Cosmography of $f(R)$ gravity

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It is nowadays accepted that the universe is undergoing a phase of accelerated expansion as tested by the Hubble diagram of type Ia supernovae (SNeIa) and several large scale structure observations. Future SNeIa surveys and other probes will make it possible to better characterize the dynamical state of the universe, renewing the interest in cosmography which allows a model independent analysis of the distance-redshift relation. On the other hand, fourth order theories of gravity, also referred to as $f(R)$ gravity, have attracted a lot of interest since they could be able to explain the accelerated expansion without any dark energy. We show here how it is possible to relate the cosmographic parameters (namely, the deceleration q_0, the jerk j_0, the snap s_0, and the lerk l_0 parameters) to the present-day values of $f(R)$ and its derivatives $f^{(n)}(R) = d^n f/dR^n$ (with $n = 1, 2, 3$), thus offering a new tool to constrain such higher order models. Our analysis thus offers the possibility to relate the model independent results coming from cosmography to the theoretically motivated assumptions of $f(R)$ cosmology.

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I. INTRODUCTION

As soon as astrophysicists realized that type Ia supernovae (SNeIa) were standard candles, it appeared evident that their high luminosity should make it possible to build a Hubble diagram, i.e. a plot of the distance-redshift relation, over some cosmologically interesting distance ranges. Motivated by this attractive consideration, two independent teams started SNeIa surveys leading to the unexpected discovery that the universe expansion is speeding up rather than decelerating [1]. This surprising result has now been strengthened by more recent data coming from SNeIa surveys [2–7], large scale structure [8], and the cosmic microwave background (CMBR) anisotropy spectrum [9–11]. This large data set coherently points toward the picture of a spatially flat universe undergoing an accelerated expansion driven by a dominant negative pressure fluid, typically referred to as *dark energy* [12].

While there is a wide consensus on the above scenario depicted by such good quality data, there is a similarly wide range of contrasting proposals to solve the dark energy puzzle. Surprisingly, the simplest explanation, namely, the cosmological constant Λ [13], is also the best one from a statistical point of view [14]. Unfortunately, the well-known coincidence and 120 orders of magnitude problems render Λ a rather unattractive solution from a theoretical point of view. Inspired by the analogy with inflation, a scalar field ϕ, dubbed *quintessence* [15], has then been proposed to give a dynamical Λ term in order to both fit the data and avoid the above problems. However, such models are still plagued by difficulties on their own, such as the almost complete freedom in the choice of the scalar field potential and the fine-tuning of the initial conditions. Needless to say, a plethora of alternative models are now on the market, all sharing the main property of being in agreement with observations, but relying on completely different physics.

Notwithstanding their differences, all the dark energy based theories assume that the observed acceleration is the outcome of the action of an up to now undetected ingredient to be added to the cosmic pie. In terms of the Einstein equations, $G_{\mu\nu} = \chi T_{\mu\nu}$, such models are simply modifying the right-hand side, including in the stress-energy tensor something more than the usual matter and radiation components.

As a radically different approach, one can also try to leave unchanged the source side, and rather modify the left-hand side. In a sense, one is therefore interpreting cosmic speedup as a first signal of the breakdown of the laws of physics as described by the standard general relativity (GR). Since this theory has been experimentally tested only up to the Solar System scale, there is no *a priori* theoretical motivation to extend its validity to extraordinarily larger scales such as the cosmological ones (e.g. the last scattering surface). Extending GR, still retaining its positive results, opens the way for a large class of alternative theories of gravity ranging from extra dimensions [16] to nonminimally coupled scalar fields [17,18]. In particular, we will be interested here in fourth order theories [19,20] based on replacing the scalar curvature R in the Hilbert-Einstein action with a generic analytic function $f(R)$ which should be reconstructed starting from data and physically motivated issues. Also referred to as $f(R)$ gravity, these models have been shown to be able to both fit the cosmological data and evade the Solar System constraints in several physically interesting cases [21–25].

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It is worth noting that both dark energy models and modified gravity theories have been shown to be in agreement with the data. As a consequence, unless higher precision probes of the expansion rate and the growth of structure will be available, these two rival approaches could not be discriminated. This confusion about the theoretical background suggests that a more conservative approach to the problem of cosmic acceleration, relying on as few model dependent quantities as possible, is welcome. A possible solution could be to come back to the cosmography [26] rather than finding solutions of the Friedmann equations and testing them. Being only related to the derivatives of the scale factor, the cosmographic parameters make it possible to fit the data on the distance-redshift relation without any *a priori* assumption on the underlying cosmological model: in this case, the only assumption is that the metric is the Robertson-Walker one (and hence does not rely on the solution of cosmic equations). Almost a century after Hubble discovery of the expansion of the universe, we can now extend cosmography beyond the search for the value of the Hubble constant. The SNeIa Hubble diagram extends up to $z = 1.7$, thus invoking the need for, at least, a fifth order Taylor expansion of the scale factor in order to give a reliable approximation of the distance-redshift relation. As a consequence, it could be, in principle, possible to estimate up to five cosmographic parameters, although the still too small data set available does not allow one to get a precise and realistic determination of all of them.

Once these quantities have been determined, one could use them to put constraints on the models. In a sense, we are reversing the usual approach consisting in deriving the cosmographic parameters as a sort of by-product of an assumed theory. Here, we follow the other way around, expressing the model characterizing quantities as a function of the cosmographic parameters. Such a program is particularly suited for the study of fourth order theories of gravity. As is well known, the mathematical difficulties entering the solution of fourth order field equations make it quite problematic to find analytical expressions for the scale factor and hence predict the values of the cosmographic parameters. A key role in $f(R)$ gravity is played by the choice of the $f(R)$ function. Under quite general hypotheses, we will derive useful relations among the cosmographic parameters and the present-day value of $f^{(n)}(R) = d^n f/dR^n$, with $n = 0, \ldots, 3$, whatever $f(R)$ is.[1] Once the cosmographic parameters are determined, this method will allow us to investigate the cosmography of $f(R)$ theories.

The layout of the paper is as follows. Sections II and III are devoted to introducing the basic notions of the cosmographic parameters and $f(R)$ gravity, respectively, summarizing the main formulas we will use later. Section IV contains the main result of the paper, demonstrating how the $f(R)$ derivatives can be related to the cosmographic parameters. Since the latter are not well determined today, we will discuss, in Sec. V, how these formulas can be adapted to a different parametrization relying on expressing the cosmographic parameters in terms of a phenomenological assumption for the dark energy equation of state (EoS). Section VI illustrates a possible application of the relation among $f(R)$ derivatives and cosmographic parameters showing how one can constrain the parameters of a given $f(R)$ model. Since future data will likely determine with a sufficient precision at least the first two cosmographic parameters, it is worth estimating how this will impact the determination of the $f(R)$ quantities, which is the argument of Sec. VII. We then summarize and conclude in Sec. VIII.

II. COSMOGRAPHIC PARAMETERS

Standard candles (such as SNeIa and, to a limited extent, gamma ray bursts) are ideal tools in modern cosmology since they make it possible to reconstruct the Hubble diagram, i.e. the redshift-distance relation up to high redshift values. It is then customary to assume a parametrized model (such as the concordance ΛCDM one, or any other kind of dark energy scenario) and contrast it against the data to check its viability and constrain its characterizing parameters. As it is clear, such an approach is model dependent so some doubts always remain on the validity of the constraints on derived quantities such as the present-day values of the deceleration parameter and the age of the universe. In order to overcome such a problem, one may resort to cosmography, i.e. expanding the scale factor in Taylor series with respect to the cosmic time [26]. Such an expansion leads to a distance-redshift relation which only relies on the assumption of the Robertson-Walker metric, thus being fully model independent since it does not depend on the particular form of the solution of cosmic equations. To this aim, it is convenient to introduce the following functions [26,28]:

$$H = \frac{1}{a}\frac{da}{dt}, \qquad q = -\frac{1}{a}\frac{d^2a}{dt^2}H^{-2}, \qquad j = \frac{1}{a}\frac{d^3a}{dt^3}H^{-3},$$
$$s = \frac{1}{a}\frac{d^4a}{dt^4}H^{-4}, \qquad l = \frac{1}{a}\frac{d^5a}{dt^5}H^{-5}, \tag{1}$$

which are usually referred to as the *Hubble, deceleration, jerk, snap*, and *lerk* parameters [29], respectively.[2] Their present-day values (which we will denote with a subscript 0) may be used to characterize the evolutionary status of the Universe. For instance, $q_0 < 0$ denotes an accelerated

[1] As an important remark, we stress that our derivation will rely on the metric formulation of $f(R)$ theories, while we refer the reader to [27] for a similar work in the Palatini approach.

[2] Note that the use of the jerk parameter to discriminate between different models was also proposed in [30] in the context of the *statefinder* parametrization.

expansion, while j_0 allows one to discriminate among different accelerating models.

It is then a matter of algebra to demonstrate the following useful relations:

$$\dot{H} = -H^2(1+q), \tag{2}$$

$$\ddot{H} = H^3(j+3q+2), \tag{3}$$

$$\dddot{H} = H^4[s-4j-3q(q+4)-6], \tag{4}$$

$$d^4H/dt^4 = H^5[l-5s+10(q+2)j+30(q+2)q+24], \tag{5}$$

where a dot denotes the derivative with respect to the cosmic time t. Equations (2)–(5) make it possible to relate the derivative of the Hubble parameter to the other cosmographic parameters. The distance-redshift relation may then be obtained starting from the Taylor expansion of $a(t)$ along the lines described in [28,31,32]. The result for the fifth order is reported in the Appendix.

It is worth stressing that the definition of the cosmographic parameters only relies on the assumption of the Robertson-Walker metric. As such, it is however difficult to state *a priori* to what extent the fifth order expansion provides an accurate enough description of the quantities of interest. Actually, the number of cosmographic parameters to be used depends on the problem one is interested in. As we will see later, here we are concerned only with the SNeIa Hubble diagram so we have to check that the distance modulus $\mu_{cp}(z)$ obtained using the fifth order expansion of the scale factor is the same (within the errors) as the one $\mu_{\rm DE}(z)$ of the underlying physical model. Since such a model is of course unknown, one can adopt a phenomenological parametrization for the dark energy[3] EoS and look at the percentage deviation $\Delta\mu/\mu_{\rm DE}$ as a function of the EoS parameters. We have carried out such an exercise using the Chevallier-Polarski-Linder (CPL) model introduced later, and verified that $\Delta\mu/\mu_{\rm DE}$ is an increasing function of z (as expected), but still remains smaller than 2% up to $z \sim 2$ over a wide range of the CPL parameter space. On the other hand, halting the Taylor expansion to a lower order may introduce significant deviation for $z > 1$ that can potentially bias the analysis if the measurement errors are as small as those predicted for future SNeIa surveys. We are therefore confident that our fifth order expansion is both sufficient to get an accurate distance modulus over the redshift range probed by SNeIa and necessary to avoid dangerous biases.

[3]Note that one can always use a phenomenological dark energy model to get a reliable estimate of the scale factor evolution even if the correct model is a fourth order one.

III. $f(R)$ GRAVITY

Much interest has been recently devoted to a form of quintessence induced by curvature according to which the present universe is filled by pressureless dust matter only and the acceleration is the result of the modified Friedmann equations obtained by replacing the Ricci curvature scalar R with a generic function $f(R)$ in the gravity action [19,20]. Under the assumption of a flat universe, the Hubble parameter is therefore determined by[4]

$$H^2 = \frac{1}{3}\left[\frac{\rho_m}{f'(R)} + \rho_{\rm curv}\right] \tag{6}$$

where the prime denotes the derivative with respect to R and $\rho_{\rm curv}$ is the energy density of an *effective curvature fluid*[5]:

$$\rho_{\rm curv} = \frac{1}{f'(R)}\left\{\frac{1}{2}[f(R) - Rf'(R)] - 3H\dot{R}f''(R)\right\}. \tag{7}$$

Assuming there is no interaction between the matter and the curvature terms (we are in the so-called *Jordan frame*), the matter continuity equation gives the usual scaling $\rho_M = \rho_M(t=t_0)a^{-3} = 3H_0^2\Omega_M a^{-3}$, with Ω_M the present-day matter density parameter. The continuity equation for $\rho_{\rm curv}$ then reads

$$\dot{\rho}_{\rm curv} + 3H(1+w_{\rm curv})\rho_{\rm curv} = \frac{3H_0^2\Omega_M\dot{R}f''(R)}{[f'(R)]^2}a^{-3} \tag{8}$$

with

$$w_{\rm curv} = -1 + \frac{\ddot{R}f''(R) + \dot{R}[\dot{R}f'''(R) - Hf''(R)]}{[f(R)-Rf'(R)]/2 - 3H\dot{R}f''(R)} \tag{9}$$

the barotropic factor of the curvature fluid. It is worth noticing that the curvature fluid quantities $\rho_{\rm curv}$ and $w_{\rm curv}$ only depend on $f(R)$ and its derivatives up to the third order. As a consequence, considering only their present-day values (which may be naively obtained by replacing R with R_0 everywhere), two $f(R)$ theories sharing the same values of $f(R_0)$, $f'(R_0)$, $f''(R_0)$, $f'''(R_0)$ will be degenerate from this point of view.[6]

Combining Eq. (8) with Eq. (6), one finally gets the following *master equation* for the Hubble parameter:

[4]We use here natural units such that $8\pi G = 1$.

[5]Note that the name *curvature fluid* does not refer to the Friedmann-Robertson-Walker curvature parameter k, but only takes into account that such a term is a geometrical one related to the scalar curvature R.

[6]One can argue that this is not strictly true since different $f(R)$ theories will lead to different expansion rates $H(t)$ and hence different present-day values of R and its derivatives. However, it is likely that two $f(R)$ functions that exactly match each other up to the third order derivative today will give rise to the same $H(t)$ at least for $t \simeq t_0$ so that $(R_0, \dot{R}_0, \ddot{R}_0)$ will be almost the same.

$$\dot{H} = -\frac{1}{2f'(R)}\{3H_0^2\Omega_M a^{-3} + \ddot{R}f''(R) + \dot{R}[\dot{R}f'''(R) - Hf''(R)]\}. \qquad (10)$$

Expressing the scalar curvature R as a function of the Hubble parameter as

$$R = -6(\dot{H} + 2H^2) \qquad (11)$$

and inserting the result into Eq. (10), one ends with a fourth order nonlinear differential equation for the scale factor $a(t)$ that cannot be easily solved also for the simplest cases [for instance, $f(R) \propto R^n$]. Moreover, although technically feasible, a numerical solution of Eq. (10) is plagued by the large uncertainties on the boundary conditions (i.e., the present-day values of the scale factor and its derivatives up to the third order) that have to be set to find the scale factor.

IV. $f(R)$ DERIVATIVES VS COSMOGRAPHY

Motivated by these difficulties, we approach now the problem from a different viewpoint. Rather than choosing a parametrized expression for $f(R)$ and then numerically solving Eq. (10) for given values of the boundary conditions, we try to relate the present-day values of its derivatives to the cosmographic parameters (q_0, j_0, s_0, l_0) so that constraining them in a model independent way gives us a hint for what kind of $f(R)$ theory could be able to fit the observed Hubble diagram.[7]

As a preliminary step, it is worth considering again the constraint equation (11). Differentiating with respect to t, we easily get the following relations:

$$\dot{R} = -6(\ddot{H} + 4H\dot{H}), \qquad \ddot{R} = -6(\dddot{H} + 4H\ddot{H} + 4\dot{H}^2),$$
$$\dddot{R} = -6(d^4H/dt^4 + 4\dddot{H}H + 12\dot{H}\ddot{H}). \qquad (12)$$

Evaluating these at the present time and using Eqs. (2)–(5), one finally gets

$$R_0 = -6H_0^2(1 - q_0), \qquad (13)$$

$$\dot{R}_0 = -6H_0^3(j_0 - q_0 - 2), \qquad (14)$$

$$\ddot{R}_0 = -6H_0^4(s_0 + q_0^2 + 8q_0 + 6), \qquad (15)$$

$$\dddot{R}_0 = -6H_0^5[l_0 - s_0 + 2(q_0 + 4)j_0 - 6(3q_0 + 8)q_0 - 24], \qquad (16)$$

which will turn out to be useful in the following.

[7] Note that a similar analysis, but in the context of the energy conditions in $f(R)$, has been presented in [33]. However, in that paper, the author gives an expression for $f(R)$ and then computes the snap parameter to be compared to the observed one. On the contrary, our analysis does not depend on any assumed functional expression for $f(R)$.

Let us now come back to the expansion rate and master equations (6) and (10). Since they have to hold along the full evolutionary history of the universe, they naively hold also at present. As a consequence, we may evaluate them in $t = t_0$, thus easily obtaining

$$H_0^2 = \frac{H_0^2\Omega_M}{f'(R_0)} + \frac{f(R_0) - R_0 f'(R_0) - 6H_0\dot{R}_0 f''(R_0)}{6f'(R_0)}, \qquad (17)$$

$$-\dot{H}_0 = \frac{3H_0^2\Omega_M}{2f'(R_0)} + \frac{\dot{R}_0^2 f'''(R_0) + (\ddot{R}_0 - H_0\dot{R}_0)f''(R_0)}{2f'(R_0)}. \qquad (18)$$

Using Eqs. (2)–(5) and (13)–(16), we can rearrange Eqs. (17) and (18) as two relations among the Hubble constant H_0 and the cosmographic parameters (q_0, j_0, s_0), on one hand, and the present-day values of $f(R)$ and its derivatives up to third order. However, two further relations are needed in order to close the system and determine the four unknown quantities $f(R_0)$, $f'(R_0)$, $f''(R_0)$, $f'''(R_0)$. The first one may be easily obtained by noting that, inserting back the physical units, the rate expansion equation reads

$$H^2 = \frac{8\pi G}{3f'(R)}[\rho_m + \rho_{\rm curv}f'(R)]$$

which clearly shows that, in $f(R)$ gravity, the Newtonian gravitational constant G is replaced by an effective (time dependent) $G_{\rm eff} = G/f'(R)$. On the other hand, it is reasonable to assume that the present-day value of $G_{\rm eff}$ is the same as the Newtonian one, so we get the simple constraint

$$G_{\rm eff}(z = 0) = G \rightarrow f'(R_0) = 1. \qquad (19)$$

In order to get the fourth relation we need to close the system, we first differentiate both sides of Eq. (10) with respect to t. We thus get

$$\ddot{H} = \frac{\dot{R}^2 f'''(R) + (\ddot{R} - H\dot{R})f''(R) + 3H_0^2\Omega_M a^{-3}}{2[\dot{R}f''(R)]^{-1}[f'(R)]^2} - \frac{\dot{R}^3 f^{(iv)}(R) + (3\dot{R}\ddot{R} - H\dot{R}^2)f'''(R)}{2f'(R)} - \frac{(\dddot{R} - H\ddot{R} + \dot{H}\dot{R})f''(R) - 9H_0^2\Omega_M H a^{-3}}{2f'(R)}, \qquad (20)$$

with $f^{(iv)}(R) = d^4f/dR^4$. Let us now suppose that $f(R)$ may be well approximated by its third order Taylor expansion in $R - R_0$, i.e. we set

$$f(R) = f(R_0) + f'(R_0)(R - R_0) + \tfrac{1}{2}f''(R_0)(R - R_0)^2 + \tfrac{1}{6}f'''(R_0)(R - R_0)^3. \qquad (21)$$

In such an approximation, $f^{(n)}(R) = d^n f/R^n = 0$ for $n \geq 4$ so that naively $f^{(iv)}(R_0) = 0$. Evaluating then Eq. (20) at present, we get

$$\ddot{H}_0 = \frac{\dot{R}_0^2 f'''(R_0) + (\ddot{R}_0 - H_0\dot{R}_0)f''(R_0) + 3H_0^2\Omega_M}{2[\dot{R}_0 f''(R_0)]^{-1}[f'(R_0)]^2} - \frac{(3\dot{R}_0\ddot{R}_0 - H\dot{R}_0^2)f'''(R_0)}{2f'(R_0)} - \frac{(\dddot{R}_0 - H_0\ddot{R}_0 + \dot{H}_0\dot{R}_0)f''(R_0) - 9H_0^3\Omega_M}{2f'(R_0)}. \quad (22)$$

We can now schematically proceed as follows. Evaluate Eqs. (2)–(5) at $z = 0$ and plug these relations into the left-hand sides of Eqs. (17), (18), and (22). Insert Eqs. (13)–(16) into the right-hand sides of these same equations so that only the cosmographic parameters (q_0, j_0, s_0, l_0) and the $f(R)$ related quantities enter both sides of these relations. Finally, solve them under the constraint (19) with respect to the present-day values of $f(R)$ and its derivatives up to the third order. After some algebra, one ends up with the desired result:

$$\frac{f(R_0)}{6H_0^2} = -\frac{\mathcal{P}_0(q_0, j_0, s_0, l_0)\Omega_M + \mathcal{Q}_0(q_0, j_0, s_0, l_0)}{\mathcal{R}(q_0, j_0, s_0, l_0)}, \quad (23)$$

$$f'(R_0) = 1, \quad (24)$$

$$\frac{f''(R_0)}{(6H_0^2)^{-1}} = -\frac{\mathcal{P}_2(q_0, j_0, s_0)\Omega_M + \mathcal{Q}_2(q_0, j_0, s_0)}{\mathcal{R}(q_0, j_0, s_0, l_0)}, \quad (25)$$

$$\frac{f'''(R_0)}{(6H_0^2)^{-2}} = -\frac{\mathcal{P}_3(q_0, j_0, s_0, l_0)\Omega_M + \mathcal{Q}_3(q_0, j_0, s_0, l_0)}{(j_0 - q_0 - 2)\mathcal{R}(q_0, j_0, s_0, l_0)}, \quad (26)$$

where we have defined

$$\mathcal{P}_0 = (j_0 - q_0 - 2)l_0 - (3s_0 + 7j_0 + 6q_0^2 + 41q_0 + 22)s_0 - [(3q_0 + 16)j_0 + 20q_0^2 + 64q_0 + 12]j_0 - (3q_0^4 + 25q_0^3 + 96q_0^2 + 72q_0 + 20), \quad (27)$$

$$\mathcal{Q}_0 = (q_0^2 - j_0q_0 + 2q_0)l_0 + [3q_0s_0 + (4q_0 + 6)j_0 + 6q_0^3 + 44q_0^2 + 22q_0 - 12]s_0 + [2j_0^2 + (3q_0^2 + 10q_0 - 6)j_0 + 17q_0^3 + 52q_0^2 + 54q_0 + 36]j_0 + 3q_0^5 + 28q_0^4 + 118q_0^3 + 72q_0^2 - 76q_0 - 64, \quad (28)$$

$$\mathcal{P}_2 = 9s_0 + 6j_0 + 9q_0^2 + 66q_0 + 42, \quad (29)$$

$$\mathcal{Q}_2 = -\{6(q_0 + 1)s_0 + [2j_0 - 2(1 - q_0)]j_0 + 6q_0^3 + 50q_0^2 + 74q_0 + 32\}, \quad (30)$$

$$\mathcal{P}_3 = 3l_0 + 3s_0 - 9(q_0 + 4)j_0 - (45q_0^2 + 78q_0 + 12), \quad (31)$$

$$\mathcal{Q}_3 = -\{2(1 + q_0)l_0 + 2(j_0 + q_0)s_0 - (2j_0 + 4q_0^2 + 12q_0 + 6)j_0 - (30q_0^3 + 84q_0^2 + 78q_0 + 24)\}, \quad (32)$$

$$\mathcal{R} = (j_0 - q_0 - 2)l_0 - (3s_0 - 2j_0 + 6q_0^2 + 50q_0 + 40)s_0 + [(3q_0 + 10)j_0 + 11q_0^2 + 4q_0 - 18]j_0 - (3q_0^4 + 34q_0^3 + 246q_0 + 104). \quad (33)$$

Equations (23)–(33) make it possible to estimate the present-day values of $f(R)$ and its first three derivatives as a function of the Hubble constant H_0 and the cosmographic parameters (q_0, j_0, s_0, l_0) provided a value for the matter density parameter Ω_M is given. This is a somewhat problematic point. Indeed, while the cosmographic parameters may be estimated in a model independent way, the fiducial value for Ω_M is usually the outcome of fitting a given data set in the framework of an assumed dark energy scenario. However, it is worth noting that different models all converge towards the concordance value $\Omega_M \simeq 0.25$ which is also in agreement with astrophysical (model independent) estimates from the gas mass fraction in galaxy clusters. On the other hand, it has been proposed that $f(R)$ theories may avoid the need for dark matter in galaxies and galaxy clusters [34]. In such a case, the total matter content of the universe is essentially equal to the baryonic one. According to the primordial element abundance and the standard big bang nucleosynthesis scenario, we therefore get $\Omega_M \simeq \omega_b/h^2$ with $\omega_b = \Omega_b h^2 \simeq 0.0214$ [35] and h the Hubble constant in units of 100 km/s/Mpc. Setting $h = 0.72$ in agreement with the results of the HST Key project [36], we thus get $\Omega_M = 0.041$ for a baryons-only universe. We will therefore consider in the following both cases when numerical estimates are needed.

It is worth noticing that H_0 only plays the role of a scaling parameter, giving the correct physical dimensions to $f(R)$ and its derivatives. As such, it is not surprising that we need four cosmographic parameters, namely, (q_0, j_0, s_0, l_0), to fix the four $f(R)$ related quantities $f(R_0)$, $f'(R_0)$, $f''(R_0)$, $f'''(R_0)$. It is also worth stressing that Eqs. (23)–(26) are linear in the $f(R)$ quantities so that (q_0, j_0, s_0, l_0) uniquely determine the former ones. On the contrary, inverting them to get the cosmographic parameters as a function of the $f(R)$ ones, we do not get linear relations. Indeed, the field equations in $f(R)$ theories are nonlinear fourth order differential equations in the scale factor $a(t)$ so that fixing the derivatives of $f(R)$ up to third order makes it possible to find a class of solutions, not a single one. Each one of these solutions will be characterized by a different set of cosmographic parameters thus explaining why the inversion of Eqs. (23)–(33) does not give a unique result for (q_0, j_0, s_0, l_0).

As a final comment, we reconsider the underlying assumptions leading to the above derived relations. While Eqs. (17) and (18) are exact relations deriving from a rigorous application of the field equations, Eq. (22) heavily relies on having approximated $f(R)$ with its third order Taylor expansion (21). If this assumption fails, the system should not be closed since a fifth unknown parameter enters the game, namely, $f^{(iv)}(R_0)$. Actually, replacing $f(R)$ with its Taylor expansion is not possible for all classes of $f(R)$ theories. As such, the above results only hold in those cases where such an expansion is possible. Moreover, by truncating the expansion to the third order, we are implicitly assuming that higher order terms are negligible over the redshift range probed by the data. That is to say, we are assuming that

$$f^{(n)}(R_0)(R-R_0)^n \ll \sum_{m=0}^{3} \frac{f^{(m)}(R_0)}{m!}(R-R_0)^m \quad \text{for } n \geq 4 \tag{34}$$

over the redshift range probed by the data. Checking the validity of this assumption is not possible without explicitly solving the field equations, but we can guess an order of magnitude estimate considering that, for all viable models, the background dynamics should not differ too much from the ΛCDM one at least up to $z \simeq 2$. Using then the expression of $H(z)$ for the ΛCDM model, it is easy to see that R/R_0 is a quickly increasing function of the redshift so, in order for Eq. (34) to hold, we have to assume that $f^{(n)}(R_0) \ll f'''(R_0)$ for $n \geq 4$. This condition is easier to check for many analytical $f(R)$ models.

Once such a relation is verified, we still have to worry about Eq. (19) relying on the assumption that the *cosmological* gravitational constant is *exactly* the same as the *local* one, i.e. the same as the one measured in the laboratory and entering the Newtonian Poisson equation. Actually, the *cosmological* gravitational constant should be identified with the one entering the perturbation equations for a given $f(R)$ model. Comparing the Newtonian G_N and this *cosmological* G, one could infer whether the G entering the background equations is the same as the local one. Although this is outside our aims here, we can, in a first reasonable approximation, argue that the condition $G_{\rm local} = G_{\rm cosmo}$ could be replaced by the weaker relation $G_{\rm eff}(z=0) = G(1+\varepsilon)$ with $\varepsilon \ll 1$. In this case, we should repeat the derivation of Eqs. (23)–(26) now using the condition $f'(R_0) = (1+\varepsilon)^{-1}$. Taylor expanding the results in ε to the first order and comparing with the above derived equations, we can estimate the error induced by our assumption $\varepsilon = 0$. The resulting expressions are too lengthy to be reported and depend in a complicated way on the values of the matter density parameter Ω_M, the cosmographic parameters (q_0, j_0, s_0, l_0), and ε. However, we have numerically checked that the errors induced on $f(R_0)$, $f''(R_0)$, $f'''(R_0)$ are much lower than 10% for a value of ε as high as an unrealistic $\varepsilon \sim 0.1$. We are therefore confident that our results are reliable also under such conditions.

V. $f(R)$ DERIVATIVES AND CPL MODELS

In order to determine the present-day values of $f(R)$ and its first three derivatives, one should first estimate the cosmographic parameters from the observational data in a model independent way. Unfortunately, even in the present era of *precision cosmology*, such a program is still too ambitious to give useful constraints on the $f(R)$ derivatives, as we will see later. On the other hand, the cosmographic parameters may also be expressed in terms of the dark energy density and EoS parameters, so we can work out the present-day values of $f(R)$ and its derivatives, giving the same (q_0, j_0, s_0, l_0) of the given dark energy model. To this aim, it is convenient to adopt a parametrized expression for the dark energy EoS in order to reduce the dependence of the results on any underlying theoretical scenario. Following the prescription of the Dark Energy Task Force [37], we will use the CPL parametrization for the EoS setting [38]:

$$w = w_0 + w_a(1-a) = w_0 + w_a z(1+z)^{-1} \tag{35}$$

so that, in a flat universe filled by dust matter and dark energy, the dimensionless Hubble parameter $E(z) = H/H_0$ reads

$$E^2(z) = \Omega_M(1+z)^3 + \Omega_X(1+z)^{3(1+w_0+w_a)} e^{-(3w_a z/1+z)} \tag{36}$$

with $\Omega_X = 1 - \Omega_M$ because of the flatness assumption. In order to determine the cosmographic parameters for such a model, we avoid integrating $H(z)$ to get $a(t)$ by noting that $d/dt = -(1+z)H(z)d/dz$. We can use such a relation to evaluate $(\dot{H}, \ddot{H}, \dddot{H}, d^4H/dt^4)$ and then solve Eqs. (2)–(5), evaluated in $z = 0$, with respect to the parameters of interest. Some algebra finally gives

$$q_0 = \tfrac{1}{2} + \tfrac{3}{2}(1-\Omega_M)w_0, \tag{37}$$

$$j_0 = 1 + \tfrac{3}{2}(1-\Omega_M)[3w_0(1+w_0) + w_a], \tag{38}$$

$$\begin{aligned} s_0 = {} & -\frac{7}{2} - \frac{33}{4}(1-\Omega_M)w_a - \frac{9}{4}(1-\Omega_M) \\ & \times [9 + (7-\Omega_M)w_a]w_0 - \frac{9}{4}(1-\Omega_M) \\ & \times (16 - 3\Omega_M)w_0^2 - \frac{27}{4}(1-\Omega_M)(3-\Omega_M)w_0^3, \end{aligned} \tag{39}$$

$$l_0 = \frac{35}{2} + \frac{1-\Omega_M}{4}[213 + (7-\Omega_M)w_a]w_a + \frac{1-\Omega_M)}{4} \times [489 + 9(82 - 21\Omega_M)w_a]w_0 + \frac{9}{2}(1-\Omega_M) \times \left[67 - 21\Omega_M + \frac{3}{2}(23 - 11\Omega_M)w_a\right]w_0^2 + \frac{27}{4}(1-\Omega_M)(47 - 24\Omega_M)w_0^3 + \frac{81}{2}(1-\Omega_M)(3 - 2\Omega_M)w_0^4. \quad (40)$$

Inserting Eqs. (37)–(40) into Eqs. (23)–(33), we get lengthy expressions (which we do not report here) giving the present-day values of $f(R)$ and its first three derivatives as a function of (Ω_M, w_0, w_a). It is worth noting that the $f(R)$ model thus obtained is not dynamically equivalent to the starting CPL one. Indeed, the two models have the same cosmographic parameters only today. As such, for instance, the scale factor is the same between the two theories only over the time period during which the fifth order Taylor expansion is a good approximation of the actual $a(t)$. It is also worth stressing that such a procedure does not select a unique $f(R)$ model, but rather a class of fourth order theories all sharing the same third order Taylor expansion of $f(R)$.

A. The ΛCDM case

With these caveats in mind, it is worth considering first the ΛCDM model which is obtained by setting $(w_0, w_a) = (-1, 0)$ in the above expressions, thus giving

$$\begin{cases} q_0 = \frac{1}{2} - \frac{3}{2}\Omega_\Lambda \\ j_0 = 1 \\ s_0 = 1 - \frac{9}{2}\Omega_M \\ l_0 = 1 + 3\Omega_M + \frac{27}{2}\Omega_M^2. \end{cases} \quad (41)$$

When inserted into the expressions for the $f(R)$ quantities, these relations give the remarkable result

$$f(R_0) = R_0 + 2\Lambda, \qquad f''(R_0) = f'''(R_0) = 0, \quad (42)$$

so we obviously conclude that the only $f(R)$ theory having exactly the same cosmographic parameters as the ΛCDM model is just $f(R) \propto R$, i.e. GR. It is worth noticing that such a result comes out as a consequence of the values of (q_0, j_0) in the ΛCDM model. Indeed, should we have left (s_0, l_0) undetermined and only fixed (q_0, j_0) to the values in (41), we should have got the same result in (42). Since the ΛCDM model fits well a large set of different data, we do expect that the actual values of (q_0, j_0, s_0, l_0) do not differ too much from the ΛCDM ones. Therefore, we plug into Eqs. (23)–(33) the following expressions:

$$q_0 = q_0^\Lambda \times (1 + \varepsilon_q), \qquad j_0 = j_0^\Lambda \times (1 + \varepsilon_j),$$
$$s_0 = s_0^\Lambda \times (1 + \varepsilon_s), \qquad l_0 = l_0^\Lambda \times (1 + \varepsilon_l),$$

with $(q_0^\Lambda, j_0^\Lambda, s_0^\Lambda, l_0^\Lambda)$ given by Eqs. (41) and $(\varepsilon_q, \varepsilon_j, \varepsilon_s, \varepsilon_l)$ quantifying the deviations from the ΛCDM values allowed by the data. A numerical estimate of these quantities may be obtained, e.g., from a Markov chain analysis, but this is outside our aims. Since here we are interested in a theoretical examination, we prefer to consider an idealized situation where the four quantities above all share the same value $\varepsilon \ll 1$. In such a case, we can easily investigate how much the corresponding $f(R)$ deviates from the GR one considering the two ratios $f''(R_0)/f(R_0)$ and $f'''(R_0)/f(R_0)$. Inserting the above expressions for the cosmographic parameters into the exact (not reported) formulas for $f(R_0)$, $f''(R_0)$, and $f'''(R_0)$, taking their ratios, and then expanding to first order in ε, we finally get

$$\eta_{20} = \frac{64 - 6\Omega_M(9\Omega_M + 8)}{[3(9\Omega_M + 74)\Omega_M - 556]\Omega_M^2 + 16} \times \frac{\varepsilon}{27}, \quad (43)$$

$$\eta_{30} = \frac{6[(81\Omega_M - 110)\Omega_M + 40]\Omega_M + 16}{[3(9\Omega_M + 74)\Omega_M - 556]\Omega_M^2 + 16} \times \frac{\varepsilon}{243\Omega_M^2}, \quad (44)$$

having defined $\eta_{20} = f''(R_0)/f(R_0) \times H_0^4$ and $\eta_{30} = f'''(R_0)/f(R_0) \times H_0^6$ which, being dimensionless quantities, are more suited to estimate the order of magnitudes of the different terms. Inserting our fiducial values for Ω_M, we get

$$\begin{cases} \eta_{20} \simeq 0.15 \times \varepsilon & \text{for } \Omega_M = 0.041 \\ \eta_{20} \simeq -0.12 \times \varepsilon & \text{for } \Omega_M = 0.250, \end{cases}$$

$$\begin{cases} \eta_{30} \simeq 4 \times \varepsilon & \text{for } \Omega_M = 0.041 \\ \eta_{30} \simeq -0.18 \times \varepsilon & \text{for } \Omega_M = 0.250. \end{cases}$$

For values of ε up to 0.1, the above relations show that the second and third derivatives are at most 2 orders of magnitude smaller than the zeroth order term $f(R_0)$. Actually, the values of η_{30} for a baryon-only model (first row) seem to argue in favor of a larger importance of the third order term. However, we have numerically checked that the above relations approximate very well the exact expressions up to $\varepsilon \simeq 0.1$ with an accuracy depending on the value of Ω_M, being smaller for smaller matter density parameters. Using the exact expressions for η_{20} and η_{30}, our conclusion on the negligible effect of the second and third order derivatives is significantly strengthened.

Such a result holds under the hypothesis that the narrower the constraints on the validity of the ΛCDM model, the smaller the deviations of the cosmographic parameters from the ΛCDM ones. It is possible to show that this is indeed the case for the CPL parametrization we are considering. On the other hand, we have also assumed that the deviations $(\varepsilon_q, \varepsilon_j, \varepsilon_s, \varepsilon_l)$ take the same values. Although such a hypothesis is somewhat *ad hoc*, we argue that the main results are not affected by giving it away. Indeed, although different from each other, we can still assume that all of them are very small so Taylor expanding to the first order should lead to additional terms into Eqs. (43) and (44) which are likely of the same order of magnitude. Therefore, if the observations confirm that the values of the cosmographic parameters agree within ~10% with those predicted for the ΛCDM model, we must conclude that the deviations of $f(R)$ from the GR case, $f(R) \propto R$, should be vanishingly small.

It is worth stressing, however, that such a conclusion only holds for those $f(R)$ models satisfying the constraint (34). It is indeed possible to work out a model having $f(R_0) \propto R_0$, $f''(R_0) = f'''(R_0) = 0$, but $f^{(n)}(R_0) \neq 0$ for some n. For such a (somewhat *ad hoc*) model, Eq. (34) is clearly not satisfied so the cosmographic parameters have to be evaluated from the solution of the field equations. For such a model, the conclusion above does not hold so one cannot exclude that the resulting (q_0, j_0, s_0, l_0) are within 10% of the ΛCDM ones.

B. The constant EoS model

Let us now take into account the condition $w = -1$, but still retain $w_a = 0$, thus obtaining the so-called *quiessence* models. In such a case, some problems arise because both the terms $(j_0 - q_0 - 2)$ and $\mathcal{R}$ may vanish for some combinations of the two model parameters (Ω_M, w_0). For instance, we find that $j_0 - q_0 - 2 = 0$ for $w_0 = (w_1, w_2)$ with

$$w_1 = \frac{1}{1 - \Omega_M + \sqrt{(1 - \Omega_M)(4 - \Omega_M)}},$$

$$w_2 = -\frac{1}{3}\left[1 + \frac{4 - \Omega_M}{\sqrt{(1 - \Omega_M)(4 - \Omega_M)}}\right].$$

On the other hand, the equation $\mathcal{R}(\Omega_M, w_0) = 0$ may have different real roots for w depending on the adopted value of Ω_M. Denoting collectively with $\mathbf{w}_{\rm null}$ the values of w_0 that, for a given Ω_M, make $(j_0 - q_0 - 2)\mathcal{R}(\Omega_M, w_0)$, taking the null value, we individuate a set of quiessence models whose cosmographic parameters give rise to divergent values of $f(R_0)$, $f''(R_0)$, and $f'''(R_0)$. For such models, $f(R)$ is clearly not defined so we have to exclude these cases from further consideration. We only note that it is still possible to work out a $f(R)$ theory reproducing the same background dynamics of such models, but a different route has to be used.

Since both q_0 and j_0 now deviate from the ΛCDM values, it is not surprising that both $f''(R_0)$ and $f'''(R_0)$ take finite non-null values. However, it is more interesting to study the two quantities η_{20} and η_{30} defined above to investigate the deviations of $f(R)$ from the GR case. These are plotted in Figs. 1 and 2 for the two fiducial Ω_M values. Note that the range of w_0 in these plots has been chosen in order to avoid divergences, but the lessons we will draw also hold for the other w_0 values.

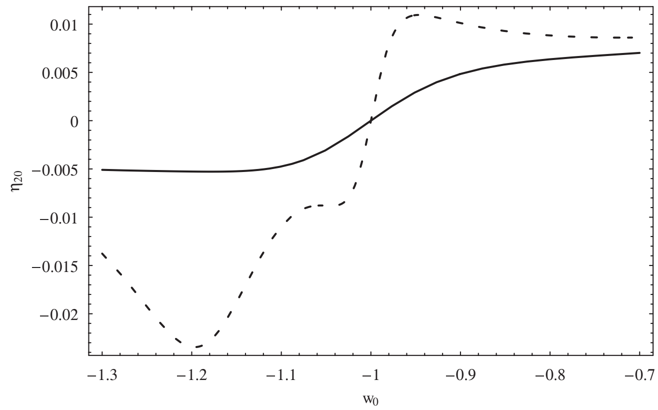

FIG. 1. The dimensionless ratio between the present-day values of $f''(R)$ and $f(R)$ as a function of the constant EoS w_0 of the corresponding quiessence model. Short dashed and solid lines refer to models with $\Omega_M = 0.041$ and 0.250, respectively.

As a general comment, it is clear that, even in this case, $f''(R_0)$ and $f'''(R_0)$ are from 2 to 3 orders of magnitude smaller than the zeroth order term $f(R_0)$. Such a result could yet be guessed from the previous discussion for the ΛCDM case. Actually, relaxing the hypothesis $w_0 = -1$ is the same as allowing the cosmographic parameters to deviate from the ΛCDM values. Although a direct mapping between the two cases cannot be established, it is nonetheless evident that such a relation can be argued, thus making the outcome of the above plots not fully surprising. It is nevertheless worth noting that, while in the ΛCDM case η_{20} and η_{30} always have opposite signs, this is not the case for quiessence models with $w > -1$. Indeed, depending on the value of Ω_M, we can have $f(R)$ theories with both η_{20} and η_{30} positive. Moreover, the lower Ω_M is, the higher the ratios η_{20} and η_{30} are for a given value of w_0. This can be explained qualitatively noticing that, for a lower Ω_M, the density parameter of the curvature fluid

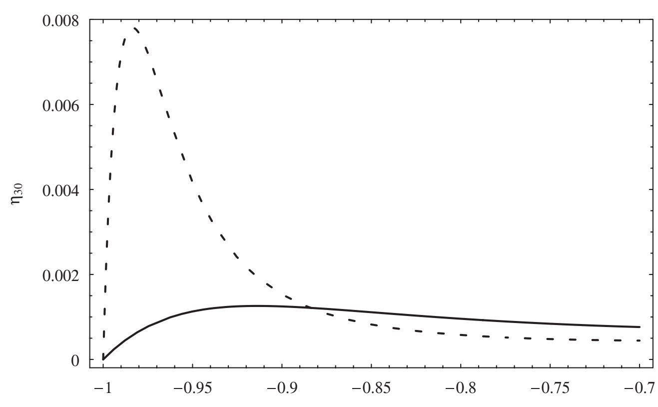

FIG. 2. The dimensionless ratio between the present-day values of $f'''(R)$ and $f(R)$ as a function of the constant EoS w_0 of the corresponding quiessence model. Short dashed and solid lines refer to models with $\Omega_M = 0.041$ and 0.250, respectively.

(playing the role of an effective dark energy) must be larger, thus claiming higher values of the second and third derivatives (see also [39] for a different approach to the problem).

C. The general case

Finally, we consider evolving dark energy models with $w_a \neq 0$. Needless to say, varying three parameters allows us to get a wide range of models that cannot be discussed in detail. Therefore, we only concentrate on evolving dark energy models with $w_0 = -1$ in agreement with some of the most recent analyses. The results on η_{20} and η_{30} are plotted in Figs. 3 and 4, as functions of w_a. Note that we are considering models with positive w_a so that $w(z)$ tends to $w_0 + w_a > w_0$ for $z \to \infty$ so the EoS dark energy can eventually approach the dust value $w = 0$. Actually, this is also the range favored by the data. We have, however, excluded values where η_{20} or η_{30} diverge. Considering how they are defined, it is clear that these two quantities diverge when $f(R_0) = 0$ so that the values of (w_0, w_a) making (η_{20}, η_{30}) diverge may be found solving

$$\mathcal{P}_0(w_0, w_a)\Omega_M + \mathcal{Q}_0(w_0, w_a) = 0$$

where $\mathcal{P}_0(w_0, w_a)$ and $\mathcal{Q}_0(w_0, w_a)$ are obtained by inserting Eqs. (37)–(40) into the definitions (27) and (28). For such CPL models, there is no $f(R)$ model having the same cosmographic parameters and, at the same time, satisfying all the criteria needed for the validity of our procedure. Actually, if $f(R_0) = 0$, the condition (34) is likely to be violated so higher than third order must be included in the Taylor expansion of $f(R)$, thus invalidating the derivation of Eqs. (23)–(26).

Under these caveats, Figs. 3 and 4 demonstrate that allowing the dark energy EoS to evolve does not change significantly our conclusions. Indeed, the second and third derivatives, although not null, are nevertheless negligible

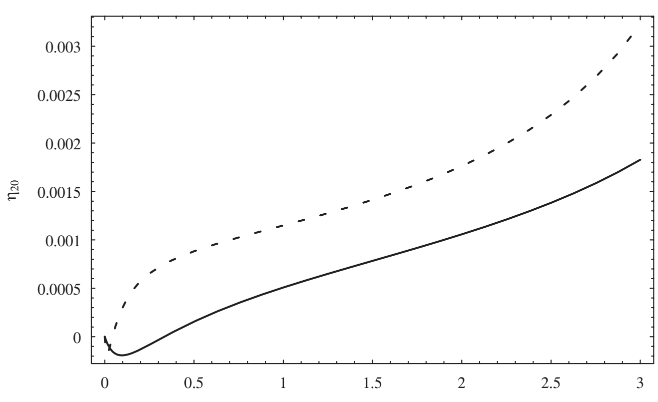

FIG. 3. The dimensionless ratio between the present-day values of $f''(R)$ and $f(R)$ as a function of the w_a parameter for models with $w_0 = -1$. Short dashed and solid lines refer to models with $\Omega_M = 0.041$ and 0.250, respectively.

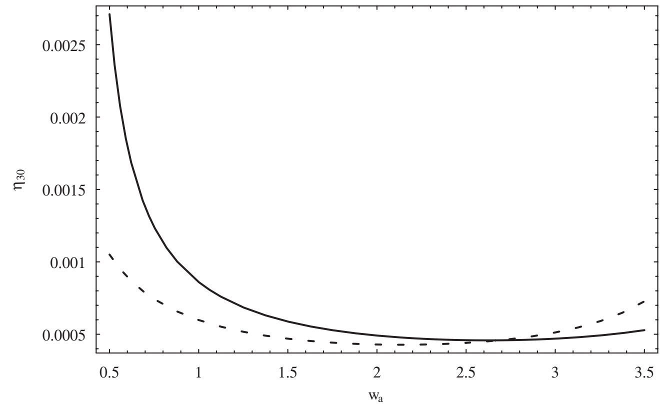

FIG. 4. The dimensionless ratio between the present-day values of $f'''(R)$ and $f(R)$ as a function of the w_a parameter for models with $w_0 = -1$. Short dashed and solid lines refer to models with $\Omega_M = 0.041$ and 0.250, respectively.

with respect to the zeroth order term, thus arguing in favor of a GR-like $f(R)$ with only very small corrections. Such a result is, however, not fully unexpected. From Eqs. (37) and (38), we see that, having set $w_0 = -1$, the q_0 parameter is the same as for the ΛCDM model, while j_0 reads $j_0^\Lambda + (3/2)(1 - \Omega_M)w_a$. As we have stressed above, the Hilbert-Einstein Lagrangian $f(R) = R + 2\Lambda$ is recovered when $(q_0, j_0) = (q_0^\Lambda, j_0^\Lambda)$, whatever the values of (s_0, l_0) are. Introducing a $w_a \neq 0$ makes (s_0, l_0) differ from the ΛCDM values, but the first two cosmographic parameters are only mildly affected. Such deviations are then partially washed out by the complicated way they enter in the determination of the present-day values of $f(R)$ and its first three derivatives.

VI. CONSTRAINING $f(R)$ PARAMETERS

In the previous section, we have worked out an alternative method to estimate $f(R_0)$, $f''(R_0)$, $f'''(R_0)$, resorting to a model independent parametrization of the dark energy EoS. However, in the ideal case, the cosmographic parameters are directly estimated from the data so that Eqs. (23)–(33) can be used to infer the values of the $f(R)$ related quantities. The latter can then be used to put constraints on the parameters entering an assumed fourth order theory assigned by a $f(R)$ function characterized by a set of parameters $\mathbf{p} = (p_1, \ldots, p_n)$ provided that the hypotheses underlying the derivation of Eqs. (23)–(33) are indeed satisfied. We show below two interesting cases which clearly highlight the potentiality and the limitations of such an analysis.

A. Double power-law Lagrangian

As a first interesting example, we set

$$f(R) = R(1 + \alpha R^n + \beta R^{-m}) \qquad (45)$$

with n and m two positive real numbers (see, for example,

[40] for some physical motivations). The following expressions are immediately obtained:

$$\begin{cases} f(R_0) = R_0(1 + \alpha R_0^n + \beta R_0^{-m}) \\ f'(R_0) = 1 + \alpha(n+1)R_0^n - \beta(m-1)R_0^{-m} \\ f''(R_0) = \alpha n(n+1)R_0^{n-1} + \beta m(m-1)R_0^{-(1+m)} \\ f'''(R_0) = \alpha n(n+1)(n-1)R_0^{n-2} \\ \qquad\qquad - \beta m(m+1)(m-1)R_0^{-(2+m)}. \end{cases}$$

Denoting by ϕ_i (with $i = 0, \ldots, 3$) the values of $f^{(i)}(R_0)$ determined through Eqs. (23)–(33), we can solve

$$\begin{cases} f(R_0) = \phi_0 \\ f'(R_0) = \phi_1 \\ f''(R_0) = \phi_2 \\ f'''(R_0) = \phi_3 \end{cases}$$

which is a system of four equations in the four unknowns (α, β, n, m) that can be analytically solved proceeding as follows. First, we solve the first and second equations with respect to (α, β), obtaining

$$\begin{cases} \alpha = \frac{1-m}{n+m}(1 - \frac{\phi_0}{R_0})R_0^{-n} \\ \beta = -\frac{1+n}{n+m}(1 - \frac{\phi_0}{R_0})R_0^{m}, \end{cases} \tag{46}$$

while, solving the third and fourth equations, we get

$$\begin{cases} \alpha = \frac{\phi_2 R_0^{1-n}[1+m+(\phi_3/\phi_2)R_0]}{n(n+1)(n+m)} \\ \beta = \frac{\phi_2 R_0^{1+n}[1-n+(\phi_3/\phi_2)R_0]}{m(1-m)(n+m)}. \end{cases} \tag{47}$$

Equating the two solutions, we get a system of two equations in the two unknowns (n, m), namely,

$$\begin{cases} \frac{n(n+1)(1-m)(1-\phi_0/R_0)}{\phi_2 R_0[1+m+(\phi_3/\phi_2)R_0]} = 1 \\ \frac{m(n+1)(m-1)(1-\phi_0/R_0)}{\phi_2 R_0[1-n+(\phi_3/\phi_2)R_0]} = 1. \end{cases} \tag{48}$$

Solving with respect to m, we get two solutions, the first one being $m = -n$ which has to be discarded since it makes (α, β) go to infinity. The only acceptable solution is

$$m = -[1 - n + (\phi_3/\phi_2)R_0] \tag{49}$$

which, inserted back into the above system, leads to a second order polynomial equation for n with solutions

$$n = \frac{1}{2}\left[1 + \frac{\phi_3}{\phi_2}R_0 \pm \frac{\sqrt{\mathcal{N}(\phi_0, \phi_2, \phi_3)}}{\phi_2 R_0(1 + \phi_0/R_0)}\right] \tag{50}$$

where we have defined

$$\begin{aligned} \mathcal{N}(\phi_0, \phi_2, \phi_3) = {} & (R_0^2\phi_0^2 - 2R_0^3\phi_0 + R_0^4)\phi_3^2 \\ & + 6(R_0\phi_0^2 - 2R_0^2\phi_0 + R_0^3)\phi_2\phi_3 \\ & + 9(\phi_0^2 - 2R_0\phi_0 + R_0^2)\phi_2^2 \\ & + 4(R_0^2\phi_0 - R_0^3)\phi_2^3. \end{aligned} \tag{51}$$

Depending on the values of (q_0, j_0, s_0, l_0), Eq. (50) may lead to one, two, or any acceptable solutions, i.e. real positive values of n. This solution then has to be inserted back into Eq. (49) to determine m and then into Eqs. (46) or (47) to estimate (α, β). If the final values of (α, β, n, m) are physically viable, we can conclude that the model in Eq. (45) is in agreement with the data, giving the same cosmographic parameters inferred from the data themselves. Exploring analytically what is the region of the (q_0, j_0, s_0, l_0) parameter space which leads to acceptable (α, β, n, m) solutions is a daunting task far outside the aim of the present paper.

B. HS model

One of the most pressing problems of $f(R)$ theories is the need to escape the severe constraints imposed by the Solar System tests. A successful model has been recently proposed by Hu and Sawicki [21] (HS), setting[8]

$$f(R) = R - R_c\frac{\alpha(R/R_c)^n}{1 + \beta(R/R_c)^n}. \tag{52}$$

As for the double power-law model discussed above, there are four parameters which can be expressed in terms of the cosmographic parameters (q_0, j_0, s_0, l_0).

As a first step, it is trivial to get

$$\begin{cases} f(R_0) = R_0 - R_c\frac{\alpha R_{0c}^n}{1+\beta R_{0c}^n} \\ f'(R_0) = 1 - \frac{\alpha n R_c R_{0c}^n}{R_0(1+\beta R_{0c}^n)^2} \\ f''(R_0) = \frac{\alpha n R_c R_{0c}^n[(1-n)+\beta(1+n)R_{0c}^n]}{R_0^2(1+\beta R_{0c}^n)^3} \\ f'''(R_0) = \frac{\alpha n R_c R_{0c}^n(An^2+Bn+C)}{R_0^3(1+\beta R_{0c}^n)^4} \end{cases} \tag{53}$$

with $R_{0c} = R_0/R_c$ and

$$\begin{cases} A = -\beta^2 R_{0c}^{2n} + 4\beta R_{0c}^n - 1 \\ B = 3(1 - \beta^2 R_{0c}^{2n}) \\ C = -2(1 - \beta R_{0c}^n)^2. \end{cases} \tag{54}$$

Equating Eqs. (53) to the four quantities $(\phi_0, \phi_1, \phi_2, \phi_3)$ defined as above, we could, in principle, solve this system of four equations in four unknowns to get (α, β, R_c, n) in terms of $(\phi_0, \phi_1, \phi_2, \phi_3)$ and then, using Eqs. (23)–(33), as functions of the cosmographic parameters. However,

[8]Note that such a model does not pass the matter instability test so some viable generalizations [41] have been proposed.

setting $\phi_1 = 1$ as required by Eq. (24) gives the only trivial solution $\alpha n R_c = 0$ so the HS model reduces to the Einstein-Hilbert Lagrangian $f(R) = R$. In order to escape this problem, we can relax the condition $f'(R_0) = 1$ to $f'(R_0) = (1+\varepsilon)^{-1}$. As we have discussed in Sec. IV, this is the same as assuming that the present-day effective gravitational constant $G_{\rm eff,0} = G_N/f'(R_0)$ only differs slightly from the usual Newtonian one which seems to be quite a reasonable assumption. Under this hypothesis, we can analytically solve for (α, β, R_c, n) in terms of $(\phi_0, \varepsilon, \phi_2, \phi_3)$. The actual values of (ϕ_0, ϕ_2, ϕ_3) will no longer be given by Eqs. (23)–(26), but we have checked that they deviate from those expressions[9] much less than 10% for values of ε as high as 10%.

With this caveat in mind, we first solve

$$f(R_0) = \phi_0, \qquad f''(R_0) = (1+\varepsilon)^{-1}$$

to get

$$\alpha = \frac{n(1+\varepsilon)}{\varepsilon}\left(\frac{R_0}{R_c}\right)^{1-n}\left(1 - \frac{\phi_0}{R_0}\right)^2,$$
$$\beta = \frac{n(1+\varepsilon)}{\varepsilon}\left(\frac{R_0}{R_c}\right)^{-n}\left[1 - \frac{\phi_0}{R_0} - \frac{\varepsilon}{n(1+\varepsilon)}\right].$$

Inserting these expressions in Eqs. (53), it is easy to check that R_c cancels out so we can no longer determine its value. Such a result is, however, not unexpected. Indeed, Eq. (52) can trivially be rewritten as

$$f(R) = R - \frac{\tilde{\alpha} R^n}{1 + \tilde{\beta} R^n}$$

with $\tilde{\alpha} = \alpha R_c^{1-n}$ and $\tilde{\beta} = \beta R_c^{-n}$, which are indeed the quantities that are determined by the above expressions for (α, β). Reversing the discussion, the present-day values of $f^{(i)}(R)$ depend on (α, β, R_c) only through the two parameters $(\tilde{\alpha}, \tilde{\beta})$. As such, the use of cosmographic parameters is unable to break this degeneracy. However, since R_c only plays the role of a scaling parameter, we can arbitrarily set its value without loss of generality.

On the other hand, this degeneracy allows us to get a consistency relation to immediately check whether the HS model is viable or not. Indeed, solving the equation $f''(R_0) = \phi_2$, we get

$$n = \frac{(\phi_0/R_0) + [(1+\varepsilon)/\varepsilon](1 - \phi_2 R_0) - (1-\varepsilon)/(1+\varepsilon)}{1 - \phi_0/R_0},$$

which can then be inserted into the equations $f'''(R_0) = \phi_3$ to obtain a complicated relation among (ϕ_0, ϕ_2, ϕ_3) which we do not report for the sake of brevity. Solving such a relation with respect to ϕ_3/ϕ_0 and Taylor expanding to first order in ε, the constraint we get reads

$$\frac{\phi_3}{\phi_0} \simeq -\frac{1+\varepsilon}{\varepsilon}\frac{\phi_2}{R_0}\left[R_0\left(\frac{\phi_2}{\phi_0}\right) + \frac{\varepsilon\phi_0^{-1}}{1+\varepsilon}\left(1 - \frac{2\varepsilon}{1 - \phi_0/R_0}\right)\right].$$

If the cosmographic parameters (q_0, j_0, s_0, l_0) are known with sufficient accuracy, one could compute the values of $(R_0, \phi_0, \phi_2 . \phi_3)$ for a given ε (eventually using the expressions obtained for $\varepsilon = 0$) and then check if they satisfied this relation. If this is not the case, one can also immediately give off the HS model without the need for solving the field equations and fitting the data. Actually, given the still large errors on the cosmographic parameters, such a test only remains in the realm of (quite distant) future applications. However, the HS model works for other tests as shown in [21], and so a consistent cosmography analysis has to be combined with them.

VII. CONSTRAINTS ON $f(R)$ DERIVATIVES FROM THE DATA

Equations (23)–(33) relate the present-day values of $f(R)$ and its first three derivatives to the cosmographic parameters (q_0, j_0, s_0, l_0) and the matter density Ω_M. In principle, therefore, a measurement of these latter quantities makes it possible to put constraints on $f^{(i)}(R_0)$, with $i = \{0, \ldots, 3\}$, and hence on the parameters of a given fourth order theory through the method shown in the previous section. Actually, the cosmographic parameters are affected by errors which obviously propagate onto the $f(R)$ quantities, and the covariance matrix for the cosmographic parameters is not diagonal so one also has to take care of this to estimate the final errors on $f^{(i)}(R_0)$. A similar discussion also holds for the errors on the dimensionless ratios η_{20} and η_{30} introduced above. As a general rule, indicating with $g(\Omega_M, \mathbf{p})$ a generic $f(R)$ related quantity depending on Ω_M and the set of cosmographic parameters $\mathbf{p}$, its uncertainty reads

$$\sigma_g^2 = \left|\frac{\partial g}{\partial \Omega_M}\right|^2 \sigma_M^2 + \sum_{i=1}^{i=4}\left|\frac{\partial g}{\partial p_i}\right|^2 \sigma_{p_i}^2 + \sum_{i \neq j} 2 \frac{\partial g}{\partial p_i}\frac{\partial g}{\partial p_j} C_{ij} \tag{55}$$

where C_{ij} are the elements of the covariance matrix (which is$C_{ii} = \sigma_{p_i}^2$). We have set $(p_1, p_2, p_3, p_4) = (q_0, j_0, s_0, l_0)$ and assumed that the error σ_M on Ω_M is uncorrelated with those on $\mathbf{p}$. Note that this latter assumption strictly holds if the matter density parameter is estimated from an astrophysical method (such as estimating the total matter in the universe from the estimated halo mass function). Alternatively, we will assume that Ω_M is constrained by the CMBR related experiments. Since the latter mainly probe the very high redshift universe ($z \simeq z_{lss} \simeq 1089$), while the cosmographic parameters are concerned with the present-day cosmos, one can argue that the determination of Ω_M is not affected by the details of the model adopted for describing the late universe. Indeed, we can reasonably assume that, whatever the dark energy candi-

[9]Note that the correct expressions for (phi_0, ϕ_2, ϕ_3) may still formally be written as Eqs. (23)–(26), but the polynomials entering them are now different and also depend on powers of ε.

date or $f(R)$ theory is, the CMBR era is well approximated by the standard GR with a model comprising only dust matter. As such, we will make the simplifying (but well motivated) assumption that σ_M may be reduced to very small values and is uncorrelated with the cosmographic parameters.

Under this assumption, the problem of estimating the errors on $g(\Omega_M, \mathbf{p})$ reduces to estimating the covariance matrix for the cosmographic parameters given the details of the data set used as observational constraints. We address this issue by computing the Fisher information matrix (see, e.g., [42] and references therein) defined as

$$F_{ij} = \left\langle \frac{\partial^2 L}{\partial\theta_i \partial\theta_j} \right\rangle \tag{56}$$

with $L = -2\ln\mathcal{L}(\theta_1, \ldots, \theta_n)$, $\mathcal{L}(\theta_1, \ldots, \theta_n)$ the likelihood of the experiment, $(\theta_1, \ldots, \theta_n)$ the set of parameters to be constrained, and $\langle \ldots \rangle$ the expectation value. Actually, the expectation value is computed by evaluating the Fisher matrix elements for fiducial values of the model parameters $(\theta_1, \ldots, \theta_n)$, while the covariance matrix $\mathbf{C}$ is finally obtained as the inverse of $\mathbf{F}$.

A key ingredient in the computation of $\mathbf{F}$ is the definition of the likelihood which depends, of course, on what experimental constraint one is using. To this aim, it is worth remembering that our analysis is based on fifth order Taylor expansion of the scale factor $a(t)$ so we can only rely on observational tests probing quantities that are well described by this truncated series. Moreover, since we do not assume any particular model, we can only characterize the background evolution of the universe, but not its dynamics which, being related to the evolution of perturbations, unavoidably need the specification of a physical model. As a result, the SNeIa Hubble diagram is the ideal test[10] to constrain the cosmographic parameters. We therefore defined the likelihood as

$$\mathcal{L}(H_0, \mathbf{p}) \propto \exp - \chi^2(H_0, \mathbf{p})/2,$$
$$\chi^2(H_0, \mathbf{p}) = \sum_{n=1}^{\mathcal{N}_{\rm SNeIa}} \left[\frac{\mu_{\rm obs}(z_i) - \mu_{\rm th}(z_n, H_0, \mathbf{p})}{\sigma_i(z_i)}\right]^2, \tag{57}$$

where the distance modulus to redshift z reads

$$\mu_{\rm th}(z, H_0, \mathbf{p}) = 25 + 5\log(c/H_0) + 5\log d_L(z, \mathbf{p}), \tag{58}$$

and $d_L(z)$ is the Hubble-free luminosity distance:

$$d_L(z) = (1+z)\int_0^z \frac{dz}{H(z)/H_0}. \tag{59}$$

Using the fifth order Taylor expansion of the scale factor, we get for $d_L(z, \mathbf{p})$ an analytical expression (reported in the Appendix) so the computation of F_{ij} does not need any numerical integration (which makes the estimate faster).

[10]See the Conclusions for further discussion on this issue.

As a last ingredient, we need to specify the details of the SNeIa survey, giving the redshift distribution of the sample and the error on each measurement. Following [43], we adopt[11]

$$\sigma(z) = \sqrt{\sigma_{\rm sys}^2 + \left(\frac{z}{z_{\rm max}}\right)^2 \sigma_m^2}$$

with $z_{\rm max}$ the maximum redshift of the survey, $\sigma_{\rm sys}$ an irreducible scatter in the SNeIa distance modulus, and σ_m to be assigned depending on the photometric accuracy.

In order to run the Fisher matrix calculation, we have to set a fiducial model, which we set according to the ΛCDM predictions for the cosmographic parameters. For $\Omega_M = 0.3$ and $h = 0.72$ (with h the Hubble constant in units of 100 km/s/Mpc), we get

$$(q_0, j_0, s_0, l_0) = (-0.55, 1.0, -0.35, 3.11).$$

As a first consistency check, we compute the Fisher matrix for a survey mimicking the recent database in [7], thus setting $(\mathcal{N}_{\rm SNeIa}, \sigma_m) = (192, 0.33)$. After marginalizing over h (which, as is well known, is fully degenerate with the SNeIa absolute magnitude $\mathcal{M}$), we get for the uncertainties

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (0.38, 5.4, 28.1, 74.0)$$

where we are still using the indexing introduced above for the cosmographic parameters. These values compare reasonably well with those obtained from a cosmographic fitting of the Gold SNeIa data set[12] [44]:

$$q_0 = -0.90 \pm 0.65, \qquad j_0 = 2.7 \pm 6.7,$$
$$s_0 = 36.5 \pm 52.9, \qquad l_0 = 142.7 \pm 320.$$

Because of the Cramer-Rao theorem, the Fisher matrix approach is known to provide the minimum variance errors a given experiment can attain, thus giving higher limits to its accuracy on the determination of a set of parameters. This is indeed the case with the comparison suggesting that our predictions are quite optimistic. It is worth stressing,

[11]Note that, in [43], the authors assume the data are separated in redshift bins so the error becomes $\sigma^2 = \sigma_{\rm sys}^2/\mathcal{N}_{\rm bin} + \mathcal{N}_{\rm bin}(z/z_{\rm max})^2\sigma_m^2$ with $\mathcal{N}_{\rm bin}$ the number of SNeIa in a bin. However, we prefer not to bin the data so that $\mathcal{N}_{\rm bin} = 1$.

[12]Actually, such estimates have been obtained by computing the mean and the standard deviation from the marginalized likelihoods of the cosmographic parameters. As such, the central values do not represent exactly the best fit model, while the standard deviations do not give a rigorous description of the error because the marginalized likelihoods are manifestly non-Gaussian. Nevertheless, we are mainly interested in an order of magnitude estimate so we do not care about such statistical details.

however, that the analysis in [44] used the Gold data set which is poorer in high z SNeIa than the one in [7] we are mimicking, so larger errors on the higher order parameters (s_0, l_0) are expected.

Rather than computing the errors on $f(R_0)$ and its first three derivatives, it is more interesting to look at the precision attainable on the dimensionless ratios (η_{20}, η_{30}) introduced above since they quantify how many deviations from the linear order are present. For the fiducial model we are considering, both η_{20} and η_{30} vanish, while, using the covariance matrix for a present-day survey and setting $\sigma_M/\Omega_M \simeq 10\%$, their uncertainties read

$$(\sigma_{20}, \sigma_{30}) = (0.04, 0.04).$$

As an application, we can look at Figs. 1 and 2 showing how (η_{20}, η_{30}) depend on the present-day EoS w_0 for $f(R)$ models sharing the same cosmographic parameters of a dark energy model with constant EoS. As it is clear, also considering only the 1σ range, the full region plotted is allowed by such large constraints on (η_{20}, η_{30}), thus meaning that the full class of corresponding $f(R)$ theories is viable. As a consequence, we may conclude that the present-day SNeIa data are unable to discriminate between a Λ dominated universe and this class of fourth order gravity theories.

As a next step, we consider a SNAP-like survey [45], thus setting $(\mathcal{N}_{\rm SNeIa}, \sigma_m) = (2000, 0.02)$. We use the same redshift distribution in Table 1 of [43] and add 300 nearby SNeIa in the redshift range (0.03, 0.08). The Fisher matrix calculation gives for the uncertainties on the cosmographic parameters

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (0.08, 1.0, 4.8, 13.7).$$

The significant improvement of the accuracy in the determination of (q_0, j_0, s_0, l_0) translates to a reduction of the errors on (η_{20}, η_{30}), which now read

$$(\sigma_{20}, \sigma_{30}) = (0.007, 0.008)$$

having assumed that, when SNAP data are available, the matter density parameter Ω_M has been determined with a precision $\sigma_M/\Omega_M \sim 1\%$. Looking again at Figs. 1 and 2, it is clear that the situation is improved. Indeed, the constraints on η_{20} make it possible to narrow the range of allowed models with low matter content (the dashed line), while models with typical values of Ω_M are still viable for w_0 covering almost the full horizontal axis. On the other hand, the constraint on η_{30} is still too weak so almost the full region plotted is allowed.

Finally, we consider a hypothetical future SNeIa survey working at the same photometric accuracy as SNAP and with the same redshift distribution, but increasing the number of SNeIa up to $\mathcal{N}_{\rm SNeIa} = 6 \times 10^4$ as expected from, e.g., DES [46], PanSTARRS [47], SKYMAPPER [48], while still larger numbers may potentially be achieved by ALPACA [49] and LSST [50]. Such a survey can achieve

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (0.02, 0.2, 0.9, 2.7)$$

so that, with $\sigma_M/\Omega_M \sim 0.1\%$, we get

$$(\sigma_{20}, \sigma_{30}) = (0.0015, 0.0016).$$

Figure 1 shows that, with such a precision on η_{20}, the region of w_0 values allowed essentially reduces to the ΛCDM value, while, from Fig. 2, it is clear that the constraint on η_{30} definitively excludes models with low matter content, further reducing the range of w_0 values to quite small deviations from $w_0 = -1$. We can therefore conclude that such a survey will be able to discriminate between the concordance ΛCDM model and all the $f(R)$ theories giving the same cosmographic parameters as quiessence models other than the ΛCDM itself.

A similar discussion may be repeated for $f(R)$ models sharing the same (q_0, j_0, s_0, l_0) values as the CPL model even if it is less intuitive to grasp the efficacy of the survey with the parameter space being multivalued. For the same reason, we have not explored what the accuracy is on the double power-law or HS models, even if this is technically possible. Actually, one should first estimate the errors on the present-day value of $f(R)$ and its three time derivatives and then propagate them on the model parameters using the expressions obtained in Sec. VI. The multiparameter space to be explored makes this exercise quite cumbersome, so we leave it for a forthcoming work where we will explore in detail how these models compare to the present and future data.

VIII. CONCLUSIONS

The recent amount of good quality data has given a new input to the observational cosmology. As often happens in science, new and better data lead to unexpected discoveries as in the case of the presently accepted evidence for cosmic acceleration. However, a fierce and strong debate is still open on what this cosmic speedup implies for theoretical cosmology. The equally impressive amount of different (more or less) viable candidates has also generated great confusion so model independent analyses are welcome. A possible solution could come from the cosmography of the universe rather than assuming *ad hoc* solutions of the cosmological Friedmann equations. Present-day and future SNeIa surveys have renewed the interest in the determination of the cosmographic parameters so it is worth investigating how these quantities can constrain cosmological models.

Motivated by this consideration, in the framework of metric formulation of $f(R)$ gravity, here we have derived the expressions of the present-day values of $f(R)$ and its first three derivatives as a function of the matter density parameter Ω_M, the Hubble constant H_0, and the cosmographic parameters (q_0, j_0, s_0, l_0). Although based on a third order Taylor expansion of $f(R)$, we have shown that such relations hold for quite a large class of models so they are valid tools to look for viable $f(R)$ models without the need of solving the mathematically difficult, nonlinear, fourth order, differential field equations.

Notwithstanding the common claim that we live in the era of *precision cosmology*, the constraints on (q_0, j_0, s_0, l_0) are still too weak to efficiently apply the program we have outlined above. As such, we have shown how it is possible to establish a link between the popular CPL parametrization of the dark energy equation of state and the derivatives of $f(R)$, imposing that they share the same values of the cosmographic parameters. This analysis has led to the quite interesting conclusion that the only $f(R)$ function able to give the same values of (q_0, j_0, s_0, l_0) as the ΛCDM model is indeed $f(R) = R + 2\Lambda$. If future observations tell us that the cosmographic parameters are those of the ΛCDM model, we can therefore rule out all $f(R)$ theories satisfying the hypotheses underlying our derivation of Eqs. (23)–(26). Actually, such a result should not be considered as a ''no way out'' situation for higher order gravity. Indeed, one could still work out a model with null values of $f''(R_0)$ and $f'''(R_0)$ as required by the above constraints, but with nonvanishing higher order derivatives. One could well argue that such a contrived model could be rejected on the basis of Occam's razor, but nothing prevents us from still taking it into account if it turns out to be both in agreement with the data and theoretically well founded.

If new SNeIa surveys determine the cosmographic parameters with good accuracy, acceptable constraints on the two dimensionless ratios $\eta_{20} \propto f''(R_0)/f(R_0)$ and $\eta_{30} \propto f'''(R_0)/f(R_0)$ could be obtained, thus allowing us to discriminate among rival $f(R)$ theories. To investigate whether such a program is feasible, we have pursued a Fisher matrix based forecast of the accuracy future SNeIa surveys can achieve on the cosmographic parameters and hence on (η_{20}, η_{30}). It turns out that a SNAP-like survey can start giving interesting (yet still weak) constraints, allowing us to reject $f(R)$ models with low matter content, while a definitive improvement is achievable with future SNeIa surveys observing $\sim 10^4$ objects, thus making it possible to discriminate between ΛCDM and a large class of fourth order theories. It is worth stressing, however, that the measurement of Ω_M should come out as the result of a model independent probe such as the gas mass fraction in galaxy clusters which, at present, is still far from the 1% requested precision. On the other hand, one can also rely on the Ω_M estimate from the CMBR anisotropy and polarization spectra even if this comes at the price of assuming that the physics at recombination is strictly described by GR so one has to limit its attention to $f(R)$ models reducing to $f(R) \propto R$ during that epoch. However, such an assumption is quite common in many $f(R)$ models available in the literature so it is not too restrictive of a limitation.

A further remark is in order concerning what kind of data can be used to constrain the cosmographic parameters. The use of the fifth order Taylor expansion of the scale factor makes it possible to not specify any underlying physical model, thus relying on the minimalist assumption that the universe is described by the flat Robertson-Walker metric. While useful from a theoretical perspective, such a generality puts severe limitations on the data set one can use. Actually, we can only resort to observational tests depending only on the background evolution so the range of astrophysical probes reduces to standard candles (such as SNeIa and possibly gamma ray bursts) and standard rods (such as the angular size-redshift relation for compact radio sources). Moreover, pushing the Hubble diagram to $z \sim 2$ may raise the question of the impact of gravitational lensing amplification on the apparent magnitude of the adopted standard candle. The magnification probability distribution function depends on the growth of perturbations [51] so one should worry about the underlying physical model in order to estimate whether this effect biases the estimate of the cosmographic parameters. However, it has been shown [4,52] that the gravitational lensing amplification does not significantly alter the measured distance modulus for $z \sim 1$ SNeIa. Although such an analysis has been done for GR based models, we can argue that, whatever the $f(R)$ model is, the growth of perturbations finally leads to a distribution of structures along the line of sight that is as similar as possible to the observed one so the lensing amplification is approximately the same. We can therefore argue that the systematic error made by neglecting lensing magnification is lower than the statistical ones expected by the future SNeIa surveys. On the other hand, one can also try further reducing this possible bias using the method of flux averaging [53] even if, in such a case, our Fisher matrix calculation should be repeated accordingly. It is also worth noting that the constraints on the cosmographic parameters may be tightened by imposing some physically motivated priors in the parameter space. For instance, we can impose that the Hubble parameter $H(z)$ always stays positive over the full range probed by the data or that the transition from past deceleration to present acceleration takes place over the range probed by the data (so that we can detect it). Such priors should be included in the likelihood definition so that the Fisher matrix is recomputed, which is left for a forthcoming paper.

Although the present-day data are still too limited to efficiently discriminate among rival $f(R)$ models, we are confident that an aggressive strategy aiming at a very precise determination of the cosmographic parameters could offer stringent constraints on higher order gravity without the need for solving the field equations or addressing the complicated problems related to the growth of perturbations. Almost 80 years after the pioneering distance-redshift diagram by Hubble, the old cosmographic approach appears nowadays as a precious observational tool to investigate the new developments of cosmology.

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APPENDIX: DISTANCE FORMULAS

We derive here some useful relations for distance related quantities as a function of the redshift z and the cosmographic parameters. Using the definitions in Eqs. (1), it is easy to get for the fifth order Taylor expansion of the scale factor

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t-t_0) - \frac{q_0}{2}H_0^2(t-t_0)^2 + \frac{j_0}{3!}H_0^3(t-t_0)^3 + \frac{s_0}{4!}H_0^4(t-t_0)^4 + \frac{l_0}{5!}H_0^5(t-t_0)^5 + O[(t-t_0)^6 \qquad \text{(A1)}$$

with t_0 the present-day age of the universe. Note that Eq. (A1) is also the fifth order expansion of $(1+z)^{-1}$, with the redshift z defined as $z = a(t_0)/a(t) - 1$. The physical distance traveled by a photon that is emitted at time t_* and absorbed at the current epoch t_0 is

$$D = c\int dt = c(t_0 - t_*)$$

so that inserting $t_* = t_0 - \frac{D}{c}$ into Eq. (A1) gives us an expression for the redshift as a function of t_0 and D/c, i.e. $z = z(D)$. Solving with respect to D up to the fifth order in z gives us the desired expansion for $D(z)$ as

$$D(z) = \frac{cz}{H_0}\{\mathcal{D}_z^0 + \mathcal{D}_z^1 z + \mathcal{D}_z^2 z^2 + \mathcal{D}_z^3 z^3 + \mathcal{D}_z^4 z^4\} \qquad \text{(A2)}$$

with

$$\mathcal{D}_z^0 = 1, \qquad \mathcal{D}_z^1 = -(1 + q_0/2),$$

$$\mathcal{D}_z^2 = 1 + q_0 + \frac{q_0^2}{2} - \frac{j_0}{6},$$

$$\mathcal{D}_z^3 = -\left(1 + \frac{3}{2}q_0 + \frac{3}{2}q_0^2 + \frac{5}{8}q_0^3 - \frac{1}{2}j_0 - \frac{5}{12}q_0 j_0 - \frac{s_0}{24}\right),$$

$$\mathcal{D}_z^4 = 1 + 2q_0 + 3q_0^2 + \frac{5}{2}q_0^3 + \frac{7}{2}q_0^4 - \frac{5}{3}q_0 j_0 - \frac{7}{8}q_0^2 j_0 - j_0 + \frac{j_0^2}{12} - \frac{1}{8}q_0 s_0 - \frac{s_0}{6} - \frac{l_0}{120}.$$

In typical applications, one is not interested in the physical distance $D(z)$, but rather in the *luminosity distance*

$$D_L = \frac{a(t_0)}{a(t_0 - D/c)}(a_0 r_0), \qquad \text{(A3)}$$

or the *angular diameter distance*

$$D_A = \frac{a(t_0 - D/c)}{a(t_0)}(a_0 r_0) \qquad \text{(A4)}$$

with $a_0 = a(t_0)$ and

$$r_0(D) = \begin{cases} \sin\int_{t_0-D/c}^{t_0}\frac{cdt}{a(t)} & k=1 \\ \int_{t_0-D/c}^{t_0}\frac{cdt}{a(t)} & k=0 \\ \sinh\int_{t_0-D/c}^{t_0}\frac{cdt}{a(t)} & k=-1. \end{cases} \qquad \text{(A5)}$$

Using Eq. (A1), some cumbersome algebra finally gives

$$\frac{r_0(D)}{D/a_0} = \mathcal{R}_D^0 + \mathcal{R}_D^1\left(\frac{H_0 D}{c}\right) + \mathcal{R}_D^2\left(\frac{H_0 D}{c}\right)^2 + \mathcal{R}_D^3\left(\frac{H_0 D}{c}\right)^3 + \mathcal{R}_D^4\left(\frac{H_0 D}{c}\right)^4 + \mathcal{R}_D^5\left(\frac{H_0 D}{c}\right)^5$$

with

$$\mathcal{R}_D^0 = 1, \qquad \mathcal{R}_D^1 = 1/2,$$

$$\mathcal{R}_D^2 = \frac{1}{6}\left[2 + q_0 - \frac{kc^2}{H_0^2 a_0^2}\right],$$

$$\mathcal{R}_D^3 = \frac{1}{24}\left[6 + 6q_0 + j_0 - 6\frac{kc^2}{H_0^2 a_0^2}\right],$$

$$\mathcal{R}_D^4 = \frac{1}{120}\left[24 + 36q_0 + 6q_0^2 + 8j_0 - s_0 - \frac{5kc^2(7 + 2q_0)}{a_0^2 H_0^2}\right],$$

$$\mathcal{R}_D^5 = \frac{24 + 48q_0 + 18q_0^2 + 4q_0 j_0 + 12j_0 - 2s_0 + 24l_0}{144} - \frac{3kc^2(15 + 10q_0 + j_0)}{144 a_0^2 H_0^2}.$$

Expressing D in Eq. (A5) as a function of z through Eq. (A2) and inserting the result into Eq. (A3), one obtains the desired fifth order approximation for the Hubble-free luminosity distance $d_L = D_L(z)/(c/H_0)$ as a function of the redshift z:

$$d_L(z) = \mathcal{D}_L^0 z + \mathcal{D}_L^1 z^2 + \mathcal{D}_L^2 z^3 + \mathcal{D}_L^3 z^4 + \mathcal{D}_L^4 z^5, \qquad \text{(A6)}$$

having defined

$$\mathcal{D}_L^0 = 1, \qquad \mathcal{D}_L^1 = -\frac{1}{2}(-1 + q_0), \qquad \mathcal{D}_L^2 = -\frac{1}{6}\left(1 - q_0 - 3q_0^2 + j_0 + \frac{kc^2}{H_0^2 a_0^2}\right),$$

$$\mathcal{D}_L^3 = \frac{2 - 2q_0 - 15q_0^2 - 15q_0^3 + 5j_0 + 10q_0 j_0 + s_0}{24} + \frac{2kc^2(1 + 3q_0)}{24H_0^2 a_0^2},$$

$$\mathcal{D}_L^4 = \frac{-6 + 6q_0 + 81q_0^2 + 165q_0^3 + 105q_0^4}{120} + \frac{10j_0^2 - 27j_0 - 110q_0 j_0 - 105q_0^2 j_0}{120} - \frac{15q_0 s_0 + 11s_0 + l_0}{120} - \frac{5kc^2(1 + 8q_0 + 9q_0^2 - 2j_0)}{120a_0^2 H_0^2}.$$

Finally, a similar procedure gives the following approximation for the Hubble-free angular diameter distance $d_A(z) = D_A(z)/(c/H_0)$ to fifth order in z:

$$d_A(z) = \mathcal{D}_A^0 z + \mathcal{D}_A^1 z^2 + \mathcal{D}_A^2 z^3 + \mathcal{D}_A^3 z^4 + \mathcal{D}_A^4 z^5, \tag{A7}$$

having set

$$\mathcal{D}_A^0 = 1, \qquad \mathcal{D}_A^1 = -\frac{1}{2}(3 + q_0), \qquad \mathcal{D}_A^2 = \frac{1}{6}\left[11 + 7q_0 + 3q_0^2 - j_0 - \frac{kc^2}{H_0^2 a_0^2}\right],$$

$$\mathcal{D}_A^3 = -\frac{50 + 46q_0 + 39q_0^2 + 15q_0^3 - 13j_0 - 10q_0 j_0 - s_0}{24} + \frac{2kc^2(5 + 3q_0)}{24H_0^2 a_0^2},$$

$$\mathcal{D}_A^4 = \frac{274 + 326q_0 + 411q_0^2 + 315q_0^3 + 105q_0^4}{120} + \frac{10j_0^2 - 137j_0 - 210q_0 j_0 - 105q_0^2 j_0 - 15q_0 s_0 - 21s_0 - l_0}{120} - \frac{5kc^2(17 + 20q_0 + 9q_0^2 - 2j_0)}{120a_0^2 H_0^2}.$$

Using such expressions (for $k = 0$ since we have assumed a flat universe in the text), it is then straightforward to compute the quantities entering the Fisher matrix so that no numerical integration or differentiation is needed.

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