Rotating Kaluza-Klein multi-black holes with Gödel parameter

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We obtain new five-dimensional supersymmetric rotating multi-Kaluza-Klein black hole solutions with the Gödel parameter in the Einstein-Maxwell system with a Chern-Simons term. These solutions have no closed timelike curve outside the black hole horizons. At infinity, the space-time is effectively four-dimensional. Each horizon admits various lens space topologies $L(n; 1) = S^3/\mathbb{Z}_n$ in addition to a round S^3. The space-time can have outer ergoregions disjointed from the black hole horizons, as well as inner ergoregions attached to each horizon. We discuss the rich structures of ergoregions.

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I. INTRODUCTION

In recent years, Kaluza-Klein black hole solutions have been studied by many authors in the context of string theory. Since Kaluza-Klein black hole solutions have compactified extra dimensions, the space-time effectively behaves as four dimensions at infinity. The first Kaluza-Klein black hole solutions with an extra twisted S^1 were found by Dobiasch and Maison [1] as vacuum solutions to the five-dimensional Einstein equation, and the features of the black hole were investigated by Gibbons and Wiltshire [2]. The static charged Kaluza-Klein black holes were also found in the five-dimensional Einstein-Maxwell theory [3] and were generalized to the rotating case as the solutions in the five-dimensional Einstein-Maxwell theory with a Chern-Simons term [4]. Supersymmetric rotating Kaluza-Klein black hole solutions were found by Gaiotto, Strominger, and Yin [5] and Elvang *et al.* [6]. Supersymmetric static multi-Kaluza-Klein black hole solutions were also constructed [7]. These solutions were constructed on the self-dual Euclidean Taub-NUT space in the framework of Gauntlett *et al.*'s classification of the five-dimensional supersymmetric solutions [8]. In these solutions, at infinity, the space-times asymptote to a twisted S^1 bundle over the four-dimensional Minkowski space-time. Exact Kaluza-Klein black hole solutions which asymptote to the direct product of the four-dimensional Minkowski space-time and an S^1 were also constructed [9–11].

The squashing transformation [3,4,12] is a useful tool to generate Kaluza-Klein black hole solutions from the class of *cohomogeneity-one* black hole solutions with the asymptotically flatness. Actually, Wang [12] regenerated the five-dimensional Kaluza-Klein black hole solution found by Dobiasch and Maison [1] from the five-dimensional Myers-Perry black hole solution with two equal angular momenta [13]. Applying the squashing transformation to the charged rotating black hole solutions with two equal angular momenta [14] in the five-dimensional Einstein-Maxwell theory with a Chern-Simons term, the present authors obtained the new Kaluza-Klein black hole solution [4] in the same theory. This is the generalization of the Kaluza-Klein black hole solutions in Refs. [1–3], and it describes a nonsupersymmetric black hole boosted in the direction of the extra dimension. One of the interesting features of the solution is that the horizon admits a prolate shape in addition to a round S^3 by the effect of the rotation of black hole.

In the previous work [15], applying this squashing transformation to nonasymptotically flat Kerr-Gödel black hole solutions [16], we also constructed a new type of rotating Kaluza-Klein black hole solutions to the five-dimensional Einstein-Maxwell theory with a Chern-Simons term. Though the Kerr-Gödel black hole solutions have closed timelike curves in the region away from the black hole, the squashed Kerr-Gödel black hole solutions have no closed timelike curve outside the black hole horizons. In addition, the solution has two kinds of rotation parameters in the same direction of the extra dimension. These two independent parameters are associated with the rotations of the black hole and the Universe. In the absence of a black hole, the solution describes the Gross-Perry-Sorkin monopole which is boosted in the direction of an extra dimension and has an ergoregion by the effect of the rotation of the Universe.

In this paper, taking a limit of parameters in the charged version of squashed Kerr-Gödel black hole solutions introduced in the appendix of Ref. [15], we construct new supersymmetric rotating Kaluza-Klein black hole solutions to the five-dimensional Einstein-Maxwell theory with a Chern-Simons term. These can be regarded as solutions generated by the squashing transformation of the supersymmetric Kerr-Newman-Gödel black hole solutions [17]. Like the squashed Kerr-Gödel black holes in [15], these Kaluza-Klein black hole solutions have no closed timelike

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curve outside the black hole horizons. The space-time is asymptotically locally flat; i.e., at infinity, the space-time approaches a twisted S^1 fiber bundle over a four-dimensional Minkowski space-time. The horizons are the round S^3, unlike known supersymmetric rotating Kaluza-Klein black hole solutions, where they are the squashed S^3. We also generalize these solutions to multi-black hole solutions. In particular, we study two-black hole solutions. As will be shown later, each horizon admits various lens space topologies $L(n; 1) = S^3/\mathbb{Z}_n$ (n: natural numbers) in addition to an S^3 and ergoregions have rich structures.

The rest of this paper is organized as follows. First, following the results of classification of solutions of the five-dimensional minimal supergravity [8], in Sec. II, we construct general solutions on the Gibbons-Hawing space, which is the Taub-NUT space in the special case. In Sec. III, we present a new supersymmetric single-black hole solution on the Taub-NUT space. In Sec. IV, we study the multi-black hole solutions, in particular, the two-black hole case. We conclude our article with a discussion in Sec. V.

II. SOLUTIONS

We consider the five-dimensional Einstein-Maxwell system with a Chern-Simons term. The action is given by

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\bigg[R - F_{\mu\nu}F^{\mu\nu} - \frac{2}{3\sqrt{3}}(\sqrt{-g})^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}A_\mu F_{\nu\rho}F_{\sigma\lambda}\bigg], \qquad (1)$$

where R is the five-dimensional scalar curvature, $\boldsymbol{F} = d\boldsymbol{A}$ is the 2-form of the five-dimensional gauge field associated with the gauge potential 1-form $\boldsymbol{A}$, and G_5 is the five-dimensional Newton constant. Varying the action (1), we can derive the Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 2(F_{\mu\lambda}F_\nu{}^\lambda - \tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}) \qquad (2)$$

and the Maxwell equation

$$F^{\mu\nu}{}_{;\nu} + \frac{1}{2\sqrt{3}\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\nu\rho}F_{\sigma\lambda} = 0. \qquad (3)$$

We construct rotating multi-black hole solutions satisfying Eqs. (2) and (3). The forms of the metric and the gauge potential 1-form are

$$ds^2 = -H^{-2}[dt + \alpha V^\beta(d\zeta + \boldsymbol{\omega})]^2 + H ds^2_{\mathrm{GH}}, \qquad (4)$$

$$\boldsymbol{A} = \frac{\sqrt{3}}{2}H^{-1}[dt + \alpha V^\beta(d\zeta + \boldsymbol{\omega})], \qquad (5)$$

where the function H and the metric ds^2_{GH} are given by

$$H = 1 + \sum_i \frac{M_i}{|\boldsymbol{R} - \boldsymbol{R}_i|}, \qquad (6)$$

$$ds^2_{\mathrm{GH}} = V^{-1}ds^2_{\mathbb{E}^3} + V(d\zeta + \boldsymbol{\omega})^2, \qquad (7)$$

$$V^{-1} = \epsilon + \sum_i \frac{N_i}{|\boldsymbol{R} - \boldsymbol{R}_i|}, \qquad (8)$$

respectively, where $ds^2_{\mathbb{E}^3} = dx^2 + dy^2 + dz^2$ is a metric on the three-dimensional Euclid space $\mathbb{E}^3$ and $\boldsymbol{R} = (x, y, z)$ denotes a position vector on $\mathbb{E}^3$. The function V^{-1} is a harmonic function on $\mathbb{E}^3$ with point sources located at $\boldsymbol{R} = \boldsymbol{R}_i := (x_i, y_i, z_i)$, where the Killing vector field ∂_ζ has fixed points in the base space. The 1-form $\boldsymbol{\omega}$, which is determined by

$$\nabla \times \boldsymbol{\omega} = \nabla V^{-1}, \qquad (9)$$

has the explicit form

$$\boldsymbol{\omega} = \sum_i N_i \frac{z - z_i}{|\boldsymbol{R} - \boldsymbol{R}_i|}\frac{(x - x_i)dy - (y - y_i)dx}{(x - x_i)^2 + (y - y_i)^2}, \qquad (10)$$

where M_i, N_i, and α are constants, $\beta = \pm 1$, and $\epsilon = 0, 1$. The base space (7) with Eqs. (8) and (10) is often called the Gibbons-Hawking space. In particular, the Gibbons-Hawking space with $\epsilon = 1$, $N_1 \neq 0$, and $N_i = 0$ ($i \geq 2$) is the self-dual Euclidean Taub-NUT space, and the space with $\epsilon = 0$, $N_1 \neq 0$, and $N_i = 0$ ($i \geq 2$) is the four-dimensional Euclid space.

The solutions (4)–(10) coincide with several known black hole solutions. For example, multi-black hole solutions on the multicentered-Taub-NUT space [7] are obtained by $\beta = -1$, $\epsilon = 1$, and $\alpha = 0$. Multiblack hole solutions on the multicentered Eguchi-Hanson spaces [18] are obtained by setting $\beta = -1$, $\epsilon = 0$, and $\alpha = 0$.

Restricting the solutions (4) to the case with $\beta = 1$ and $\epsilon = 1$, we can obtain new black hole solutions. To avoid the existence of singularities and closed timelike curves outside the black hole horizons, we choose the parameters such that

$$M_i > 0, \qquad N_i > 0, \qquad 0 \leq \alpha^2 < 1. \qquad (11)$$

See Appendix A about the detail discussion.

III. SINGLE BLACK HOLE WITH GÖDEL ROTATION

First, we study the case of a single black hole, i.e., the case of $M_1 = M$, $N_1 = N$, and $M_i = N_i = 0$ ($i \geq 2$).

A. Metric and gauge potential

In the single-black hole case, the metric (4) and the gauge potential 1-form (5) are expressed in the form

$$ds^2 = -H^{-2}[dt + \alpha V(d\zeta + N\cos\theta d\phi)]^2 + H[V^{-1}(dR^2 + R^2 d\Omega^2_{S^2}) + V(d\zeta + N\cos\theta d\phi)^2], \qquad (12)$$

$$\boldsymbol{A} = \frac{\sqrt{3}}{2}H^{-1}[dt + \alpha V(d\zeta + N\cos\theta d\phi)], \qquad (13)$$

where the functions H and V^{-1} can be written as

$$H = 1 + \frac{M}{R}, \qquad V^{-1} = 1 + \frac{N}{R}, \tag{14}$$

respectively. $d\Omega_{S^2}^2 = d\theta^2 + \sin^2\theta d\phi^2$ denotes the metric of the unit two-sphere. The coordinates run the ranges of $-\infty < t < \infty$, $-M < R < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, and $0 \leq \zeta \leq 2\pi L$. From the requirements for the absence of naked singularities and closed timelike curves outside the black hole horizon, the parameters are restricted to the region

$$M > 0, \qquad N > 0, \qquad 0 \leq \alpha^2 < 1. \tag{15}$$

The constant N is related to the size of the compactified radius L at infinity by

$$N = \frac{L}{2} n, \tag{16}$$

where n is a natural number.

When the parameter α vanishes, the metric (12) and the gauge potential 1-form (13) coincide with the extreme case of the static charged Kaluza-Klein black hole solution [3].

It should be noted that this solution (12) with $n = 1$ coincides with a limiting solution of the squashed Kerr-Newman-Gödel black hole solution [15] by putting the parameters as

$$m = -q, \qquad a = -2jq \tag{17}$$

and by identifying the coordinates and parameters as

$$t = \frac{r_\infty^2 - m}{r_\infty^2} T, \qquad R = \frac{r_\infty}{2} \frac{r^2 - m}{r_\infty^2 - r^2}, \qquad M = \frac{m}{2r_\infty},$$
$$N = \frac{r_\infty}{2}, \qquad \alpha = \frac{2j(r_\infty^2 - m)^2}{r_\infty^3}. \tag{18}$$

About the explicit form of the squashed Kerr-Newman-Gödel black hole solution, readers should see Appendix B in this article.

B. Asymptotic structure and asymptotic charge

Introduce a new coordinate $\psi := 2\zeta/L$ with the periodicity of $\Delta\psi = 4\pi$. In the neighborhood of infinity, $R = \infty$, the metric (12) behaves as

$$ds^2 \simeq -d\tilde{t}^2 + dR^2 + R^2 d\Omega_{S^2}^2 + \frac{L^2}{4}(1 - \alpha^2)\left(\frac{d\tilde{\psi}}{n} + \cos\theta d\phi\right)^2, \tag{19}$$

where we introduced the following coordinates $(\tilde{t}, \tilde{\psi})$:

$$\tilde{t} = \frac{t}{\sqrt{1 - \alpha^2}}, \qquad \tilde{\psi} = \psi - \frac{\alpha t}{N(1 - \alpha^2)}, \tag{20}$$

which are chosen so that they are in the rest frame at the infinity. The asymptotic structure of the solution (12) is an asymptotically locally flat; i.e., the metric asymptotes to a twisted constant S^1 fiber bundle over the four-dimensional Minkowski space-time, and the spatial infinity has the structure of an S^1 bundle over an S^2 such that it is the lens space $L(n; 1) = S^3/\mathbb{Z}_n$.

The Komar mass associated with the timelike Killing vector field $\partial_{\tilde{t}}$ at infinity M, the charge at infinity Q, and the angular momenta associated with the spacelike Killing vector fields ∂_ϕ and $\partial_{\tilde{\psi}}$ at infinity J_ϕ and $J_{\tilde{\psi}}$ can be obtained as

$$\mathcal{M} = \frac{3L[(\alpha^2 + 2)M + \alpha^2 N]}{8\pi\sqrt{1 - \alpha^2} G_5} \frac{\mathcal{A}_{S^3}}{n}, \tag{21}$$

$$Q = \frac{\sqrt{3} L M}{2\pi G_5} \frac{\mathcal{A}_{S^3}}{n}, \tag{22}$$

$$J_\phi = 0, \tag{23}$$

$$J_{\tilde{\psi}} = -\frac{\alpha L^2 (3M + \alpha^2 N)}{8\pi G_5} \frac{\mathcal{A}_{S^3}}{n}, \tag{24}$$

where $\mathcal{A}_{S^3}$ denotes the area of a unit S^3.

C. Horizon

A black hole horizon exists at the position of the source for the harmonic functions H and V^{-1}, i.e., $R = 0$. In the coordinate system $(t, R, \theta, \phi, \zeta)$, the metric (12) diverges apparently at $R = 0$. In order to remove this apparent divergence, we introduce a new coordinate v such that

$$dv = dt - \sqrt{\left(1 + \frac{M}{R}\right)^3 \left(1 + \frac{N}{R}\right)} dR. \tag{25}$$

Then, near $R = 0$, the metric (12) behaves as

$$ds^2 \simeq -2\sqrt{\frac{N}{M}} dv dR + MN\left[d\Omega_{S^2}^2 + \left(\frac{d\zeta}{N} + \cos\theta d\phi\right)^2\right] + \mathcal{O}(R). \tag{26}$$

This metric well behaves at the null surface $R = 0$. The Killing vector field $V = \partial_v$ becomes null at $R = 0$, and V is hypersurface orthogonal from $V_\mu dx^\mu = g_{vR} dR$ at the place. Therefore the hypersurface $R = 0$ is a Killing horizon. In the coordinate system $(v, R, \theta, \phi, \zeta)$, each component of the metric is analytic in the region of $R \geq 0$. Hence the space-time has no curvature singularity on and outside the black hole horizon.

The induced metric on the three-dimensional spatial cross section of the black hole horizon located at $R = 0$ with the time slice is obtained as

$$ds^2|_{R=0, v=\mathrm{const}} = \frac{LMn}{2}\left[d\Omega_{S^2}^2 + \left(\frac{d\psi}{n} + \cos\theta d\phi\right)^2\right] = 2LMn d\Omega_{S^3/\mathbb{Z}_n}^2, \tag{27}$$

where $d\Omega^2_{S^3/\mathbb{Z}_n}$ denotes the metric on the lens space $L(n;1) = S^3/\mathbb{Z}_n$ with a unit radius. In particular, in the case of $n = 1$, the shape of the horizon is a round S^3 in contrast to Gaiotto *et al.*'s supersymmetric black holes [5].

D. Ergoregions

Here we investigate the number and the structure of the ergoregions. As is discussed in Ref. [15], the space-time admits a considerable rich structure by two kinds of rotations of black holes and the background.

For the solution (12), the ergosurfaces are located at R satisfying the equation $f(R) := (1-\alpha^2)(R+M)^2 \times (R+N)^2 g_{\bar{t}\bar{t}} = 0$, where the explicit form of the function $f(R)$ is given by

$$f(R) = -(1-\alpha^2)R^4 + [3(M+N)\alpha^2 - 2N]R^3 + [3M\alpha^2(M+N) - N^2(1-\alpha^2)^2]R^2 + M^2(M+3N)\alpha^2 R + M^3 N\alpha^2. \tag{28}$$

Note that $f(0) = M^3N\alpha^2 > 0$ and $f(\infty) < 0$ for the regions of parameters (15). Hence there always exists an ergoregion around the black hole horizon. Furthermore, within the parameter region (M, N, α) satisfying the inequalities $f(R_1)f(R_2)f(R_3) < 0$ and $f'(R_+)f'(R_-) < 0$, which is shown in Fig. 1, there are two disconnected ergoregions, inner $(0 \le R \le R_{\rm II})$ and outer $(R_{\rm III} \le R \le R_{\rm IV})$ ergoregions, where R_i $(i = 1, 2, 3,\ R_1 < R_2 < R_3)$, $R_\pm$ $(R_- < R_+)$, and R_γ $(\gamma = {\rm I}, \ldots, {\rm IV},\ R_{\rm I} < R_{\rm II} < R_{\rm III} < R_{\rm IV})$ are three different positive roots of $f'(R) = 0$, two different positive roots of $f''(R) = 0$, and four different roots of $f(R) = 0$, respectively. The inner ergoregion is inside a sphere which contains the black hole horizon. The outer ergoregion has a shape of shell *ergoshell*, which is disjoint from the inner ergoregion. There exists a normal region between the inner and the outer ergoregions.

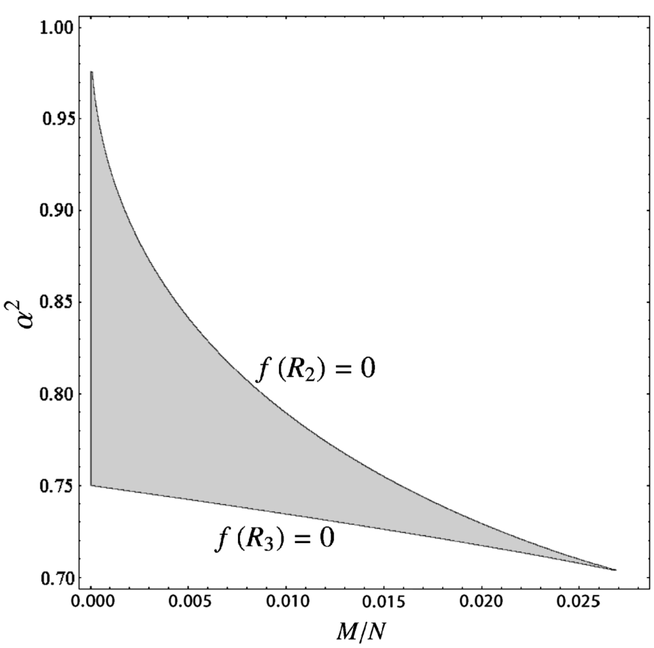

FIG. 1. This figure shows a region of parameters for the solution (12) with two disconnected ergoregions.

The angular velocities of the locally nonrotating observers are obtained as

$$\Omega_\phi = 0,$$
$$\Omega_{\tilde{\psi}} = -2\frac{\alpha^2NR^3 + M(R+N)(M^2+3MR+3R^2)}{L\sqrt{1-\alpha^2}[(R+M)^3(R+N) - \alpha^2R^4]}\alpha. \tag{29}$$

From these equations, two disconnected ergoregions of the solution (12) always rotate in the same direction in contrast to the squashed Kerr-Gödel black hole solutions [15], which two ergoregions can also rotate in opposite directions. The angular velocities of the horizon $R = 0$ are

$$\Omega_{{\rm H}\phi} = 0, \qquad \Omega_{{\rm H}\tilde{\psi}} = -\frac{2\alpha}{L\sqrt{1-\alpha^2}}. \tag{30}$$

IV. TWO ROTATING BLACK HOLES

For simplicity, we restrict ourselves to the two-black hole case, i.e., $M_i = N_i = 0$ $(i \ge 3)$. Without loss of generality, we can put the locations of two point sources as $\boldsymbol{R}_1 = (0, 0, d)$ and $\boldsymbol{R}_2 = (0, 0, -d)$, where the constant $2d$ denotes the separation between two black holes.

A. Metric

In this case, the metric is given by

$$ds^2 = -H^{-2}[dt + \alpha V(d\zeta + \boldsymbol{\omega})]^2 + H[V^{-1}(dR^2 + R^2 d\Omega^2_{S^2}) + V(d\zeta + \boldsymbol{\omega})^2], \tag{31}$$

where the functions H and V^{-1} and the 1-form $\boldsymbol{\omega}$ are

$$H = 1 + \frac{M_1}{\sqrt{R^2 - 2dR\cos\theta + d^2}} + \frac{M_2}{\sqrt{R^2 + 2dR\cos\theta + d^2}}, \tag{32}$$

$$V^{-1} = 1 + \frac{N_1}{\sqrt{R^2 - 2dR\cos\theta + d^2}} + \frac{N_2}{\sqrt{R^2 + 2dR\cos\theta + d^2}}, \tag{33}$$

$$\boldsymbol{\omega} = \left(N_1\frac{R\cos\theta - d}{\sqrt{R^2 - 2dR\cos\theta + d^2}} + N_2\frac{R\cos\theta + d}{\sqrt{R^2 + 2dR\cos\theta + d^2}}\right)d\phi. \tag{34}$$

The constants N_i $(i = 1, 2)$ are related to the size of the compactified radius L at infinity by

$$N_i = \frac{L}{2} n_i, \tag{35}$$

where n_i $(i = 1, 2)$ are the natural numbers.

B. Near horizon

The metric diverges at the locations of two point sources, i.e., $\boldsymbol{R} = \boldsymbol{R}_1$ and $\boldsymbol{R} = \boldsymbol{R}_2$. We make the coordinate transformation so that $\boldsymbol{R}_1 = 0$ and $\boldsymbol{R}_2 = (0, 0, -2d)$. Then the functions H and V^{-1} and the 1-form $\boldsymbol{\omega}$ in the metric and the gauge potential take the following forms:

$$H = 1 + \frac{M_1}{R} + \frac{M_2}{\sqrt{R^2 + 4dR\cos\theta + 4d^2}}, \tag{36}$$

$$V^{-1} = 1 + \frac{N_1}{R} + \frac{N_2}{\sqrt{R^2 + 4dR\cos\theta + 4d^2}}, \tag{37}$$

$$\boldsymbol{\omega} = \left(N_1 \cos\theta + N_2 \frac{R\cos\theta + 2d}{\sqrt{R^2 + 4dR\cos\theta + 4d^2}}\right) d\phi, \tag{38}$$

respectively.

In order to remove this apparent divergence at $\boldsymbol{R} = \boldsymbol{R}_1 = 0$, we introduce new coordinates (v, ζ') such that

$$dv = dt - \left[\frac{[3(2d + M_2)N_1 + M_1(2d + N_2)]\sqrt{M_1}}{4d\sqrt{N_1}R} + \frac{\sqrt{M_1^3 N_1}}{R^2}\right] dR, \tag{39}$$

$$d\zeta' = d\zeta + N_2 d\phi. \tag{40}$$

Then, near $R = 0$, the metric (31) behaves as

$$ds^2 \simeq -2\sqrt{\frac{N_1}{M_1}} dv dR + M_1 N_1 \left[\left(\frac{d\zeta'}{N_1} + \cos\theta d\phi\right)^2 + d\Omega_{S^2}^2\right] + \left[\frac{3N_1(2d + M_2)[N_1(2d + M_2) + 2M_1(2d + N_2)] - M_1^2(2d + N_2)^2}{16d^2 M_1 N_1} - \frac{3M_2 N_1 + M_1 N_2}{4d^2}\cos\theta\right] dR^2. \tag{41}$$

This metric well behaves at the null surface $R = 0$. In this case, $g_{vv} = 0$ and $(\partial_v)_\mu dx^\mu = g_{vR} dR$ also hold at $R = 0$. Therefore, the Killing vector field ∂_v becomes null and is hypersurface orthogonal at this place. So the hypersurface $R = 0$ is a Killing horizon. From the same discussion, the other point source $\boldsymbol{R} = \boldsymbol{R}_2$ also corresponds to a Killing horizon.

Note that $\partial_\zeta = \partial_{\zeta'}$. So the periodic coordinate ζ' has the same periodicity as ζ. Then the induced metric on the ith horizon $(i = 1, 2)$ is

$$ds^2|_{i\text{th horizon}} = \frac{LM_i n_i}{2}\left[\left(\frac{d\psi'}{n_i} + \cos\theta d\phi\right)^2 + d\Omega_{S^2}^2\right], \tag{42}$$

where $0 \le \psi' = 2\zeta'/L \le 4\pi$. Hence the horizon is topologically the lens space $L(n_i; 1) = S^2/\mathbb{Z}_{n_i}$.

C. Asymptotic structure

In the neighborhood of infinity, $R = \infty$, the harmonic functions H and V^{-1} behave as ones with a single point source, i.e.,

$$H \simeq 1 + \frac{\sum_i M_i}{R} + \mathcal{O}\left(\frac{1}{R^2}\right), \tag{43}$$

$$V^{-1} \simeq 1 + \frac{\sum_i N_i}{R} + \mathcal{O}\left(\frac{1}{R^2}\right). \tag{44}$$

Then the 1-form $\boldsymbol{\omega}$ is asymptotically

$$\boldsymbol{\omega} \simeq \left(\sum_i N_i\right)\cos\theta d\phi + \mathcal{O}\left(\frac{1}{R}\right). \tag{45}$$

Hence the metric asymptotically behaves as

$$ds^2 \simeq -d\tilde{t}^2 + dR^2 + R^2 d\Omega_{S^2}^2 + \frac{L^2}{4}(1 - \alpha^2)\left(\frac{d\bar{\psi}}{\sum_i n_i} + \cos\theta d\phi\right)^2, \tag{46}$$

where we introduced the following coordinates $(\tilde{t}, \bar{\psi})$:

$$\tilde{t} = \frac{t}{\sqrt{1 - \alpha^2}}, \qquad \bar{\psi} = \psi - \frac{\alpha t}{\sum_i N_i (1 - \alpha^2)}, \tag{47}$$

which are chosen so that they are in the rest frame at infinity. The asymptotic structure of the solution (31) is asymptotically locally flat; i.e., the metric asymptotes to a twisted constant S^1 fiber bundle over the four-dimensional Minkowski space-time, and the spatial infinity has the structure of an S^1 bundle over an S^2 such that it is the lens space $L(n; 1) = S^3/\mathbb{Z}_n$, where $n = \sum_i n_i$ is the natural number.

From the asymptotic behavior of the metric, we can obtain the Komar mass, the charge, and the Komar angular momenta at spatial infinity as

$$M = \frac{3L[(\alpha^2 + 2)\sum_i M_i + \alpha^2 \sum_i N_i]}{8\pi\sqrt{1 - \alpha^2} G_5} \frac{\mathcal{A}_{S^3}}{n}, \tag{48}$$

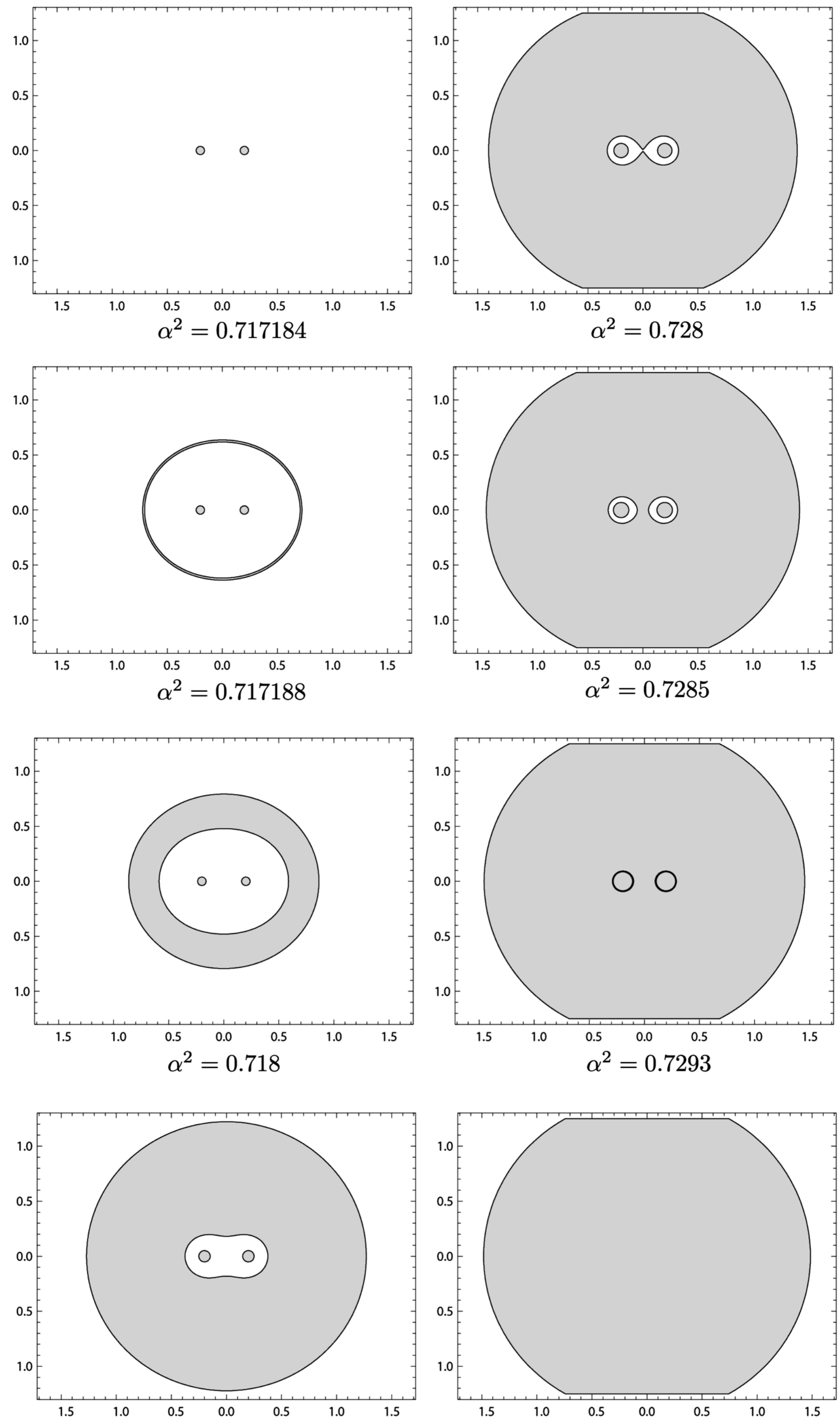

FIG. 2. Ergoregions (shaded regions) in varying α^2. $d = 0.2$ and $\alpha^2 = 0.717\,184 \sim 0.73$.

$$Q = \frac{\sqrt{3}L\sum_i M_i}{2\pi G_5}\frac{\mathcal{A}_{S^3}}{n}, \tag{49}$$

$$J_\phi = 0, \tag{50}$$

$$J_{\tilde{\psi}} = -\frac{\alpha L^2(3\sum_i M_i + \alpha^2 \sum_i N_i)}{8\pi G_5}\frac{\mathcal{A}_{S^3}}{n}. \tag{51}$$

D. Ergoregions

For simplicity, assume that two black holes have equal mass and the horizon topology of S^3, which correspond to the choice of the parameters $M_1 = M_2$ and $N_1 = N_2 = L/2$. The ergosurfaces are located at R satisfying the equation

$$-H^{-2}\left(\sqrt{1-\alpha^2} + \frac{\alpha^2}{\sqrt{1-\alpha^2}}V\right)^2 + \frac{\alpha^2}{1-\alpha^2}HV = 0. \tag{52}$$

Introduce the coordinates (x, y) defined by $x = R\cos\theta$ and $y = R\sin\theta$. Figures 2 and 3 show how the ergoregions change the shapes in a (x, y) plane as the rotation parameter α varies with the other parameters fixed in the cases of the separation parameter $d = 0.2$ and $d = 1$, respectively. Here the horizontal axis and the vertical axis denote the x axis and the y axis, respectively. In these figures, the shaded regions denote the ergoregions, i.e., the regions such that the left-hand side of Eq. (52) is positive. Two black holes are located at $(\pm d, 0)$ in this plane.

First, see Fig. 2—the case of $d = 0.2$. There exists an ergoregion around each rotating black hole when α is small. When $\alpha^2 \simeq 0.717\,188$, a new ergoregion in the

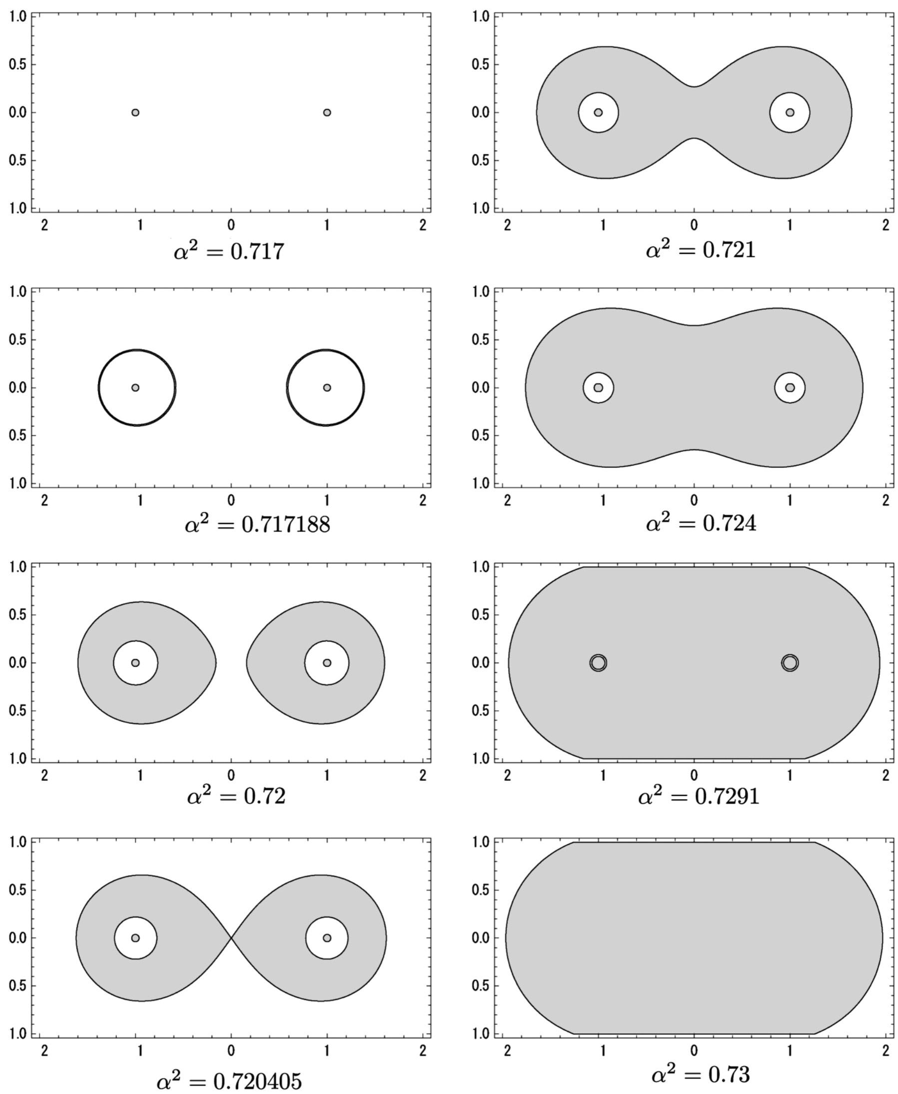

FIG. 3. Ergoregions in varying α^2. $d = 1$ and $\alpha^2 = 0.717 \sim 0.73$.

shape of a shell enclosing two black holes appears far away from them. There is an inner normal region between the ergoshell and the two inner ergoregions. As the value of α gets larger, the outer ergoshell becomes thick. When $\alpha^2 \simeq 0.728$, the inner normal region becomes disconnected. When $\alpha^2 \simeq 0.7293$, the outer ergoshell and the two inner ergoregions merge, and the inner normal region disappears.

See also Fig. 3—the case of $d = 1$. In this case, when $\alpha^2 \simeq 0.717\,188$, in contrast to the previous case, two disconnected outer ergoshells appear around each black hole. When $\alpha^2 \simeq 0.720\,405$ they merge together.

Next, Fig. 4 shows how the shapes of ergoregions change with varying the separation parameter d, where the other parameters are kept unchanged. When the separation is large enough, there exist two disconnected outer ergoshells and two inner ergoregions around each black hole. When two black holes become closer, two outer ergoshells are connected with each other into a single large outer ergoshell. There are two inner normal regions around each black hole. When the separation becomes smaller, the inner normal regions join together. Finally, two inner ergoregions around two black holes also join together.

V. SUMMARY AND DISCUSSION

We have considered the limiting case given by (17) in the Kaluza-Klein-Kerr-Newman-Gödel black hole solutions [15] and have presented supersymmetric Kaluza-

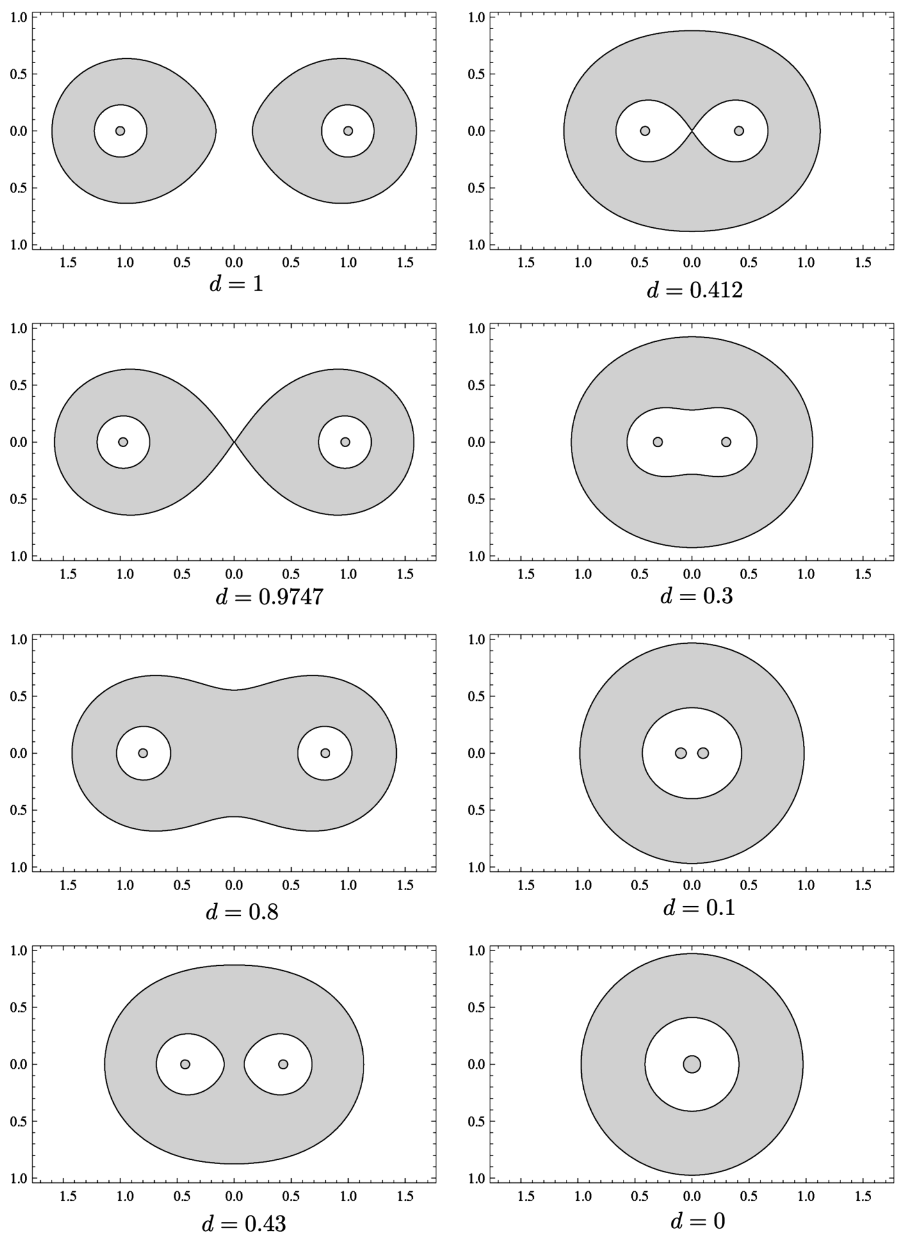

FIG. 4. Ergoregions in varying d. $d = 1 - 0$ and $\alpha^2 = 0.72$.

Klein-Kerr-Newman-Gödel multi-black hole solutions, in the five-dimensional Einstein-Maxwell theory with a Chern-Simons term. The new solutions have no closed timelike curve everywhere outside the black hole horizons. At infinity, the metric asymptotically approaches a twisted S^1 bundle over the four-dimensional Minkowski space-time.

Though the Kerr parameter and the Gödel parameter are related, each black hole can have an inner ergoregion and an outer ergoshell depending on the parameter. We have explicitly presented the various shapes of ergoregions in the case of two black holes.

The solutions (4)–(10) can be easily generalized to solutions with a positive cosmological constant $\Lambda > 0$. In this solution, the harmonic function (6) is replaced by

$$H = \lambda t + \sum_i \frac{M_i}{|\boldsymbol{R} - \boldsymbol{R}_i|}, \tag{53}$$

where the constant λ is related to the cosmological constant by $\lambda = \pm 2\sqrt{\Lambda/3}$. In particular, in the case with $\beta = -1$ and $\epsilon = 0$, the solutions coincide with five-dimensional Kastor-Traschen solutions ($\alpha = 0$) [19] or Klemm-Sabra solutions ($\alpha \neq 0$) [20], which describes the coalescences of black holes with the horizon topologies of S^3 into a single black hole with the horizon topology of S^3. In the case of two black holes with $M_i \neq 0$, $N_i \neq 0$ ($i = 1, 2$), $M_j = N_j = 0$ ($j \geq 3$), $\beta = -1$, and $\epsilon = 0$, the solution describes the coalescence of two rotating black holes with the horizon topologies of S^3 into a single rotating black hole with the horizon topology of the lens space $L(2; 1) = S^3/\mathbb{Z}_2$ [21]. Cosmological nonrotating multi-black hole solutions on the multicentered-Taub-NUT space are obtained by setting $\alpha = 0$, $\beta = -1$, and $\epsilon = 1$ [22]. This solution is not static even in a single-black hole case. In the case of $\beta = 1$, the solution with the harmonic function (53) would also describe the coalescence of black holes. We leave the analysis for the future.

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APPENDIX A: PARAMETER REGION

Here we show that the inequality $0 \leq \alpha^2 < 1$ under $M_i > 0$ and $N_i > 0$ is the necessary and sufficient condition for the absence of closed timelike curves outside the horizons.

Assume that all point sources are located at the z axis, i.e., $\theta = 0, \pi$ on the three-dimensional Euclid space in the Gibbons-Hawking space. In this case, the 1-form $\boldsymbol{\omega}$ is proportional to $d\phi$. The condition of the absence of closed timelike curves outside the horizons is equivalent to the condition that the two-dimensional (ϕ, ζ) part of the metric

$$ds^2|_{(\phi,\zeta)} = A(d\zeta + \boldsymbol{\omega})^2 + Bd\phi^2 \tag{A1}$$

is positive-definite, where

$$A = HV - \alpha^2 H^{-2} V^2, \tag{A2}$$

$$B = HV^{-1} R^2 \sin^2\theta. \tag{A3}$$

This metric is positive-definite if and only if the following two-dimensional matrix is positive-definite:

$$M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \tag{A4}$$

Therefore, noting that $B > 0$, we obtain the condition

$$M > 0 \Leftrightarrow A > 0. \tag{A5}$$

As a result, it is enough to prove that $A > 0$. At infinity $R \to \infty$, the function behaves as

$$A \simeq 1 - \alpha^2 + \mathcal{O}\left(\frac{1}{R}\right). \tag{A6}$$

Hence $0 \leq \alpha^2 < 1$ is necessary. Next we show that if $0 \leq \alpha^2 < 1$ is satisfied with $M_i > 0$ and $N_i > 0$, there is no closed timelike curve outside the horizons. Noting that $1 < H < \infty$ and $1 < V^{-1} < \infty$ under the conditions $M_i > 0$ and $N_i > 0$, the inequality

$$A = HV(1 - \alpha^2 H^{-3} V) > HV(1 - \alpha^2) > 0 \tag{A7}$$

holds everywhere outside the horizons.

APPENDIX B: SQUASHED KERR-NEWMAN-GÖDEL BLACK HOLES

The metric and the gauge potential of the squashed Kerr-Newman-Gödel black hole solution [15] is given by

$$\begin{aligned} ds^2 = {}& -f(r)dT^2 - 2g(r)dT(d\psi + \cos\theta d\phi) \\ & + h(r)(d\psi + \cos\theta d\phi)^2 + \frac{k^2(r)}{V(r)}dr^2 \\ & + \frac{r^2}{4}[k(r)d\Omega_{S^2}^2 + (d\psi + \cos\theta d\phi)^2], \end{aligned} \tag{B1}$$

and

$$\boldsymbol{A} = \frac{\sqrt{3}}{2}\left[\frac{q}{r^2}dT + \left(jr^2 + 2jq - \frac{qa}{2r^2}\right)(d\psi + \cos\theta d\phi)\right], \tag{B2}$$

respectively, where the metric functions are

$$f(r) = 1 - \frac{2m}{r^2} + \frac{q^2}{r^4}, \tag{B3}$$

$$g(r) = jr^2 + 3jq + \frac{(2m-q)a}{2r^2} - \frac{q^2 a}{2r^4}, \tag{B4}$$

$$h(r) = -j^2 r^2(r^2 + 2m + 6q) + 3jqa + \frac{(m-q)a^2}{2r^2} - \frac{q^2 a^2}{4r^4}, \tag{B5}$$

$$V(r) = 1 - \frac{2m}{r^2} + \frac{8j(m+q)[a + 2j(m+2q)]}{r^2} + \frac{2(m-q)a^2 + q^2[1 - 16ja - 8j^2(m+3q)]}{r^4}, \tag{B6}$$

$$k(r) = \frac{V(r_\infty)r_\infty^4}{(r^2 - r_\infty^2)^2}. \tag{B7}$$

In the limit of $r_\infty \to \infty$, i.e., $k(r) \to 1$, the solution coincides with the Kerr-Newman-Gödel black hole solution in Ref. [23].

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