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Solving Einstein field equations in observational coordinates with cosmological data functions: Spherically symmetric universes with a cosmological constant

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Extending the approach developed by Araújo and Stoeger [M. E. Araújo and W. R. Stoeger, Phys. Rev. D **60**, 104020 (1999)] and improved in Araújo *et al.* [M. E. Araújo, S. R. M. M. Roveda, and W. R. Stoeger, Astrophys. J. **560**, 7 (2001)], we have shown how to construct dust-filled $\Lambda \neq 0$ Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models from FLRW cosmological data on our past light cone. Apart from being of interest in its own right—demonstrating how such data fully determines the models—it is also illustrated in the flat case how the more general spherically symmetric Einstein field equations can be integrated in observational coordinates with data fit to FLRW forms arrayed on our past light cone, thus showing how such data determines an FLRW universe—which is not *a priori* obvious. It is also shown how to integrate these exact sypherically symmetric equations, in cases where the data are not FLRW, and the space-time is not known to be flat. It is essential for both flat and nonflat cases to have data giving the maximum of the observer area (angular-diameter) distance, and the redshift $z_{\max}$ at which that occurs. This enables the determination of the vacuum-energy density μ_Λ, which would otherwise remain undetermined.

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I. INTRODUCTION

The recent WMAP results (see Spergel *et al.* [1] and references therein) strongly support the inflationary scenario and are consistent with a nearly flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe with a cosmological constant Λ and dust ($\Omega \approx 1$, with $\Omega_m \approx 0.27$ and $\Omega_\Lambda \approx 0.73$), and with an almost scale-invariant spectrum for the primordial perturbations. It is obviously very important to test and confirm this result. One observationally sensitive theoretical approach to doing so begins by establishing a more general framework than FLRW—say a general perturbed spherically symmetric space-time—and then using the data itself to determine the more specific model. Can we obtain perturbed FLRW by doing this?

In two papers Araújo and Stoeger [2] and Araújo, Roveda and Stoeger [3] demonstrated in detail how to solve exactly the spherically symmetric (SS) Einstein field equations for dust in observational coordinates without assuming FLRW and with cosmological data representing galaxy redshifts, observer area distances and galaxy-number counts as functions of redshift. These data are given, not on a spacelike surface of constant time, but rather on our past light cone $C^-(p_0)$, which is centered at our observational position p_0 "here and now" on our world line $\mathcal{C}$. These results demonstrate how cosmologically relevant astronomical data can be used to determine the space-time structure of the Universe—the cosmological model which best fits it. This has been the aim of a series of papers going back to the *Physics Reports* paper by Ellis *et al.* [4]. The motivation and history of this "observational cosmology (OC) program" is summarized in Araújo and Stoeger [2]. All these papers assumed that $\Lambda = 0$. In this paper we demonstrate how this program may be carried out when $\Lambda \neq 0$. As a simple, and very relevant example, we take a flat SS universe, and suppose that the redshift, observer area-distance and number-count data can be fit to FLRW functional forms (these are very special forms the data must take, if the Universe is FLRW). Then we show how such data determines a *bona fide* FLRW universe—which is not *a priori* obvious. We then go on to indicate how the solution can be obtained in the more general nonflat case, with no constrains on the functional form of the data functions. In doing this for $\Lambda \neq 0$, we also need another necessary piece of data, the maximum of the observer area distance, and the redshift at which it occurs. Without these extra observables, we do not have enough independent data to determine the model—in particular to determine the extra parameter Λ.

The primary aim of the OC program is to strengthen the connections between astronomical observations and cosmological theory. We do this by allowing observational data to determine the geometry of space-time as much as possible, *without* relying on *a priori* assumptions more than is necessary or justified. Basically, we want to find out not only how far our observable Universe is from being isotropic and spatially homogeneous (that, is describable by an FLRW cosmological model) on various length scales, but also to give a dynamic account of those deviations (Stoeger *et al.* [5]).

By using observational coordinates, we can thus formulate Einstein's equations in a way which reflects both the geodesic flow of the cosmological fluid and the null geometry of $C^-(p_0)$, along which practically all of our information about the distant reaches of our Universe comes to us—in photons. In this formulation the field equations split naturally into two sets, as can be easily seen: a set of equations which can be solved on $C^-(p_0)$, that is on our past light cone, specified by $w = w_0$, where w is the observational time coordinate; and a second set which evolves these solutions off $C^-(p_0)$ to other light cones into the past or into the future. The solution to the first set is directly determined from the data, and those solutions constitute the "initial conditions" for the solution of the second set.

There are many reasons for investigating FRLW in observational coordinates from this more general starting point. It is clear from the cosmic background radiation (CBR) anisotropies measured by the Cosmic Backgroung Explorer (COBE) and by WMAP that the Universe is not exactly described by the FLRW models (this follows from the analysis by Sachs and Wolfe [6], and see Stoeger, Araújo and Gebbie [7] for an analysis related to the viewpoint of this paper). But on the largest scales its deviations from FLRW are small. So on those scales the Universe can be described by an *almost* FLRW model. Therefore, our first step towards a strictly observationally based approach to this realistic model involves a complete understanding of the inner workings of the integration procedure in observational coordinates for FLRW data.

In this paper, for completeness, we review some aspects of the problem of determining the solution of the exact spherically symmetric Einstein equations for dust in observational coordinates and then integrate the field equations with FLRW data to obtain the FLRW ($k = 0$ and $\Lambda \neq 0$) solution explicitly. We refer the reader to Ellis *et al.* [4], Kristian and Sachs [8], Araújo and Stoeger [2] and references therein for a complete account of the philosophy and the foundations of the OC approach leading to the integration of Einstein field equations in observational coordinates.

In the next section we define observational coordinates, write the general spherically symmetric metric using them and present the very important central conditions for the metric variables. Section III summarizes the basic observational parameters we shall be using and presents several key relationships among the metric variables. Section IV presents the full set of field equations for the spherically symmetric case, with dust and with $\Lambda \neq 0$. Section V shows the integration procedure for the flat case, with FLRW data. In Section VI, we present the integration for the general, nonflat case, and in Sec. VII we briefly discuss our conclusions.

II. THE SPHERICALLY SYMMETRIC METRIC IN OBSERVATIONAL COORDINATES

We are using observational coordinates (which were first suggested by Temple [9]). As described by Ellis *et al.* [4] the observational coordinates $x^i = \{w, y, \theta, \phi\}$ are centered on the observer's world line C and defined in the following way:

(i) w is constant on each past light cone along C, with $u^a \partial_a w > 0$ along C, where u^a is the 4-velocity of matter ($u^a u_a = -1$). In other words, each $w =$ constant specifies a past light cone along C. Our past light cone is designated as $w = w_0$.

(ii) y is the null radial coordinate. It measures distance down the null geodesics—with affine parameter ν—generating each past light cone centered on C. $y = 0$ on C and $dy/d\nu > 0$ on each null cone—so that y increases as one moves down a past light cone away from C.

(iii) θ and ϕ are the latitude and longitude of observation, respectively—spherical coordinates based on a parallelly propagated orthonormal tetrad along C, and defined away from C by $k^a \partial_a \theta = k^a \partial_a \phi = 0$, where k^a is the past-directed wave vector of photons ($k^a k_a = 0$).

There are certain freedoms in the specification of these observational coordinates. In w there is the remaining freedom to specify w along our world line C. Once specified there it is fixed for all other world lines. There is considerable freedom in the choice of y—there are a large variety of possible choices for this coordinate—the affine parameter, z, the area-distance $C(w, y)$ itself. We normally choose y to be comoving with the fluid, that is $u^a \partial_a y = 0$. Once we have made this choice, there is still a little bit of freedom left in y, which we shall use below. The remaining freedom in the θ and ϕ coordinates is a rigid rotation at *one* point on C.

In observational coordinates the spherically symmetric metric takes the general form:

$$ds^2 = -A(w, y)^2 dw^2 + 2A(w, y)B(w, y)dwdy + C(w, y)^2 d\Omega^2, \tag{1}$$

where we assume that y is comoving with the fluid, so that the fluid 4-velocity is $u^a = A^{-1}\delta^a_w$.

The remaining coordinate freedom which preserves the observational form of the metric is a scaling of w and of y:

$$w \longrightarrow \tilde{w} = \tilde{w}(w), \qquad y \longrightarrow \tilde{y} = \tilde{y}(y) \qquad \left(\frac{d\tilde{w}}{dw} \neq 0 \neq \frac{d\tilde{y}}{dy}\right). \tag{2}$$

The first, as we mentioned above, corresponds to a freedom to choose w as any time parameter we wish along C, along our world line at $y = 0$. This is usually effected by choosing $A(w, 0)$. The second corresponds to the freedom to choose y as any null distance parameter on an initial light cone—typically our light cone at $w = w_0$. Then that choice is effectively dragged onto other light cones by the fluid flow—y is comoving with the fluid 4-velocity, as we have already indicated. We shall use this freedom to choose y by setting

$$A(w_0, y) = B(w_0, y). \tag{3}$$

We should carefully note here that setting $A(w, y) = B(w, y)$ off our past light cone $w = w_0$ is too restrictive.

In general, these freedoms in w and y imply the metric scalings:

$$A \to \tilde{A} = \frac{dw}{d\tilde{w}} A, \qquad B \to \tilde{B} = \frac{dy}{d\tilde{y}} B. \tag{4}$$

It is important to specify the central conditions for the metric variables $A(w, y)$, $B(w, y)$ and $C(w, y)$ in Eq. (1)—that is, their proper behavior as they approach $y = 0$. These are

$$\begin{aligned} \text{as } y \to 0\text{: } A(w, y) &\to A(w, 0) \neq 0, \\ B(w, y) &\to B(w, 0) \neq 0, \\ C(w, y) &\to B(w, 0)y = 0, \\ C_y(w, y) &\to B(w, 0). \end{aligned} \tag{5}$$

III. THE BASIC OBSERVATIONAL QUANTITIES

The basic observable quantities on C are the following:

(i) Redshift. The redshift z at time w_0 on C for a comoving source a null radial distance y down $C^-(p_0)$ is given by

$$1 + z = \frac{A(w_0, 0)}{A(w_0, y)}. \tag{6}$$

This is just the observed redshift, which is directly determined by source spectra, once they are corrected for the Doppler shift due to local motions.

(ii) Observer Area Distance. The observer area distance, often written as r_0, measured at time w_0 on C for a source at a null radial distance y is simply given by

$$r_0 = C(w_0, y), \tag{7}$$

provided the central condition (5), determining the relation between $C(w, y)$ and $B(w, y)$ for small values of y, holds. This quantity is also measurable as the luminosity distance d_L because of the reciprocity theorem of Etherington [10] (see also Ellis [11]),

$$d_L = (1 + z)^2 C(w_0, y). \tag{8}$$

(iii) The Maximum of Observer Area Distance. Generally speaking, $C(w_0, y)$ reaches a maximum $C_{\max}$ for a relatively small redshift $z_{\max}$ (Hellaby [12]; see also Ellis and Tivon [13] and Araújo and Stoeger [14]). At $C_{\max}$, of course, we have

$$\frac{dC(w_0, z)}{dz} = \frac{dC(w_0, y)}{dy} = 0, \tag{9}$$

further conditioned by

$$\frac{d^2 C(w_0, z)}{dz^2} < 0. \tag{10}$$

Furthermore, of course, as we shall review below, the data set will give us $y = y(z)$, from which we shall be able to find $y_{\max} = y_{\max}(z_{\max})$. These $C_{\max}$ and $z_{\max}$ data provide additional independent information about the cosmology. Without $C_{\max}$ and $z_{\max}$ we cannot constrain the value of Λ.

(iv) Galaxy Number Counts. The number of galaxies counted by a central observer out to a null radial distance y is given by

$$N(y) = 4\pi \int_0^y \mu(w_0, \tilde{y}) m^{-1} B(w_0, \tilde{y}) C(w_0, \tilde{y})^2 d\tilde{y}, \tag{11}$$

where μ is the mass-energy density and m is the average galaxy mass. Then the total energy density can be written as

$$\mu(w_0, y) = mn(w_0, y) = M_0(z) \frac{dz}{dy} \frac{1}{B(w_0, y)}, \tag{12}$$

where $n(w_0, y)$ is the number density of sources at (w_0, y), and where

$$M_0 \equiv \frac{m}{J} \frac{1}{d\Omega} \frac{1}{r_0^2} \frac{dN}{dz}. \tag{13}$$

Here $d\Omega$ is the solid angle over which sources are counted, and J is the completeness of the galaxy count; that is, the fraction of sources in the volume that are counted is J. The effects of dark matter in biasing the galactic distribution may be incorporated via m and/or J. In particular, strong biasing is needed if the number counts have a fractal behavior on local scales (Humphreys *et al.* [15]). In order to effectively use number counts to constrain our cosmology, we shall also need an adequate model of galaxy evolution. We shall not discuss this important issue in this paper. But, fundamentally, it would give us an expression for $m = m(z)$ in Eqs. (12) and (13) above.

There are a number of other important quantities which we catalogue here for completeness and for later reference.

First, there are the two fundamental four-vectors in the problem, the fluid 4-velocity u^a and the null vector k^a, which points down the generators of past light cones. These are given in terms of the metric variables as

$$u^a = A^{-1} \delta^a{}_w, \qquad k^a = (AB)^{-1} \delta^a{}_y. \tag{14}$$

Then, the rate of expansion of the dust fluid is $3H = \nabla_a u^a$, so that, from the metric (1) we have

$$H = \frac{1}{3A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{C}}{C}\right), \tag{15}$$

where a "dot" indicates $\partial/\partial w$ and a "prime" indicates $\partial/\partial y$, which will be used later. For the central observer H is precisely the Hubble expansion rate. In the homogeneous (FLRW) case, H is constant at each instant of time t. But in the general inhomogeneous case, H varies with radial distance from $y = 0$ on $t = t_0$. From our central conditions above (3), we find that the central behavior of H is given by

$$\text{as } y \rightarrow 0\text{: } H(w, y) \rightarrow \frac{1}{A(w, 0)} \frac{\dot{B}(w, 0)}{B(w, 0)} = H(w, 0). \tag{16}$$

At any given instant $w = w_0$ along $y = 0$; this expression is just the Hubble constant $H_0 \equiv H(w_0, 0) = A_0^{-1}B_0^{-1}(\dot{B})_0$ as measured by the central observer. In the above we have also written $A_0 \equiv A(w_0, 0)$ and $B_0 \equiv B(w_0, 0)$.

Finally, from the normalization condition for the fluid 4-velocity, we can immediately see that it can be given (in covariant vector form) as the gradient of the proper time t along the matter world lines: $u_a = -t_{,a}$. It is also given by (1) and (14) as

$$u_a = g_{ab}u^b = -Aw_{,a} + By_{,a}. \tag{17}$$

Comparing these two forms implies

$$dt = Adw - Bdy \Leftrightarrow A = t_w, \qquad B = -t_y, \tag{18}$$

which shows that the surfaces of simultaneity for the observer are given in observational coordinates by $Adw = Bdy$. The integrability condition of Eq. (18) is simply then

$$A' + \dot{B} = 0. \tag{19}$$

This turns out precisely to be the momentum conservation equation, which is the key equation in the system and essential to finding a solution.

IV. THE SPHERICALLY SYMMETRIC FIELD EQUATIONS IN OBSERVATIONAL COORDINATES

Using the fluid-ray tetrad formulation of the Einstein's equations developed by Maartens [16] and Stoeger *et al.* [17], one obtains the spherically symmetric field equations in observational coordinates with $\Lambda \neq 0$ (see Stoeger *et al.* [5] for a detailed derivation). Besides the momentum conservation Eq. (19), they are as follows.

A set of two very simple fluid-ray tetrad time-derivative equations:

$$\dot{\mu}_m = -2\mu_m\left(\frac{\dot{B}}{2B} + \frac{\dot{C}}{C}\right), \tag{20}$$

$$\dot{\omega} = -3\frac{\dot{C}}{C}\left(\omega + \frac{\mu_\Lambda}{6}\right), \tag{21}$$

where μ_m again is the relativistic mass-energy density of the dust, including dark matter, and

$$\omega(w, y) \equiv -\frac{1}{2C^2} + \frac{\dot{C}}{AC}\frac{C'}{BC} + \frac{1}{2}\left(\frac{C'}{BC}\right)^2$$

is a quantity closely related to $\mu_{m_0}(y) \equiv \mu_m(w_0, y)$ (see Eq. (30) below).

Equations. (20) and (21) can be quickly integrated to give

$$\mu_m(w, y) = \mu_{m_0}(y)\frac{B(w_0, y)}{B(w, y)}\frac{C^2(w_0, y)}{C^2(w, y)}; \tag{22}$$

$$\omega(w, y) = \left(\omega_0(y) + \frac{\mu_\Lambda}{6}\right)\frac{C^3(w_0, y)}{C^3(w, y)} - \frac{\mu_\Lambda}{6}$$
$$= -\frac{1}{2C^2} + \frac{\dot{C}}{AC}\frac{C'}{BC} + \frac{1}{2}\left(\frac{C'}{BC}\right)^2, \tag{23}$$

where $\omega_0(y) \equiv \omega(w_0, y)$ and the last equality in (23) follows from the definition of ω given above. In deriving and solving these equations, and those below, we have used the typical Λ equation of state, $p_\Lambda = -\mu_\Lambda$, where p_Λ and $\mu_\Lambda \equiv \frac{\Lambda}{8\pi G}$ are the pressure and the energy density due to the cosmological constant. Both ω_0 and μ_0 are specified by data on our past light cone, as we shall show. μ_Λ will eventually be determined from the measurement of $C_{\max}$ and $z_{\max}$.

The fluid-ray tetrad radial equations are

$$\frac{C''}{C} = \frac{C'}{C}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{2}B^2\mu_m; \tag{24}$$

$$\left[\left(\omega_0(y) + \frac{\mu_\Lambda}{6}\right)C^3(w_0, y)\right]' = -\frac{1}{2}\mu_{m_0}B(w_0, y)C^2(w_0, y) \times\left(\frac{\dot{C}}{A} + \frac{C'}{B}\right); \tag{25}$$

$$\frac{\dot{C}'}{C} = \frac{\dot{B}}{B}\frac{C'}{C} - \left(\omega + \frac{\mu_\Lambda}{2}\right)AB. \tag{26}$$

The remaining "independent" time-derivative equations given by the fluid-ray tetrad formulation are

$$\frac{\ddot{C}}{C} = \frac{\dot{C}}{C}\frac{\dot{A}}{A} + \left(\omega + \frac{\mu_\Lambda}{2}\right)A^2; \tag{27}$$

$$\frac{\ddot{B}}{B} = \frac{\dot{B}}{B}\frac{\dot{A}}{A} - 2\omega A^2 - \frac{1}{2}\mu_m A^2. \tag{28}$$

From Eq. (25) we see that there is a naturally defined "potential" (see Stoeger *et al.* [5]) depending only on the radial null coordinate y—since the left-hand side depends only on y, the right-hand side can only depend on y:

$$F(y) \equiv \frac{N_\star{}'}{N'} = \frac{\dot{C}}{A} + \frac{C'}{B}, \tag{29}$$

where $N_\star(y)$ is an arbitrary function, whose central behavior is the same as that of number counts (Stoeger *et al.* [5]). Thus,

$$\omega_0(y) = -\frac{\mu_\Lambda}{6} - \frac{1}{2C^3(w_0, y)} \times \int \mu_{m_0}(y)B(w_0, y)C^2(w_0, y)F(y)dy. \quad (30)$$

Connected with this relationship is Eq. (23), which we rewrite as

$$\frac{\dot{C}}{C}\frac{C'}{C} + \frac{A}{2B}\frac{C'^2}{C^2} - \frac{AB}{2C^2} = \frac{AB}{C^3}\left[C_0^3\left(\omega_0 + \frac{\mu_\Lambda}{6}\right) - \frac{\mu_\Lambda}{6}C^3\right], \quad (31)$$

where $C_0 \equiv C(w_0, y)$.

Stoeger *et al.* [5] and Maartens *et al.* [18] have shown that Eqs. (29) and (31) can be transformed into equations for A and B, thus reducing the problem to determining C:

$$A = \frac{\dot{C}}{[F^2 - 1 - 2(\omega_0 + \mu_\Lambda/6)C_0^3/C + (\mu_\Lambda/3)C^2]^{1/2}} \quad (32)$$

$$B = \frac{C'}{F \pm [F^2 - 1 - 2(\omega_0 + \mu_\Lambda/6)C_0^3/C + (\mu_\Lambda/3)C^2]^{1/2}}. \quad (33)$$

The Lemaître-Tolman-Bondi (LTB) exact solution (Lemaître [19]), Tolman [20], Bondi [21]; and cf. Humphreys [22] and references therein) is obtained by integration of (32) along the matter flow $y =$ constant using (18)

$$t - T(y) = \int \frac{dC}{[F^2 - 1 - 2(\omega_0 + \mu_\Lambda/6)C_0^3/C + (\mu_\Lambda/3)C^2]^{1/2}}, \quad (34)$$

where $T(y)$ is arbitrary, provided we identify

$$F^2 = 1 - kf^2, \qquad k = 0, \pm 1. \quad (35)$$

Here $f = f(y)$ is a function commonly used in describing LTB models in the 3 + 1 coordinates (Bonnor [23]).

V. INTEGRATION WITH FLRW ($k = 0$ AND $\Lambda \neq 0$) DATA

In this section and the next we use a generalization (to incorporate the $C_{\max}$ and $z_{\max}$ data) of the integration procedure described in detail in Araújo *et al.* [3], which in turn is an improvement of the integration scheme developed by Araújo and Stoeger [2], to solve the above system of SS field equations when $\Lambda \neq 0$. First, we consider a concrete, simplified, but very relevant example. Suppose that we know that the Universe is flat. Then $F(y) = 1$. This means that we need only the observer area distance, or the galaxy-number counts—not both. $F = 1$ establishes a relation between these data functions. Suppose also that, though we do not know that the Universe is FLRW, we find that our observer-area-distanceand galaxy-number-count data can be fit—or very closely approximated—by the FLRW, $\Lambda \neq 0$ observational relationships as functions of the redshift z. For the flat case these are (Araújo and Stoeger [14])

$$r_0(z) = \frac{[Q(z) - 1]}{\sqrt{\Omega_\Lambda \sqrt{3}}H_0[(1 + \sqrt{3})Q(z) - (1 - \sqrt{3})]} \times \{cn^{-1}[Q(z), \kappa] - cn^{-1}[Q(0), \kappa]\},$$

$$Q(z) \equiv \frac{(1 - \sqrt{3}) + (1 + z)\sqrt[3]{\frac{1}{\Omega_\Lambda} - 1}}{(1 + \sqrt{3}) + (1 + z)\sqrt[3]{\frac{1}{\Omega_\Lambda} - 1}}, \quad (36)$$

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H_0{}^2}, \qquad \kappa = \sqrt{\frac{2 + \sqrt{3}}{4}},$$

and

$$M_0(z) = \frac{\mu_{m_0}(1 + z)^2}{A_0 H_0 \sqrt{[\Omega_\Lambda + (1 + z)^3(1 - \Omega_\Lambda)]}}. \quad (37)$$

$cn^{-1}(u, \kappa)$ is the inverse of the Jacobi elliptic function $cn(u, \kappa)$, where κ is the modulus. Equation (36) can clearly also be written in terms of elliptic integrals of the first kind (Araújo and Stoeger [14]).

Equations (36) and (37) are the $\Lambda \neq 0$ analogues of the familiar characteristic FLRW $r_0 = r_0(z)$ and $M_0 = M_0(z)$ relationships for $\Lambda = 0$ (Ellis and Stoeger [24]; Stoeger, *et al.* [5]): If the Universe is FLRW and $\Lambda = 0$, $r_0(z)$ and $M_0(z)$ will have those functional forms. Equations (36) and (37)—or their elliptic-integral equivalents—are the corresponding characteristic functional forms for flat FLRW, $\Lambda \neq 0$ cases. As in the $\Lambda = 0$ cases, however, it is not a trivial conclusion that data which satisfies Eqs. (36) and (37) implies an FLRW, $\Lambda \neq 0$ universe. This must be demonstrated—and can be demonstrated—by using Eqs. (36) and (37) as data functions to solve the field equations and obtain an FLRW solution. This was done for the flat $\Lambda = 0$ case by Araújo and Stoeger [2]. We now do the same for these flat $\Lambda \neq 0$ cases.

Because of the additional parameter (Λ) in the equations, though, in the general (nonflat) case we need more data than supplied simply by Eqs. (36) and (37). When we know the Universe is flat, this extra data can be supplied by Eq. (37), which will then automatically enable us to calculate Ω_Λ—from the flatness condition. But also on $w = w_0$ we can observationally determine where $C_0 \equiv r_0$ reaches its maximum value $C_{0\max}$ and the redshift $z_{\max}$ at which this occurs. These measurements provide the needed extra data in the general case, and can also be used, instead of the data in Eq. (37), in the flat case. From Eq. (36), we can immediately determine the equation for this maximum redshift, which will be

$$dr_0(z)/dz = 0. \tag{38}$$

Plugging the observationally determined values of z_{max} into this equation, we obtain a unique relationship between z_{max} and Ω_Λ (Araújo and Stoeger [14]), since H_0 cancels out of Eq. (38). Using this relationship along with $C_{0\,max}$ in Eq. (36) will also determine H_0. The precise interpretation and definition of these parameters, e.g. that Ω_Λ—depending not only on Λ but also on H_0, the FLRW Hubble parameter—is the density parameter for Λ, is in reference to a supposed FLRW universe (H_0, and therefore Ω_Λ, cannot be unambiguously defined in a general exactly spherically symmetric—also often referred to as a Lemaître-Tolman-Bondi —universe; the definition of the rate of expansion given in Eq. (15) is not the only one that could be chosen, or measured). That assumption is validated by continuing with the integration and showing that a universe with such data is indeed FLRW.

Of course, this is a somewhat contrived case, since we would not attempt to fit the data to such a functional form (36) unless we already suspected that the Universe may be FLRW, and that therefore Ω_Λ and H_0, the FLRW density parameter and the Hubble parameter at $w = w_0$ and $y = 0$, can be defined in terms of an FLRW model. But besides being a very relevant simple case, it serves to illustrate the integration procedure with definite meaningful data input functions.

Solving the null Raychaudhuri Eq. (24) on $w = w_0$ with this data (see Stoeger *et al.* [5], and also Araújo and Stoeger [2]) yields the following relation between redshift and the null coordinate y:

$$\begin{aligned} 1 + z &= \frac{(1 - \sqrt{3}) - (1 + \sqrt{3})cn(Ly + \sigma)}{\sqrt[3]{\frac{1}{\Omega_\Lambda} - 1}[cn(Ly + \sigma) - 1]}, \\ L &\equiv -\sqrt{\Omega_\Lambda \sqrt{3}} A_0 H_0 \sqrt[3]{\frac{1}{\Omega_\Lambda} - 1} = -\frac{1}{w_0}, \\ \sigma &\equiv cn^{-1}[Q(0), \kappa]. \end{aligned} \tag{39}$$

Using Eq. (6) we can now write $A(w_0, y)$ as a function of y as

$$A(w_0, y) = -A_0\left(\sqrt[3]{\frac{1}{\Omega_\Lambda} - 1}\right) \times \frac{[1 - Cn(Ly + \sigma)]}{[(1 - \sqrt{3}) - (1 + \sqrt{3})Cn(Ly + \sigma)]}. \tag{40}$$

$A_0 \equiv A(w_0, 0)$ is a constant scaling factor which can be chosen arbitrarily.

If we wish to continue using our assumption that this will be an FLRW universe, then the observer area distance $C(w_0, y)$ as a function of y can be written as

$$C(w_0, y) = -A_0 y\left(\sqrt[3]{\frac{1}{\Omega_\Lambda} - 1}\right) \times \frac{[1 - Cn(Ly + \sigma)]}{[(1 - \sqrt{3}) - (1 + \sqrt{3})Cn(Ly + \sigma)]}. \tag{41}$$

If we do not wish to assume FLRW here, we can still find $C(w_0, y)$ by using Eq. (36) in conjunction the result given in Eq. (39).

Furthermore, we can clearly now determine what y_{max} is, corresponding to z_{max}, from Eq. (39).

Now we begin to move our solution off our past light cone, $w = w_0$. Since y is chosen to be a comoving radial coordinate, the functional dependence of $A(w, y)$ with respect to y cannot change as we move off our light cone. We have already mentioned the freedom we have, temporally setting aside central-condition considerations, to rescale the time coordinate w, which is affected by choosing $A(w, 0)$. Therefore, this freedom effectively corresponds to choosing the functional dependence of $A(w, y)$ with respect to w in any way we like, constrained only by the form of $A(w_0, y)$ (later this choice may have to be adjusted to satisfy all the central conditions, those on $C(w, y)$ and $B(w, y)$). In our expression for $A(w_0, y)$ is hidden an implicit dependence on w. We need to extract that dependence and make it explicit, so that we can then determine the general dependence of A on w and proceed with the integration. In general, this is not simply achieved by replacing w_0 with w, because—besides the w_0 dependence arising from setting $w = w_0$ when we write Eq. (6)—there is another part of the w_0 dependence which derives from integration constants of the null Raychaudhuri equation and remains through the entire problem.

At this point, accordingly to what we just pointed out, we arbitrarily set the w dependence for A and proceed with the integration. The next step is then the solution of Eqs. (19) and (32) to determine B and C respectively. Their formal general solutions are:

$$B = -\int A' dw + l(y), \tag{42}$$

where $l(y)$ is determined from the condition $A(w_0, y) = B(w_0, y)$, and

$$C = \left[-\left(\frac{6\omega_0}{\mu_\Lambda} + 1\right)C_0{}^3 \sinh^2\left(\sqrt{\frac{3\mu_\Lambda}{4}}\int A dw\right) + h(y)\right]^{1/3}, \tag{43}$$

where $h(y)$ is determined from the data $r_0 = C(w_0, y)$. Since we know $C(w_0, y)$ from Eq. (41), or from Eq. (36) with Eq. (39), $\omega_0(y)$ is obtained from Eq. (33) and is given by

$$\omega_0 = -\frac{1}{2C^2}\left[1 - \frac{(1 + z)C'}{A_0}\right]^2, \tag{44}$$

where we have used (3) and (6) to write

$$B(w_0, y) = A(w_0, 0)/[1 + z(y)]. \tag{45}$$

Here $z(y)$ is given by Eq. (39). When we do not know that the Universe is flat, we must, of course, first determine $\dot{C}(w_0, y)$, in order to determine $F(y)$ from Eq. (29). This can be easily done, as explained in Maartens *et al.* [18], in Araújo and Stoeger [14] and in the next section.

$B(w, y)$ and $C(w, y)$ are then determined by integrating Eqs. (19) and (32) with respect to w. $B(w, y)$ and $C(w, y)$ are further constrained, as discussed above, by the fact that they have to satisfy the central conditions (5). Now, it is clear from an examination of these equations that unless $A(w, y)$ has a very specific functional dependence on w the resulting solutions $B(w, y)$ and $C(w, y)$ will not satisfy the central conditions. That implies that, although we can find solutions to the field equations, it does not guarantee that the null surface on which we assume we have the data is a past light cone of our world line (Ellis *et al.* [4]). So we conclude that given the fulfilment of the following conditions:

(1) *$A(w_0, y)$ is determined by the data and the central conditions;*
(2) *The coordinate y is chosen to be a comoving radial coordinate;*
(3) *The central conditions* (5);

we can remove the freedom of rescaling the time coordinate w and completely determine $A(w, y)$. Thus, all the coordinate freedom in y and w has been used up at this stage.

Therefore, following the above analysis we find that the appropriate form for $A(w, y)$ is

$$A(w, y) = -A_0\left(\sqrt[3]{\frac{1}{\Omega_\Lambda} - 1}\right) \times \frac{1 - Cn[(\sigma - 1) + \frac{w-y}{w_0}]}{(1 - \sqrt{3}) - (1 + \sqrt{3})Cn[(\sigma - 1) + \frac{w-y}{w_0}]}, \tag{46}$$

where $z(y)$ is given by Eq. (39). Now, observing that

$$\frac{dz}{dy} = A_0 H_0 \sqrt{\Omega_\Lambda + (1 + z)^3(1 - \Omega_\Lambda)}, \tag{47}$$

we substitute $A'(w, y)$ and $A(w, y)$ into Eqs. (42) and (43) and determine the arbitrary functions of y that arise from these integrations by the conditions $B(w_0, y) = A(w_0, y)$ and $C(w_0, y) = r_0(y)$, respectively. Thus,

$$\begin{aligned} B(w, y) &= A(w, y) \\ &= -A_0\left(\sqrt[3]{\frac{1}{\Omega_\Lambda} - 1}\right) \\ &\quad \times \frac{1 - Cn[(\sigma - 1) + \frac{w-y}{w_0}]}{(1 - \sqrt{3}) - (1 + \sqrt{3})Cn[(\sigma - 1) + \frac{w-y}{w_0}]}, \end{aligned} \tag{48}$$

and

$$\begin{aligned} C(w, y) &= A(w, y)y \\ &= -A_0 y\left(\sqrt[3]{\frac{1}{\Omega_\Lambda} - 1}\right) \\ &\quad \times \frac{1 - Cn[(\sigma - 1) + \frac{w-y}{w_0}]}{(1 - \sqrt{3}) - (1 + \sqrt{3})Cn[(\sigma - 1) + \frac{w-y}{w_0}]}, \end{aligned} \tag{49}$$

which are the FLRW form of the solutions for $\Lambda \neq 0$ in observational coordinates. One can easily check (after some algebra) that the central conditions (5) are all satisfied, which in turn guarantees that the null surface on which we assume we have the data is indeed a past light cone of our world line.

VI. THE GENERAL SOLUTION IN THE NONFLAT CASE

We now outline the integration procedure in the case where we do not know whether the Universe is flat or not, and where data gives us redshifts z, observer area distances (angular-diameter distances) $r_0(z)$, "mass source densities" $M_0(z)$ which cannot be fit by the FLRW functional form and the angular-distance maximum $C_{\max}(w_0, z)$ at $z_{\max}$. It is important to specify the latter, because, as we have already emphasized, without them, we do not have enough information to determine all the parameters of the space-time in the $\Lambda \neq 0$ case. For instance, although we can determine $C(w_0, z)$ with good precision (by obtaining luminosity distances d_L and employing the reciprocity theorem, Eq. (8)) out to relatively high redshifts, at present we do not yet have reliable data deep enough to determine $C_{\max}$ and $z_{\max}$. But this has just recently become possible with precise space-telescope distance measurements of distance for supernovae Ia.

In pursuing the general integration with these data, we use the framework and the intermediate results we have presented in Sec. IV. Obviously, one of the key steps we must take now is the determination of the "potential" $F(y)$, given by Eq. (29). This was done in a similar way for $\Lambda = 0$ by Araújo and Stoeger [2], as indicated above. This means we need to determine $C'(w_0, y)$ and $\dot{C}(w_0, y)$, which we now write as C_0' and $\dot{C}_0$, respectively. We also need $A(w_0, y)$. We remember, too, that on $w = w_0$ we have chosen $B(w_0, y) = A(w_0, y)$, which we have the freedom to do.

Clearly, C_0' can be determined from the $r_0(z) \equiv C(w_0, z)$ data, through fitting, along with the solution of the null Raychaudhuri Eq. (24), as indicated in Sec. V, to obtain $z = z(y)$. $A(w_0, y)$, too, is obtained from redshift data along with this same $z(y)$ result. $\dot{C}_0$ is somewhat more difficult to determine. But the procedure is straightforward.

We determine $\dot{C}_0$ by solving Eq. (26) for it on $w = w_0$. Using Eqs. (3) and (19), we can write this now as

$$\frac{\dot{C}_0'(y)}{C_0(y)} = -\frac{A_0'(y)C_0'(y)}{A_0(y)C_0(y)} - A_0^2(y)(\omega_0(y) + \mu_\Lambda/2). \quad (50)$$

But, from Eq. (23) we can write $\omega_0(y)$ in terms of $C_0(y)$, $C_0'(y)$ and $\dot{C}_0(y)$. So Eq. (50) becomes

$$\dot{C}_0'(y) + \frac{C_0'(y)\dot{C}_0(y)}{C_0(y)} = \frac{A_0^2(y)}{2C_0(y)} - \frac{A_0'(y)}{A_0(y)}C_0'(y) - \frac{(C_0'(y))^2}{2C_0(y)} + \frac{A_0^2(y)C_0(y)}{2}\mu_\Lambda. \quad (51)$$

This is a linear differential equation for $\dot{C}_0(y)$, where from data we know everything on our past light cone, $w = w_0$, (once the null Raychaudhuri Eq. (24) has been solved) except $\dot{C}_0(y)$ itself and μ_Λ, which is a constant that can be carried along and determined subsequently from $C(w_0, z_{max})$ and z_{max} measurements (see below). Thus, we can easily solve Eq. (51) for $\dot{C}_0(y)$, which will also depend on the unknown constant μ_Λ.

However, introducing this result back into Eq. (51), and evaluating it at y_{max}, which corresponds to z_{max}, we have simply

$$\dot{C}'_0(y_{max}) = \frac{A_0^2(y_{max})}{2C_0(y_{max})} - \frac{A_0^2(y_{max})C_0(y_{max})}{2}\mu_\Lambda, \quad (52)$$

where $\dot{C}_0'(y_{max})$, as we have already emphasized, also depends on the unknown μ_Λ. Since everything else is now known, Eq. (52) is now an algebraic equation for $\mu_\Lambda(y_{max})$, or equivalently for $\mu_\Lambda(z_{max})$. Obviously, we could have simply worked out this result in terms of z_{max} to begin with.

With this determination of μ_Λ, we know $\dot{C}_0(y)$ completely, and can now determine $F(y)$ from Eq. (29). From there on, we can follow the solution off $w = w_0$ for all w, as we have outlined in Sec. V. Obviously, we shall obtain very different results than we did there for flat FLRW data—depending on the exact character of our more general data. This completes the framework for solving these exact spherically symmetric field equations for adequate data when $\Lambda \neq 0$ and the space-time is not flat.

VII. CONCLUSION

In this paper we have shown in detail how to construct flat dust-filled $\Lambda \neq 0$ Friedmann-Lemaître-Robertson-Walker cosmological models from FLRW cosmological data on our past light cone, by integrating the exact spherically symmetric Einstein field equations in observational coordinates, extending the approach developed by Araújo and Stoeger [2] and improved in Araújo *et al.* [3]. Besides being of interest in its own right—demonstrating how such data fully determines the models—it also illustrates in a simple case how the more general SS Einstein equations can be integrated in observational coordinates with data arrayed on our past light cone, $w = w_0$. Then, we have gone on to show how to integrate these exact spherically symmetric (LBT) equations, also for $\Lambda \neq 0$ in cases where the data are not FLRW, and the space-time is not known to be flat. It is essential for these to have data giving the maximum of the observer area (angular-diameter) distance, $C_0(w_0, z_{max})$, and the redshift z_{max} at which that occurs. This enables the determination of the vacuum-energy density μ_Λ, which would otherwise remain undetermined.

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