where $f(R)$ is a function of the Ricci scalar R , and \mathcal{L}_m is the Lagrangian of matter fields. Variation with respect to metric leads to the field equations

$$f_R R_{\mu\nu} - \nabla_\mu \nabla_\nu f_R + (\square f_R - \frac{1}{2}f)g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (2)$$

where $f_R := df/dR$ and $T_{\mu\nu} := -2\delta \mathcal{L}_m / \delta g^{\mu\nu} + g_{\mu\nu} \mathcal{L}_m$. The trace of Eq. (2) reduces to

$$\square f_R = \frac{8\pi G}{3} T + \frac{1}{3}(2f - f_R R). \quad (3)$$

We now introduce an effective scalar degree of freedom by defining $\chi := f_R$. Inverting this relation, the Ricci scalar can be expressed in terms of χ : $R = Q(\chi)$. Thus, Eqs. (2) and (3) are equivalently rewritten as

$$\chi G_\mu^\nu = 8\pi G T_\mu^\nu + (\nabla_\mu \nabla^\nu - \delta_\mu^\nu \square)\chi - \chi^2 V(\chi) \delta_\mu^\nu, \quad (4)$$

$$\square \chi = \frac{8\pi G}{3} T + \frac{2\chi^3}{3} \frac{dV}{d\chi}, \quad (5)$$

where the effective potential V is given by

$$V(\chi) := \frac{1}{2\chi^2} [\chi Q(\chi) - f(Q(\chi))], \quad (6)$$

and

$$\frac{dV}{d\chi} = \frac{1}{2\chi^3} [2f(Q(\chi)) - \chi Q(\chi)]. \quad (7)$$

Equations (4) and (5) are equivalent to the Jordan frame equations of motion in the Brans-Dicke theory with $\omega = 0$, if we ignore the potential term $V(\chi)$. One can move to the Einstein frame by performing the conformal transformation $\tilde{g}_{\mu\nu} = \chi g_{\mu\nu}$ with $\chi = \exp(\sqrt{16\pi G/3}\phi)$, where ϕ is the canonical scalar field. The potential for ϕ is then given by $V(\chi(\phi))$. Although the Einstein frame equations of motion are sometimes convenient, we shall not use this and work in the Jordan frame directly throughout the paper, so as to avoid confusion concerning the coupling between matter fields and the effective scalar degree of freedom.

B. The model

In order to be explicit, we take $f(R)$ in the following form [19]:

$$f(R) = R + \lambda R_0 \left[\left(1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right], \quad (8)$$

where $n, \lambda(>0)$, and $R_0(>0)$ are parameters. This model is carefully constructed so that it gives viable cosmology and

satisfies solar system and laboratory tests. Suppose that the de Sitter solution in this theory is expressed as $R = R_1 = \text{constant} = x_1 R_0$. We may define the ‘‘cosmological constant’’ as $\Lambda_{\text{eff}} := R_1/4$. Since the de Sitter solution follows from $dV/d\chi = 0$, one has

$$\lambda = \frac{1}{2} \frac{x_1(1+x_1^2)^{n+1}}{(1+x_1^2)^{n+1} - 1 - (n+1)x_1^2}. \quad (9)$$

Thus, we may use x_1 as a model parameter instead of λ . We will take as x_1 the maximal root of Eq. (9) for a given λ , because it corresponds to the de Sitter minimum of the potential.

The scalar field χ is written in terms of R as

$$\chi = 1 - 2n\lambda \frac{R}{R_0} \left(1 + \frac{R^2}{R_0^2} \right)^{-n-1}. \quad (10)$$

One sees that $\chi \rightarrow 1$ as $R \rightarrow \pm\infty$ and $R \rightarrow 0$. Note that curvature singularities, $R = \pm\infty$, is mapped to the finite value of $\chi = 1$. The value of χ at the de Sitter minimum is given by

$$\chi_1 = 1 - \frac{nx_1^2}{(1+x_1^2)^{n+1} - 1 - (n+1)x_1^2}. \quad (11)$$

A typical form of the potential $V(\chi)$ around the de Sitter minimum is shown in Fig. 1. In fact, the potential is a multivalued function of χ , and its shape is complicated, probably even pathological, away from the plotted region. However, since our discussion here focuses on the behavior of χ around the de Sitter minimum, there is no difficulty with such a complicated potential.

Though we shall confine ourselves to the specific model defined by Eq. (8), the result will apply to other $f(R)$ models as well. In particular, the models proposed by Hu and Sawicki [20] and by Appleby and Battye [21] fall into

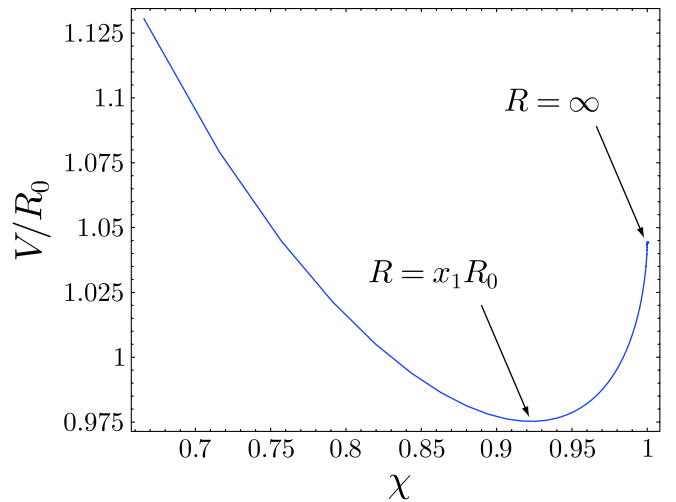


FIG. 1 (color online). The effective potential $V(\chi)$ for Starobinsky’s $f(R)$ model with $n = 1$ and $x_1 = 3.6$ ($\lambda \simeq 2$). χ is the effective scalar degree of freedom defined by $\chi := df/dR$.

the same class as [19] in the sense that the potential around the de Sitter minimum has the same structure.

III. SPHERICALLY SYMMETRIC STARS IN $f(R)$ GRAVITY

A. Basic equations

To study stellar configurations in $f(R)$ gravity, we take the ansatz of a spherically symmetric and static metric:

$$ds^2 = -N(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (12)$$

The energy-momentum tensor of matter fields is given by

$$T^\nu_\mu = \text{diag}(-\rho, p, p, p). \quad (13)$$

From the energy-momentum conservation, $\nabla_\nu T^\nu_\mu = 0$, we obtain

$$p' + \frac{N'}{2N}(\rho + p) = 0. \quad (14)$$

Here and hereafter a prime denotes differentiation with respect to r . The (tt) and (rr) components of the field equations (4) yield, respectively,

$$\begin{aligned} \frac{\chi}{r^2}(-1 + B + rB') &= -8\pi G\rho - \chi^2 V \\ &\quad - B\left[\chi'' + \left(\frac{2}{r} + \frac{B'}{2B}\right)\chi'\right], \end{aligned} \quad (15)$$

$$\frac{\chi}{r^2}\left(-1 + B + rB\frac{N'}{N}\right) = 8\pi Gp - \chi^2 V - B\left(\frac{2}{r} + \frac{N'}{2N}\right)\chi'. \quad (16)$$

The equation of motion for χ [Eq. (5)] gives

$$\begin{aligned} B\left[\chi'' + \left(\frac{2}{r} + \frac{N'}{2N} + \frac{B'}{2B}\right)\chi'\right] &= \frac{8\pi G}{3}(-\rho + 3p) \\ &\quad + \frac{2\chi^3}{3} \frac{dV}{d\chi}. \end{aligned} \quad (17)$$

We will not integrate the angular components of the field equations. Instead, we will use them to check the accuracy of our numerical results, because those are derived from other equations via the Bianchi identity.

To specify the boundary conditions at the center of a star, assuming the regularity, we expand the variables in the power series of r as

$$\begin{aligned} N(r) &= 1 + N_2 r^2 + \dots, & B(r) &= 1 + B_2 r^2 + \dots, \\ \chi(r) &= \chi_c \left(1 + \frac{C_2}{2} r^2 + \dots\right), \\ \rho(r) &= \rho_c + \frac{\rho_2}{2} r^2 + \dots, & p(r) &= p_c + \frac{p_2}{2} r^2 + \dots, \end{aligned} \quad (18)$$

where χ_c , ρ_c , and p_c are the central values of the scalar field, the energy density, and the pressure, respectively. Note that using the scaling freedom of the t coordinate, we set $N(0) = 1$. From Eqs. (15)–(17), we obtain

$$3B_2 = -8\pi\hat{G}\rho_c - \chi_c V_c - 3C_2, \quad (19)$$

$$B_2 + 2N_2 = 8\pi\hat{G}p_c - \chi_c V_c - 2C_2, \quad (20)$$

$$3C_2 = \frac{8\pi\hat{G}}{3}(-\rho_c + 3p_c) + \frac{2\chi_c^2}{3}V_{\chi_c}, \quad (21)$$

where $\hat{G} := G/\chi_c$ is the effective gravitational constant, $V_c := V(\chi_c)$, and $V_{\chi_c} = dV/d\chi|_{\chi=\chi_c}$. These three equations are rearranged to give

$$B_2 = -\frac{8\pi\hat{G}}{9}(2\rho_c + 3p_c) - \frac{\chi_c}{3}V_c - \frac{2\chi_c^2}{9}V_{\chi_c}, \quad (22)$$

$$N_2 = \frac{8\pi\hat{G}}{9}(2\rho_c + 3p_c) - \frac{\chi_c}{3}V_c - \frac{\chi_c^2}{9}V_{\chi_c}, \quad (23)$$

$$C_2 = \frac{8\pi\hat{G}}{9}(-\rho_c + 3p_c) + \frac{2\chi_c^2}{9}V_{\chi_c}. \quad (24)$$

Then, p_2 is derived from the conservation equation:

$$p_2 + N_2(\rho_c + p_c) = 0. \quad (25)$$

The Ricci scalar is given by $R = R_c + \mathcal{O}(r^2)$ with $R_c = -6(B_2 + N_2)$ near $r = 0$.

If the energy density is constant inside the star, $\rho = \rho_0$, Eq. (14) immediately gives

$$N(r) = \left[\frac{\rho_0 + p_c}{\rho_0 + p(r)}\right]^2. \quad (26)$$

In the rest of the paper, we focus on constant density stars for simplicity.

B. A classical mechanics picture

Given ρ_0 , p_c , and χ_c (or, equivalently, R_c), Eqs. (14)–(17) can be integrated outwards from the center to the surface of the star, $r = \mathcal{R}$, which is defined by $p(\mathcal{R}) = 0$. Then, one integrates the vacuum field equations from the star surface to sufficiently large r , finding the exterior profile of the metric and the scalar field. Unlike in general relativity, we have the extra scalar degree of freedom corresponding to the choice of χ_c . However, not all values of χ_c can lead to a physically reasonable solution inside and outside a star. The desired solution is such that $\chi(r) \rightarrow \chi_1$ as $r \rightarrow \infty$, i.e., the asymptotically de Sitter solution with the cosmological constant Λ_{eff} .

We can formulate the problem of finding the physical configuration of stars as follows. Equation (17) can be written as

$$\frac{d^2}{dr^2}\chi + \frac{2}{r}\frac{d}{dr}\chi = -\frac{dU}{d\chi} + \mathcal{F}, \quad (27)$$

where

$$\frac{dU}{d\chi} := \frac{1}{3}[f_R Q - 2f] = -\frac{2\chi^3}{3}\frac{dV}{d\chi} \quad (28)$$

and

$$\mathcal{F} := -\frac{8\pi G}{3}(\rho - 3p). \quad (29)$$

Here we neglect the effect of the metric for the moment to comprehend the essential point. [Later we will solve the full set of Eqs. (14)–(17) numerically.]

Regarding r as a time coordinate, we find that Eq. (27) is a “dynamical” equation describing the motion of a particle in the potential U under the time-dependent force \mathcal{F} . The second term in the left-hand side of Eq. (27) represents frictional force, which may also affect the dynamics in some cases. The potential for the particle $U(\chi)$, defined by Eq. (28), is different from the inverted potential $-V(\chi)$. However, the structure of U around $\chi = \chi_1$ is quite similar to $-V$, as is shown in Figs. 2 and 3: the point A ($\chi = 1$) corresponds to a curvature singularity, $R \rightarrow \infty$, and the point B ($\chi = \chi_1$) is the de Sitter extremum.

In Ref. [26], Frolov similarly introduced the potential $-U(\chi)$ and the force term which is essentially given by the trace of the energy-momentum tensor. For the purpose of solving for the radial profile, it is more convenient to consider the inverted potential $-(-U) = U$, as in the cases of bubble nucleation [33] and of the chameleon model [10]

Suppose that the initial position of the particle, χ_c , lies between points A and B. The particle starts at rest since $\chi'|_{r=0} = 0$. The force term \mathcal{F} depends on the matter

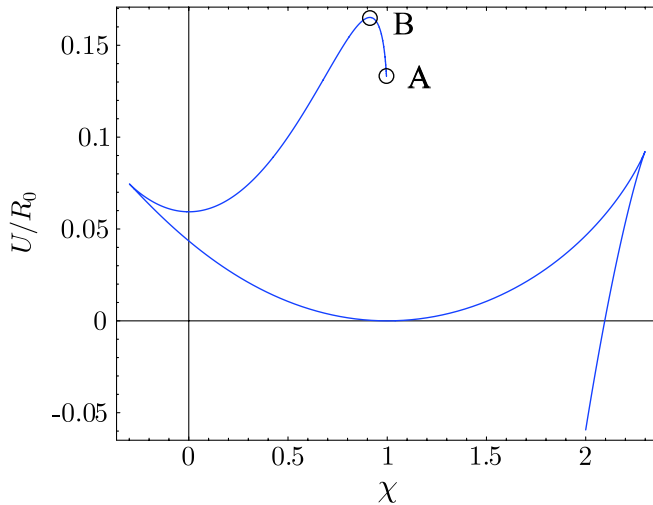


FIG. 2 (color online). The (inverted) potential $U(\chi)$ for Starobinsky’s $f(R)$ model with $\lambda = 2$ and $n = 1$. The point A corresponds to a curvature singularity ($R = +\infty$), and the point B is the de Sitter extremum. (See also Fig. 1 of Ref. [26].)

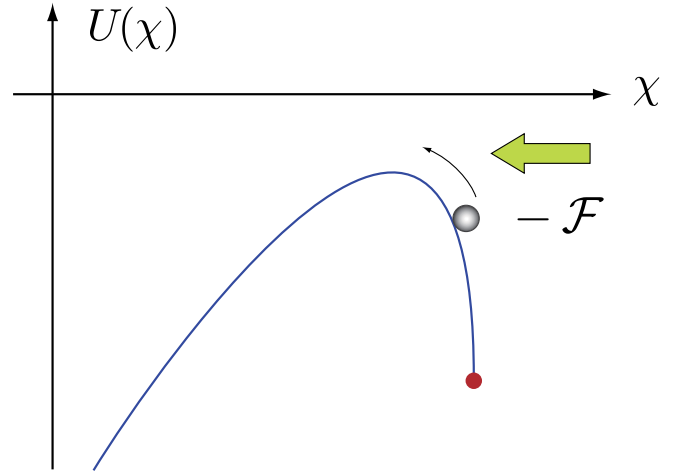


FIG. 3 (color online). Motion of a particle near the de Sitter extremum of $U(\chi)$. The particle feels the force $\mathcal{F} (< 0)$ which arises from the trace of the energy-momentum tensor of the matter, $\mathcal{F} \propto T$.

configuration inside the star and plays a crucial role in this problem. When $\rho > 3p$, we have $\mathcal{F} < 0$. Since the pressure becomes smaller for larger r , the force $|\mathcal{F}|$ is stronger near the surface than in the central region of the star. The force vanishes for $r > \mathcal{R}$ (if one assumes the vacuum exterior).

For fixed ρ_0 and p_c , the behavior of the particle depends on its initial position. If χ_c is sufficiently close to 1, the slope of the potential is bigger than the force term \mathcal{F} initially, so that χ will rapidly roll down to the curvature singularity, $\chi = 1$. Let χ_s be the minimum value of χ_c for which this occurs. Specifically, χ_s is found by solving the equation $[\mathcal{F} - dU/d\chi]_{\chi_s} = 0$. If $\chi_c < \chi_s$, the particle initially climbs up the potential hill under the force $\mathcal{F} < 0$. The force vanishes at $r = \mathcal{R}$, but the particle keeps climbing up the potential for the moment. For $r > \mathcal{R}$, the scenario one can easily deduce is as follows: The particle cannot reach the top of the potential hill and turns around at some r , ending up with the curvature singularity $\chi = 1$, or goes through the top and rolls down to the left, depending on the initial position χ_c [34]. In this case, one finds a critical value χ_{crit} between the “turn-around” and “-rolling-down” solutions. By fine-tuning the initial position so that $\chi_c = \chi_{\text{crit}}$, we can realize the asymptotically de Sitter solution for which $\chi \rightarrow \chi_1$ as $r \rightarrow \infty$. Thus, the problem reduces to a boundary value problem.

However, there is another possibility that the particle inevitably overshoots the potential even for the possible maximum value of χ_c (i.e., χ_s). In this case, one cannot obtain the desired solution: the particle rolls down into the curvature singularity right after it starts to move, or overshoots the potential. Since the gravitational potential produced by a star is proportional to $(\rho_0 \mathcal{R}^3)/\mathcal{R} = \rho_0 \mathcal{R}^2$, a stronger gravitational field implies stronger force and/or a longer period during which the force term survives effec-

tively. As a result, the fine-tuned initial location χ_{crit} goes toward the right as the star accommodates a larger gravitational potential, and χ_{crit} will eventually reach the point χ_s . Therefore, it is expected that there is a maximum value of the gravitational potential for a star to exist.

IV. ANALYTIC ARGUMENT

Before solving the full set of Eqs. (14)–(17) numerically, in this section we shall provide some analytic arguments. The analysis with approximate solutions will help to understand our numerical results presented in Sec. V.

First let us consider the interior of a star: $r < \mathcal{R}$. To give a tractable argument, we assume that

$$|B - 1|, |N - 1| \ll 1, \quad |B'/B|, |N'/N| \ll r^{-1}. \quad (30)$$

Then, the solution to Eq. (17) which is regular at the center is given by

$$\chi' \simeq -\frac{2G\mu}{3\mathcal{R}^3} r, \quad \chi \simeq \chi_c - \frac{G\mu}{3\mathcal{R}^3} r^2, \quad (31)$$

where we have defined

$$\mu := \frac{4\pi}{3} \left(\rho_0 - \frac{\chi_c^3}{4\pi G} V_{\chi_c} \right) \mathcal{R}^3. \quad (32)$$

Here we have ignored the pressure p relative to ρ_0 , and made a rough approximation $\chi^3 V_\chi \approx \chi_c^3 V_{\chi_c}$. Using Eqs. (15) and (31), we obtain

$$B \simeq 1 - \frac{2\hat{G}(M - \mu/3)}{\mathcal{R}^3} r^2, \quad (33)$$

where $M := 4\pi\rho_0\mathcal{R}^3/3$. Here we have neglected the ‘‘cosmological constant’’ $\chi^2 V$. (We remind the reader that $\hat{G} := G/\chi_c$.)

To derive the exterior solution, we approximate

$$\frac{2\chi^3}{3} \frac{dV}{d\chi} \simeq \frac{\chi - \chi_1}{\lambda_\chi^2}, \quad (34)$$

where $\lambda_\chi^{-2} := (2\chi_1^3/3)d^2V/d\chi^2|_{\chi_1}$, and analyze the behavior of χ around χ_1 . The exterior solution to Eq. (17) is found to be

$$\chi \simeq \chi_1 + \mathcal{C} \frac{e^{-(r-\mathcal{R})/\lambda_\chi}}{r}. \quad (35)$$

Matching this to the interior solution (31) at $r = \mathcal{R}$, we obtain

$$\mathcal{C} \simeq \frac{2G\mu}{3}, \quad (36)$$

$$\chi_c \simeq \chi_1 + \frac{G\mu}{\mathcal{R}}, \quad (37)$$

where we have used $\lambda_\chi \approx \mathcal{O}(\Lambda_{\text{eff}}^{-1/2}) \gg \mathcal{R}$. Given ρ_0 , Eq. (37) determines χ_c . Solving Eq. (15) and matching the solution to the interior one (33) at the surface of the

star, we find

$$B \simeq 1 - \frac{2\hat{G}(M - \mu/3)}{r}. \quad (38)$$

Then, Eq. (16) implies that

$$N \simeq N_\infty \left[1 - \frac{2\hat{G}(M + \mu/3)}{r} \right], \quad (39)$$

where the asymptotic value N_∞ is not unity because of our boundary condition at the center, but it can be set to unity by rescaling of the time coordinate. From Eqs. (38) and (39), the PPN parameter γ is found to be

$$\gamma = \frac{3M - \mu}{3M + \mu}. \quad (40)$$

Now let us define the gravitational potential evaluated at the surface of the star:

$$\Phi := \frac{\hat{G}(M + \mu/3)}{\mathcal{R}}. \quad (41)$$

In terms of this, Eq. (37) can be written as

$$\frac{\Delta}{\Phi} = \frac{3\mu}{3M + \mu} \quad \text{with} \quad \Delta := \frac{\chi_c - \chi_1}{\chi_c}. \quad (42)$$

The ‘‘thin-shell’’ condition is given by $\Delta \ll \Phi$ [10]. This is equivalent to $\mu \ll M$, which leads to $\mathcal{C} \ll GM$ and indeed suppresses the deviation of $\chi(r)$ from χ_1 outside the star. This situation is realized if $4\pi G\rho_0 - \chi_c^3 V_{\chi_c} \approx 0$. On the other hand, if $4\pi G\rho_0 \gg \chi_c^3 V_{\chi_c}$, and hence $\mu \simeq M$, the thin-shell condition does not hold. It is easy to see that $\gamma \simeq 1/2$. In this ‘‘thick-shell’’ case, χ_c must be not too far from χ_1 so as not to fall into the curvature singularity, $\chi = 1$. Therefore, we require that $\chi_c < 1$. This condition together with Eq. (37) gives the bound

$$\Phi < \Phi_{\text{max}} = \frac{4}{3}(1 - \chi_1), \quad (43)$$

i.e., stars with $\Phi > \Phi_{\text{max}}$ cannot exist. As shown later by numerical solutions, the thin-shell condition is violated as long as the exterior is vacuum.

Specifically, for $n = 1$ one needs $x_1 > \sqrt{3}$ in order to have a de Sitter minimum. This leads to $1 - \chi_1 < 1/3$ ($\Phi_{\text{max}} < 4/9$). For $x_1 = 3.6$ ($\lambda = 2.088$), we have $1 - \chi_1 = 0.07716$ ($\Phi_{\text{max}} = 0.1029$). For $n = 2$, one needs $x_1 > \sqrt{\sqrt{13} - 2}$, giving $1 - \chi_1 \lesssim 0.2705$ ($\Phi_{\text{max}} \lesssim 0.3606$).

V. NUMERICAL RESULTS

A. Stars in $f(R)$ gravity

We numerically integrate Eqs. (14)–(17) outwards from the center, imposing the appropriate boundary conditions given by Eq. (18) with Eqs. (22)–(25). We reconstruct $R = R(r)$ from the metric and compare it with the solution for the scalar degree of freedom, $R = Q(\chi(r))$, to make sure

that numerical errors are sufficiently small. The angular component of the field equations is also used for the same purpose.

The sets of the model parameters we use are: (i) $n = 1$ and $x_1 = 3.6$ ($\lambda = 2.088$), for which $\chi_1 = 0.9228$; (ii) $n = 2$ and $x_1 = 3.6$ ($\lambda = 1.827$), for which $\chi_1 = 0.9903$. For numerical solutions we shall take $4\pi G\rho_0 = 10^6 \Lambda_{\text{eff}}$. This is not a realistic value, e.g., for neutron stars, but we do not have to be concerned about this point because the properties of stellar solutions which we are interested in are basically characterized by the gravitational potential rather than the energy density itself, provided that $G\rho_0 \gg \Lambda_{\text{eff}}$. This is expected from the analytic result, and we have confirmed that it is indeed true for our numerical solutions.

1. $n = 1$ and $x_1 = 3.6$

Taking $4\pi G\rho_0 = 10^6 \Lambda_{\text{eff}}$ and $p_c = 10^{-4} \rho_0$, we find an asymptotically de Sitter solution of a star for $R_c = 2.001 \times 10^{-6} \times 8\pi G\rho_0$. The solution agrees well with the analytic approximation in Sec. IV.

A numerical calculation has been performed also in the case of $4\pi G\rho_0 = 10^6 \Lambda_{\text{eff}}$ and $p_c = 5 \times 10^{-2} \rho_0$. We find a solution by tuning $R_c = 3.462 \times 10^{-6} \times 8\pi G\rho_0$ ($\chi_c = 0.9836$), for which

$$\frac{4\pi}{3} \hat{G}\rho_0 \mathcal{R}^2 = \frac{\hat{G}M}{\mathcal{R}} \simeq 0.06687. \quad (44)$$

Our numerical result is shown in Figs. 4–6. One can see from Fig. 5 that $R \rightarrow x_1 R_0$ and $\chi \rightarrow \chi_1$ as $r \rightarrow \infty$. Since $\Delta = 0.06176$ is almost the same as the gravitational potential (44), the thin-shell does not form. A numerical fitting leads to the approximate expression for the exterior metric:

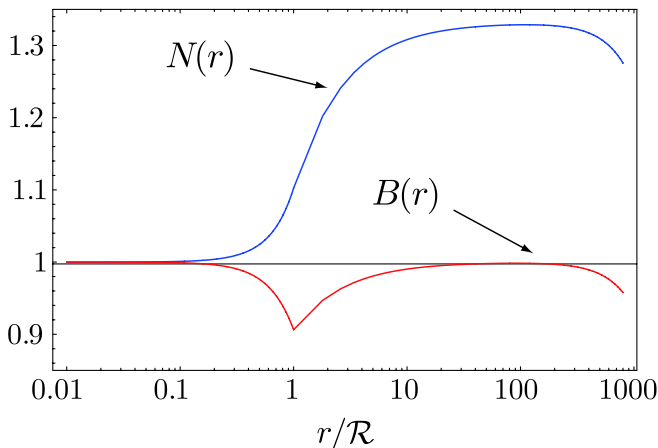


FIG. 4 (color online). Metric for a nonrelativistic star. Parameters are given by $n = 1$, $x_1 = 3.6$, $4\pi G\rho_0 = 10^6 \Lambda_{\text{eff}}$, and $p_c = 5 \times 10^{-2} \rho_0$. The central value of the Ricci scalar is tuned to be $R_c = 3.462 \times 10^{-6} \times 8\pi G\rho_0$. The radial coordinate is normalized by the radius of the star \mathcal{R} .

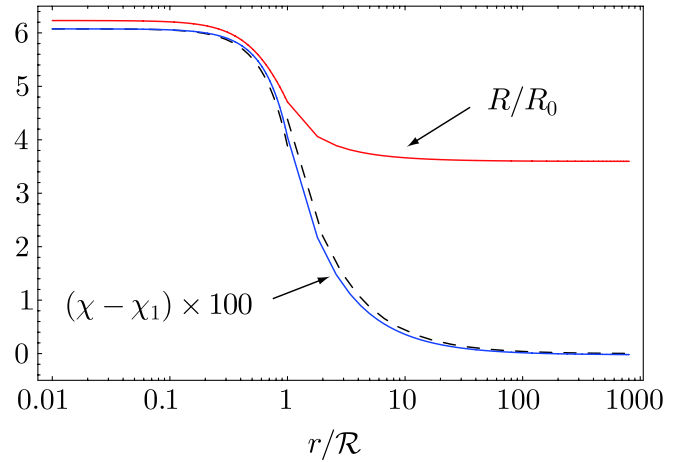


FIG. 5 (color online). Numerical solutions of the Ricci scalar and χ for a nonrelativistic star. Parameters are the same as those in Fig. 4. Dashed line is a plot of the analytic approximation (31) and (35). Since $R \rightarrow x_1 R_0$ and $\chi \rightarrow \chi_1$ as $r \rightarrow \infty$, this is the desired solution with asymptotically de Sitter geometry.

$$N \simeq N_\infty \left(1 - 2c_1 \frac{\mathcal{R}}{r} - \frac{c_2}{3} \Lambda_{\text{eff}} r^2 \right), \quad (45)$$

$$B \simeq 1 - 2c_3 \frac{\mathcal{R}}{r} - \frac{c_4}{3} \Lambda_{\text{eff}} r^2, \quad (46)$$

with

$$\begin{aligned} N_\infty &= 1.332, & c_1 &= 0.08716, & c_2 &= 0.9973, \\ c_3 &= 0.04747, & c_4 &= 0.9993. \end{aligned} \quad (47)$$

The PPN parameter turns out to be $\gamma \simeq c_3/c_1 \simeq 0.5446$, which is close to $1/2$, as expected.

To explore stars with larger $\hat{G}M/\mathcal{R}$, we have tried to find numerical solutions for $4\pi G\rho_0 = 10^6 \Lambda_{\text{eff}}$ and $p_c = 0.1 \times \rho_0$. A solution which is regular inside the star is obtained, e.g., for $R_c = 0.7000 \times 8\pi G\rho_0$, but this is the rolling-down solution (see Fig. 7). Hence it is unphysical

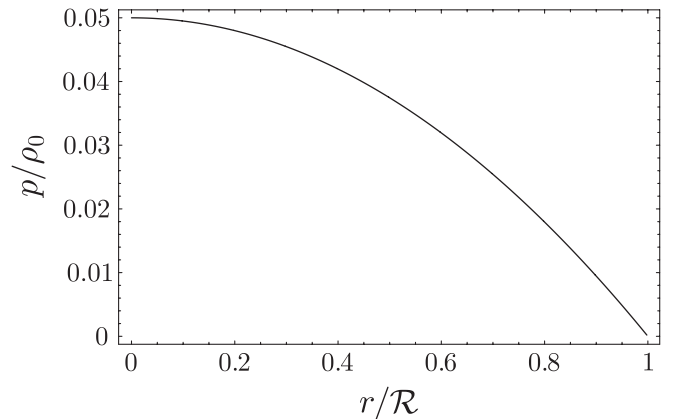


FIG. 6. The pressure profile $p(r)$ of a nonrelativistic star. Parameters are the same as those in Fig. 4.

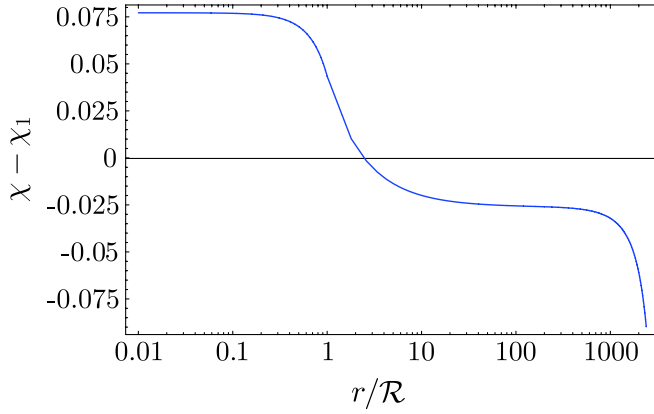


FIG. 7 (color online). A rolling-down solution for a would-be relativistic star. Parameters are given by $n = 1$, $x_1 = 3.6$, $4\pi G\rho_0 = 10^6 \Lambda_{\text{eff}}$, and $p_c = 0.1 \times \rho_0$. The central value of the Ricci scalar is $R_c = 0.7000 \times 8\pi G\rho_0$. The solution clearly overshoots the de Sitter extremum, $\chi = \chi_1$.

($N(r) \rightarrow 0$ and $B(r) \rightarrow \infty$ as $r \rightarrow \infty$), though the gravitational potential is as “large” as

$$\frac{\hat{G}M}{\mathcal{R}} \simeq 0.1203. \quad (48)$$

Taking a slightly larger value of the central curvature, $R_c = 0.7001 \times 8\pi G\rho_0$, the Ricci scalar rapidly diverges inside the star. This corresponds to the case in which χ starts to move to the right toward the curvature singularity, $\chi = 1$. Note that since $\chi^3 V \approx R/2$, $R_c = 0.7 \times 8\pi G\rho_0$ implies $4\pi G(\rho_0 - 3p_c) - \chi_c^3 V_{\chi_c} \approx 0$.

2. $n = 2$ and $x_1 = 3.6$

For $4\pi G\rho_0 = 10^6 \Lambda_{\text{eff}}$ and $p_c = 5 \times 10^{-4} \rho_0$, we find the stellar configuration with de Sitter asymptotic behavior by tuning $R_c = 2.035 \times 10^{-6} \times 8\pi G\rho_0$ ($\chi_c = 0.9911$). In this case, $\hat{G}M/\mathcal{R} = 7.491 \times 10^{-4}$. The thin-shell condition does not hold ($\Delta = 7.448 \times 10^{-4}$), and $\gamma = 0.5005$.

Looking for stars with stronger gravitational fields, we take $4\pi G\rho_0 = 10^6 \Lambda_{\text{eff}}$ and $p_c = 10^{-2} \rho_0$. When $R_c = 0.9700 \times 8\pi G\rho_0$, a rolling-down solution is found with $\hat{G}M/\mathcal{R} = 0.01465$, while a slightly larger value $R_c = 0.9701 \times 8\pi G\rho_0$ leads to a curvature singularity $R \rightarrow \infty$ inside the star.

From the above numerical analysis, we conclude that stars with strong gravitational fields cannot be present in this class of $f(R)$ theories.

B. Stars in surrounding media

So far we have examined the case of the vacuum exterior. Although we can indeed construct a stellar configuration provided that gravity is weak, the exterior metric is not given by the de Sitter–Schwarzschild solution in general relativity and the PPN parameter is found to be $\gamma \simeq 1/2$. This simply reflects the fact that the thin-shell condition is

violated in the present case. However, we can make the chameleon mechanism effective by taking into account the effect of surrounding media (e.g., dark matter). Let us make a brief comment on this point.

In a “realistic” situation, exterior matter is present around a star, giving rise to the force term $\mathcal{F}_{\text{ext}} \simeq -8\pi G\rho_{\text{ext}}/3$ there. Then, to obtain a viable stellar configuration, one has to take a shot at the point χ_* satisfying the equation $\mathcal{F}_{\text{ext}} - dU/d\chi = 0$ rather than the top of the potential hill, χ_1 . Since $\chi_* > \chi_1$, the difference in χ between inside and outside the star becomes smaller, and hence it is easier to satisfy the thin-shell condition. Indeed, the chameleon mechanism has been shown to work in the $f(R)$ model which is very similar to the current one, reproducing $\gamma \simeq 1$ in the solar vicinity [20].

The above argument, however, only applies to stars with weak gravitational fields. When gravity is strong, for any initial condition of χ_c which avoids the curvature singularity, the scalar field χ inevitably overshoots the potential. It is clear from this fact that one cannot stop χ at any value of $\chi_* > \chi_1$. Therefore, a relativistic star cannot be present even with surrounding medium.

VI. CONCLUSIONS

In this paper, we have studied the strong gravity aspect of $f(R)$ modified gravity models that reproduce the conventional cosmological evolution and evade solar system and laboratory tests [19–21]. It is known that $f(R)$ theories can be recasted simply in the Brans–Dicke theory with $\omega = 0$, but the potential for the effective scalar degree of freedom may play a complicated and nontrivial role. Moreover, the presence of matter may affect dynamics of the scalar field, possibly mimicking the chameleon model [10].

We have explored uniform density, spherically symmetric stars and their exterior geometry in the $f(R)$ model of [19]. The main result of the present paper is summarized as follows: given model parameters, there is a maximum value of the gravitational potential produced by a star, above which no asymptotically de Sitter stellar configurations can be constructed. We show this both analytically and numerically. For example, the model with $n = 1$ and $\lambda \approx 2$ gives $\Phi_{\text{max}} \approx 0.1$. This raises a warning sign for a class of $f(R)$ theories, because neutron stars cannot be present in such gravity models.

The underlying mechanism that hinders strong gravitational fields around matter is explained essentially as follows [26]. Consider a static matter distribution. The Newtonian potential obeys the Poisson equation $\nabla^2 \Phi \sim G\rho$, while the equation of motion for the scalar field implies $\nabla^2 \chi \sim G\rho$. From this, one can evaluate the excitation of the scalar degree of freedom around the matter distribution as $\delta\chi \sim \mathcal{O}(\Phi)$. If the de Sitter minimum is located very close to the point $\chi = 1$, which corresponds to $R = \infty$ in the effective potential, a slightly strong gravita-

tional field will cause the problem of appearance of a curvature singularity.

Bearing the above evaluation in mind, let us comment on the other specific models of $f(R)$ gravity. The model of Hu and Sawicki [20] and Starobinsky's one share the same structure of $f(R)$ in the high-curvature regime, i.e., $f(R) \approx R - 2\Lambda_{\text{eff}} + C/R^\alpha$ with $\alpha > 0$. Therefore, we expect that the same problem arises in the Hu and Sawicki's model. The model by Appleby and Battye is characterized by [21]

$$f(R) = \frac{R}{2} + \frac{1}{2a} \ln[\cosh(aR) - \tanh(b) \sinh(aR)], \quad (49)$$

where a and b are parameters. Since $\chi = df/dR = [1 + \tanh(aR - b)]/2$, a positive curvature singularity corresponds to $\chi = 1$. Taking, for example, $b = 1.5$, we find that $\chi_1 \approx 0.93$ at the de Sitter minimum, which is very close to the dangerous curvature singularity (the result is independent of a). Since also in this model the effective potential is finite at $R = +\infty$, we anticipate the same singularity problem.

Our choice of the parameters in the present paper gave $\Phi_{\text{max}} \approx 0.1$, for which neutron stars are unlikely to exist.

However, there still remains a possibility that more realistic stellar environments and matter profiles weaken the bound on the potential by a factor of 2 or 3, and at the same time make the chameleon mechanism work [35]. It is technically much more difficult to construct stellar configurations with a realistic equation of state, realistic energy densities, and realistic stellar environments. Such an elaborated modeling of relativistic stars might allow for Φ_{max} as large as, say, 0.3, but the parameter space of the theory will be very restricted. To conclude, $f(R)$ theories that reproduce the correct behavior of weak gravity in the solar vicinity do not admit neutron star solutions without special care.

ACKNOWLEDGMENTS

T. K. would like to thank Kenta Kiuchi for valuable discussions on neutron stars. This work was partially supported by the JSPS under Contact No. 19-4199, by the Grant-in-Aid for Scientific Research Fund of the JSPS (No. 19540308) and by the Japan-U.K. Research Cooperative Program.

-
- [1] A. G. Riess *et al.* (Supernova Search Team Collaboration), *Astron. J.* **116**, 1009 (1998); S. Perlmutter *et al.* (Supernova Cosmology Project Collaboration), *Astrophys. J.* **517**, 565 (1999); A. G. Riess *et al.* (Supernova Search Team Collaboration), *Astrophys. J.* **607**, 665 (2004).
 - [2] A. A. Starobinsky, *Phys. Lett.* **91b**, 99 (1980).
 - [3] For a review, see T. P. Sotiriou and V. Faraoni, arXiv:0805.1726.
 - [4] B. Whitt, *Phys. Lett.* **145b**, 176 (1984); K. I. Maeda, *Phys. Rev. D* **39**, 3159 (1989).
 - [5] T. Chiba, *Phys. Lett. B* **575**, 1 (2003).
 - [6] A. L. Erickcek, T. L. Smith, and M. Kamionkowski, *Phys. Rev. D* **74**, 121501 (2006).
 - [7] T. Chiba, T. L. Smith, and A. L. Erickcek, *Phys. Rev. D* **75**, 124014 (2007).
 - [8] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, *Phys. Rev. D* **70**, 043528 (2004).
 - [9] C. M. Will, *Living Rev. Relativity* **9**, 3 (2005).
 - [10] J. Khoury and A. Weltman, *Phys. Rev. Lett.* **93**, 171104 (2004); *Phys. Rev. D* **69**, 044026 (2004).
 - [11] D. F. Mota and J. D. Barrow, *Mon. Not. R. Astron. Soc.* **349**, 291 (2004); T. Clifton, D. F. Mota, and J. D. Barrow, *Mon. Not. R. Astron. Soc.* **358**, 601 (2005).
 - [12] D. F. Mota and D. J. Shaw, *Phys. Rev. Lett.* **97**, 151102 (2006); *Phys. Rev. D* **75**, 063501 (2007).
 - [13] J. A. R. Cembranos, *Phys. Rev. D* **73**, 064029 (2006).
 - [14] A. W. Brookfield, C. van de Bruck, and L. M. H. Hall, *Phys. Rev. D* **74**, 064028 (2006).
 - [15] I. Navarro and K. Van Acoleyen, *J. Cosmol. Astropart. Phys.* **02** (2007) 022.
 - [16] T. Faulkner, M. Tegmark, E. F. Bunn, and Y. Mao, *Phys. Rev. D* **76**, 063505 (2007).
 - [17] B. Li and J. D. Barrow, *Phys. Rev. D* **75**, 084010 (2007).
 - [18] L. Amendola and S. Tsujikawa, *Phys. Lett. B* **660**, 125 (2008).
 - [19] A. A. Starobinsky, *JETP Lett.* **86**, 157 (2007).
 - [20] W. Hu and I. Sawicki, *Phys. Rev. D* **76**, 064004 (2007).
 - [21] S. A. Appleby and R. A. Battye, *Phys. Lett. B* **654**, 7 (2007).
 - [22] S. Tsujikawa, *Phys. Rev. D* **77**, 023507 (2008).
 - [23] S. Capozziello and S. Tsujikawa, *Phys. Rev. D* **77**, 107501 (2008).
 - [24] P. Brax, C. van de Bruck, A. C. Davis, and D. J. Shaw, arXiv:0806.3415.
 - [25] S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **68**, 123512 (2003); M. C. B. Abdalla, S. Nojiri, and S. D. Odintsov, *Classical Quantum Gravity* **22**, L35 (2005); S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **652**, 343 (2007); **657**, 238 (2007); *Phys. Rev. D* **77**, 026007 (2008); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani, and S. Zerbini, *Phys. Rev. D* **77**, 046009 (2008); S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **78**, 046006 (2008).
 - [26] A. V. Frolov, *Phys. Rev. Lett.* **101**, 061103 (2008).
 - [27] K. Kainulainen, J. Piilonen, V. Reijonen, and D. Sunhede, *Phys. Rev. D* **76**, 024020 (2007).
 - [28] K. Kainulainen and D. Sunhede, arXiv:0803.0867.
 - [29] T. Multamaki and I. Vilja, *Phys. Rev. D* **76**, 064021 (2007); K. Henttunen, T. Multamaki, and I. Vilja, *Phys. Rev. D* **77**, 024040 (2008); T. Multamaki and I. Vilja, *Phys. Lett. B* **659**, 843 (2008).
 - [30] A. J. Bustelo and D. E. Barraco, *Classical Quantum*

- Gravity **24**, 2333 (2007).
- [31] E. Barausse, T.P. Sotiriou, and J.C. Miller, Classical Quantum Gravity **25**, 105008 (2008); **25**, 062001 (2008).
- [32] S.A. Appleby and R.A. Battye, J. Cosmol. Astropart. Phys. 05 (2008) 019.
- [33] S.R. Coleman, Phys. Rev. D **15**, 2929 (1977); **16**, 1248(E) (1977).
- [34] The particle which climbs up the potential initially never turns around before $r = \mathcal{R}$. This is because $|\mathcal{F}| \propto \rho_0 - 3p$ never decreases inside the star, and the potential is less steep for smaller χ . This will be true for more realistic profiles of the density and pressure, except in the vicinity of the star surface. Note, however, that it is possible for χ to pass through χ_1 before $r = \mathcal{R}$.
- [35] The neutron star–white dwarf system PSR J1141–6545 can put strong constraints on alternative theories of gravity around relativistic stars [9,36]. For this reason, we need to invoke the chameleon mechanism to describe such stars.
- [36] G. Esposito-Farese, arXiv:gr-qc/0402007.