

**Dynamic vacuum variable and equilibrium approach in cosmology**

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A modified-gravity theory is considered with a four-form field strength  $F$ , a variable gravitational coupling parameter  $G(F)$ , and a standard matter action. This theory provides a concrete realization of the general vacuum variable  $q$  as the four-form amplitude  $F$  and allows for a study of its dynamics. The theory gives a flat Friedmann-Robertson-Walker universe with rapid oscillations of the effective vacuum energy density (cosmological “constant”), whose amplitude drops to zero asymptotically. Extrapolating to the present age of the Universe, the order of magnitude of the average vacuum energy density agrees with the observed near-critical vacuum energy density of the present universe. It may even be that this type of oscillating vacuum energy density constitutes a significant part of the so-called cold dark matter in the standard Friedmann-Robertson-Walker framework.

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**I. INTRODUCTION**

In a previous article [1], we proposed to characterize a Lorentz-invariant quantum vacuum by a nonzero conserved relativistic “charge”  $q$ . This approach allowed us to discuss the *thermodynamics* of the quantum vacuum, in particular, thermodynamic properties as stability and compressibility. We found that the vacuum energy density appears in two guises.

The *microscopic* vacuum energy density is characterized by an ultraviolet energy scale,  $\epsilon(q) \sim E_{UV}^4$ . For definiteness, we will take this energy scale  $E_{UV}$  to be close to the Planck energy scale  $E_{\text{Planck}} \equiv \sqrt{\hbar c^5/G_N} \approx 1.22 \times 10^{19}$  GeV. The *macroscopic* vacuum energy density is, however, determined by a particular thermodynamic quantity  $\tilde{\epsilon}_{\text{vac}}(q) \equiv \epsilon - qd\epsilon/dq$ , and it is this energy density that contributes to the effective gravitational field equations at low energies. For a self-sustained vacuum in full thermodynamic equilibrium and in the absence of matter, the effective (coarse-grained) vacuum energy density  $\tilde{\epsilon}_{\text{vac}}(q)$  is automatically nullified (without fine-tuning) by the spontaneous adjustment of the vacuum variable  $q$  to its equilibrium value  $q_0$ , so that  $\tilde{\epsilon}_{\text{vac}}(q_0) = 0$ . This implies that the effective cosmological constant  $\Lambda$  of a perfect quantum vacuum is strictly zero, which is consistent with the requirement of Lorentz invariance.

The presence of thermal matter makes the vacuum state Lorentz noninvariant and leads to a readjustment of the variable  $q$  to a new equilibrium value,  $q'_0 = q_0 + \delta q$ , which shifts the effective vacuum energy density away from zero,  $\tilde{\epsilon}_{\text{vac}}(q_0 + \delta q) \neq 0$ . The same happens with

other types of perturbations that violate Lorentz invariance, such as the existence of a spacetime boundary or an interface. According to this approach, the present value of  $\tilde{\epsilon}_{\text{vac}}$  is nonzero but small because the Universe is close to equilibrium and Lorentz-noninvariant perturbations of the quantum vacuum are small (compared with the ultraviolet scale, which sets the microscopic energy density  $\epsilon$ ).

The situation is different for Lorentz-invariant perturbations of the vacuum, such as the formation of scalar condensates as discussed in Ref. [1] or quark/gluon condensates derived from quantum chromodynamics (cf. Ref. [2]). In this case, the variable  $q$  shifts in such a way that it completely compensates the energy density of the perturbation and the effective cosmological constant is again zero in the new Lorentz-invariant equilibrium vacuum.

The possible origin of the conserved vacuum charge  $q$  in the perfect Lorentz-invariant quantum vacuum was discussed in Ref. [1] in general terms. But a specific example was also given in terms of a four-form field strength  $F$  [3–8]. Here, we use this explicit realization with a four-form field  $F$  to study the *dynamics* of the vacuum energy, which describes the relaxation of the vacuum energy density  $\tilde{\epsilon}_{\text{vac}}$  (effective cosmological “constant”) from its natural Planck-scale value at early times to a naturally small value at late times. In short, the present cosmological constant is small because the Universe happens to be old.<sup>1</sup>

The results of the present article show that, for the type of theory considered, the decay of  $\tilde{\epsilon}_{\text{vac}}$  is accompanied by

<sup>1</sup>An extensive but nonexhaustive list of references to research papers and reviews on the so-called “cosmological constant problem(s)” can be found in Ref. [1]. A recent review on cosmic “dark energy” is given in Ref. [9].

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rapid oscillations of the vacuum variable  $F$  and that the relaxation of  $\tilde{\epsilon}_{\text{vac}}$  mimics the behavior of cold dark matter (CDM) in a standard Friedmann-Robertson-Walker (FRW) universe. This suggests that part of the inferred CDM may come from dynamic vacuum energy density and may also give a clue to the solution of the so-called coincidence problem [9], namely, why the approximately constant vacuum energy density is precisely now of the same order as the time-dependent CDM energy density.

These results are obtained by the following steps. In Sec. II, a modified-gravity theory with a four-form field  $F$  is defined in terms of general functions for the microscopic energy density  $\epsilon(F)$  and variable gravitational coupling parameter  $G(F)$ . In Sec. III, the dynamics of the corresponding de Sitter universe without matter is discussed and, in Sec. IV, the dynamics of a flat FRW universe with matter, using simple *Ansätze* for the functions  $\epsilon(F)$  and  $G(F)$ . In Sec. V, the approach to equilibrium in such a FRW universe is studied in detail and the above-mentioned vacuum oscillations are established. In Sec. VI, the main results are summarized.

## II. GRAVITY WITH F FIELD AND VARIABLE GRAVITATIONAL COUPLING

Here, and in the following, the vacuum variable  $q$  is represented by a four-form field  $F$ .<sup>2</sup> The corresponding action is given by a generalization of the action in which only a quadratic function of  $F$  is used (see, e.g., Refs. [3–8]). Such a quadratic function gives rise to a gas-like vacuum [1]. But a gas-like vacuum cannot exist in equilibrium without external pressure, as the equilibrium vacuum charge vanishes,  $q_0 = 0$ . A self-sustained vacuum requires a more complicated function  $\epsilon(F)$  in the action, so that the equilibrium at zero external pressure occurs for  $q_0 \neq 0$ . An example of an appropriate function  $\epsilon(F)$  will be given in Sec. IV B.

The action is chosen as in Ref. [1], but with one important modification: Newton's constant  $G_N$  is replaced by a gravitational coupling parameter  $G$ , which is taken to depend on the state of the vacuum and thus on the vacuum variable  $F$ . Such a  $G(F)$  dependence is natural and must, in principle, occur in the quantum vacuum. Moreover, a  $G(F)$  dependence allows the cosmological “constant” to change with time, which is otherwise prohibited by the Bianchi identities and energy-momentum conservation [10,11].

Specifically, the action considered takes the following form ( $\hbar = c = 1$ ):

<sup>2</sup>To clarify our notation, a four-form field has components  $F_{\kappa\lambda\mu\nu}(x)$ , which can always be written as  $e_{\kappa\lambda\mu\nu}\sqrt{|g(x)|}F(x)$ , in terms of the constant Levi-Civita symbol  $e_{\kappa\lambda\mu\nu}$ , the determinant of the metric  $g(x) \equiv \det g_{\mu\nu}(x)$ , and a real scalar field  $F(x)$ . Hence, we can simply write  $F$  if we speak about the four-form field. However, this scalar field  $F(x)$  is not fundamental, as will become clear later.

$$S[A, g, \psi] = - \int_{\mathbb{R}^4} d^4x \sqrt{|g|} \left( \frac{R}{16\pi G(F)} + \epsilon(F) + \mathcal{L}^M(\psi) \right), \quad (2.1a)$$

$$F^2 \equiv - \frac{1}{24} F_{\kappa\lambda\mu\nu} F^{\kappa\lambda\mu\nu}, \quad F_{\kappa\lambda\mu\nu} \equiv \nabla_{[\kappa} A_{\lambda\mu\nu]}, \quad (2.1b)$$

$$F_{\kappa\lambda\mu\nu} = F e_{\kappa\lambda\mu\nu} \sqrt{|g|} \quad F^{\kappa\lambda\mu\nu} = F e^{\kappa\lambda\mu\nu} / \sqrt{|g|}, \quad (2.1c)$$

where  $\nabla_\mu$  denotes a covariant derivative and a square bracket around spacetime indices complete antisymmetrization. The functional dependence on  $g$  has been kept implicit on the right-hand side of (2.1a) showing only the dependence on  $F = F(A, g)$  and  $\psi$ . The field  $\psi$  in (2.1a) stands, in fact, for a generic low-energy matter field with a scalar Lagrange density  $\mathcal{L}^M(\psi)$ , which is assumed to be without  $F$ -field dependence (this assumption can be relaxed later by changing the low-energy constants in  $\mathcal{L}^M$  to  $F$ -dependent parameters). It is also assumed that a possible constant term  $\Lambda^M$  in  $\mathcal{L}^M(\psi)$  has been absorbed in  $\epsilon(F)$ , so that, in the end,  $\mathcal{L}^M(\psi)$  contains only  $\psi$ -dependent terms. In this section, the low-energy fields are indicated by lower-case letters, namely,  $g_{\mu\nu}(x)$  and  $\psi(x)$ , whereas the fields originating from the microscopic theory are indicated by upper-case letters, namely,  $A(x)$  and  $F(x)$  [later also  $\Phi(x)$ ]. Throughout, we use the conventions of Ref. [10], in particular, those for the Riemann tensor and the metric signature  $(-+++)$ .

The variation of the action (2.1a) over the three-form gauge field  $A$  gives the generalized Maxwell equation

$$\nabla_\nu \left( \sqrt{|g|} \frac{F^{\kappa\lambda\mu\nu}}{F} \left( \frac{d\epsilon(F)}{dF} + \frac{R}{16\pi} \frac{dG^{-1}(F)}{dF} \right) \right) = 0, \quad (2.2)$$

and the variation over the metric  $g_{\mu\nu}$  gives the generalized Einstein equation

$$\begin{aligned} \frac{1}{8\pi G(F)} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{16\pi} F \frac{dG^{-1}(F)}{dF} R g_{\mu\nu} \\ + \frac{1}{8\pi} (\nabla_\mu \nabla_\nu G^{-1}(F) - g_{\mu\nu} \square G^{-1}(F)) \\ - \tilde{\epsilon}(F) g_{\mu\nu} + T_{\mu\nu}^M = 0, \end{aligned} \quad (2.3)$$

where  $\square$  is the invariant d'Alembertian,  $T_{\mu\nu}^M$  the energy-momentum tensor of the matter field  $\psi$ , and  $\tilde{\epsilon}$  the effective vacuum energy density

$$\tilde{\epsilon}(F) \equiv \epsilon(F) - F \frac{d\epsilon(F)}{dF}, \quad (2.4)$$

whose precise form has been argued on thermodynamic grounds in Ref. [1].

At this point two remarks may be helpful. First, observe that the action (2.1a) is not quite the one of Brans-Dicke theory [10,12], as the argument of  $G(F)$  is not a funda-

mental scalar field but involves the inverse metric [needed to change the covariant tensor  $F_{\kappa\lambda\mu\nu}$  into a contravariant tensor  $F^{\kappa\lambda\mu\nu}$  for the definition of  $F \equiv \sqrt{F^2}$  according to (2.1b)]. This implicit metric dependence of  $G(F)$  explains the origin of the second term on the left-hand side of (2.3). Second, observe that the three-form gauge field  $A$  does not propagate physical degrees of freedom in flat spacetime [3,8]. Still,  $A$  has gravitational effects, both classically in the modified-gravity theory with  $G = G(F)$  as discussed in the present article (see, in particular, Sec. V E) and quantum mechanically already in the standard gravity theory with  $G = G_N$  (giving, for example, a nonvanishing gravitational trace anomaly [3]).

Using (2.1c) for  $F^{\kappa\lambda\mu\nu}$ , we obtain the Maxwell Eq. (2.2) in the form

$$\partial_\nu \left( \frac{d\epsilon(F)}{dF} + \frac{R}{16\pi} \frac{dG^{-1}(F)}{dF} \right) = 0. \quad (2.5)$$

The solution is simply

$$\frac{d\epsilon(F)}{dF} + \frac{R}{16\pi} \frac{dG^{-1}(F)}{dF} = \mu, \quad (2.6)$$

with an integration constant  $\mu$ . Hence, the constant  $\mu$  is seen to emerge dynamically. In a thermodynamic equilibrium state, this constant becomes a genuine chemical potential corresponding to the conservation law obeyed by the vacuum ‘‘charge’’  $q \equiv F$ . Indeed, the integration constant  $\mu$  is, according to (2.6), thermodynamically conjugate to  $F$  in an equilibrium state with vanishing Ricci scalar  $R$ .

Eliminating  $dG^{-1}/dF$  from (2.3) by use of (2.6), the generalized Einstein equation becomes

$$\frac{1}{8\pi G(F)} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{8\pi} (\nabla_\mu \nabla_\nu G^{-1}(F) - g_{\mu\nu} \square G^{-1}(F)) - (\epsilon(F) - \mu F) g_{\mu\nu} + T_{\mu\nu}^M = 0, \quad (2.7)$$

which will be used in the rest of this article, together with (2.6).

Equations (2.6) and (2.7) can also be obtained if we use, instead of the original action, an effective action in terms of a Brans-Dicke-type scalar field  $\Phi(x)$  with mass dimension 2, setting  $\Phi(x) \rightarrow F(x)$  afterwards. Specifically, this effective action is given by

$$S_{\text{eff}}[\Phi, \mu, g, \psi] = - \int_{\mathbb{R}^4} d^4x \sqrt{|g|} \left( \frac{R}{16\pi G(\Phi)} + (\epsilon(\Phi) - \mu\Phi) + \mathcal{L}^M(\psi) \right). \quad (2.8)$$

The potential term in (2.8) contains, different from a conventional Brans-Dicke potential  $V(\Phi)$ , a linear term  $-\mu\Phi$  for a constant  $\mu$  of mass dimension 2. This linear term reflects the fact that our effective scalar field  $\Phi$  is not an arbitrary field but should be a conserved quantity, for

which the constant parameter  $\mu$  plays the role of a chemical potential that is thermodynamically conjugate to  $\Phi$ .

Indeed, if  $\Phi$  in (2.8) is replaced by a four-form field  $F$  given in terms of the three-form potential  $A$ , the resulting  $\mu F$  term in the effective action does not contribute to the equations of motion (2.2), because it is a total derivative

$$\int_{\mathbb{R}^4} d^4x \sqrt{|g|} \mu F = -\frac{\mu}{24} e^{\kappa\lambda\mu\nu} \int_{\mathbb{R}^4} d^4x F_{\kappa\lambda\mu\nu} \quad (2.9)$$

where the constant  $\mu$  plays the role of a Lagrange multiplier related to the conservation of vacuum ‘‘charge’’  $F$  (see also the discussion in Refs. [4,6], where  $\mu$  is compared with the  $\theta$  parameter of quantum chromodynamics).

Instead of the large microscopic energy density  $\epsilon(F)$  in the original action (2.1a), the potentially smaller macroscopic vacuum energy density  $\rho_V \equiv \epsilon(F) - \mu F$  enters the effective action (2.8). Precisely this macroscopic vacuum energy density gravitates and determines the cosmological term in the gravitational field Eq. (2.7).

Equations (2.6) and (2.7) are universal: they do not depend on the particular origin of the vacuum field  $F$ . The  $F$  field can be replaced by any conserved variable  $q$ , as discussed in Ref. [1]. Observe that, for thermodynamics, the parameter  $\mu$  is the quantity that is thermodynamically conjugate to  $q$  and that, for dynamics,  $\mu$  plays the role of a Lagrange multiplier. The functions  $\epsilon(q)$  and  $G(q)$  can be considered to be phenomenological parameters in an effective low-energy theory (see also the general discussion in the Appendix of Ref. [13]).

Before we turn to the cosmological solutions of our particular  $F$  theory (2.1), it may be useful to mention the connection with so-called  $f(R)$  models, which have recently received considerable attention (see, e.g., Refs. [14,15] and references therein). The latter are purely phenomenological models, in which the linear function of the Ricci scalar  $R$  from the Einstein-Hilbert action term is replaced by a more general function  $f(R)$ . This function  $f(R)$  can, in principle, be adjusted to fit the astronomical observations and to produce a viable cosmological model. Returning to our  $F$  theory, we can express  $F$  in terms of  $R$  by use of (2.6) and substitute the resulting expression  $F(R)$  into (2.7). This gives an equation for the metric field, which is identical to the one of  $f(R)$  cosmology. (The latter result is not altogether surprising as the metric  $F(R)$  model is known to be equivalent to a Brans-Dicke model without kinetic term [15], and the same holds for our effective action (2.8) at the classical level.) In this way, the  $F$  theory introduced in this section (or, more generally,  $q$  theory as mentioned in the previous paragraph) may give a microscopic justification for the phenomenological  $f(R)$  models used in theoretical cosmology and may allow for a choice between different classes of model functions  $f(R)$  based on fundamental physics.

### III. DE SITTER EXPANSION

Let us, first, consider *stationary* solutions of the generalized Maxwell-Einstein equations from the effective action (2.8). At this moment, we are primarily interested in the class of spatially flat, homogeneous, and isotropic universes. In this class, only the matter-free de Sitter universe is stationary.

The de Sitter universe is characterized by a time-independent Hubble parameter  $H$  (that is, a genuine Hubble constant  $H$ ), which allows us to regard this universe as a thermodynamic equilibrium system. Using

$$R_{\mu\nu} = \frac{1}{4}g_{\mu\nu}R, \quad R = -12H^2, \quad (3.1)$$

we get from (2.6) and (2.7) two equations for the constants  $F$  and  $H$ :

$$\left(\frac{d\epsilon(F)}{dF} - \mu\right) = \frac{3H^2}{4\pi} \frac{dG^{-1}(F)}{dF}, \quad (3.2a)$$

$$(\epsilon(F) - \mu F) = \frac{3H^2}{8\pi} G^{-1}(F), \quad (3.2b)$$

with  $\mu$  considered given.

Eliminating the chemical potential  $\mu$  from the above equations, we find the following equation for  $F$ :

$$\begin{aligned} \tilde{\epsilon}(F) &\equiv \epsilon(F) - F \frac{d\epsilon(F)}{dF} \\ &= \frac{3H^2}{8\pi} \left( G^{-1}(F) - 2F \frac{dG^{-1}(F)}{dF} \right), \end{aligned} \quad (3.3)$$

where the functions  $\epsilon(F)$  and  $G^{-1}(F)$  are assumed to be known.

The perfect quantum vacuum corresponds to  $H = 0$  and describes Minkowski spacetime. The corresponding equilibrium values  $F = F_0$  and  $\mu = \mu_0$  in the perfect quantum vacuum are determined from the following equations:

$$\epsilon(F_0) - F \frac{d\epsilon(F)}{dF} \Big|_{F=F_0} = 0, \quad \mu_0 = \frac{d\epsilon(F)}{dF} \Big|_{F=F_0}, \quad (3.4)$$

which are obtained from (2.6) and (2.7) by recalling that the perfect quantum vacuum is the equilibrium vacuum in the absence of matter and gravity fields ( $T_{\mu\nu}^M = R = 0$ ).

If  $H$  is nonzero but small compared with the Planck energy scale, the  $H^2$  term on the right-hand side of (3.3) can be considered as a perturbation. Then, the correction  $\delta F = F - F_0$  due to the expansion is given by

$$\frac{\delta F}{F_0} = -\frac{3}{8\pi} \chi(F_0) H^2 \left( G^{-1}(F_0) - 2F_0 \frac{dG^{-1}(F)}{dF} \Big|_{F=F_0} \right), \quad (3.5)$$

where  $\chi(F_0)$  is the vacuum compressibility introduced in Ref. [1],

$$\chi(F_0) \equiv \left( F^2 \frac{d^2 \epsilon(F)}{dF^2} \Big|_{F=F_0} \right)^{-1}. \quad (3.6)$$

Equally, the chemical potential is modified by the expansion ( $H \neq 0$ )

$$\mu = \mu_0 + \delta\mu = \frac{d\epsilon(F)}{dF} \Big|_{F=F_0} - \frac{3H^2}{8\pi G(F_0)F_0}. \quad (3.7)$$

But, instead of fixing  $H$ , it is also possible to fix the integration constant  $\mu$ . From (3.2), we then obtain the other parameters as functions of  $\mu$ :  $H(\mu)$ ,  $F(\mu)$ , and  $\rho_V(\mu) \equiv \epsilon(\mu) - \mu F(\mu)$ . The cosmological constant  $\Lambda(\mu) \equiv \rho_V(\mu)$  is zero for  $\mu = \mu_0$ , which corresponds to thermodynamic equilibrium in the absence of external pressure and expansion [ $P_{\text{external}} = P_{\text{vac}}(\mu_0) = -\Lambda(\mu_0) = 0$ ]. From now on, the physical situation considered will be the one determined by having a fixed chemical potential  $\mu$ .

The de Sitter universe is of interest because it is an equilibrium system and, therefore, may serve as the final state of a dynamic universe with matter included (see Sec. V).

### IV. DYNAMICS OF A FLAT FRW UNIVERSE

#### A. General equations

The discussion of this section and the next is restricted to a spatially flat FRW universe, because of two reasons. The first reason is that flatness is indicated by the data from observational cosmology (cf. Refs. [9,16–20] and references therein). The second reason is that flatness is a natural property of the quantum vacuum in an emergent gravity theory (cf. Ref. [1] and references therein). In addition, the matter energy-momentum tensor for the model universe is taken as that of a perfect fluid characterized by the energy density  $\rho_M$  and isotropic pressure  $P_M$ . As mentioned in the previous section, the physics of the  $F$  field is considered to be specified by a fixed chemical potential  $\mu$ .

For a spatially flat ( $k = 0$ ) FRW universe [10] with expansion factor  $a(t)$ , the homogenous matter has, in general, a time-dependent energy density  $\rho_M(t)$  and pressure  $P_M(t)$ . Equally, the scalar field entering the four-form field-strength tensor (2.1c) is taken to be homogenous and time dependent,  $F_{\kappa\lambda\mu\nu} = F(t)|a(t)|^3 e_{\kappa\lambda\mu\nu}$ .

With a time-dependent Hubble parameter  $H(t) \equiv (da/dt)/a$ , we then have from the reduced Maxwell Eq. (2.6)

$$\frac{3}{8\pi} \frac{dG^{-1}}{dF} \left( \frac{dH}{dt} + 2H^2 \right) = \frac{d\epsilon}{dF} - \mu, \quad (4.1)$$

and from the Einstein Eq. (2.7)



$$H^2 = \frac{8\pi}{3} G \rho_{\text{tot}} - HG \frac{dG^{-1}}{dt}, \quad (4.2a)$$

$$2 \frac{dH}{dt} + 3H^2 = -8\pi G P_{\text{tot}} - 2HG \frac{dG^{-1}}{dt} - G \frac{d^2 G^{-1}}{dt^2}, \quad (4.2b)$$

with total energy density and pressure

$$\rho_{\text{tot}} \equiv \rho_{\text{V}} + \rho_{\text{M}}, \quad P_{\text{tot}} \equiv P_{\text{V}} + P_{\text{M}}, \quad (4.3)$$

for the effective vacuum energy density

$$\rho_{\text{V}}(F) = -P_{\text{V}}(F) = \epsilon(F) - \mu F. \quad (4.4)$$

With definition (4.4), the reduced Maxwell Eq. (4.1) can be written as

$$\dot{\rho}_{\text{V}} = \frac{3}{8\pi} \frac{dG^{-1}}{dt} (\dot{H} + 2H^2), \quad (4.5)$$

where the overdot stands for differentiation with respect to cosmic time  $t$ . The above equations give automatically energy conservation of matter

$$\dot{\rho}_{\text{M}} + 3H(P_{\text{M}} + \rho_{\text{M}}) = 0, \quad (4.6)$$

as should be the case for a standard matter field  $\psi$  (recall that  $\nabla^\mu T_{\mu\nu}^{\text{M}} = 0$  follows from the invariance of  $S^{\text{M}}[g_{\mu\nu}, \psi]$  under general coordinate transformations; cf. Appendix E of Ref. [11]).

### B. Model for $\epsilon(F)$

The equations of Sec. IVA allow us to study the development of the Universe from very small (near-Planckian) time scales to macroscopic time scales. Because the results do not depend very much on the details of the functions  $\epsilon(F)$  and  $G(F)$ , it is possible to choose the simplest functions for an exploratory investigation. The only requirements are that the vacuum is self-sustained [i.e., (3.4) has a solution with nonzero  $F_0$ ] and that the vacuum is stable [i.e., the vacuum compressibility (3.6) is positive,  $\chi(F_0) > 0$ ].

A simple choice for the function  $\epsilon(F)$  is

$$\epsilon(F) = \frac{1}{2\chi} \left( -\frac{F^2}{F_0^2} + \frac{F^4}{3F_0^4} \right), \quad (4.7)$$

where  $\chi > 0$  is a constant parameter (vacuum compressibility) and  $F_0$  the value of  $F$  in a particular equilibrium vacuum satisfying (3.4). The equilibrium value of the chemical potential  $\mu$  in the perfect vacuum is then given by

$$\mu_0 = -\frac{1}{3\chi F_0}. \quad (4.8)$$

The microscopic parameters  $F_0$  and  $\chi$  are presumably determined by the Planck energy scale,  $|F_0| \sim E_{\text{Planck}}^2$  and  $\chi \sim 1/\epsilon(F_0) \sim 1/E_{\text{Planck}}^4$ . From (4.8), we then see that  $|\mu_0| \sim |F_0|$ . Let us now rewrite our equations in microscopic (Planckian) units by introducing appropriate dimen-

sionless variables  $f$ ,  $y$ ,  $u$ ,  $k$ ,  $h$ , and  $\tau$

$$F = fF_0, \quad y \equiv f - 1, \quad (4.9a)$$

$$\mu = \frac{u}{\chi F_0}, \quad G^{-1}(F) = k(f)|F_0|, \quad (4.9b)$$

$$H = h/\sqrt{\chi|F_0|}, \quad t = \tau\sqrt{\chi|F_0|}, \quad (4.9c)$$

where the variable  $y$  has been introduced in anticipation of the calculations of Sec. V. The corresponding normalized vacuum and matter energy densities are defined as follows:

$$\rho_{\text{V,M}} = \frac{r_{\text{V,M}}}{\chi}, \quad (4.10)$$

and Ansatz (4.7) gives

$$r_{\text{V}} = \frac{1}{2} \left( -f^2 + \frac{1}{3} f^4 \right) - uf, \quad (4.11)$$

with  $u = u_0 = -1/3$  from (4.8).

From the Maxwell Eq. (4.1), the Friedmann Eq. (4.2a), and the matter conservation Eq. (4.6), we finally obtain a closed system of three ordinary differential equations (ODEs) for the three dimensionless variables  $h$ ,  $f$ , and  $r_{\text{M}}$

$$\frac{3}{8\pi} \frac{dk}{df} \left( \frac{dh}{d\tau} + 2h^2 \right) = \frac{dr_{\text{V}}}{df}, \quad (4.12a)$$

$$\frac{3}{8\pi} \left( h \frac{dk}{df} \frac{df}{d\tau} + kh^2 \right) = r_{\text{V}} + r_{\text{M}}, \quad (4.12b)$$

$$\frac{dr_{\text{M}}}{d\tau} + 3h(1 + w_{\text{M}})r_{\text{M}} = 0, \quad (4.12c)$$

with matter equation-of-state (EOS) parameter  $w_{\text{M}} \equiv P_{\text{M}}/\rho_{\text{M}}$ .

### C. Model for $G(F)$

Next, we need an appropriate Ansatz for the function  $G(F)$  or the dimensionless function  $g(f) \equiv 1/k(f)$  in microscopic units. There are several possible types of behavior for  $G(F)$ , but we may reason as follows.

It is possible that for  $F^2 \ll F_0^2$  (i.e., in the gas-like vacuum) the role of the Planck scale is played by  $E_{\text{p}}(F) \equiv |\epsilon(F)|^{1/4} \sim |F|^{1/2}$ . The gravitational coupling parameter would then be given by

$$\frac{1}{G(F)} \sim E_{\text{p}}^2(F) \sim |F|, \quad |F| \ll |F_0|. \quad (4.13)$$

This equation also gives the correct estimate for  $G(F)$  in the equilibrium vacuum  $1/G(F_0) \sim E_{\text{Planck}}^2(F_0) \sim |F_0|$ , according to the estimates given a few lines below (4.8). Thus, a simple choice for the function  $G^{-1}(F)$  is

$$G^{-1}(F) = s|F|, \quad k(f) = sf, \quad (4.14)$$

with  $f$  taken positive (in fact,  $f \sim 1$  for  $F \sim F_0$ ) and a single time-independent dimensionless parameter  $s$  also taken positive.

Assuming (4.14), the three ODEs (4.12) become

$$\sigma \left( \frac{dh}{d\tau} + 2h^2 \right) = \frac{dr_V}{df}, \quad (4.15a)$$

$$\sigma \left( h \frac{df}{d\tau} + fh^2 \right) = r_V + r_M, \quad (4.15b)$$

$$\frac{dr_M}{d\tau} + 3h(1 + w_M)r_M = 0, \quad (4.15c)$$

with  $r_V = r_V(f)$  given by (4.11) and a single free parameter  $\sigma \equiv 3s/8\pi$ . This dimensionless parameter  $\sigma$  is of order 1 if the physics of  $F$  field is solely determined by the Planck energy scale (i.e., for  $F_0^2 \sim 1/\chi \sim \mu_0^2 \sim E_{\text{Planck}}^4$ ). Anyway, the parameter  $\sigma$  can be absorbed in  $h$  and  $\tau$  by the redefinition  $h \rightarrow h/\sqrt{\sigma}$  and  $\tau \rightarrow \tau\sqrt{\sigma}$ . Henceforth, we set  $\sigma = 1$  in (4.15), so that there are no more free parameters except for the EOS parameter  $w_M$  (taken to be time independent in the analysis of the next section).

## V. EQUILIBRIUM APPROACH IN A FLAT FRW UNIVERSE

### A. Equations at the equilibrium point $\mu = \mu_0$

Equations (4.15a)–(4.15c) allow us to study the evolution of the flat FRW universe toward a stationary state, if the initial universe was far away from equilibrium. The final state can be either the de Sitter universe of Sec. III with  $\rho_M = 0$  and  $\rho_V \neq 0$  or the perfect quantum vacuum (Minkowski spacetime) with  $H = \rho_M = \rho_V = 0$  and  $f = 1$ . Here, we consider the latter possibility where the system approaches one of the two perfect quantum vacuum states with  $f = 1$ , which correspond to either  $F = +|F_0|$  or  $F = -|F_0|$  for vacuum energy density (4.7).

Such an equilibrium vacuum state can be reached only if the chemical potential  $\mu$  corresponds to full equilibrium  $\mu = \mu_0$  as given by (4.8) or  $u = u_0 = -1/3$  in microscopic units (4.9b). Since  $\mu$  is an integration constant, there may be a physical reason for the special value  $\mu_0$ . Indeed, the starting nonequilibrium state could, in turn, be obtained by a large perturbation of an initial equilibrium vacuum. In this case, the integration constant would remember the original perfect equilibrium. (The evolution toward a de Sitter universe for  $\mu \neq \mu_0$  will be only briefly discussed in Sec. VD.)

In order to avoid having to consider quantum corrections to the Einstein equation, which typically appear near the time  $\tau \sim 1$  (or  $t \sim t_{\text{Planck}} \equiv \hbar/E_{\text{Planck}}$ ), we consider times  $\tau \gg 1$ , where the quantum corrections can be expected to be small. For these relatively large times,  $f$  is close to unity, and we may focus on the deviation from equilibrium as given by the variable  $y$  defined in (4.9a).

Taking the time derivative of (4.15b) for  $\sigma = 1$  and using (4.15a) and (4.15c), we obtain

$$\ddot{y} - \dot{y}h + 2(1 + y)\dot{h} = -3(1 + w_M)r_M, \quad (5.1)$$

where, from now on, the overdot stands for differentiation

with respect to  $\tau$ . Next, eliminate the matter density  $r_M$  from Eqs. (4.15b) and (5.1), in order to obtain a system of two equations for the two variables  $y$  and  $h$ :

$$\ddot{y} - \dot{y}h + 2(1 + y)\dot{h} = -3(1 + w_M)[\dot{y}h + (1 + y)h^2 - r_V], \quad (5.2a)$$

$$\dot{h} + 2h^2 = \frac{dr_V}{dy}, \quad (5.2b)$$

where the last equation corresponds to (4.15a) for  $\sigma = 1$ . The dimensionless vacuum energy density (4.11) for the dimensionless equilibrium chemical potential  $u = u_0 = -1/3$  is given by

$$r_V = \frac{1}{2}y^2 + \frac{2}{3}y^3 + \frac{1}{6}y^4, \quad (5.3)$$

which obviously vanishes in the equilibrium state  $y = 0$ .

In order to simplify the analysis, we, first, consider matter with a nonzero time-independent EOS parameter

$$w_M > 0, \quad (5.4)$$

so that the matter energy density from (4.15c) can be neglected asymptotically, as will become clear later on.

### B. Vacuum oscillations

Close to equilibrium, Eqs. (5.2a) and (5.2b) can be linearized

$$\ddot{y} + 2\dot{h} = 0, \quad \dot{h} = y. \quad (5.5)$$

The solution of these equations describes rapid oscillations near the equilibrium point

$$y = y_0 \sin \omega \tau, \quad h = h_0 - \frac{y_0}{\omega} \cos \omega \tau, \quad (5.6a)$$

$$r_V = \frac{1}{2}y_0^2 \sin^2 \omega \tau, \quad \omega^2 = 2. \quad (5.6b)$$

The (dimensionless) oscillation period of  $y$  and  $h$  is given by

$$\tau_0 = 2\pi/\omega = \pi\sqrt{2} \approx 4.44. \quad (5.7)$$

The corresponding oscillation period of the vacuum energy density  $r_V$  is smaller by a factor 2, so that numerically this period is given by  $\tau_0/2 \approx 2.22$ . Both oscillation periods will be manifest in the numerical results of Sec. VD.

### C. Vacuum energy decay

The neglected quadratic terms in Eqs. (5.2a) and (5.2b) provide the slow decay of the amplitudes in (5.6), namely, the  $f$ -field oscillation amplitude  $y_0(\tau)$ , the Hubble term  $h_0(\tau)$ , and the vacuum energy density averaged over fast oscillations  $\langle r_V \rangle = y_0^2(\tau)/4$ .

The explicit behavior is found by expanding the functions  $y(\tau)$  and  $h(\tau)$  in powers of  $1/\tau$  and keeping terms up to  $1/\tau^2$

$$y = \frac{b(\tau)}{\tau} + \frac{c(\tau)}{\tau^2}, \quad h = \frac{l(\tau)}{\tau} + \frac{m(\tau)}{\tau^2}, \quad (5.8a)$$

$$\dot{y} = \frac{\dot{b}}{\tau} + \frac{\dot{c} - b}{\tau^2}, \quad \dot{h} = \frac{\dot{l}}{\tau} + \frac{\dot{m} - l}{\tau^2}, \quad (5.8b)$$

$$\ddot{y} = \frac{\ddot{b}}{\tau} + \frac{\ddot{c} - 2\dot{b}}{\tau^2}, \quad (5.8c)$$

where the equality sign has been used rather freely. Collecting the  $1/\tau$  terms, we get homogeneous linear equations for  $b(\tau)$  and  $l(\tau)$ , which are actually the same as the linear ODEs (5.5) with  $y$  replaced by  $b$  and  $h$  replaced by  $l$ . The solution of these equations is given by (5.6) with the same replacements

$$b(\tau) = b_0 \sin \omega \tau, \quad l(\tau) = l_0 - \frac{b_0}{\omega} \cos \omega \tau, \quad \omega^2 = 2, \quad (5.9)$$

where  $l_0$  and  $b_0$  are numerical coefficients, which ultimately determine the decay of  $h(\tau)$  and  $r_V(\tau)$ .

In order to obtain these coefficients, we must collect the  $1/\tau^2$  terms. This leads to inhomogeneous linear equations for the functions  $m(\tau)$  and  $c(\tau)$ . The consistency of these equations determines the coefficients  $l_0$  and  $b_0$ . It suffices to keep only the zeroth and first harmonics in the functions  $m(t)$  and  $c(t)$ :

$$m(\tau) = m^{(1)} \sin \omega \tau, \quad c(\tau) = c^{(0)} + c^{(1)} \cos \omega \tau. \quad (5.10)$$

As a result, we obtain the following equations for  $m(\tau)$  and  $c(\tau)$

$$\dot{m} - c = \left[ l_0 - 2l_0^2 + \frac{1}{2}b_0^2 \right] + \frac{b_0}{\omega} (4l_0 - 1) \cos \omega \tau, \quad (5.11a)$$

$$2\dot{m} + \ddot{c} = \left[ 2l_0 - \frac{3}{2}b_0^2 - 3(1 + w_M) \left( l_0^2 - \frac{1}{2}b_0^2 \right) \right] + b_0 \omega (l_0 + 1) \cos \omega \tau. \quad (5.11b)$$

From the consistency of these equations for the first harmonics of  $m$  and  $c$ , we obtain

$$4l_0 - 1 = l_0 + 1, \quad (5.12)$$

which gives  $l_0 = 2/3$ . Similarly, we find from the zeroth harmonic of (5.11b)

$$w_M \left( \frac{4}{3} - \frac{3}{2}b_0^2 \right) = 0, \quad (5.13)$$

which, for  $w_M \neq 0$ , gives  $b_0 = 2\sqrt{2}/3 = \omega l_0$ .

The above results for the coefficients  $l_0$  and  $b_0$  hold for the generic case  $w_M > 0$ , as stated in (5.4). For the special case  $w_M = 0$ , inspection of (4.15) shows that the same *Ansätze* for  $y(\tau)$  and  $h(\tau)$  can be used, but with the following coefficients:

$$l_0 = 2/3, \quad b_0 = d_M \omega l_0, \quad (5.14a)$$

$$d_M = 1 + \delta_{w_M,0} (\sqrt{1 - (9/4)r_{M\infty}} - 1), \quad (5.14b)$$

where a Kronecker delta has been employed in the expression for the damping factor  $d_M$  and the coefficient  $r_{M\infty}$  of the  $w_M = 0$  asymptotic energy density  $r_M \sim r_{M\infty}/\tau^2$  has been assumed to be less than  $4/9$ .

Altogether, we have the following behavior of  $y(\tau)$ ,  $h(\tau)$ , and  $r_V(\tau)$  for  $\tau \rightarrow \infty$ :

$$y \sim \frac{2}{3} d_M \frac{\sqrt{2}}{\tau} \sin \omega \tau, \quad (5.15a)$$

$$h \sim \frac{2}{3} \frac{1}{\tau} (1 - d_M \cos \omega \tau), \quad (5.15b)$$

$$r_V \sim \frac{4}{9} d_M^2 \frac{1}{\tau^2} \sin^2 \omega \tau, \quad (5.15c)$$

with dimensionless frequency  $\omega = \sqrt{2}$  and damping factor  $d_M$  given by (5.14b). This asymptotic solution has some remarkable properties (in a different context, the same oscillatory behavior of  $h$  has been found in Ref. [21]; see also the discussion in the last paragraph of Sec. II). First, the solution depends rather weakly on the parameter  $w_M$  of the matter EOS, which is confirmed by the numerical results of the next subsection. Second, the average value of the vacuum energy density decays as  $\langle r_V \rangle \propto 1/\tau^2$  and the average value of the Hubble parameter as  $\langle h \rangle \propto 1/\tau$ , while the average scale parameter increases as  $\langle a(\tau) \rangle \propto \tau^{2/3}$ . Combined, the average vacuum energy density is found to behave as  $\langle r_V \rangle \propto 1/a^3$ , which is the same behavior as that of CDM in a standard FRW universe, as will be discussed further in Sec. V E.

## D. Numerical results

For ultrarelativistic matter ( $w_M = 1/3$ ), chemical potential  $\mu = \mu_0$ , and parameter  $\sigma = 1$ , the numerical solution of the coupled ODEs (4.15a)–(4.15c) is given in Figs. 1 and 2. The behavior near  $\tau \sim 1$  is only indicative, as significant quantum corrections to the classical Einstein equation can be expected (cf. Sec. V A). Still, the numerical results show clearly that

- (i) the equilibrium vacuum is approached asymptotically ( $f \rightarrow 1$  for  $\tau \rightarrow \infty$ );
- (ii) the FRW universe (averaged over time intervals larger than the Planck-scale oscillation period) does not have the expected behavior  $a \propto \tau^{1/2}$  for ultrarelativistic matter but rather  $a \propto \tau^{2/3}$ ;
- (iii) the same  $a \propto \tau^{2/3}$  behavior occurs if there is initially nonrelativistic matter, as demonstrated by Figs. 3 and 4 for a relatively small initial energy density and by Fig. 5 for a relatively large initial energy density;
- (iv) for a chemical potential  $\mu$  slightly different from the equilibrium value  $\mu_0$ , the vacuum decay is displayed in Fig. 6.

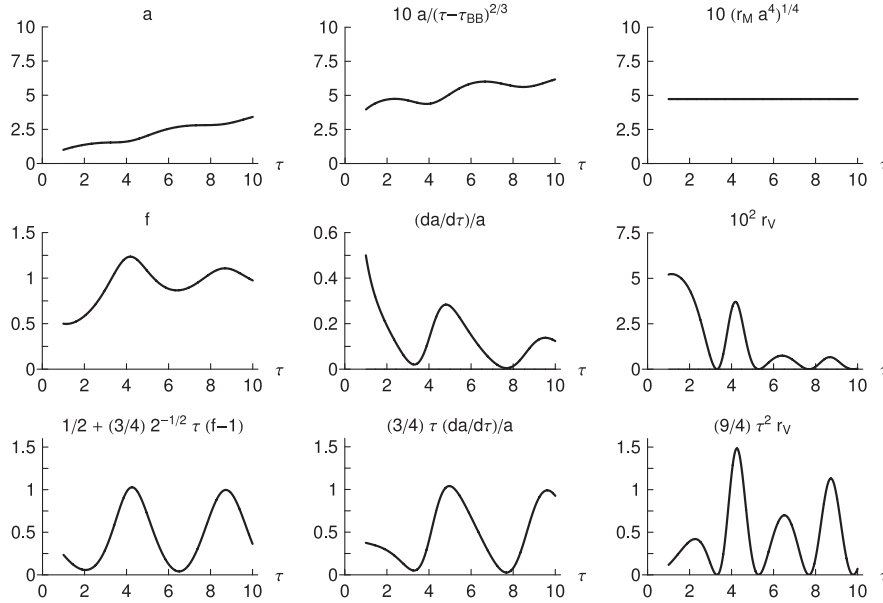


FIG. 1. Flat FRW universe with scale factor  $a(\tau)$ , Hubble parameter  $h(\tau) \equiv (da/d\tau)/a$ , ultrarelativistic-matter energy density  $\rho_M(\tau)$ , dynamic vacuum energy density  $\rho_V(F)$  controlled by the vacuum variable  $F = F(\tau)$ , and variable gravitational coupling parameter  $G = G(F)$ . All variables are scaled to become dimensionless and are denoted by lower-case Latin letters, for example,  $r_V = r_V(f)$  and  $g = g(f)$ . The specific choices for  $r_V(f)$  and  $g(f)$  are given by (4.11) at chemical potential  $u = u_0 = -1/3$  and by (4.14), respectively. The parameters of the coupled ODEs (4.15) are chosen as  $(\sigma, w_M) = (1, 1/3)$  and the boundary conditions at  $\tau = 1$  are  $(a, h, f, r_M) = (1, 1/2, 1/2, 1/20)$ . The effective parameter  $\tau_{BB}$  in the middle panel of the top row has been set to the value  $-3$ .

The first three items of the above list of numerical results confirm the previous asymptotic analytic results of Sec. VC (these asymptotic results predict, in fact, oscillations between  $[0, 1]$  for the particular combinations shown on the bottom-row panels of Figs. 1–5), while the last item shows that, after an initial oscillating stage, the model universe approaches a de Sitter stage (see, in particular, the middle panel of the second row of Fig. 6).

**E. Effective CDM-like behavior**

The main result of the previous two subsections can be summarized as follows: the oscillating vacuum energy density  $\rho_V[F(t)]$  and the corresponding oscillating gravitational coupling parameter  $G[F(t)]$  conspire to give the same Hubble expansion as pressureless matter (e.g., CDM) in a standard FRW universe with fixed gravitational coupling constant  $G = G_N$ . Recall that the standard behavior

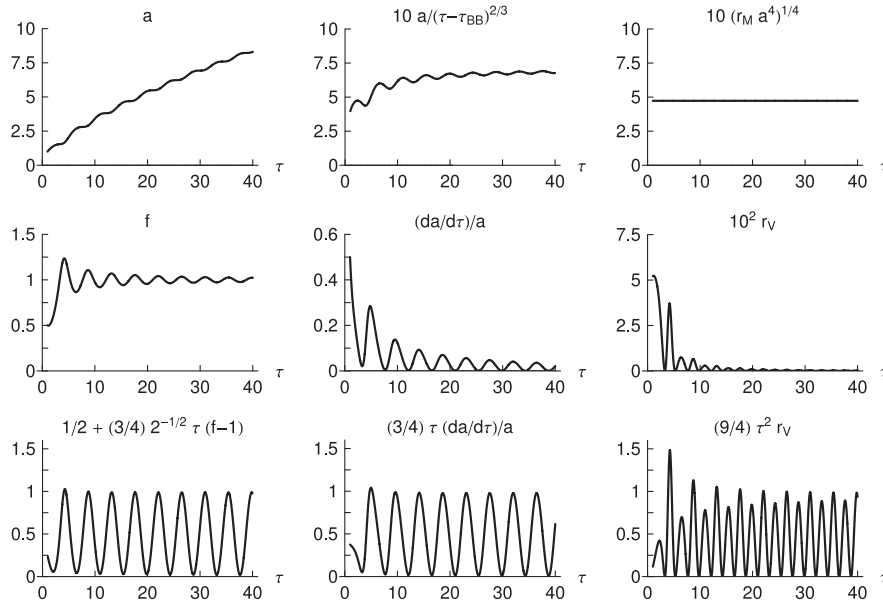


FIG. 2. Same as Fig. 1 but over a longer time.



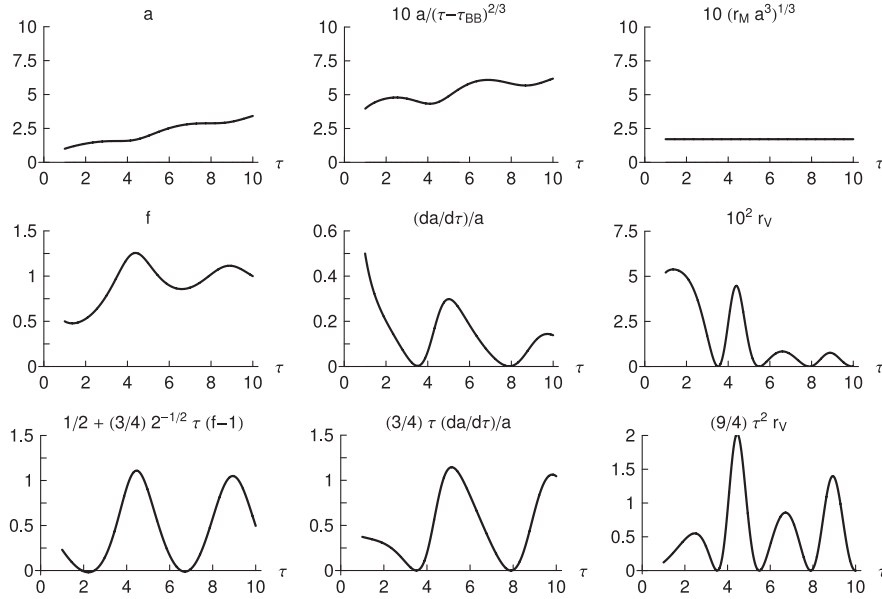


FIG. 3. Flat FRW universe with nonrelativistic-matter energy density  $r_M$ , dynamic vacuum energy density  $r_V(f)$  controlled by the dimensionless vacuum variable  $f$ , and variable gravitational coupling parameter  $g = g(f)$ . The parameters are chosen as  $(u, \sigma, w_M) = (-1/3, 1, 0)$  and the boundary conditions at  $\tau = 1$  are  $(a, h, f, r_M) = (1, 1/2, 1/2, 1/200)$ . The effective parameter  $\tau_{BB}$  in the middle top-row panel has been set to the value  $-3$ . The matter effects on the oscillation amplitudes of the bottom-row panels are small, as the damping factor  $d_M$  is close to 1, namely,  $d_M \approx 0.986$  from (5.14b) with  $r_{M\infty} \approx 0.0125$  [compare to the values of Fig. 5].

of the CDM energy density is given by  $\rho_{CDM}(t) \propto a(t)^{-3} \propto t^{-2}$ , which matches the average behavior found in (5.15).

The explanation is as follows. The average values of the rapidly oscillating vacuum energy density and vacuum pressure act as a source for the slowly varying gravitational field. The rapidly oscillating parts of  $h$  and  $y \equiv f - 1$  in the linearized Eq. (5.5) correspond to a dynamic system with Lagrangian density  $\frac{1}{2}(\dot{y})^2 - \frac{1}{2}\omega^2 y^2$  for a time-dependent homogenous field  $y = y(t)$ . The  $F$  (or  $y$ ) field has no

explicit kinetic term in the action (2.1a), but derivatives of  $F$  appear in the generalized Einstein Eq. (2.7) via terms with covariant derivatives of  $G^{-1}(F)$ , which trace back to the Einstein-Hilbert-like term  $R/G(F)$  in (2.1a). In a way, the effective Lagrange density  $\frac{1}{2}(\dot{y})^2 - \frac{1}{2}\omega^2 y^2$  can be said to be induced by gravity. The pressure of this rapidly oscillating field  $y$  is now given by  $P = \frac{1}{2}(\dot{y})^2 - \frac{1}{2}\omega^2 y^2$ . In turn, this implies that the rapidly oscillating vacuum pressure is zero on average and that the main contribution of the oscillating

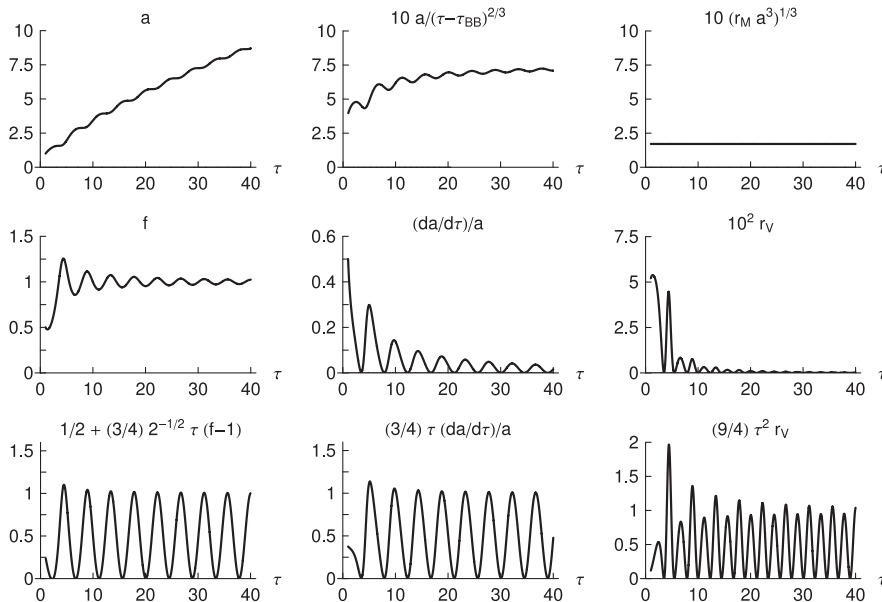


FIG. 4. Same as Fig. 3 but over a longer time.

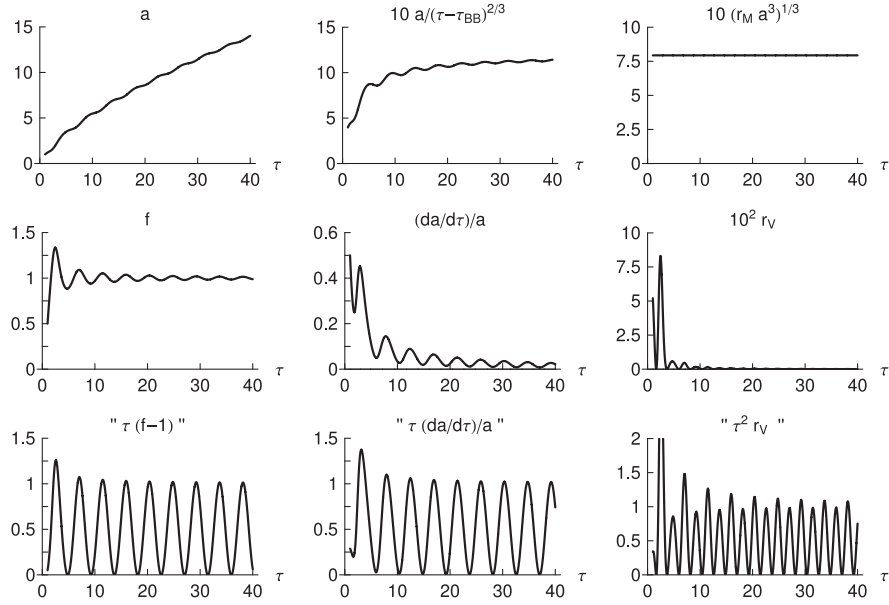


FIG. 5. Same as Figs. 3 and 4 but with a larger initial density of nonrelativistic matter. Specifically, the parameters are  $(u, \sigma, w_M) = (-1/3, 1, 0)$  and the boundary conditions at  $\tau = 1$  are  $(a, h, f, r_M) = (1, 1/2, 1/2, 1/2)$ . Plotted on the bottom row are in the left panel:  $1/2 + (1/d_M)(3/4)(1/\sqrt{2})\tau(f-1)$ , in the middle panel:  $[(3/2)\tau(da/d\tau)/a + d_M - 1]/(2d_M)$ , and in the right panel:  $(1/d_M^2) \times (9/4)\tau^2 r_V$ , for damping factor  $d_M \approx 0.590$  from (5.14b) with  $r_{M\infty} \approx 0.290$ .

vacuum energy density behaves effectively as cold dark matter.<sup>3</sup>

Observe that, while the  $F$  field itself has an EOS parameter  $w = -1$  corresponding to vacuum energy density, the net effect of the dampened  $F$  oscillations is to mimic the evolution of cold dark matter with  $w = 0$  in a standard flat FRW universe. As mentioned before, this effective EOS parameter  $w = 0$  is induced by the interaction of the  $F$  and gravity fields.

An outstanding task is to establish the clustering properties of this type of oscillating vacuum energy density. *A priori*, we may expect the same properties as CDM, because the relevant astronomical length scales are very much larger than the ultraviolet length scales that determine the microscopic dynamics of the vacuum energy density. But surprises are, of course, not excluded.

### F. Extrapolation to large times

In Secs. VC and VD, we have established that the average vacuum energy density decreases quadratically with cosmic time. This behavior follows, analytically, from (5.15c) and, numerically, from the bottom-right panels of Figs. 2, 4, and 5.

<sup>3</sup>It is known that a rapidly oscillating homogeneous scalar field in a standard FRW universe corresponds to pressureless matter (cf. Sec. 5.4.1 of Ref. [16]), but, in our case, matter plays only a secondary role compared with vacuum energy. Moreover, the oscillating scalar field gives an oscillating term in  $h(\tau)$ , which is subleading (of order  $1/\tau^2$ ), whereas the oscillating term in (5.15b) is already of order  $1/\tau$ .

Extrapolating this evolution to the present age of the Universe ( $t_{\text{now}} \approx 10$  Gyr) and using  $|F_0| = s^{-1}G^{-1}(F_0) \sim 3/(8\pi G_N)$  for  $\sigma \equiv 3s/8\pi = 1$ , the numerical value of the average vacuum energy density is given by

$$\begin{aligned} \langle \rho_V(t_{\text{now}}) \rangle &\sim \frac{|F_0|}{t_{\text{now}}^2} \sim \frac{E_{\text{Planck}}^2}{t_{\text{now}}^2} = \left( \frac{t_{\text{Planck}}}{t_{\text{now}}} \right)^2 E_{\text{Planck}}^4 \\ &\approx (4 \times 10^{-3} \text{ eV})^4 \left( \frac{10^{10} \text{ yr}}{t_{\text{now}}} \right)^2, \end{aligned} \quad (5.16)$$

for  $t_{\text{Planck}} = 1/E_{\text{Planck}} \approx 5 \times 10^{-44}$  s. The order of magnitude of the above estimate is in agreement with the observed vacuum energy density of the present universe, which is close to the critical density of a standard FRW universe (cf. Refs. [17,20] and references therein). If the behavior found had been  $\langle \rho_V \rangle \propto t^{-n}$  for an integer  $n \neq 2$ , this agreement would be lost altogether. In other words, the dynamic behavior established in (5.15c) is quite nontrivial.

Let us expand on the previous remarks. For a *standard* flat FRW universe, the *total* energy density is, of course, always equal to the critical density  $\rho_c \equiv 3H^2/(8\pi G_N)$ . But, here, the gravitational coupling parameter is variable,  $G = G(t)$ , and there are rapid oscillations, so that, for example,  $\langle H \rangle^2 \neq \langle H^2 \rangle$ . This explains the following result for the case of a nonzero matter EOS parameter ( $w_M > 0$ ):

$$\lim_{t \rightarrow \infty} \frac{\langle \rho_V \rangle}{3 \langle H \rangle^2 / (8\pi \langle G \rangle)} = \frac{1}{2}, \quad (5.17)$$

which is of order 1 but not exactly equal to 1. For nonrelativistic matter ( $w_M = 0$ ), the right-hand side of (5.17) is

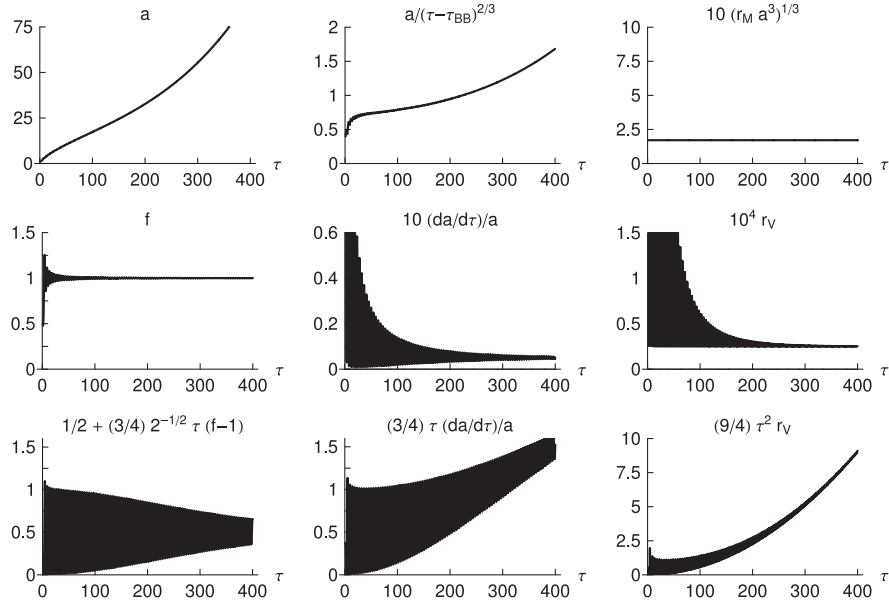


FIG. 6. Same as Figs. 3 and 4 but with a small perturbation of the dimensionless chemical potential away from the equilibrium value,  $u = u_0 + \delta u = -1/3 - 1/40000$ , and evolved over an even longer time.

multiplied by a further reduction factor  $d_M^2 = 1 - (9/4)r_{M\infty}$ , according to the results of Sec. VC.

Even though the order of magnitude of (5.16) or (5.17) appears to be relevant to the observed universe, the  $1/t^2$  behavior of  $\langle \rho_V \rangle$  contradicts the current astronomical data on “cosmic acceleration” [9,18,19]. A related problem is the CDM-like expansion of the model universe,  $a \propto t^{2/3}$ , whereas big-bang nucleosynthesis requires radiation-like expansion,  $a \propto t^{1/2}$ , at least for the relevant temperature range. Clearly, there are many other processes that intervene between the very early (Planckian) phase of the Universe and later phases such as the nucleosynthesis era and the present epoch. An example of a relevant process may be particle production (e.g., by parametric resonance [16,22]), which can be expected to be effective because of the very rapid (small-amplitude) oscillations.<sup>4</sup> A further possible source of modified vacuum energy behavior may be the change of EOS parameter  $w_M = 1/3$  to  $w_M = 0$ , which occurs when the expanding universe leaves the radiation-dominated epoch. Still, there is a possibility that these and other processes are only secondary effects and that the main mechanism of dark-energy dynamics at

the early stage is the decay of vacuum energy density by oscillations.

Another aspect of the large-time extrapolation concerns the variation of Newton’s “constant.” For the theory (2.1) and the particular *Ansatz* (4.14), the gravitational coupling parameter  $G(t)$  is found to relax to an equilibrium value in the following way:

$$G^{-1}(t) \sim G_\infty^{-1} \left( 1 + c_0 \frac{t_{UV}}{t} \sin\left(\frac{t}{t_{UV}}\right) \right), \quad (5.18)$$

with  $c_0$  a constant of order unity,  $G_\infty$  a gravitational constant presumably very close to the Cavendish-type value for Newton’s constant  $G_N$ , and  $t_{UV} = \sqrt{\chi|F_0|/2}$  an ultraviolet timescale of the order of the corresponding Planckian time scale  $\sqrt{G_\infty \hbar/c^5}$ . The behavior (5.18), shown qualitatively by the  $f$  panels in Figs. 1–6, is very different from previous suggestions for the dynamics of  $G(t)$ , including Dirac’s original suggestion  $G \propto 1/t$  (cf. Sec. 16.4 of Ref. [10]). For the present universe and the solar system in it, the gravitational coupling parameter (5.18) would have minuscule oscillations. Combined with the Planck-scale mass of the  $F$  degree of freedom (cf. the discussion in Sec. VE), this would suggest that all solar-system experimental bounds are satisfied, but, again, surprises are not excluded.<sup>5</sup>

<sup>4</sup>Energy exchange between dark matter and dark energy (for example, between cold dark matter and dynamic vacuum energy), may, in fact, be essential to explain the current epoch of cosmic acceleration; cf. Ref. [23]. The model considered in the present paper does not allow for energy exchange between dark matter and dark energy, as (4.6) makes clear. However, it may be that the effects of such an interaction can be partially incorporated in our model as a small perturbation of the chemical potential away from the value (4.8). The resulting behavior with exponential expansion setting in at large times (for small negative perturbations  $\delta u$ ) is shown in Fig. 6.

<sup>5</sup>After a first version of the present article was completed, we became aware of earlier work on a rapidly varying gravitational coupling parameter  $G(t)$ ; see, e.g., Refs. [24–26] and references therein. These articles discuss, in particular, solar-system experimental bounds and the possibility that the effective density ratio in a flat FRW universe may be less than unity, which corresponds to our result (5.17) for the case of dynamic vacuum energy and ultrarelativistic matter with  $\langle \rho_V \rangle \gg \langle \rho_M \rangle \geq 0$  asymptotically.

## VI. CONCLUSION

The considerations of the present article and its predecessor [1] by no means solve the cosmological constant problems, but may provide hints. Specifically, the new results are

- (i) a mechanism of vacuum-energy decay, which, starting from a “natural” Planck-scale value at very early times, leads to the correct order of magnitude (5.16) for the present cosmological constant;
- (ii) the realization from result (5.15) that a substantial part of the inferred CDM may come from an oscillating vacuum energy density;
- (iii) the important role of oscillations of the vacuum variable  $q$  (here,  $F$ ), which drive the vacuum energy density oscillations responsible for the first two results.

Expanding on the last point, another consequence of  $q$  oscillations is that they naturally lead to the creation of

hot (ultrarelativistic) matter from the vacuum. This effective mechanism of energy exchange between vacuum and matter deserves further study.

## ACKNOWLEDGMENTS

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*Note added in proof.*—Following up on the remarks in the last paragraph of Sec. II, we have recently shown [27] that, close to equilibrium, the  $q$  theory of the quantum vacuum gives rise to an effective  $f(R)$  model, which belongs to the  $R + R^2/M^2$  class of models with a Planck-scale mass  $M \sim E_{UV}$ . We have also extended our analysis to a quantum vacuum containing several conserved  $q$  fields, which allows for the coexistence of different vacua.

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